Quenched sub–exponential tail estimates for one-dimensional random walk in random environment

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Abstract

Suppose that the integers are assigned i.i.d. random variables $\{\omega_x\}$ (taking values in the unit interval), which serve as an environment. This environment defines a random walk $\{X_n\}$ (called a RWRE) which, when at x, moves one step to the right with probability ω_x, and one step to the left with probability $1-\omega_x$. Solomon (1975) determined the almost-sure asymptotic speed v_α (=rate of escape) of a RWRE. Greven and den Hollander (1994) have proved a large deviation principle for X_n/n, conditional upon the environment, with deterministic rate function. For certain environment distributions where the drifts $2\omega_x - 1$ can take both positive and negative values, their rate function vanishes on an interval $(0, v_\alpha)$. We find the rate of decay on this interval and prove it is a stretched exponential of appropriate exponent, that is the absolute value of the log of the probability that the empirical mean X_n/n is smaller than v, $v \in (0, v_\alpha)$, behaves roughly like a fractional power of n. The annealed estimates of Dembo, Peres and Zeitouni (1996) play a crucial role in the proof.

We also deal with the case of positive and zero drifts, and prove there a quenched decay of the form $\exp(-cn/(\log n)^2)$.

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1. Introduction

In this paper, we continue the study, initiated in [4] and [2], of tail estimates for a nearest-neighbor random walk on $\mathbb{Z}$ with site-dependent transition probabilities.

Let $\omega = (\omega_x)_{x \in \mathbb{Z}}$ be an i.i.d. collection of $(0,1)$-valued random variables, with marginal distribution α such that supp $\alpha \subset (0,1)$. For every fixed ω, let $X = (X_n)_{n \geq 0}$ be the Markov chain on $\mathbb{Z}$ starting at $X_0 = 0$ (unless explicitly stated otherwise), and with transition probabilities

$$\mathbf{P}_\omega(X_{n+1} = y \mid X_n = x) = \begin{cases} \omega_x & \text{if } y = x+1 \\ 1-\omega_x & \text{if } y = x-1 \\ 0 & \text{otherwise} \end{cases} . \tag{1}$$

The symbol $\mathbf{P}_\omega$ denotes the measure on path space given the environment ω, and is referred to as the "quenched" setting. The process (X, ω) is an example of a *random walk in random environment* (RWRE), and X has law $\mathbf{P} = \int \alpha^{\mathbb{Z}}(d\omega)\mathbf{P}_\omega$, referred to as the "annealed" law. When no confusion arises, we use $\mathbf{P}$ also to denote the law of (X, ω). We use in various places, when confusion does not occur, P to denote the probability of events constructed from random variables unrelated to the RWRE.

For a discussion of the different regimes that the RWRE X_n exhibits, we refer to the introduction in [2].

Abbreviate $\rho = \rho(x, \omega) = (1 - \omega_x)/\omega_x$ and $\langle f \rangle = \int f(\omega)\alpha^{\mathbb{Z}}(d\omega)$ for any function f of the environment. Let $\rho_{\max}$ denote the maximum of ρ over the closed support of α, and let $\rho_{\min}$ denote the corresponding minimum. We will be interested here in the case $\langle \rho \rangle < 1$ and $\rho_{\max} \geq 1$, in which case (c.f. [7]) the RWRE is transient and, $\mathbf{P}$-a.s.,

$$\lim_{n \to \infty} n^{-1} X_n = v_\alpha := \frac{1 - \langle \rho \rangle}{1 + \langle \rho \rangle} . \tag{2}$$

Tail estimates for X_n/n have been derived for the quenched setting in [4]. In particular, it was shown there that, $\mathbf{P}$-a.s, the random variables X_n/n satisfy with respect to $\mathbf{P}_\omega$ a large deviation principle of speed n and explicit, deterministic, rate function $I(v)$, defined as follows (see [4, Theorem 2 and Corollary 1]). Let $f(r, \omega), r \geq 0$ denote the continued fraction function

$$f(r, \omega) = \frac{1|}{e^r(1 + \rho(0, \omega))} - \frac{\rho(0, \omega)|}{e^r(1 + \rho(1, \omega))} - \frac{\rho(1, \omega)|}{\cdots} ,$$

and let $\lambda(r) = \exp\langle \log f(r, \omega) \rangle$. Let $r(v) = 0$ for $v \leq v_\alpha$, and for $v \in (v_\alpha, 1]$, let $r(v)$ be the unique solution of the equation $v^{-1} = -\lambda'(r)/\lambda(r)$. Then,

$$I(v) = \begin{cases} -r(v) - v \log \lambda(r(v))\,, & v \in [0,1] \\ I(-v) + v\langle \log \rho \rangle\,, & v \in [-1,0] \\ \infty\,, & v \notin [-1,1]\,. \end{cases}$$

Furthermore, $I(v) = 0$ for $v \in [0, v_\alpha]$ and I is strictly positive elsewhere.

Our goal in this paper is to study in greater details the regime $v \in (0, v_\alpha)$ under $\mathbf{P}_\omega$. In the annealed setting, i.e. when one is interested in $\mathbf{P}(X_n \le nv)$, $v \in (0, v_\alpha)$, sub–exponential rates of decay were derived in [2]. We summarize now the main results of [2] relevant to us. Recall (c.f. [2]) that when $\langle \rho \rangle < 1$, there exists a unique $s > 1$ satisfying $\langle \rho^s \rangle = 1$.

Theorem 1 (see [2]) *Let $v \in (0, v_\alpha)$.*

(a) Positive and negative drifts *Suppose that $\langle \rho \rangle < 1$ and $\rho_{\max} > 1$. Then,*

$$\lim_{n\to\infty} \log \mathbf{P}(X_n \le nv)/\log n = 1 - s\,.$$

(b) Positive and zero drifts *Suppose that $\langle \rho \rangle < 1$ but $\rho_{\max} = 1$ and $\alpha(1/2) > 0$. Then, with $C_1 = \frac{3}{2}|\frac{\pi \log \alpha(1/2)}{2}|^{2/3}$ and $C_2 = |\frac{\pi(\log\langle\rho\rangle)}{8}|^{2/3}$,*

$$\begin{aligned} -C_1(1-\frac{v}{v_\alpha})^{1/3} &\le \liminf_{n\to\infty} \frac{1}{n^{1/3}}\log \mathbf{P}(X_n \le nv) \\ &\le \limsup_{n\to\infty} \frac{1}{n^{1/3}}\log \mathbf{P}(X_n \le nv) \le -C_2(1-\frac{v}{v_\alpha})^{1/3}\,. \end{aligned} \tag{3}$$

Maybe surprisingly, it turns out that the annealed estimates are key to understanding the quenched asymptotics. The next theorems are our main results. They quantify the fact that the annealed probabilities of large deviations are of bigger order than their quenched counterparts, due to the possibility of rare fluctuations in the environment which may slow down the RWRE.

Theorem 2 (Positive and negative drifts) *Suppose that $\langle \rho \rangle < 1$, $\rho_{\max} > 1$, and let $v \in (0, v_\alpha)$. Then, for $\mathbf{P}$-a.a. ω, the following statements hold:*

1. For any $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{n^{1-1/s-\delta}} \log \mathbf{P}_\omega\left(X_n < nv\right) = -\infty\,. \tag{4}$$

2. For any $\delta > 0$,

$$\liminf_{n\to\infty} \frac{1}{n^{1-1/s+\delta}} \log \mathbf{P}_\omega\left(X_n < nv\right) = 0\,. \tag{5}$$

Furthermore,

$$\limsup_{n\to\infty} \frac{1}{n^{1-1/s}} \log \mathbf{P}_\omega (X_n < nv) = 0 \,. \tag{6}$$

One should compare the rate of decay obtained in Theorem 2 with the annealed polynomial rate of decay (see Theorem 1) $\mathbf{P}(X_n < nv) \simeq n^{1-s}$.

As in [2], tail estimates are different when the drift cannot be negative:

Theorem 3 (Positive and zero drifts) *Suppose that* $\langle\rho\rangle < 1$, $\rho_{\max} = 1$, *and* $\alpha(\{1/2\}) > 0$. *Then, for* $\mathbf{P}$*-a.a.* ω, *and for* $v \in (0, v_\alpha)$,

$$-c_1(1-\frac{v}{v_\alpha}) \le \liminf_{n\to\infty} \frac{(\log n)^2}{n} \log \mathbf{P}_\omega(X_n < nv) \le \limsup_{n\to\infty} \frac{(\log n)^2}{n} \log \mathbf{P}_\omega(X_n < nv) \le -c_2(1-\frac{v}{v_\alpha})^2 \,. \tag{7}$$

Here, $c_1 = |\pi \log \alpha(\{1/2\})|^2/8$ *and* $c_2 = |\pi \log\langle\rho\rangle|^2/24^3$.

Again, the rate in Theorem 3 should be compared with the annealed rate (c.f. Theorem 1) $\mathbf{P}(X_n < nv) \simeq \exp(-C_i n^{1/3})$.

Remarks

1. As in [2], we have not covered the case of $\langle\rho\rangle < 1$, $\rho_{\max} = 1$, while $\alpha(\{1/2\}) = 0$. The tail estimates in the annealed case were conjectured in [2, pg. 681] to be of the form $\exp(-D_i n^\beta)$, $i = 1, 2$, for some $\beta \in (1/3, 1)$ determined by the tails of $\alpha(\cdot)$ near $1/2$. The same proof as in Theorem 3 then shows that the upper quenched estimates in Theorem 3 become $\exp(-dn/(\log n)^\gamma)$, with $\gamma = 1/\beta - 1$.

2. In the setting of Theorem 2, we conjecture that actually

   $$\liminf_{n\to\infty} \frac{1}{n^{1-1/s}} \log \mathbf{P}_\omega (X_n < nv) = -\infty \,,$$

   In fact, the derivation of the lower bound in (6) hints at such a limit.

   In the setting of Theorem 3, we conjecture, as in [2], that the lower bound is sharp, that is

   $$\lim_{n\to\infty} \frac{(\log n)^2}{n} \log \mathbf{P}_\omega(X_n < nv) = -c_1(1 - \frac{v}{v_\alpha}) \,.$$

   In fact, it was shown recently (see [6]) that the lower bound is sharp in the annealed setting, that is one may replace C_2 in the right hand side of (3) by C_1. This however does not suffice for closing the gap in our Theorem 3, see the comment following the proof of the theorem.

3. In the setting of Theorem 2, it is natural to attempt to improve on (4), (5) by allowing for $\delta_n \to_{n\to\infty} 0$. Such improvement is possible if in Theorem 1.1 of [2], one refines the convergence, that is one proves bounds of the form

$$\limsup_{n\to\infty} g_n n^{s-1} \mathbf{P}(X_n < nv) < \infty$$

for appropriate $g_n \to_{n\to\infty} 0$ sub–polynomialy , which is possible albeit tedious. It seems however impossible by this way to completely close the gap between the upper and lower bounds exhibited in (4) and (5).

We conclude this introduction with two technical lemmas, borrowed from [2], whose proof follows readily from the explicit computations for inhomogeneous random walk of [1, pg 66–71]. Let X_n denote a RWRE and let $\bar{X}_n$ denote a RWRE with $\omega_0 = 1$. Let $\bar{\tau}_k = \min\{n : \bar{X}_n = k\}$, let $R_k = k^{-1}\sum_{i=1}^{k} \log \rho(i)$, and let $L_0 = \max_{n\geq 0}\{-X_n\}$.

Lemma 1 ([2], Lemma 2.1) *For all n, k,*

$$\mathbf{P}_\omega(\bar{\tau}_k \geq n) \geq (1 - e^{-(k-1)R_{k-1}})^n \,.$$

Lemma 2 ([2], Lemma 2.2) *For any $k \geq 1$,*

$$\mathbf{P}(L_0 \geq k) \leq \frac{\langle\rho\rangle^k}{1-\langle\rho\rangle} \,.$$

2. Proofs

Proof of Theorem 2. Since the lower bound of Theorem 2 is relatively simple, and the key ideas are already explained in [2], we postpone the discussion of it and begin by providing a sketch of the proof of the upper bound leading to (4), that is, with

$$\tau_n = \inf\{t :\, X_t = n\} \,, \tag{8}$$

we will explain why

$$\lim_{n\to\infty} \frac{1}{n^{1-1/s-\delta}} \log \mathbf{P}_\omega\left(\tau_n > n/v\right) = -\infty \,.$$

The required upper bound follows readily.

We will omit subsequences, etc. in this sketch, and thus the reader interested in a complete proof should take the next few paragraphs with somewhat of a grain of salt. The precise statement of the required estimate is contained in the statement of Proposition 1.

Divide the interval $[0, nv]$ into blocks of size roughly $k = k_n := n^{1/s+\delta}$. Let X_n^x denote the RWRE started at x, and define

$$T_k^{(i)} = \inf\{t > 0 : X_t^{ik} = (i+1)k\}\,, \quad i = 0, \pm 1, \dots\,. \tag{9}$$

By slight abuse of notation, we continue to use $\mathbf{P}_\omega$ for the quenched law of the $\{X_n^x\}$. By using the annealed bounds of [2], see Theorem 1, one knows that $\mathbf{P}(\tau_k > k/v) \sim k^{1-s}$. Hence, taking appropriate subsequences, one applies a Borel-Cantelli argument to control the probability, conditioned on the environment, of the time spent in each such block being large, i.e. one exhibits a uniform estimate on $\mathbf{P}_\omega(T_k^{(i)} > k/v)$, c.f. Lemma 5.

The next step involves a decoupling argument. Let

$$\overline{T}_k^{(i)} = \inf\{t > 0 : X_t^{ik} = (i+1)k \text{ or } X_t^{ik} = (i-1)k\}\,. \tag{10}$$

Then, using Lemma 2, and the Borel-Cantelli lemma, one shows that for all relevant blocks, that is $i = \pm 1, \pm 2, \dots, \pm n/k$, $\mathbf{P}_\omega(\overline{T}_k^{(i)} \neq T_k^{(i)})$ is small enough. Therefore, we can consider the random variables $\overline{T}_k^{(i)}$ instead of $T_k^{(i)}$, which have the advantage that their dependence on the environment is well localized. This allows us to obtain (c.f. Lemma 7) a uniform bound on the tails of $\overline{T}_k^{(i)}$, *for all relevant* i.

The final step involves estimating how many of the k-blocks will be traversed from right to left before the RWRE hits the point nv. This is done by constructing a simple random walk (SRW) S_t whose probability of jump to the left dominates $\mathbf{P}_\omega(T_k^{(i)} \neq \overline{T}_k^{(i)})$ for all relevant i. The analysis of this SRW will allow us to claim (c.f. Lemma 9) that the number of visits to a k-block after entering its right neighbor is negligible. Thus, the original question on the tail of τ_n is replaced by a question on the sum of (dominated by i.i.d.) random variables $\overline{T}_k^{(i)}$, which is resolved by means of the tail estimates obtained in the second step.

A slight complication is presented by the need to work with subsequences in order to apply the Borel-Cantelli lemma at various places. Going from subsequences to the original n sequence is achieved by means of monotonicity arguments.

Turning now to the complete proof, we first note that it is actually enough to prove a weaker statement. For $\delta \in (0, 1 - 1/s)$, let $C_n = n^\delta$ and let $n_j = [j^{2/\delta}]$. Recall that $\tau_n = \inf\{t : X_t = n\}$, and let $\mu := v^{-1} > v_\alpha^{-1}$. The key to the upper bound is the following proposition, whose proof is postponed.

Proposition 1.

$$\lim_{j\to\infty} \frac{C_{n_j}}{n_j^{1-1/s}} \log \mathbf{P}_\omega \left(\tau_{n_j} > n_j \mu\right) = -\infty\,. \tag{11}$$

Assuming the proposition holds true, let us show how to complete the proof of the upper bound (4). Note that, for j large, $n_{j+1}/n_j \le \dfrac{(j+1)^{2/\delta}+1}{j^{2/\delta}-1} \underset{j\to\infty}{\longrightarrow} 1$. Let j_n be such that $n_{j_n} \le n < n_{j_{n+1}}$. Then, for any n,

$$\mathbf{P}_\omega\left(\tau_n > n\mu\right) \le \mathbf{P}_\omega\left(\tau_{n_{j_{n+1}}} > n_{j_n}\mu\right) = \mathbf{P}_\omega\left(\tau_{n_{j_{n+1}}} > n_{j_{n+1}}\,\mu^{(n)}\right)\,,$$

where $\mu^{(n)} = \dfrac{\mu n_{j_n}}{n_{j_{n+1}}}$.

Let N be large such that $\inf_{n\ge N} \dfrac{\mu n_{j_n}}{n_{j_{n+1}}} > \mu_\alpha$, and consider only $n > N$. One concludes from Proposition 1 that for all $\delta > 0$, $\mathbf{P}$ a.s.,

$$\limsup_{n\to\infty} \frac{1}{n^{1-\frac{1}{s}+\delta}} \log\, \mathbf{P}_\omega(\tau_n > n\mu) = -\infty\,. \tag{12}$$

To prove (4), let $v < v' < v_\alpha$ and define $L^{[nv']} = \max\{[nv'] - X_k^{[nv']}; k \ge 0\}$. Then,

$$\mathbf{P}_\omega(X_n < nv) \le \mathbf{P}_\omega(\tau_{[nv']} > n) + \mathbf{P}_\omega(L^{[nv']} \ge [nv'] - nv)\,. \tag{13}$$

By Lemma 2,

$$\mathbf{P}(L^{[nv']} \ge [nv'] - nv) = \mathbf{E}(\mathbf{P}_\omega(L^{[nv']} \ge [nv'] - nv)) \le \frac{\langle\rho\rangle^{[nv']-[nv]-1}}{1-\langle\rho\rangle}\,.$$

Hence, one may find some $\varepsilon > 0, \theta > 0$ such that

$$\mathbf{P}(\mathbf{P}_\omega(L^{[nv']} \ge [nv'] - nv) \ge e^{-\varepsilon n}) \le e^{-\theta n}\,.$$

Applying now the Borel-Cantelli lemma, one concludes that $\mathbf{P}$-a.s.,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}_\omega(L^{[nv']} \ge [nv'] - nv) < -\varepsilon < 0\,. \tag{14}$$

(4) follows from (13), (14) and (12).

As mentioned before, the proof of the lower bounds (5) and (6) follows the ideas of [2] (see in particular Remark 4, pg. 682). Indeed, it is already explained there why, for any $\delta > 0$

$$\liminf_{n\to\infty} \frac{1}{n^{1-1/s+\delta}} \log\, \mathbf{P}_\omega\left(\frac{X_n}{n} < v\right) = 0\,.$$

In order to see the refined estimate in (6) , we recall the following notations from [2]. Let $R_k(m) = \frac{1}{k}\sum_{i=m+1}^{m+k} \log\rho(i)$. Define $\tau_k^x = \inf\{t : X_t^x = k+x\}$ and $\overline{\tau}_k^x = \inf\{t : \overline{X}_t^x = k+x\}$, where $\overline{X}_t^x$ is the RWRE with $\omega(x) = 1$, initiated at x. It follows from Lemma 1 that

$$\mathbf{P}_\omega\Big(\tau_{k+1}^x \geq n\Big) \geq \mathbf{P}_\omega\Big(\overline{\tau}_{k+1}^x \geq n\Big) \geq \Big(1 - e^{-kR_k(x)}\Big)^n . \tag{15}$$

For $n = 1, 2, \ldots,$ define

$$M_n(x) = \max_{\substack{x \leq m \leq x+n \\ k \leq x+n-m}} kR_k(m) .$$

In particular, it follows from (15) that for any $c > 0$ and $l = [n/c]$,

$$\mathbf{P}_\omega(\tau_{l+1}^x \geq n) \geq \mathbf{P}_\omega(\overline{\tau}_{l+1}^x \geq n) \geq \Big(1 - e^{-M_l(x)}\Big)^n . \tag{16}$$

We recall the following exceedence bounds, due to Iglehart. For this version, see [5], Theorem A.

Lemma 3. *There exist constants K_1, K_2, such that for any $z \in \mathbb{R}$,*

$$\begin{aligned}\exp\Big(-K_1\exp(-sz)\Big) \quad &\leq \quad \liminf_{l\to\infty} \mathbf{P}\Big(M_l(x) - \frac{\log l}{s} \leq z\Big) \\ &\leq \quad \limsup_{l\to\infty} \mathbf{P}\Big(M_l(x) - \frac{\log l}{s} \leq z\Big) \leq \exp\Big(-K_2\exp(-sz)\Big) .\end{aligned}$$

A corollary of Lemma 3 and (16) (taking $y = e^z$) is the following:

Lemma 4. *For any $y > 0$ there exists a $c_y > 0$ such that, for any $v' < v_\alpha$,*

$$\liminf_{n\to\infty} \mathbf{P}\Big(\mathbf{P}_\omega(\tau_{[nv']}^x \geq n) \geq e^{-\frac{n^{1-1/s}}{y(v')^{1/s}}}\Big) \geq c_y$$

and the convergence is uniform in x.

Equipped with Lemma 4, we have completed all the preliminaries required for proving (6). Indeed, fix $y > 0$, and let $n_k = 2^{2^k}$. Note that

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}_\omega(X_n \leq nv)}{n^{1-1/s}} \quad \geq \quad \limsup_{k\to\infty} \frac{\log \mathbf{P}_\omega(X_{n_k} \leq n_k v)}{n_k^{1-1/s}}$$

$$
\begin{aligned}
&\geq \limsup_{k\to\infty} \frac{\log \mathbf{P}_\omega(\tau^0_{[n_k v]} \geq n_k)}{n_k^{1-1/s}} \\
&\geq \limsup_{k\to\infty} \frac{\log \mathbf{P}_\omega(\tau^{n_{k-1}}_{[n_k v]-n_{k-1}} \geq n_k)}{n_k^{1-1/s}} \\
&\geq \limsup_{k\to\infty} \frac{\log \mathbf{P}_\omega(\tau^{n_{k-1}}_{[n_k v']} \geq n_k)}{n_k^{1-1/s}}
\end{aligned}
$$

where $v' = v - \varepsilon$ for arbitrary ε. By Lemma 4, and the Borel-Cantelli lemma, for any $z > 0$

$$
\mathbf{P}_\omega\left(\tau^{n_{k-1}}_{[n_k v']} \geq n_k\right) \geq e^{-\frac{n_k^{1-1/s}}{z}}
$$

infinitely often. The conclusion follows by taking $z \to \infty$. This completes the proof of Theorem 2, except that we still have to show Proposition 1. $\square$

Proof of Proposition 1 Let $k = k_j = \dfrac{C_{n_j} n_j^{1/s}}{1-\varepsilon}$ for some $1 > \varepsilon > 0$. For X_n^x the RWRE started at x, recall that

$$
T_k^{(i)} = \inf\{t > 0 : X_t^{ik} = (i+1)k\}, \quad i = 0, \pm 1, \ldots .
$$

By slight abuse of notation, we continue to use $\mathbf{P}_\omega$ for the quenched law of the $\{X_n^x\}$.

Finally, let $b_n = C_n^{-\delta}$ and $I_j = \left\{-\left[\frac{n_j}{k_j}\right] - 1, \cdots, \left[\frac{n_j}{k_j}\right] + 1\right\}$. Fix $\mu' > \mu$.

Lemma 5. *For $\mathbf{P}$ – a.e. ω, there exists a $J_0(\omega)$ such that for all $j > J_0(\omega)$, and all $i \in I_j$,*

$$
\mathbf{P}_\omega\left(\frac{T_{k_j}^{(i)}}{k_j} > \mu'\right) \leq b_{n_j} .
$$

Proof of Lemma 5. By Chebycheff's bound,

$$
\begin{aligned}
\mathbf{P}\left(\mathbf{P}_\omega\Big(\frac{T_{k_j}^{(i)}}{k_j} > \mu'\Big) > b_{n_j}\right) &\leq \frac{1}{b_{n_j}} \mathbf{P}\Big(\frac{T_{k_j}^{(i)}}{k_j} > \mu'\Big) \\
&\leq \frac{1}{b_{n_j}} k_j^{1-s+o(1)} ,
\end{aligned}
$$

where the last inequality follows from Theorem 1(a), and $o(1) \underset{j\to\infty}{\longrightarrow} 0$.

Hence,

$$\begin{aligned}\mathbf{P}\left(\mathbf{P}_\omega\Big(\frac{T^{(i)}_{k_j}}{k_j}>\mu'\Big)>b_{n_j}\text{ for some } i\in I_j\right) &\le 3\left[\frac{n_j}{k_j}\right]\cdot\frac{1}{b_{n_j}}\cdot k_j^{1-s+o(1)}\\ &\le \frac{3}{n_j^{\delta(s-o(1)-\delta)}}\le\frac{4}{j^{2(s-o(1)-\delta)}}\end{aligned}$$

and the conclusion follows from the Borel-Cantelli lemma. □

Let $0<\theta<-\frac{\log\langle\rho\rangle}{1-\varepsilon}$, $d_n^\theta=e^{-\theta n^{1/s}C_n}$, and recall that

$$\overline{T}^{(i)}_k=\inf\left\{t>0:\ X^{ik}_t=(i+1)k\text{ or }X^{ik}_t=(i-1)k\right\}.$$

Lemma 6. *For* $\mathbf{P}$ *– a.e.* ω*, there is a* $J_1(\omega)$ *s.t. for all* $j\ge J_1(\omega)$*,*

$$\mathbf{P}_\omega\Big(\overline{T}^{(i)}_{k_j}\neq T^{(i)}_{k_j}\,,\text{ some } i\in I_j\Big)\le d^\theta_{n_j}\,.$$

Proof of Lemma 6. Again, we use the Chebycheff bound:

$$\begin{aligned}&\mathbf{P}\Big(\mathbf{P}_\omega\ \Big(\overline{T}^{(i)}_{k_j}\neq T^{(i)}_{k_j}\,,\text{ some }\quad i\in I_j\Big)>d^\theta_{n_j}\Big)\\ &\le\frac{1}{d^\theta_{n_j}}\cdot\frac{3n_j}{k_j}\,\mathbf{P}\Big(\overline{T}^{(0)}_{k_j}\neq T^{(0)}_{k_j}\Big)\\ &\le\frac{1}{d^\theta_{n_j}}\cdot\frac{3n_j}{k_j}\cdot\frac{\langle\rho\rangle^{k_j}}{1-\langle\rho\rangle}\\ &\le\quad\frac{3}{(1-\langle\rho\rangle)}\,n_j^{1-\frac{1}{s}-\delta}\exp\left(n_j^{\frac{1}{s}+\delta}\ \left(\frac{\log\langle\rho\rangle}{(1-\varepsilon)}+\theta\right)\right)\,,\end{aligned}$$

where the second inequality follows from Lemma 2. The conclusion follows from the Borel-Cantelli lemma. □

We actually need to iterate the estimates of Lemma 5.

Lemma 7. *For* $\mathbf{P}$ *– a.e.* ω*, for all* $j>J_0(\omega)$, and each $i\in I_j$*, and for* $x\ge1$

$$\mathbf{P}_\omega\left(\frac{\overline{T}^{(i)}_{k_j}}{k_j}>\mu'x\right)\le(2b_{n_j})^{[x/2]\vee1}\,.$$

Proof of Lemma 7. For $1 \le x < 4$, the claim follows from Lemma 5. Assume thus that $x \ge 4$. Then,

$$\mathbf{P}_\omega\left(\frac{\overline{T}^{(i)}_{k_j}}{k_j} > \mu' x\right) \le \mathbf{P}_\omega\Big(\quad \frac{\overline{T}^{(i)}_{k_j}}{k_j} > \mu'(x-2)\ ,$$

$$(i-1)k_j < X^{ik_j}_{[\mu' k_j (x-2)]+1} < (i+1)k_j\ ,$$

$$\min\{t:\ t \ge [\mu' k_j (x-2)] + 2\,,\ X^{ik_j}_t = (i+1)k_j\} \ge x\mu' k_j\Big)\ .$$

Hence, by the Markov property,

$$\begin{aligned}
\mathbf{P}_\omega\Big(\frac{\overline{T}^{(i)}_{k_j}}{k_j} > \mu' x\Big) &\le \mathbf{P}_\omega\left(\frac{\overline{T}^{(i)}_{k_j}}{k_j} > \mu'(x-2)\right) \\
&\quad \times \sup_{(i-1)k_j < y < (i+1)k_j} \mathbf{P}_\omega\Big(\inf\{t:\ X^y_t = (i+1)k_j\} \ge 2\mu' k_j\Big) \\
&\le \mathbf{P}_\omega\Big(\frac{\overline{T}^{(i)}_{k_j}}{k_j} > \mu'(x-2)\Big) \cdot \mathbf{P}_\omega\Big(T^{(i)}_{k_j} + T^{(i-1)}_{k_j} > 2\mu' k_j\Big) \\
&\le \mathbf{P}_\omega\left(\frac{\overline{T}^{(i)}_{k_j}}{k_j} > \mu'(x-2)\right) \Big[\mathbf{P}_\omega\left(T^{(i)}_{k_j} > \mu' k_j\right) + \mathbf{P}_\omega\left(T^{(i-1)}_{k_j} > \mu' k_j\right)\Big] \\
&\le 2b_{n_j}\,\mathbf{P}_\omega\left(\frac{\overline{T}^{(i)}_{k_j}}{k_j} > \mu'(x-2)\right)\ ,
\end{aligned}$$

where the last inequality is a consequence of Lemma 5. The lemma follows by induction. $\square$

We need one more preliminary computation related to the bounds in Lemma 7. Let $\{Z^{(i)}_{k_j}\}$, $i = 1, 2, \ldots$ denote a sequence of i.i.d. positive random variables, with

$$P\left(\frac{Z^{(i)}_{k_j}}{k_j} < \mu'\right) = 0\,, \quad P\left(\frac{Z^{(i)}_{k_j}}{k_j} > \mu' x\right) = \left(2b_{n_j}\right)^{[x/2]\vee 1}\ , \quad x \ge 1\ .$$

Lemma 8. *For any $\lambda > 0$, and any $\varepsilon > 0$,*

$$E\left(\exp\left(\lambda \frac{Z^{(i)}_{k_j}}{k_j}\right)\right) \le e^{\lambda\mu'(1+\varepsilon)} + g_j$$

where $g_j \underset{j\to\infty}{\longrightarrow} 0$.

Proof of Lemma 8.

$$\begin{aligned}
E\left(\exp\left(\lambda\frac{Z^{(i)}_{k_j}}{k_j}\right)\right) &= \int_0^\infty P\Big(\frac{Z^{(i)}_{k_j}}{k_j} > \frac{\log u}{\lambda}\Big)\,du \\
&\le e^{\lambda\mu'(1+\varepsilon)} + \int_{e^{\lambda\mu'(1+\varepsilon)}}^\infty (2b_{n_j})^{\left[\frac{\log u}{2\lambda\mu'(1+\varepsilon)}\right]\vee 1}\,du \\
&= e^{\lambda\mu'(1+\varepsilon)} + g_j
\end{aligned}$$

where $g_j \underset{j\to\infty}{\longrightarrow} 0$. $\square$

In order to control the number of repetitions of visits to k_j–blocks, we introduce an auxiliary random walk. Let S_t, $t=0,1,\dots,$ denote a simple random walk with $S_0=0$ and

$$P\Big(S_{t+1}=S_t+1\Big|\,S_t\Big) = 1-P\Big(S_{t+1}=S_t-1\Big|\,S_t\Big) = 1-d_n^\theta\ .$$

Set $M_{n_j} = \dfrac{1}{C_{n_j}}\,n_j^{1-\frac{1}{s}}$.

Lemma 9. *For θ as in Lemma 6, and n large enough,*

$$P\Big(\inf\{t:\ S_t=\big[\tfrac{n_j}{k_j}\big]\} > M_{n_j}\Big) \le \exp\Big(-\frac{\theta\varepsilon}{2}\,n_j\Big)\ .$$

Proof of Lemma 9.

$$P\left(\inf\Big\{t:\ S_t=\big[\tfrac{n_j}{k_j}\big]\Big\} > M_{n_j}\right) \le P\left(\frac{S_{[M_{n_j}]}}{M_{n_j}} < \frac{n_j}{k_j\,M_{n_j}}\right) = P\left(\frac{S_{[M_{n_j}]}}{M_{n_j}} < 1-\varepsilon\right) \le 2\,e^{-M_{n_j}h_{n_j}(1-\varepsilon)}\ ,$$

where the last inequality is a consequence of Cramèr's theorem (c.f. [3]), and the fact that $d_n^\theta < \varepsilon$. Here,

$$h_n(1-x) = (1-x)\log\Big(\frac{1-x}{1-d_n^\theta}\Big) + x\log\frac{x}{d_n^\theta}\ .$$

Using $h_n(1-x) \ge -\frac{2}{e} - x\log d_n^\theta$, we get

$$P\left(\frac{S_{[M_{n_j}]}}{M_{n_j}} < 1-\varepsilon\right) \le 2\,e^{2M_{n_j}/e}\,e^{+\varepsilon M_{n_j}\log d_{n_j}^\theta} \le e^{-\frac{\varepsilon}{2}\theta\,n_j}$$

$\square$

We are now ready to prove (11). Note that, for all $j > J_0(\omega)$, and all $i \in I_j$, we may, due to Lemma 7, construct $\{Z_{k_j}^{(i)}\}$ and $\{\overline{T}_{k_j}^{(i)}\}$ on the same probability space such that $\mathbf{P}_\omega\Big(Z_{k_j}^{(i)} \geq \overline{T}_{k_j}^{(i)} \quad \forall\, i \in I_j\Big) = 1$. Fix $\mu_\alpha < \mu' < \mu$ and $\varepsilon > 0$ small enough. Recalling that, under $\mathbf{P}_\omega$, the $\overline{T}_{k_j}^{(i)}$ are independent, we obtain, with $\{S_t\}$ defined before Lemma 9, and j large enough,

$$
\begin{aligned}
\mathbf{P}_\omega(\tau_{n_j} > n_j\mu) &\leq P\Big(\inf\Big\{t:\, S_t = \Big[\frac{n_j}{k_j}\Big]\Big\} > M_{n_j}\Big) + P\Big(\sum_{i=1}^{M_{n_j}} Z_{k_j}^{(i)} > n_j\mu\Big) \\
&\leq e^{-\theta\varepsilon n_j/2} + P\Big(\frac{1}{M_{n_j}}\sum_{i=1}^{M_{n_j}} \frac{Z_{k_j}}{k_j} > \mu(1-\varepsilon)\Big) \\
&\leq e^{-\theta\varepsilon n_j/2} + \Big[E\Big(\exp\left(\lambda\,\frac{Z_{k_j}^{(i)}}{k_j^{(i)}}\right)\Big)\cdot e^{-\lambda\mu(1-\varepsilon)}\Big]^{M_{n_j}} \\
&\leq e^{-\theta\varepsilon n_j/2} + \Big(e^{\lambda(\mu'+2\varepsilon\mu-\mu)} + g_j e^{-\lambda\mu(1-\varepsilon)}\Big)^{M_{n_j}} \\
&\leq e^{-\theta\varepsilon n_j/2} + \Big(e^{-\lambda\varepsilon\mu}\Big)^{M_{n_j}}\,,
\end{aligned}
$$

where Lemma 9 was used in the second inequality and Lemma 8 in the fourth. Since $\lambda > 0$ is arbitrary, (11) follows. $\square$

Proof of Theorem 3. We begin by giving a quick sketch of the lower bound in (7), based on [2]. By the Erdös-Renyi strong law for longest run of heads, (or the asymptotics for long rare segments in random walks, see e.g. [3, pg. 69]), there is a segment $I = (i_{\min}, i_{\max})$, with $i_{\min} \geq n(v-\varepsilon)$, $i_{\max} < nv$ and $i_{\max} - i_{\min} = \log n/(-\log\alpha(\{1/2\}))(1+o(1))$, such that $\omega_i = 1/2$ for $i \in I$. Let $\tilde{X}_n$ denote the RWRE started at $(i_{\min}+i_{\max})/2$. Let $\tau = \min\{t : \tilde{X}_t = i_{\min} \text{ or } \tilde{X}_t = i_{\max}\}$. Then, τ possesses the same law as the exit time, denoted $\bar{\tau}$, of simple symmetric random walk from the interval $[-(i_{\max}-i_{\min})/2, (i_{\max}-i_{\min})/2]$. As before, we let $\tau_k = \min\{t : X_t = k\}$. We have,

$$
\begin{aligned}
\mathbf{P}_\omega(X_n < nv) &\geq \mathbf{P}_\omega(\tau_{n(v-\varepsilon)} \geq n\frac{v-2\varepsilon}{v_\alpha})\mathbf{P}_\omega(\tau > n(1-\frac{v}{v_\alpha}+\frac{2\varepsilon}{v_\alpha})) \\
&= \mathbf{P}_\omega(\tau_{n(v-\varepsilon)} \geq n\frac{v-2\varepsilon}{v_\alpha})P(\bar{\tau} > n(1-\frac{v}{v_\alpha}+\frac{2\varepsilon}{v_\alpha}))\,. \qquad (17)
\end{aligned}
$$

By Solomon's law of large numbers, c.f. (2),

$$
\lim_{n\to\infty}\mathbf{P}_\omega(\tau_{n(v-\varepsilon)} \geq n\frac{v-2\varepsilon}{v_\alpha}) = 1\,. \qquad (18)
$$

By standard eigenvalue estimates for simple random walk (c.f. [8, p. 243]),

$$\lim_{n\to\infty} \frac{(\log n)^2}{n(1-\frac{v}{v_\alpha}-\frac{2\varepsilon}{v_\alpha})(\log\alpha(1/2))^2}\log P(\bar\tau>n) = -\pi^2/8\,. \tag{19}$$

Combining (19), (17), and (18), the lower bound in (7) follows.

The proof of the upper bound in (7) follows the proof of part 1 of Theorem 2, except that there is no need for subsequences here. With $\mu = v^{-1} > v_\alpha^{-1} = \mu_\alpha$ and $t \in (0,1)$, define $\bar\mu = t\mu_\alpha + (1-t)\mu$. Fix $1/2 > \varepsilon > 0, \delta > 2$, $b_n = n^{-(\delta/2)}$ and

$$k = k(n) := \frac{(\log n)^3(1+\delta)^3}{C_2^3(\bar\mu-\mu_\alpha)(1-\varepsilon)^3}\,,$$

where C_2 was defined in Theorem 1. We define $I_n = \left\{-\left[\frac{n}{k}\right]-1\,,\cdots,\,\left[\frac{n}{k}\right]+1\right\}$, and use $T_k^{(i)}$ as in (9). Then, following the outline of the proof of Lemma 5,

$$\mathbf{P}(\mathbf{P}_\omega(T_k^{(i)} > \bar\mu k) > b_n) \le \frac{\exp(-C_2(\bar\mu-\mu_\alpha)^{1/3}k^{1/3}(1-\varepsilon))}{b_n}\,, \tag{20}$$

where we have used the bound

$$\mathbf{P}(T_k^{(i)} > \bar\mu k) \le \exp(-k^{1/3}C_2(\bar\mu-\mu_\alpha)^{1/3})\,,$$

which follows from Theorem 1 using the inequalities

$$P(T_k^{(i)} > \bar\mu k) \le P(X_{[\bar\mu k]} < k) \le P(X_{[\bar\mu k]} < ([\bar\mu k]+1)/\bar\mu)\,.$$

Thus, by the Borel-Cantelli lemma, for $\mathbf{P}$-a.e. ω, there exists an $N_0(\omega)$ such that for all $n > N_0(\omega)$,

$$\mathbf{P}_\omega(T_k^{(i)} > \bar\mu k\,, \text{some } i\in I_n) \le b_n\,. \tag{21}$$

Define $\overline{T}_k^{(i)}$ as in (10). Set $0 < \gamma < (1+\delta)^3|\log\langle\rho\rangle|/C_2^3(\bar\mu-\mu_\alpha)$. With $d_n = \exp(-\gamma(\log n)^3)$, the Borel-Cantelli lemma yields, as in the proof of Lemma 6, that for $\mathbf{P}$-a.e. ω, there exists an $N_1(\omega)$ such that for $n \ge N_1(\omega)$,

$$\mathbf{P}_\omega(T_k^{(i)} \ne \overline{T}_k^{(i)}\,, \text{some } i\in I_n) < d_n\,. \tag{22}$$

Using (21), one concludes as in Lemma 7 that for $\mathbf{P}$-a.e. ω, for $n > N_0(\omega)$, and each $i \in I_n$,

$$\mathbf{P}_\omega(\overline{T}_k^{(i)} > k\bar\mu x) \le (2b_n)^{[x/2]\vee 1}\,. \tag{23}$$

Let $Z_k^{(i)}$, $i = 1, 2, \dots$ denote a sequence of positive, i.i.d random variables with

$$P\Big(\frac{Z_k^{(i)}}{k} < \bar{\mu}\Big) = 0 \,, \quad P\Big(\frac{Z_k^{(i)}}{k} > \bar{\mu} x\Big) = (2b_n)^{[x/2] \vee 1} \,, \quad x \geq 1 \,.$$

The following Lemma takes the place of Lemma 8 in the proof of Theorem 2:

Lemma 10. *For each $\varepsilon' > 0$, we have, for $\lambda_n = -\log(2b_n)/2\bar{\mu}(1+\varepsilon')$,*

$$E \exp\left(\lambda_n Z_k^{(i)}/k\right) \leq e^{\lambda_n \bar{\mu}} + g_n \,,$$

where $g_n \underset{n\to\infty}{\longrightarrow} 0$.

Proof of Lemma 10. Exactly as in the course of the proof of Lemma 8, for n large enough,

$$\begin{aligned} E \exp\left(\lambda_n Z_k^{(i)}/k\right) &= \int_0^\infty P\Big(\frac{Z_k^{(i)}}{k} > \frac{\log u}{\lambda_n}\Big)\, du \\ &\leq e^{\lambda_n \bar{\mu}} + \int_{e^{\lambda_n \bar{\mu}}}^\infty (2b_n)^{\frac{\log u}{2\lambda_n \bar{\mu}}}\, du = e^{\lambda_n \bar{\mu}} + g_n \,, \end{aligned}$$

where

$$g_n = \int_{e^{\lambda_n \bar{\mu}}}^\infty u^{\frac{\log 2b_n}{2\lambda_n \bar{\mu}}}\, du = \int_{e^{\lambda_n \bar{\mu}}}^\infty u^{-(1+\varepsilon')}\, du \underset{n\to\infty}{\longrightarrow} 0 \,.$$

□

Let S_t, $t = 0, 1, \dots$, denote the simple random walk with $S_0 = 0$ and

$$P(S_{t+1} = S_t + 1 | S_t) = 1 - P(S_{t+1} = S_t - 1 | S_t) = 1 - d_n \,,$$

and let

$$M_n = \frac{n C_2^3 (\bar{\mu} - \mu_\alpha)(1-\varepsilon)^2}{(\log n)^3 (1+\delta)^3} \,.$$

Mimicking the proof in Lemma 9, we obtain that

$$P(\inf\{t : S_t = [n/k]\} > M_n) \leq \exp(-n\theta\varepsilon) \,, \tag{24}$$

where $\theta = \gamma C_2^3 (\bar{\mu} - \mu_\alpha)(1-\varepsilon)^2 / (3(1+\delta)^3)$.

Following the proof of Theorem 2, we have

$$\mathbf{P}_\omega[\tau_n > n\mu] \;\leq\; P\Big(\inf\Big\{t:\, S_t = \Big[\frac{n}{k}\Big]\Big\} > M_n\Big) + P\Big(\sum_{i=1}^{M_n} Z_k^{(i)} > n\mu\Big)$$

$$\begin{aligned}
&\leq \; e^{-n\theta\varepsilon} + P\Big(\frac{1}{M_n}\sum_{i=1}^{M_n}\frac{Z_k^{(i)}}{k} > \mu(1-\varepsilon)\Big)\\
&\leq \; e^{-n\theta\varepsilon} + \Big(E\exp\Big(\lambda_n\, Z_k^{(i)}/k\Big)\; e^{-\lambda_n\mu(1-\varepsilon)}\Big)^{M_n}\\
&\leq \; e^{-n\theta\varepsilon} + e^{-\lambda_n M_n(\mu(1-\varepsilon)-\bar{\mu}-\varepsilon)}\,,
\end{aligned}$$

where the second inequality is due to (24) and the last due to Lemma 10.

Plug in the definition of M_n and λ_n to get

$$\limsup_{n\to\infty}\frac{(\log n)^2}{n}\log\mathbf{P}_\omega\left(\tau_n > n\mu\right) \;\leq\; -\frac{C_2^3\,(\bar{\mu}-\mu_\alpha)(1-\varepsilon)^2\,\frac{\delta}{2}\,\Big(\mu(1-\varepsilon)-\bar{\mu}-\varepsilon\Big)}{2(1+\delta)^3\,\bar{\mu}(1+\varepsilon')}\,.$$

Letting ε and $\varepsilon'\to 0$ and $\delta\to 2$, one gets

$$\begin{aligned}
\limsup_{n\to\infty}\frac{(\log n)^2}{n}\log\mathbf{P}_\omega(\tau_n > n\mu) \;&\leq\; -C_2^3(\bar{\mu}-\mu_\alpha)\,\frac{1}{2\cdot 3^3}\,\frac{\mu-\bar{\mu}}{\bar{\mu}}\\
&=\; -C_2^3\,\frac{1}{2\cdot 3^3}\,(\mu-\mu_\alpha)^2\,\frac{t(1-t)}{(1-t)\mu+t\mu_\alpha}\,,
\end{aligned}\tag{25}$$

where we used the definition of $\bar{\mu}$ in the last equality. Optimizing over $t\in(0,1)$ yields

$$\limsup_{n\to\infty}\frac{(\log n)^2}{n}\log\mathbf{P}_\omega(\tau_n > n\mu) \leq -C_2^3\,\frac{1}{2\cdot 3^3}\,(\mu-\mu_\alpha)^2\,\frac{1}{(\sqrt{\mu}+\sqrt{\mu_\alpha})^2}\,.$$

To prove the upper bound in (7), observe that for $v < v' < v_\alpha$, by the same argument as in (14),

$$\begin{aligned}
\limsup_{n\to\infty}\frac{(\log n)^2}{n}\log\mathbf{P}_\omega\Big(\frac{X_n}{n} < v\Big) \;&\leq\; \limsup_{n\to\infty}\frac{(\log n)^2}{n}\log\mathbf{P}_\omega\Big(\tau_{[nv']} > [nv']\,\frac{1}{v'}\Big)\\
&=\; \limsup_{n\to\infty}\, v'\,\frac{(\log[nv'])^2}{[nv']}\log\mathbf{P}_\omega\Big(\tau_{[nv']} > [nv']\,\frac{1}{v'}\Big)\\
&\leq\; -C_2^3\,\frac{1}{2\cdot 3^3}\,v'\Big(\frac{1}{v'}-\frac{1}{v_\alpha}\Big)^2\frac{1}{\Big(\frac{1}{\sqrt{v'}}+\frac{1}{\sqrt{v_\alpha}}\Big)^2}\\
&\leq\; -C_2^3\,\frac{1}{2\cdot 3^3}\,\Big(1-\frac{v'}{v_\alpha}\Big)^2\frac{v_\alpha}{\Big(\sqrt{v'}+\sqrt{v_\alpha}\Big)^2}\,.
\end{aligned}$$

Letting $v'\to v$, and using $v_\alpha/(\sqrt{v}+\sqrt{v_\alpha})^2 \geq 1/4$, we get

$$\limsup_{n\to\infty}\frac{(\log n)^2}{n}\log\mathbf{P}_\omega\Big(\frac{X_n}{n} < v\Big) \leq -C_2^3\,\frac{1}{8\cdot 3^3}\,\Big(1-\frac{v}{v_\alpha}\Big)^2,\tag{26}$$

completing the proof of the upper bound in (7). $\square$

Remark: Even when one uses the results of [6] and replaces C_2 by C_1 in the right hand side of (26), the behaviour of the exponent in the upper bound is quadratic in $(v_\alpha - v)$, which is far from the linear behaviour exhibited by the exponent of the corresponding lower bound. While the constant in the upper bound can be slightly further improved (e.g., by using subsequences in the proof), it seems that a new approach is needed to completely close the gap.

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