ON THE WAVELET TRANSFORM OF FRACTIONAL BROWNIAN MOTION

J. Ramanathan[1]
O. Zeitouni[2]

The MITRE Corporation
Massachusetts Institute of Technology

§Introduction.

The wavelet transform of a function $f(t)$ is defined by the formula:

$$\mathcal{W}f(t,a) = \mathcal{W}_a f(t) = \frac{1}{\sqrt{a}} \int f(s) g(\frac{t-s}{a})\, ds$$

where $g(t)$ is a fixed function, $t \in \mathbb{R}$ and $a \in \mathbb{R}^+$. This transform yields a joint time-scale representation the original input function that has been of great recent interest. (See for example [**D1**] [**D2**] and [**HW**]).

In a recent correspondence, Flandrin [**F**] proposed the use of the wavelet transform to analyze the behavior of fractional Brownian motion, a highly nonstationary random process. (For a background on fractional Brownian motion and some of it's applications, see [**M1**] and [**MV**]). The wavelet transform of a stochastic process, $X(t)$, is a random field $\mathcal{W}X(t,a)$ on the upper half plane. The process $t \mapsto \mathcal{W}_a X(t)$ can be thought of as the component of the original process at scale a. A consequence of Flandrin's computation is that fractional Brownian motion is stationary at each fixed scale. In particular, when $X(t)$ is a fractional Brownian motion, the covariance of the process $t \mapsto \mathcal{W}_a X(t)$ is of the form

$$E[\mathcal{W}_a X(t)\ \mathcal{W}_a X(s)] = a^\lambda \rho(\frac{t-s}{a}) \tag{1}$$

where ρ is a positive definite function determined in an explicit manner by the order or the fractional Brownian motion and the defining function $g(t)$ of the wavelet transform. This fact is used by Flandrin to make rigorous sense of the spectral content of fractional Brownian motion.

It is natural to ask whether there are other Gaussian processes whose wavelet transforms have such a natural covariance structure. In addition, are there any Gaussian processes whose wavelet transform is stationary with respect to the affine group (i.e. the statistics of the wavelet transform do not depend on translations and

[1]The work of this author was supported by the MITRE Sponsored Research Program.
[2]The work of this author was partially supported by the Army Research Office under contract DAAL03-86-K-0171.

dilations of the process)? The purpose of this paper is to point out that the answers to both these questions are negative. In particular, fractional Brownian motion is characterized by the property that its wavelet transform has the form shown above in equation (1). A consequence is that there are no nontrivial Gaussian processes on the real line whose wavelet transform produces a random field that is stationary with respect to the affine group. It should be remarked that the results presented assume some fairly nonrestrictive growth conditions on the covariance of the process $X(t)$ and the kernel $g(t)$ used to define the wavelet transform - c.f. remark 2 at the end of the paper.

§**A Characterization of Fractional Brownian Motion.**

Let $X(t)$ be a Gaussian random process defined for $t \in \mathbb{R}$ such that the covariance $R(s,t) = E[X(s)X(t)]$ is continuous and satisfies

$$\begin{aligned} R(0,0) &= 0 \\ |R(s,t)| &\le C(1+|s|^2+|t|^2)^{N/2} \end{aligned} \tag{2}$$

where $N \in Z$ is fixed. In addition, we also impose the following conditions on the analysing function $g(t)$:

a) $\int_{-\infty}^{\infty} |g(t)|(1+|t|^2)^{N/2}\, dt < \infty$,
b) $\int_{-\infty}^{\infty} \frac{|\hat{g}(\xi)|^2}{|\xi|}\, d\xi < \infty$ and
c) $\hat{g}$ is smooth and has a simple zero at the origin.

(The Fourier transform of g is denoted by $\hat{g}$.) The growth conditions for the process $X(t)$ and the analysing wavelet $g(t)$ insure that the wavelet transform produces a random field with finite covariance.

Theorem. *Suppose the wavelet transform $\mathcal{W}X$ has the form*

$$E[\mathcal{W}_a X(t)\ \mathcal{W}_a X(s)] = a^{\lambda}\rho(\frac{t-s}{a})$$

where $\lambda \in \mathbb{R}$ and ρ is a positive semidefinite function on the real line. Then $X(t)$ is fractional Brownian motion of order $H = (\lambda-1)/2$. In particular, $\lambda \in (1,3)$.

We begin with the following lemma.

Lemma. *The Fourier transform $\hat{R}(\xi,\eta)$ of the covariance $R(s,t)$ is a distribution such that*

$$\hat{R}(\xi,\eta) = C\frac{\delta_{\eta+\xi=0}}{|\xi|^{\lambda}}$$

on the set $\mathbb{R}^2\backslash\{\xi = 0 \text{ or } \eta = 0\}$, where C is some constant.

Proof. Apply the two dimensional Fourier transform to equation (1) to obtain

$$a^{\lambda}\delta_{\xi+\eta=0}d\mu_a(\xi) = a\hat{g}(a\xi)\hat{g}(a\eta)\ \hat{R}(d\xi,d\eta),$$

where $d\mu_a$ is the Fourier transform of the positive definite function $\rho(t/a)$. Since this equation holds in the sense of distributions,

$$a^{\lambda-1}\int \phi(\xi,-\xi)\, d\mu_a(\xi) = \left(\hat{R}, \hat{g}(a\xi)\hat{g}(a\eta)\phi(\xi,\eta)\right).$$

for any smooth function $\phi(\xi, \eta)$ with compact support. Fix a point (ξ_0, η_0) with $\xi_0 + \eta_0 = 0$ and $\hat{g}(a\xi_0) \neq 0$ for all $a \in (0, a_0)$. Let U be a neighborhood of (ξ_0, η_0) whose closure does not intersect the zero sets of the functions $\hat{g}(a\xi)\hat{g}(a\eta)$ for all $a \in (0, a_0)$. Such a point ξ_0 and a neighborhood U can be chosen since $\hat{g}$ has a simple zero at the origin. Set $\phi_a(\xi, \eta) = \psi(\xi, \eta)/ (\hat{g}(a\xi)\hat{g}(a\eta))$ where $\psi(\xi, \eta)$ is any smooth function compactly supported in U. Therefore

$$a^{\lambda-1} \int \psi(\xi, -\xi) \frac{d\mu_a(\xi)}{|\hat{g}(a\xi)|^2} = \left(\hat{R}, \psi(\xi, \eta)\right)$$

for all $a \in (0, a_0)$. This implies that for any $a, b \in (0, a_0)$,

$$b^{\lambda-1} \int \psi(\xi, -\xi) \frac{d\mu_b(\xi)}{|\hat{g}(b\xi)|^2} = a^{\lambda-1} \int \psi(\xi, -\xi) \frac{d\mu_a(\xi)}{|\hat{g}(a\xi)|^2}.$$

By applying a limiting argument, the above identity holds for all bounded Borel measurable functions ψ. A consequence of this scaling identity is that $d\mu_b$ is absolutely continuous with respect to Lebesgue measure in a neighborhood of ξ_0. Otherwise there would exist a set A with zero Lebesgue measure and $\mu_a(A) > 0$. By taking $\psi(\xi, -\xi) = \mathbf{1}_A$ in the above identity, one easily achieves a contradiction. Therefore we may write $d\mu_b = m(\xi)\, d\xi$. Set $b = (1 + \alpha)a$. The preceding computation implies that for a fixed b and all sufficiently small α,

$$\int \frac{\psi(\xi, -\xi)}{|\hat{g}(b\xi)|^2} m(\xi) d\xi = (1 + \alpha)^{-\lambda} \int \frac{\psi(\xi, -\xi)}{|\hat{g}(b\xi/(1 + \alpha))|^2} m(\xi/(1 + \alpha)) d\xi.$$

This implies that

$$(1 + \alpha)^{-\lambda} \frac{m(\xi/(1 + \alpha))}{|\hat{g}(b\xi/(1 + \alpha))|^2}$$

is a constant independent of α. Therefore, we have that $m(\xi) = C|\hat{g}(b\xi)|^2/|\xi|^\lambda$ in a neighborhood of ξ_0. The lemma follows. ✓

It follows that on $\mathbb{R}^2 \backslash \{\xi = 0 \text{ or } \eta = 0\}$

$$\hat{R} = \frac{C}{|\xi|^\lambda} \delta_{\xi+\eta=0}.$$

Let r be a tempered distribution of one variable such that $r(t) = C|t|^{\lambda-1}$ on $\mathcal{R}\backslash\{0\}$ and $\hat{r}(\xi) = \frac{C}{|\xi|^\lambda}$ on $\mathcal{R}\backslash\{0\}$. The distribution $R(t, s) - r(t - s)$ has its Fourier transform supported on the set $\{\xi = 0 \text{ or } \eta = 0\}$. Moreover, an application of the two-variable Fourier transform to equation (1) yields

$$\hat{g}(a\xi)\hat{g}(a\eta) \left(\hat{R}(\xi, \eta) - \hat{r}(\xi)\delta_{\xi+\eta=0}\right) \equiv 0. \tag{3}$$

Since $\hat{g}$ has a simple zero at the origin, it is straight-forward to check that equation (3) implies that

$$\xi\eta \left(\hat{R}(\xi, \eta) - \hat{r}(\xi)\delta_{\xi+\eta=0}\right) \equiv 0. \tag{4}$$

This implies the differential equation

$$D_{ts}\left(R(t,s) - r(t-s)\right) \equiv 0.$$

As a consequence $R(t,s) - r(t-s)$ is a distribution of the form $r_1(t) + r_2(s)$. On the other hand the symmetry of the function $R(t,s)$ forces the distributions r_1 and r_2 to be the same up to a constant. However, by redefining r_1 and r_2, we may actually assume that they are the same. Write $r_0 = r_1 = r_2$. We now have that

$$R(t,s) = r(t-s) + r_0(t) + r_0(s). \tag{5}$$

Since $R(t,s) - r(t-s)$ is a continuous function away from the diagonal, r_0 is a continuous function. Substituting $t = s = 0$ implies that $r_0(0) = 0$. Now, letting $s = 0$ yields $r_0(t) = -r(t)$. Therefore, we have

$$R(t,s) = C(|t-s|^{\lambda-1} - |t|^{\lambda-1} - |s|^{\lambda-1}). \tag{6}$$

This function cannot satisfy the growth conditions of equation 2 if $\lambda < 1$. On the other hand, $R(0,0) \neq 0$ if $\lambda = 1$. It is also easy to check that $R(s,t)$ cannot be a positive definite function if $\lambda > 3$; the matrix

$$\begin{pmatrix} R(1,1) & R(2,1) \\ R(1,2) & R(2,2) \end{pmatrix}$$

is not positive definite for $\lambda \geq 3$. We have as a consequence that $\lambda \in (1,3)$. It is well known that Gaussian processes with covariance given by equation 6 with $\lambda \in (1,3)$, are fractional Brownian motion.

Remarks.

(1) Note that since we have shown that $\lambda = 1$ isn't possible under our assumptions, it follows that there does not exist a process satisfying (2) and a wavelet satisfying a)-c) such that the wavelet transform of the process using the wavelet is stationary under translations and dilations.

(2) Condition c) on the wavelet function is somewhat annoying: indeed, we note that [**D1**] has presented examples of wavelets where several moments vanish, which translates to a multiple zero at the origin of the Fourier transform $\hat{g}$. To illustrate what the difficulty is, let $\hat{g}$ possess a zero of order two at the origin. Using the analysis we have applied above, one can only conclude that $R(t,s) = C(|t-s|^{\lambda-1} - |t|^{\lambda-1} - |s|^{\lambda-1} + tf(s) + sf(t))$ for some function $f(t)$. To get the conclusion of the theorem, one would have to show that such $R(t,s)$ is positive definite only if $f(t) = 0$. We have not been able to prove that.

References

[**D1**] I[redacted]abechies, *Orthonormal Bases of Compactly Supported Wavelets*, Comm. Pure and Appl. Math. **41** (1988), 909 –996.

[D2] I[redacted]aubechies, *The Wavelet Transform, Time-Frequency Localization and Signal Analysis*, IEEE Transactions on Information Theory (to appear).

[F] P[redacted]Flandrin, *On the Spectrum of Fractional Brownian Motions*, IEEE Transactions on Information Theory **IT-35 no. 1** (1989), 197 –199.

[HW] H[redacted]il and D. Walnut, *Continuous and Discrete Wavelet Transforms*, SIAM Review **31** (1989), 628 –666.

[M1] B[redacted] Mandelbrot, *Some Noises with $1/f$ Spectrum, a Bridge Between Direct Current and White Noise*, IEEE Transactions on Information Theory **IT-13 no. 2** (1967), 289–298.

[MV] B[redacted]Mandelbrot and J.W. Van Ness, *Fractional Brownian Motions, Fractional Noises and Applocations*, SIAM Review **10 no. 4** (1968), 422–437.