The exit problem for a class of density dependent branching systems

Fima Klebaner*
Department of Statistics
University of Melbourne, Parkville
Victoria, AUSTRALIA 3052

Ofer Zeitouni†
Department of Electrical Engineering
Technion- Israel Institute of Technology
Haifa 32000, ISRAEL

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Abstract

The influence of noise on a class of discrete time systems arising from models of density dependent branching processes is investigated. By considering iterates of the basic map, the time to escape from a stable orbit is investigated as a (nonstandard) problem of exit from a domain.

1 Introduction

This paper deals with small random perturbations of certain one dimensional maps of the form

$$x_{n+1} = f(x_n)\,. \tag{1.1}$$

Here, $f : [0,1] \to [0,1]$ is such that (1.1) possesses a single stable periodic orbit. The much studied logistic model with $f(x) = rx(1-x)$, $3 \le r \le r_{\rm cr} < 4$ serves as a representative example, where $r_{\rm cr}$ denotes the onset of chaos. It is used to describe various real life populations such as insect populations and predator-prey situations, see [10, 11, 12]. Models of population dynamics of the form (1.1) are used to model situations where due to lack of resources, individual reproduction declines as population density increases. Populations, however, grow by a basic branching mechanisms, but classical branching models are not suitable for modeling such a situation as they do not allow dependence of offspring distribution on the population size. In population dependent branching models such dependence is allowed, consequently they can serve as stochastic analogues of the models of population dynamics.

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Our main motivation for introducing random noise into the system (1.1) is a study of density dependent branching processes which are defined inductively as

$$z_{n+1} = \frac{1}{K} \sum_{j=1}^{Kz_n} \xi_j(z_n)\,, \tag{1.2}$$

where K is the threshold, Kz_n denotes the number of individuals in the nth generation, and ξ_j, $j = 1, \cdots, Kz_n$, denote the number of offsprings of individuals in the nth generation. The law of $\xi_j(z_n)$ is assumed to depend only on the population density z_n. Such models can be put into the form

$$x_{n+1}^K = f(x_n^K) + \nu_{n+1}^K\,, \tag{1.3}$$

where K is a large parameter and given x_n^K, the noise ν_{n+1}^K is independent of the past (see section 3). It is easily seen that $x_n^K = z_n$ satisfy (1.3) with $f(x) = xm(x)$, $m(x) = E\xi(x)$.

It follows from general Markov chains theory that integer valued models (1.2) in which the individual reproduction declines to 0 as the population density increases, and the only absorbing state is 0, eventually become extinct. In [7] a LLN and CLT, when $K \to \infty$, were shown for density dependent branching processes, and it was also observed from simulations that for large values of K the perturbed system tracks the corresponding deterministic system for long periods of time prior to extinction. Estimates for the time to extinction, as the parameter K becomes large, may be obtained from the version of the Freidlin-Wentzell estimates for discrete time systems due to Kifer [5], and we comment on those below. Here, we are interested in a somewhat different phenomenon. We show that if the deterministic system has a stable periodic orbit, then the perturbed system will follow approximately a limit cycle for an exponentially in K long time before switching to another cycle (which actually represent a "phase", or initial condition, change for the cycle).

The key to our analysis is the following observation. Note that if f possesses a stable limit cycle consisting of k points, then its kth iterate $f^{(k)}$ possesses k stable fixed points. Thus, by looking at $f^{(k)}$, the problem of tracking a stable trajectory transforms into a problem of exit time from a basin of attraction of a stable fixed point of $f^{(k)}$. On the other hand, investigating $f^{(k)}$ complicates the analysis of the effects of noise, due to the action of the iterations on the noise, and due to the fact that one has now to analyse exit from stable basins of attraction. In particular, the assumptions of [5] and [8] do not directly apply to this analysis, and some technical modification of their results is required.

The organization of the paper is as follows: in section 2, definitions and basic large deviations estimates are presented. We also comment on the relation of these estimates with [5] and [8]. In section 3, we check that the assumptions of section 2 are satisfied for a class of noise perturbed periodic systems which includes independent additive noise or noise created by density dependent branching processes. We remark that the latter model does not, in general, satisfy the conditions in either [5] or [8].

2 Notations and basic estimates

Let x_n^K, $n = 0, 1, 2 \cdots$ be a time-homogeneous Markov chain, indexed by a parameter K, with state space $\mathcal{G}$, a convex closed subset of $\mathbb{R}$, and with Markov transition kernel $P_x^K(\cdot)$. Let $E_x^K(\cdot)$ denote the conditional expectation $E(\cdot | x_o^K = x)$. Let $f^K(x) = E_x^K(x_1^K)$. We assume throughout that for each $x \in \mathcal{G}$, $f^K(x) \underset{K\to\infty}{\longrightarrow} f(x)$, and both $f^K(\cdot)$ and $f(\cdot)$ are Lipschitz continuous and bounded.

Consider next the dynamical system $z_{n+1} = f(z_n)$. We say that z^* is a *stable point* of the dynamical system generated by $f(\cdot)$ if $z^* = f(z^*)$ and there exists a neighborhood B of z^* such that if $z_o \in B$, then $f(z_o) \in B$ and $z_n \to z^*$. The largest such neighborhood is called the *fundamental basin of attraction* of z^* and is denoted $\overline{B}_{z^*}$. Note that one may well have $z_o \notin \overline{B}_{z^*}$ and still $z_n \to z^*$. The *basin of attraction* of z^*, which consists of all points x such that $f^{(n)}(x) \to z^*$, is denoted B_{z^*}. Since we need to consider functions f whose associated systems possess more than one stable point, we denote such points by $z_1^*, \cdots z_k^*$, and their basins of attraction by $B_{z_1^*}, \cdots, B_{z_k^*}$.

Our interest lies in the problem of exit from a basin of attraction. Specifically, suppose $x_o^K \in B_{z_i^*}$. If the system were propagated according to the dynamical system associated with f, then the chain x_n^K would stay in $B_{z_i^*}$. But there are random perturbations, and thus we define

$$\tau_i^{x,K} = \inf\{n > 0 : x_n^K \notin B_{z_i^*}, x_o^K = x \in B_{z_i^*}\}.$$

We will show below that, under suitable conditions on the chain x_n^K, one has, for any $\delta > 0$,

$$P(\exp(K(\bar{\tau}_i - \delta)) \le \tau_i^{x,K} \le \exp(K(\bar{\tau}_i + \delta))) \underset{K\to\infty}{\longrightarrow} 1, \tag{2.1}$$

where $\overline{\tau}_i$, which we characterize, does not depend on the particular $x \in B_{z_i^*}$ used. A similar limit also holds for the exit from a union of basins of attractions, but we will not deal with that here.

Fix w.l.o.g. some i, and denote by B^* the basin of attraction of $z^* \equiv z_i^*$. Define now

$$g^K(x,t) \triangleq \frac{1}{K} \log E_x(e^{tKx_1^K}); g(x,t) \triangleq \lim_{K\to\infty} \frac{1}{K} \log E_x(e^{tKx_1^K}) \tag{2.2}$$

$$I(x,y) \triangleq \sup_{t\in R}[ty - g(x,t)] \tag{2.3}$$

where in (2.2) we assumed that both expressions exist and are finite. Following [5, 8], let $[u]_T$ stand for a specific path $(u_o, u_1, \cdots, u_T), T < \infty$. Define next

$$S(T, [u]_T) \triangleq \sum_{i=0}^{T-1} I(u_i, u_{i+1}) \tag{2.4}$$

$$V(x,y) \triangleq \inf\{S(T,[u]_T) :\ u_o = x,\ u_T = y,\ T < \infty\} \tag{2.5}$$

$$V(x) \triangleq \inf_{y\notin B^*} V(x,y) \tag{2.6}$$

A sequence of probability measures P_x^K indexed by x will be said to satisfy the large deviations principle (LDP) with rate function (a nonnegative, lower semicontinuous function in y) $I(x,y)$ if, for every measurable set A,

$$-\inf_{y\in A^o} I(x,y) \le \liminf_{K\to\infty} \frac{1}{K}\log P_x^K(A) \le \limsup_{K\to\infty} \frac{1}{K}\log P_x^K(A) \le -\inf_{y\in \overline{A}} I(x,y)\,,$$

where A^o ($\overline{A}$) denote the interior (closure) of A. $I(x,y)$ is called a *good* rate function if its level sets (in y) are compact.

Definition 2.1 *$\{P_x^K\}$ satisfies uniformly in G the LDP with rate function $I(x,y)$ if, for every measurable sets A and compact $B\subset G$,*

$$-\sup_{x\in B}\inf_{y\in A^o} I(x,y) \le \liminf_{K\to\infty}\inf_{x\in B} \frac{1}{K}\log P_x(A) \le \limsup_{K\to\infty}\sup_{x\in B} \frac{1}{K}\log P_x^K(A) \le -\inf_{x\in B}\inf_{y\in \overline{A}} I(x,y)\,.$$

The following assumptions are used in the sequel. As usual, A^δ denotes the δ-blow-up of a set A, i.e., $A^\delta=\{x:\ |x-y|<\delta \text{ for some } y\in A\}$, and $A^{-\delta}=\{x\in A:\ |x-y|\ge\delta\ \forall y\in A^c\}$. G denotes a compact, closed set which is the closure of its interior. Finally, $\delta_o>0$ is some constant to be chosen below.

(A1) $V\equiv V(z^*)<\infty$.

(A2) $I(\cdot,y)$ is lower semicontinuous in G^{δ_o}.

(A3) For each $\delta>0$ and $\epsilon>0$ small enough, there exists a function $C(\epsilon,T,\delta)$, such that any path $[u]_T$ with $u_0\in(G)^{-\delta}$, $u_i\in G$ and $|u_i-z^*|>\epsilon\ \forall i$, satisfies $S(T,[u]_T)\ge C(\epsilon,T,\delta)$, and $C(\epsilon,T,\delta)\xrightarrow[T\to\infty]{}\infty$.

(A4) For all $\delta>0$ small enough, and for all $x\in B^*\backslash(B^*)^{-\delta}$, there exists a $y\in[(B^*)^\delta]^c$ with $I(x,y)\le g_1(\delta)$, and $g_1(\delta)\xrightarrow[\delta\to0]{}0$ independently of x. Further, $P_x(|x_1^K-y|<\delta)\ge e^{-K(g_1(\delta)+o(1))}$, uniformly in such x.

(A5) Uniformly in $x\in G^{\delta_o}$, $P_x(x_1^K\in\cdot)$ satisfies the LDP with the good rate function $I(x,y)$.

Our main result is the following:

Theorem 2.1 *Assume* **(A1)–(A5)** *holds for every $\eta>0$ with $G=(B^*)^{-\eta}$ and some $\delta_o<\eta$. Then (2.1) holds, with $\overline{\tau}_i=V$.*

Remarks 1) It is useful at this point to compare our results with those of [5] and [8]. Motivated by the analysis of small noise in the case of diffusions due to Freidlin and Wentzell, Kifer [5] essentially works with the assumption that, for open sets U, $\frac{1}{K}\log P_x^K(U)\xrightarrow[K\to\infty]{}-\inf_{y\in U} I(x,y)$, uniformly in x. This assumption is actually stronger than **(A5)**, and in particular does not seem to be satisfied (in general) for the branching mechanism of (1.2). If $I(x, y)$ is assumed to be jointly continuous in x, y, then our assumptions essentially reduce to Kifer's. However, such an assumption is somewhat restrictive and, in particular, does not allow for dealing with the case of $\xi_i(x)$ in (1.2) of bounded support. Moreover, when dealing with the problem of exit from a domain, Kifer also uses the assumption that the orbits of the deterministic dynamical system enter the domain (see (4.3) in [5]). This precludes looking at the exit from basins of attraction as we do here.

A more direct approach to the exit problem is used in [8], relying on an analysis of the moment generating function. Our proof borrows from their techniques, however their assumptions fail to hold for the iterates of density-dependent branching processes that are of concern to us. In particular, one needs to get away from their contraction condition (1.15).

2) As in the case of the standard Freidlin-Wentzell theory, Theorem 2.1 may be extended to the analysis of the exit from a domain which may include several basins of attraction. Under somewhat different assumptions, such an extension is presented in [5].

The proof of Theorem 2.1 follows directly from the sequence of lemmas below. Although many steps in the proof are similar to those in the above references, due to its technical nature we present it in some details for completeness.

For the sake of shorter notations, we let $\tau^x \equiv \tau_i^{x,K}$. In both parts (a) and (b) of Lemma 2.1 below, $o(1)$ means a function $g(K,T)$ such that $g(K,T)/K \to_{K\to\infty} 0$.

Lemma 2.1 (Large Deviations Bounds) *Assume* **(A1), (A2), (A5)**, *and fix $T < \infty$ independently of K.*
(a) For every path $[u]_T$ with $u_i \in G^{\delta_o}$, where δ_o is as in **(A2)**, *and every $\delta > 0$,*

$$\inf_{x\in G^{\delta_o}} P_x\big(\sup_{1\le n\le T} |x_n^K - u_n| < \delta\big) \ge e^{-K(S(T,[u]_T)+o(1))}$$

(b) Let $B^{\alpha,T} = \{[u]_T : S(T,[u]_T) \ge \alpha, u_i \in G^{\delta_o} \text{ for all } i < T\}$. Then, for any closed set $A \subset B^{\alpha,T}$,

$$\sup_{x\in G^{\delta_o}} P_x([x^K]_T \in A) \le e^{-K(\alpha+o(1))},$$

where $[x^K]_T = (x_o^K, x_1^K, \ldots, x_T^K)$.

The proof of Lemma 2.1 paraphrases the proof of [6, Theorem 5.2 and Corollary 5.2] and is therefore omitted.

Lemma 2.2 *Assume* **(A1)–(A5)** *for every $\eta > 0$ with $G = (B^*)^{-\eta}$. Then, for all $\delta > 0$ and all $x \in B^*$ there exists a constant α such that $P_x(\tau^x \ge e^{K(V+\delta)}) \le e^{-K\alpha}$.*

Lemma 2.3 *Assume* **(A1)–(A5)** *for every $\eta > 0$ with $G = (B^*)^{-\eta}$. Then, for all $\delta > 0$ and all $x \in B^*$ there exists a constant α such that $P_x(\tau^x \le e^{K(V-\delta)}) \le e^{-K\alpha}$.*

Proof of Lemma 2.2
Define

$$V^\eta = \inf_{y\notin(B^*)^{-\eta}} V(z^*, y)\,.$$

Then $V^\eta \underset{\eta\to 0}{\longrightarrow} V$ by **(A4)**. Choose now η small enough such that $|V^\eta - V| < \frac{\delta}{4}$, $g_1(\eta) < \delta/8$, and

$$g_\eta \stackrel{\triangle}{=} \sup_{x,x',y,y' \in G^{\delta_o}, |x-x'|<\eta, |y-y'|<\eta} |I(x,y) - I(x',y')| < \delta/8.$$

Let T^η be the maximal T such that $C(\eta, T-1, \eta) < V+1$, (see **(A3)**), and let $[u]_T$ be a path with $u_0 = z^*$, $u_i \in (B^*)^{-\eta}$ for all $i < T$, $u_T \in B^* \setminus (B^*)^{-\eta}$ and $S(T, [u]_T) \leq V + \frac{\delta}{4}$. (Such a path exists for small enough η by **(A1)**). We now show that

$$\inf_{x \in B^*} P_x \text{ (exit up to time } \overline{T}) \geq e^{-K\left(V + \frac{3\delta}{4}\right)}, \tag{2.7}$$

where $\overline{T} = T^\eta + T + 1$. By the Markov structure of the chain, it then follows that

$$\begin{aligned} &P_x \text{ (no exit up to time } e^{K(V+\delta)}) \\ &\leq \left[1 - \inf_{x\in B^*} P_x \text{ (exit up to time } \overline{T})\right]^{e^{K(V+\delta)}/\overline{T}} \\ &\leq \left[1 - e^{-K\left(V+\frac{3\delta}{4}\right)}\right]^{e^{K(V+\delta)}/\overline{T}} < e^{-K\alpha} \end{aligned}$$

for some $\alpha > 0$.

We thus turn to the proof of the basic estimate (2.7). First, note that for $x \in B^* \backslash (B^*)^{-\eta}$, by **(A4)**,

$$P_x \text{ (exit in one step)} \geq e^{-Kg_1(\eta)} \geq e^{-K\,\delta/8} \tag{2.8}$$

whereas, for $x \notin B^* \backslash (B^*)^{-\eta}$ and K large enough, by **(A3)** and part (b) of Lemma 2.1,

$$\sup_{x \notin B^* \backslash (B^*)^{-\eta}} P_x(x_n^K \text{ satisfies } x_n^K \in (B^*)^{-\eta}, |x_n^K - z^*| \geq \eta \text{ for all } n \geq 1 \text{ up to } T^\eta) \leq e^{-K(V+1-o(1))}.$$

Hence, by the union of events bound,

$$\begin{aligned} \inf_{x\in B^*} P_x \text{ (exit up to } \overline{T}) \;\geq\;& \inf_{\substack{x \\ |x-z^*|<2\eta}} P_x\left(|x_{n+1} - u_n| < \frac{\eta}{2},\, T \geq n \geq 0\right) \cdot \inf_{x \in B^* \backslash (B^*)^{-\eta}} P_x(x_1^K \in (B^*)^c) \\ &- \sup_{x \notin B^* \backslash (B^*)^{-\eta}} P_x(x_n^K \text{ satisfies } x_n^K \in (B^*)^{-\eta}, |x_n^K - z^*| > \eta \text{ for all } 1 \leq n \leq T^\eta\;) \\ \geq\;& e^{-K\left(V + g_1(\eta) + g_\eta + \frac{\delta}{4} + o(1)\right)} - e^{-K(V+1-o(1))} \\ \geq\;& e^{-K\left(V + \frac{3\delta}{4} + o(1)\right)} \end{aligned} \tag{2.9}$$

where we used part (a) of Lemma 2.1 and (2.8) in the second inequality. □

Proof of Lemma 2.3

Note first that, again by an application of **(A3)**,

$$P_x(x_n^K \text{ hits } (B^*)^c \text{ before hitting a given neighborhood of } z^*) \underset{K\to\infty}{\longrightarrow} 0$$

exponentially fast. Hence, one may assume that $|x_o^K - z^*| < \eta$ for any given fixed η. Choose η as in the proof of Lemma 2.2. Let

$$\tau^{x,\eta} \triangleq \inf\{t > 0 : x_t^K \notin (B^*)^{-\eta}\}.$$

Note that $\tau^{x,\eta} \leq \tau^x$, and define the stopping times

$$\begin{aligned}
\tau_0 &= 0 \\
\tau_k &= \inf\{t \geq \tau_{k-1} + 1 : \ x_n^K \notin (B^*)^{-\eta} \text{ or } |x_n - z^*| \leq \eta\}.
\end{aligned}$$

We show below that

$$\sup_{x:\,|x-z^*|\leq\eta} P_x(x_{\tau_1}^K \notin (B^*)^{-\eta}) < e^{-K\left(V-\frac{\delta}{4}+o(1)\right)}. \tag{2.10}$$

Hence, again by the Markov property, and taking $|x_o^K - z^*| < \eta$,

$$\begin{aligned}
P_x(\tau^x \leq e^{K(V-\delta)}) &\leq P_x(\tau^{x,\eta} \leq e^{K(V-\delta)}) \\
&\leq P_x(x_{\tau_1} \notin (B^*)^{-\eta}) + E_x(P_{x_{\tau_1}^K}(\tau^x < e^{K(V-\delta)} - \tau_1^K)1_{x_{\tau_1}^K \in (B^*)^{-\eta}}) \\
&\leq P_x(x_{\tau_1} \notin (B^*)^{-\eta}) + \sup_{x:\,|x-z^*|\leq\eta} P_x(\tau^x \leq e^{K(V-\delta)} - 1) \\
&\leq \cdots \\
&\leq e^{K(V-\delta)} \sup_{x:\,|x-z^*|\leq\eta} P_x(x_{\tau_1}^K \notin (B^*)^{-\eta}) \\
&\leq e^{K(V-\delta)} e^{-K\left(V-\frac{\delta}{4}\right)} \leq e^{-K\delta/4}.
\end{aligned}$$

We return now to the proof of (2.10). Let T^η be as in the proof of Lemma 2.2. Then, since to exit $(B^*)^{-\eta}$ before returning to a neighborhood of z^*, either x_n^K remains away from this neighborhood up to τ_1 or x_n^K hits $((B^*)^{-\eta})^c$ before T^η,

$$\begin{aligned}
\sup_{x:|x-z^*|\leq 2\eta} P_x(x_{\tau_1}^K \notin (B^*)^{-\eta}) &\leq \sup_{x:|x-z^*|\leq 2\eta} P_x(|x_n^K - z^*| > \eta, x_n^K \in (B^*)^{-\eta} \quad \forall T^\eta \geq n \geq 1) \\
&\quad + P_x(x_n^K \notin (B^*)^{-\eta} \text{ for some } T^\eta \geq n \geq 1) \\
\sup_{x:|x-z^*|\leq 2\eta} &\leq e^{-K\left(V+\frac{\delta}{4}+o(1)\right)} + e^{-K\left(V-\frac{\delta}{4}+o(1)\right)}
\end{aligned}$$

where we used the definition of T^η in the first inequality and the upper bound of Lemma 2.1, coupled with **(A3)**, in the second. $\square$

Remark: As is obvious from the proof, the exit from B_{z^*} occurs, with probability approaching 1, at a neighborhood of the endpoints of the minimizing paths in (2.5) and (2.6). Such paths exist (maybe not uniquely) due to the lower semicontinuity of $I(x,y)$ on the minimizing paths, which is ensured by **(A2)**.

We conclude this section with the following lemmas, which are borrowed, respectively, from [8] and [3]. Their main usefulness lies in checking the uniformity and continuity assumptions **(A2)** and **(A5)**.

Lemma 2.4 *Assume that, for some compact set $\mathcal{K}$,*

$$\limsup_{|t|\to\infty} \sup_{x,z\in\mathcal{K}} (tz - g(x,t)) = -\infty\,. \tag{2.11}$$

Further, assume that for all $s,t \in [-M,M]$ and $x,x' \in \mathcal{K}$,

$$|g(x,t) - g(x',s)| \leq c_{M,\mathcal{K}}(|x-x'| + |s-t|)\,, \tag{2.12}$$

where $c_{M,\mathcal{K}}$ depends only on $M,\mathcal{K}$. Then, for each $x,x',y,y' \in \mathcal{K}$,

$$|I(x,y) - I(x',y')| \leq c(|y-y'| + |x-x'|)\,,$$

where c depends only on $M,\mathcal{K}$.

Proof: Note first that since, by Hölder's inequality, $g(x,t)$ is continuous in t in the interior of its domain, (2.11) implies that for every $z \in \mathcal{K}$, there exists a $t_0(x,z)$ such that $I(x,z) = (t_0(x,z)z - g(x,t_0(x,z)))$, and, furthermore, $|t_0(x,z)| < k$ for some k independent of $x,z \in \mathcal{K}$. Therefore,

$$I(x,y) - I(x',y') \leq t_0(x,y)y - g(x,t_0(x,y)) - t_0(x,y)y' + g(x',t_0(x,y)) \leq k|y-y'| + c_{k,\mathcal{K}}|x-x'|\,.$$

Since the same inequality holds also when reversing the role of (x,y) and (x',y'), the assertion follows. $\square$

Lemma 2.5 *Let $\mathcal{K}$ be a compact set. Assume that the convergence in (2.2) is uniform in $x \in \mathcal{K}$. Further assume that $g(x,t) < \infty$ for each $t \in \mathbb{R}$, that it is differentiable in t and continuous in $x \in \mathcal{K}$. Then $P_x(x_1^K \in \cdot)$ satisfies the LDP uniformly in $x \in \mathcal{K}$.*

Proof: Due to our assumptions on $g(x,t)$, the LDP holds for each fixed x by an application of the Gärtner-Ellis theorem (Theorem 2.3.6 in [3]). By the same argument as in the proof of Corollary 5.6.15 in [3], in order to prove the uniformity of Definition 2.1, it is enough to show that, for each $x \in \mathcal{K}$ and measurable set A,

$$-\inf_{y\in A^o} I(x,y) \leq \liminf_{K\to\infty, z\to x} \frac{1}{K}\log P_z(x_1^K \in A) \leq \limsup_{K\to\infty, z\to x} \frac{1}{K}\log P_z(x_1^K \in A) \leq -\inf_{y\in\bar{A}} I(x,y)\,.$$

To this end, enough to show that for any sequence $x_K \to x$, $P_{x_K}(x_1^K \in \cdot)$ satisfies the LDP with the rate function $I(x,y)$. But, by the continuity in x of $g(x,t)$ and the uniform convergence,

$$\lim_{K\to\infty} \frac{1}{K}\log E_{x_K}(e^{tx_1^K}) = g(x,t)\,,$$

and the conclusion follows by another application of the Gärtner-Ellis theorem. $\square$

Lemma 2.6 *Let G be an open set. Assume that for every sequence $\{x_K\}$ such that $x_K \to x \in G$ one may construct probability measures P^K on $\mathcal{G}\times\mathcal{G}$ such that $P^K(X_1 \in \cdot) = P_x(x_1^K \in \cdot)$ and $P^K(X_2 \in \cdot) = P_{x_K}(x_1^K \in \cdot)$. Further, assume that for all $\delta > 0$,*

$$\lim_{K\to\infty} \frac{1}{K}\log P^K(|X_1 - X_2| > \delta) = -\infty\,.$$

Finally, assume that $P_x(x_1^K \in \cdot)$ satisfy the LDP with rate function $I(x,y)$. Then, it satisfies the LDP uniformly in $x \in G$ in the sense of Definition 2.1.

Proof: The proof paraphrases the proofs of Theorem 5.6.12 and Corollary 5.6.15 in [3]. $\square$

3 Applications

In this section, we assume that the dynamical system (1.1) possesses a unique stable periodic orbit of period k. Our prototype example is the logistic equation $f(x) = f_r(x) = rx(1-x)1_{x\in[0,1]}$. For this equation, depending on the value of r, the stable orbits of the system are either isolated points or stable orbits of period 2^i, integer i-s, up to $r = r_{\rm cr}$, where the nature of the stable attractor changes (see [11, 13]). To each value of r in the range $r < r_{\rm cr}$, one may attach an integer $k = k_r = 2^i$ which is the period of the stable orbit. In particular, it follows that the k_r-th iterate of f_r, denoted $f_r^{(k_r)}$, possesses k_r stable fixed points. Thus, the question of transition from one orbit to the other may be phrased in terms of the exit from the stable points of $f_r^{(k_r)}$.

We describe below two types of random perturbations of the dynamical system governed by f. In the first, which is the simpler, independent noise is added at each step. In the second, whose motivation comes from population dynamics and is described in the introduction, the noise comes from the fact that the actual value of the next iterate is a function of the current value via a random, population dependent, branching mechanism.

Throughout this section, f is a smooth function on $[0,1]$, with $f(0) = f(1) = 0$, $f(x) > 0$ for $x \in (0,1)$, and, whenever it is necessary to extend $f(z)$ for $z \notin [0,1]$, we take $f(z) = 0$. Finally, we assume that the fundamental basin of attraction of each stable point z^* of the map $f^{(k)}$ is separated from 0 by a positive quantity.

3.1 Additive noise

We consider the system

$$x_{n+1} = f(x_n) + \theta_{n+1}^K \tag{3.1}$$

where $\{\theta_n^K\}$ is, for each K, a sequence of zero mean, i.i.d. random variables such that $\Lambda(\lambda) \triangleq \lim_{K\to\infty} K^{-1} \log E e^{K\lambda\theta_n^K}$ is finite for each λ and non-trivial. A particularly important case is when θ_n^K is a sequence of i.i.d. zero mean, variance $1/K$, Gaussian random variables, in which case $\Lambda(\lambda) = \lambda^2/2$.

Define $x_n^K \triangleq x_{kn}$. We now check that, under suitable conditions on $\Lambda(\lambda)$, assumptions **(A1)**–**(A5)** hold for the chain x_n^K, and hence the results of section 2 apply to estimate the time of switch between stable points of the chain x_n^K, which is closely related (and equals asymptotically) the time between switches of orbits for the chain x_n.

Note first that if $g(x,t)$ exists then

$$g(x,t) = \lim_{K\to\infty} \frac{1}{K} \log E_x(e^{Ktx_1^K}) = \Lambda(t) + \lim_{K\to\infty} \frac{1}{K} \log E_x(e^{Ktf(x_{k-1})}) \tag{3.2}$$

from where it follows, by the boundedness of f, that $g(x,t)$, if existing, takes values in $(-\infty,\infty)$ as soon as the same applies for $\Lambda(\cdot)$. The following assumptions are sufficient to ensure the convergence of $g(x,t)$.

(B1) $\Lambda(t)<\infty$ for all $t \in \mathbb{R}$.

(B2) For all $t \geq 0$,

$$\lim_{K\to\infty} \frac{1}{K} \log E(e^{tK\min(|\theta_1^K|,2)^2}) = 0\,. \tag{3.3}$$

(B3) $\liminf_{|t|\to\infty} \Lambda(t)/|t| = \infty$.

Note that **(B1)** implies that $\Lambda(t)$ is infinitely differentiable. Note also that **(B2)** is satisfied if θ_1^K is Gaussian, zero mean, variance $1/2K$. We now claim:

Lemma 3.1 *Assume* **(B1)**–**(B2)**. *Then*

$$g(x,t) = \Lambda(t) + tf^{(k)}(x) + \sum_{i=1}^{k-1} \Lambda(t(f^{(i)})'(f^{(k-i)}(x)))\,. \tag{3.4}$$

Moreover, $g(x,t)$ is differentiable in t and, for every bounded M, uniformly Lipschitz continuous in $[0,1]\times[-M,M]$.

Proof: Due to the fact that $f\equiv 0$ outside $[0,1]$, when expanding the rightmost term in (3.2), the random variables θ_i^K, $i=1,\ldots,k-1$ may all be truncated such that $|\theta_i^K|\leq 2$. Thus,

$$\begin{aligned} E_x(e^{Ktf(x_{k-1})}) &= E_x(e^{Kt(f(f(x_{k-2}))+\theta_{k-1}^K)}) \\ &= E_x(e^{Kt(f^{(2)}(x_{k-2})+f'(f(x_{k-2}))\theta_{k-1}^K+c(x_{k-1},\theta_{k-1}^K)(\overline{\theta_{k-1}^K})^2)})\,, \end{aligned}$$

where $c(\cdot,\cdot)$ is bounded, and $\overline{\theta_{k-1}^K}$ is the truncation of θ_{k-1}^K as described above. It follows from (3.3) and Hölder's inequality that

$$\lim_{K\to\infty}\frac{1}{K}\log E_x(e^{Ktf(x_{k-1})}) = \lim_{K\to\infty}\frac{1}{K}\log E_x(e^{Kt(f^{(2)}(x_{k-2})+f'(f(x_{k-2}))\theta_{k-1}^K)})\,. \tag{3.5}$$

Iterating this equality and using (3.2), one obtains both (3.4) and the differentiability and Lipschitz continuity of $g(x,t)$ asserted in the lemma. □

We are now ready to claim:

Theorem 3.1 *Assume that Λ satisfies* **(B1)**–**(B3)**. *Then the system (3.1) satisfies* **(A1)**–**(A5)** *(with $G=(B^*)^{-\delta}$, any $\delta>0$ and $\delta_o<\delta$), and Theorem 2.1 applies, with $g(x,t)$ given by Lemma 3.1.*

Proof: By Lemma 3.1, $g(x,t)$ is bounded and differentiable in t for each fixed x. It follows from the Gärtner-Ellis theorem (see [3, Theorem 2.3.6]) that for each fixed x, $P_x(x_1^K \in \cdot)$ satisfies the LDP with rate function $I(x,y)$ given by (2.3).

To see the uniformity, note that x_1^K may be constructed in a deterministic fashion from x and the random vector θ_i^K, $i = 1, 2, \ldots k$. Let X_1, X_2 be thus constructed on the same probability space as in Lemma 2.6. Note that, due to the uniform continuity of $f(\cdot)$, $|X_1 - X_2| < g(|x - x_K|)$, where $g(\cdot)$ is a deterministic function satisfying $g(x) \to_{|x|\to 0} 0$. It thus follows from Lemma 2.6 that the LDP for x_1^K actually holds uniformly as well, and **(A5)** holds.

Next, assumption **(B3)** implies the goodness of the rate function $I(x,y)$. Coupled with Lemma 2.4, it also yields the continuity of $I(x,y)$ in x. Thus, **(A2)** and **(A1)** follow.

We next turn to proving **(A4)**. Since $f^{(k)}$ is continuous, one may find a function $g_2(\delta)$ such that $d(f^{(k)}(x), B^* \setminus (B^*)^{-\delta}) < g_2(\delta)$ for all $x \in B^* \setminus (B^*)^{-\delta}$, and $g_2(\delta) \to_{\delta\to 0} 0$. Using Lemma 3.1, one obtains for such x and some $y \in ((B^*)^\delta)^c$ with $|f^{(k)}(x) - y| < 2\delta$,

$$\begin{aligned} I(x,y) &= \sup_{t\in\mathbb{R}}\{ty - g(x,t)\} \le \sup_{t\in\mathbb{R}}\{t(y - f^{(k)}(x)) - \Lambda(t)\} \\ &\le \max\{\sup_{t\in\mathbb{R}}\{t(2\delta + g_2(\delta)) - \Lambda(t)\}, \sup_{t\in\mathbb{R}}\{-t(2\delta + g_2(\delta)) - \Lambda(t)\}\}\,. \end{aligned} \tag{3.6}$$

Let $\Lambda^*(x) \stackrel{\triangle}{=} \sup_{t\in\mathbb{R}}\{tx - \Lambda(t)\}$ be the Fenchel-Legendre transform of $\Lambda(t)$. Since $E(\theta_1^K) = 0$, it follows that $\Lambda^*(0) = 0$. Moreover, **(B3)** implies that $\Lambda^*(\cdot)$ is finite in a neighborhood of the origin. Hence, due to its convexity, it follows that $\Lambda^*(\cdot)$ is continuous in a neighborhood of the origin. This, combined with (3.6), yields the first part of **(A4)**, while the second part follows from the bound, valid for small enough m,

$$P_x(|x_1^K - y| < \delta) \ge P(|\theta_i^K| < \delta/m \;\forall i < k) P(|\theta_k^K - y - f^{(k)}(x)| < \delta/2)\,.$$

We finally turn to checking **(A3)**. Let $t_{\max} > 0$ be arbitrary (to be specified below). Since $\Lambda(\cdot)$ is infinitely differentiable, $|\partial^2 g(x,t)/\partial t^2| < c = c(t_{\max})$ for all $x \in B^*$ and $|t| \le t_{\max}$. It follows that

$$\begin{aligned} I(x,y) &= \sup_t(ty - g(x,t)) \ge \sup_{|t|\le t_{\max}}(ty - g(x,t)) \\ &= \sup_{|t|\le t_{\max}}(ty - g(x,0) - \frac{\partial g}{\partial t}(x,0)t - \frac{\partial^2 g}{\partial t^2}(x,\zeta_t)t^2/2) \\ &\ge \sup_{|t|\le t_{\max}}(t(y - f^{(k)}(x)) - ct^2/2) \\ &\ge (y - f^{(k)}(x))^2/2c\,, \end{aligned} \tag{3.7}$$

where $|\zeta_t| \le t_{\max}$ in the second equality, and the last inequality is obtained by taking $t = (y - f^{(k)}(x))/c(t_{\max})$ for some large enough $t_{\max}$ such that $|t| < t_{\max}$.

Next, let ϵ and δ be given. Then there exists a k_δ large enough, with $k_\delta = \overline{k}_\delta k$ and $\overline{k}_\delta$ integer, such that for every $x \in (B^*)^{-\delta}$, and for some $\alpha > 0$,

$$|f^{(k_\delta)}(x) - z^*| \le (1-\alpha)|x - z^*|\,. \tag{3.8}$$

Indeed, since z^* is attracting, it follows that for each $x \in (B^*)^{-\delta}$ there exists a k_x such that $|f^{(k_x)}(x) - z^*| < \epsilon$. Let ϵ be small enough if necessary such that

$$|f^{(k)'}(y)| < (1-\alpha) \quad \text{for } |y - z^*| < \epsilon \tag{3.9}$$

(this is possible since $|f^{(k)'}(z^*)| < 1$ and $f^{(k)'}(\cdot)$ is continuous). Since $f^{(k_x)}(\cdot)$ is continuous, it follows that $|f^{(k_x)}(x) - z^*| < \epsilon$ holds with the same k_x for an open neighborhood G_x of x. Using compactness, one may thus find one k_δ such that $|f^{(k_\delta - k)}(x) - z^*| < \epsilon$ holds for all $x \in (B^*)^{-\delta}$. Using now (3.9), (3.8) follows for all x such that $|x - z^*| \geq \epsilon$. Finally, (3.8) follows for x satisfying $|x - z^*| < \epsilon$ by iterating (3.9).

We prove **(A3)** by contradiction. Assume that there is some $c(\epsilon, \delta)$ such that for arbitrary large T, there exists some $[u]_T$ with $|u_i - z^*| > \epsilon$, and $u_i \in (B^*)^{-\delta}$ satisfies

$$\sum_{i=1}^{T} I(u_{i-1}, u_i) \leq c(\epsilon, \delta)\,. \tag{3.10}$$

In what follows we use c_i, $i = 0, 1, \ldots$ to denote various constants which may depend on ϵ, δ, k but not on $[u]_T$ or T. Then by (3.7) and (3.10),

$$\sum_{i=1}^{T} |u_i - f^{(k)}(u_{i-1})|^2 \leq c_0\,. \tag{3.11}$$

Note that

$$|u_n - f^{(k)}(u_{n-1})| + |f^{(k)}(u_{n-1}) - f^{(2k)}(u_{n-2})| \geq |u_n - f^{(2k)}(u_{n-2})|\,.$$

Hence,

$$\begin{aligned} |u_n - f^{(k)}(u_{n-1})|^2 &\geq \frac{1}{2}|u_n - f^{(2k)}(u_{n-2})|^2 - |f^{(k)}(u_{n-1}) - f^{(2k)}(u_{n-2})|^2 \\ &\geq \frac{1}{2}|u_n - f^{(2k)}(u_{n-2})|^2 - c_1|u_{n-1} - f^{(k)}(u_{n-2})|^2\,, \end{aligned}$$

where we have used the Lipschitz continuity of $f^{(k)}$ in the last inequality. Iterating this inequality, we arrive at

$$|u_n - f^{(k)}(u_{n-1})|^2 \geq c_2|u_n - f^{(k_\delta)}(u_{n-\bar{k}_\delta})|^2 - c_3 \sum_{i=1}^{\bar{k}_\delta} |u_{n-i} - f^{(k)}(u_{n-i-1})|^2\,. \tag{3.12}$$

Summing both sides of (3.12) over n, one concludes that (3.11) implies that

$$\sum_{n=\bar{k}_\delta+1}^{T} |u_n - f^{(k_\delta)}(u_{n-\bar{k}_\delta})|^2 < c_4\,. \tag{3.13}$$

Now,

$$|u_{n+\bar{k}_\delta} - z^*| \leq |u_{n+\bar{k}_\delta} - f^{(k_\delta)}(u_n)| + |f^{(k_\delta)}(u_n) - z^*| \leq |u_{n+\bar{k}_\delta} - f^{(k_\delta)}(u_n)| + (1-\alpha)|u_n - z^*|\,,$$

where we have used (3.8) in the last inequality. Using the inequality $(x+(1-\alpha)y)^2 \leq x^2/\alpha+(1-\alpha)y^2$, one gets, for any $0<\alpha<1$,

$$|u_{n+\overline{k}_\delta}-z^*|^2 \leq \frac{|u_{n+\overline{k}_\delta}-f^{(k_\delta)}(u_n)|^2}{\alpha}+(1-\alpha)|u_n-z^*|^2 .$$

Summing over n, it follows that

$$\sum_{n=0}^{T-1}|u_n-z^*|^2 \leq c_5\sum_{n=0}^{T-1}|u_{n+\overline{k}_\delta}-f^{(k_\delta)}(u_n)|^2 \leq c_6 ,$$

where we have used (3.13) in the last inequality. Note however that by assumption, $|u_n-z^*|\geq\epsilon$, and hence

$$\sum_{n=0}^{T-1}|u_n-z^*|^2 \geq \epsilon^2 T ,$$

a contradiction if T is large enough. $\square$

Remark: Due to the strict convexity of the rate function, and the fact that $\overline{B}_{z^*}$ is separated from 0, the endpoint of the minimizing paths in (2.5),(2.6) actually belong to $\overline{B}_{z^*}$. Hence, cycle slip occurs before extinction with overwhelming probability.

3.2 Branching systems

As in the introduction, we take

$$z_{n+1}=\frac{1}{K}\sum_{j=1}^{Kz_n}\xi_j(z_n) \tag{3.14}$$

where $\xi_j(x)$ are i.i.d., integer valued, random variables whose law, denoted P_x, depends on the parameter x. Expectations with respect to P_x are denoted E_x. To fit in the model (1.3), it is assumed that $E_x\xi_j(x)=f(x)/x$, with $\xi_j(x)=0$ for $x\notin(0,1)$ (which corresponds to extinction).

As in the case of additive noise, let k denote the period of the stable orbit. Define $x_n^K \stackrel{\triangle}{=} z_{nk}$. Let $\Lambda(x,t)=\log E_x(e^{t\xi_1(x)})$. Note that, if all limits exist,

$$\begin{aligned} g(x,t)=\lim_{K\to\infty}\frac{1}{K}\log E_x(e^{Ktx_1^K}) &= \lim_{K\to\infty}\frac{1}{K}\log E_x(E_{z_{k-1}}e^{t\sum_{j=1}^{Kz_{k-1}}\xi_j(z_{k-1})}) \\ &= \lim_{K\to\infty}\frac{1}{K}\log E_x(e^{Kz_{k-1}\Lambda(z_{k-1},t)}) . \end{aligned} \tag{3.15}$$

The following assumptions are used to ensure the convergence in the definition of $g(x,t)$.

(C1) $\Lambda(x,t)<\infty$ for all $t\in\mathbb{R}$ and $x\in(0,1)$.

(C2) $\Lambda(x,t)$ is continuous in x and differentiable in t.

Define $\Lambda^*(x,y)=\sup_{t\in\mathbb{R}}(ty-x\Lambda(x,t))\geq 0$.

Lemma 3.2 *Assume* **(C1)**–**(C2)**. *Define*

$$\begin{aligned} g_1(x,t) &= x\Lambda(x,t) \\ g_2(x,t) &= \sup_y (g_1(y,t) - \Lambda^*(x,y)) \\ \dots &= \dots \\ g_j(x,t) &= \sup_y (g_{j-1}(y,t) - \Lambda^*(x,y)) \,. \end{aligned} \tag{3.16}$$

and assume that $g_j(x,t)$ is continuous in x, $j \le k$. Then $g(x,t) = g_k(x,t)$. Moreover,

$$g(x,t) \ge f^{(k-1)}(x)\Lambda(f^{(k-1)}(x),t). \tag{3.17}$$

Proof: As in (3.15), note that

$$\begin{aligned} E_x(e^{Ktx_1^K}) &= E_x(e^{Kg_1(z_{k-1},t)}) \\ &= E_x(E_{z_{k-2}}(e^{Kg_1(z_{k-1},t)})) = E_x(e^{Kg_2(z_{k-2},t)+o(K)}) \end{aligned}$$

where $o(K)$ is uniform in x and we used in the last equality the continuity of $g_1(x,t)$ in x and a version of Varadhan's lemma (see [3], Theorem 4.3.1). It follows that $g_2(x,t)$ is bounded and is also continuous by assumption. Iterating this procedure, the first part of the lemma follows. To see the second part, note that $\Lambda^*(x, f(x)) = 0$. Hence, $g_2(x,t) \ge g_1(f(x),t)$, and the claim follows by iterating this inequality.

□

Remark: Note that $\Lambda^*(x,y)$ is convex and lower-semicontinuous in y, and hence continuous in y in the interior of its domain. In addition, $g_1(x,t) \ne 0$ only for $x \in (0,1)$. Thus, if one knows also that $\Lambda^*(x,y)$ is continuous in $x \in (0,1)$, one could deduce from the continuity of $g_1(x,t)$ in (x,t) the same continuity for $g_2(x,t)$. Iterating this yields the continuity of $g_j(x,t)$ required in the lemma. Thus, a sufficient condition for the applicability of Lemma 3.2 is the continuity in x of $\Lambda^*(x,t)$ throughout $[0,1]$.

We need the following assumptions in order to have the analogue of Theorem 3.1.

(C3) $g(x,t)$ is Lipschitz continuous in (x,t) and twice differentiable in t.

(C4) $\Lambda^*(x,y) < \infty$ for all $y \in [0,\infty)$ and $x \in (0,1)$.

Remarks: 1) As we have seen in Lemma 3.2, **(C3)** is implied by **(C1)**–**(C2)** and a smoothness assumption on the $g_j(x,t)$ defined in (3.16).
2) Note that $\Lambda^*(x,y) = \infty$ for $y \in (-\infty, 0)$. Assumption **(C4)** implies, in particular, that the support of $\xi_j(x)$ is not bounded for all $x \in (0,1)$.

We are now ready to claim:

Theorem 3.2 *Assume that $\Lambda(x,t)$ satisfies* **(C1)**–**(C4)**. *Then the system generated by x_n^K satisfies* **(A1)**–**(A5)** *(with $G = (B^*)^{-\delta}$, any $\delta > 0$ and $\delta_o < \delta$), and Theorem 2.1 applies, with $g(x,t)$ given by Lemma 3.2.*

Proof: The proof parallels that of Theorem 3.1. Note first that the boundedness of $g(x,t)$ obtained in Lemma 3.2 implies, again by the Gärtner-Ellis theorem, that x_n^K satisfy the LDP with the good rate function $I(x,y)$. The continuity of $g(x,t)$ in x implies the lower semicontinuity of $I(x,y)$ needed for **(A2)**. Furthermore, by (3.17), $g(x,t) \geq c(x)\Lambda(c(x),t)$ and, in particular, $I(x,y) \leq \Lambda^*(c(x), y/c(x)) < \infty$ for all $x \in (0,1)$ and $y \in (0,\infty)$. The boundedness of V needed for **(A1)** follows immediately. To see the uniformity of the LDP, consider first the case $k=1$. Let $x_K \to x$. Fix M large, and note that $P(\xi_i(x) > M \,\mathrm{or}\, \xi_i(x_K) > M) < 2e^{-Kg_M}$ where $g_M \to_{M\to\infty} \infty$ by Chebycheff's bound and the continuity of $g(x,t)$ in x (here, g_M does not depend on x_K!). Construct a sequence of random variable z_i^K taking values in $\mathbb{Z}$ such that, for all integer j, $P_x(\xi_i(x) + z_i^K = j) = P_{x_K}(\xi_i(x_K) = j)$, $j = 1,2,\ldots$ (this can always be acomplished since $\xi_i(x)$ takes only countably many values). Moreover, since M is finite, due to the continuity of $g(x,t)$, one can construct these z_i^K such that

$$P(z_i^K \neq 0) \leq 2 \max_{j=1,\ldots,M} |P_x(\xi_i(x) = j) - P_{x_K}(\xi_i(x_K) = j)| \stackrel{\triangle}{=} c_M^K \to_{K\to\infty} 0\,.$$

Using the notations of Lemma 2.6, fix $t > 0$ and apply truncation and then the Chebycheff bound to get

$$P^K(|X_1 - X_2| > \delta) \leq 2e^{-Kg_M} + e^{-t\delta K} e^{(tM + \log c_M^K)K}\,.$$

Taking $K \to \infty$, $t \to \infty$ and $M \to \infty$ in such a way that $M + \log c_M^K/t \to_{K\to\infty} 0$ allows Lemma 2.6 to be applied, and concludes the proof of uniformity for $k=1$. The case of general k is obtained by iterating the above argument. This concludes the proof of **(A5)**.

The proof of **(A4)** is identical to the one given for the additive noise case, while the proof of **(A3)** relies on the explicit expression for $g(x,t)$ provided in Lemma 3.2 and on assumption **(C3)** in the same way that it relied on Lemma 3.1 in the additive noise case. □

Remark: An alternative proof of theorem 3.2 could proceed by using Lemma 2.5 instead of Lemma 2.6, and showing uniform convergence in the definition of $g(x,t)$. In particular, Theorem 3.2 holds true if **(C4)** is replaced by the assumption that $\xi_i(x)$ take values in the finite set $1,\ldots,M$.

4 An open problem

Consider the logistic model $f_r(x) = rx(1-x)$. A direct consequence of Theorem 3.2 is that as $r \to r_{\mathrm{cr}}$, one has that $\sup_{z^*\in\mathcal{O}_k} V(z^*) \to 0$, where $\mathcal{O}_k$ denotes the set of stable points of the map $f_r^{(k_r)}$. Thus, as expected, at the onset of chaos the noisy system does not follow closely the path of the unperturbed system. On the other hand, as long as $r < 4$, the positive distance between the atractor and 0 lead to the conclusion that the time to extinction still grows exponentially with K. We conjecture, but have been unable to show, that as $r \to 4$, the time to extinction becomes shorter and eventually, at $r = 4$, does not grow exponentially in K . Motivated by [1, 2, 14], it is expected that the extinction time in this regime grows with an exponent K^β, some $\beta < 1$.

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