Large and Moderate Deviations for the Local Time of a Recurrent Markov Chain on $\mathbb{Z}^2$

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Abstract Let (X_n) be a recurrent Markov chain on $\mathbb{Z}^2$ with $X_0 = (0,0)$ such that for some constant C, $P[X_k = (0,0)] \leq \frac{C}{k}$, and whose truncated Green function is slowly varying at infinity. Let L_n^0 denote the local time at zero of such a Markov chain. We prove various moderate and large deviation statements and limit laws for rescaled versions of L_n^0, including functional versions of these. A version of Strassen's functional law of the iterated logarithm, recently discovered by E. Csáki, P. Révész and J. Rosen, can be derived as a corollary.

Résumé Soit (X_n) une chaîne de Markov récurrente sur $\mathbb{Z}^2$, avec $X_0 = (0,0)$, telle que pour une constante C, $P[X_k = (0,0)] \leq \frac{C}{k}$, et telle que la fonction de Green est de variation lente à l'infini. Avec L_n^0 le temps local de (X_n) à zero, nous démontrons des résultats de grandes déviations et de déviations modérées pour certains changements d'échelle de L_n^0, ainsi qu'une version fonctionelle. Comme corollaire, on note un théorème du logarithme itéré fonctionnel de type Strassen, demontré récemment par E. Csáki, P. Révész, et J. Rosen.

Key words: Local time, Markov chain, large deviations, Strassen's law.

AMS Classifications: 60J10, 60J55, 60F10.

1 Introduction and statement of results

Let (X_n) be a recurrent Markov chain on $\mathbb{Z}^2$ with $X_0 = (0,0)$, and let $g(n) := \sum_{k=0}^{n} P[X_k = (0,0)]$ be the truncated Green function. We can extend g to a continuous, increasing function $g(t), t \geq 0$. Since (X_n) is recurrent, $g(t) \to \infty$ for $t \to \infty$.

We will assume throughout that, for some positive constant C,

$$P[X_k = (0,0)] \leq \frac{C}{k}\,, \tag{1}$$

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hence $g(n) \leq C \log n$. We will also assume throughout that

$$g \text{ is slowly varying at } \infty\,, \tag{2}$$

that is $g(tx)/g(t) \underset{t\to\infty}{\longrightarrow} 1$ for any $x > 0$. Note that (1) is satisfied for symmetric random walks on $\mathbb{Z}^2$, i.e. if $P[X_1 = (y,z)] = P[X_1 = -(y,z)]$, see [6, Proposition 2.14]. Since our results depend only on (1) and (2), they might also apply to symmetric recurrent random walks on $\mathbb{Z}$ in the domain of attraction of a Cauchy random variable.

We denote by L_n^0 the local time of X at $(0,0)$, i.e. $L_n^0 := |\{0 \leq k \leq n : X_k = (0,0)\}|$, and $L_0^0 = 0$. Let $\rho_0 = 0$, $\rho_k = \min\{j : j > \rho_{k-1}, X_j = (0,0)\}$, $k = 1,2,3,\cdots$. It is known, see [6], (and will follow from the proof of Theorem 1), that $L_n^0/g(n)$ converges in distribution to an exponential distribution, i.e.

$$P\left[\frac{L_n^0}{g(n)} \geq y\right] \underset{n\to\infty}{\longrightarrow} e^{-y} \text{ for } y \geq 0\,. \tag{3}$$

Our goal is to investigate the fluctuations of L_n^0, and associated functional laws.

Theorem 1 (Moderate Deviations) *Let $\psi(n)$ be a positive, non-decreasing function such that*

$$\gamma_n := \frac{n}{\psi(n)g(n)} \underset{n\to\infty}{\longrightarrow} \infty\,.$$

Then $L_n^0/g(n)\psi(n)$ satisfies a large deviation principle with speed $\psi(n)g(n)/g(\gamma_n)$ and rate function y.

We refer to [2] for the definition of a large deviation principle. Here, it will be enough to show that

$$\frac{g(\gamma_n)}{\psi(n)g(n)} \log P\left[\frac{L_n^0}{g(n)\psi(n)} \geq y\right] \underset{n\to\infty}{\longrightarrow} -y\,. \tag{4}$$

Theorem 1 is a moderate deviation principle since the speed can vary without changing the rate function. Further, the rate function does not depend on the distribution of ρ_1.

The next theorem gives a large deviation principle for the distributions of L_n^0/n, with rate function which does depend on the distribution of ρ_1.

Theorem 2 (Large Deviations) *Let $\Lambda^*(y) = \sup_{\lambda\leq 0}(\lambda y - \log E[e^{\lambda\rho_1}])$ and*

$$J(y) = \begin{cases} y\Lambda^*\left(\frac{1}{y}\right), & 0 < y \leq 1 \\ 0, & y = 0 \\ +\infty, & \textit{otherwise} \end{cases}$$

Then the distributions of L_n^0/n satisfy a LDP with speed n and rate function J.

Remarks

1. Comparing with Theorem 1, the large deviation principle holds for $\psi(n) = \frac{n}{g(n)}$. In this case, $\gamma_n = 1$ and Theorem 1 does not apply. Considering the proof of Theorem 2, it is easy to show that we have a LDP whenever $\gamma_n \underset{n\to\infty}{\longrightarrow} \alpha$, $0 < \alpha < 1$.

2. Let $p_0 := P[X_1 = (0,0)]$. Then we have $J(1) = -\log p_0$ if $p_0 > 0$ and $J(1) = \infty$ otherwise.

3. Let $L^0(\cdot)$ be the linear interpolation of $L^0_\cdot$ between integer points. We believe (but have not checked the details) that the standard argument (see e.g. [2, Section 5.1]) allows one to conclude that the distributions of $(\frac{L^0(nt)}{n})_{0\le t\le 1}$ satisfy a large deviation principle (in $C[0,1]$) with rate function

$$\tilde{J}(f) = \begin{cases} \int_0^1 J(f'(s))ds, & f \text{ absolutely continuous with derivative} f' \\ +\infty, & \text{otherwise.} \end{cases}$$

As usual, we can derive an Erdös-Renyi law from the large deviation principle:

Corollary 1 *Let $c > 0$ and $\eta_{n,j} := \frac{1}{c\log g(n)}\left(L^0_{j+\lfloor c\log g(n)\rfloor} - L^0_j\right)$, $j = 0,1,2,\cdots,n-\lfloor c\log g(n)\rfloor$. Then $\lim_{n\to\infty}\sup_{j=0,1,\cdots,n-\lfloor c\log g(n)\rfloor}\eta_{n,j} = d_c$, a.s., where $d_c = \inf\{y:\ J(y)\ge \frac{1}{c}\}$.*

For a random walk on $\mathbb{Z}$, this complements results of [5].

We next turn to the appropriate functional statements. Let $\psi(n)$ and γ_n be as in the statement of Theorem 1, and let $t(n,x)$ be a sequence of positive, increasing (in n,x) functions satisfying, for any $x \in]0,1]$,

$$\lim_{n\to\infty}\frac{g\left(\frac{t(n,x)}{g(n)\psi(n)}\right)}{g(\gamma_n)} = x > 0\,. \tag{5}$$

For example, if $g(n) \sim C\log n$, and $\frac{\log\psi(n)}{\log n}\underset{n\to\infty}{\longrightarrow} 0$, we can take $t(n,x) = n^x$. If $g(n) \sim C\log n$ and $\psi(n) = n^\beta$, $(0<\beta<1)$, we can take $t(n,x) = n^{x(1-\beta)+\beta}$. If $g(n)\sim C\log_2 n$ and $\frac{\log\psi(n)}{\log n}\underset{n\to\infty}{\longrightarrow} 0$, we can take $t(n,x) = e^{(\log n)^x}$ (here and throughout, $\log_k n$ denotes the $k-$th iterated logarithm function). If $g(n)\sim C\log_2 n$ and $\psi(n) = n^\beta$, $(0<\beta<1)$, we can take $t(n,x) = n^\beta e^{(\log n)^x}$.

It is straightforward to check, using (5), that for $0\le x_1 < x_2 \le 1$, we have

$$\frac{t(n,x_1)}{t(n,x_2)}\underset{n\to\infty}{\longrightarrow} 0\,. \tag{6}$$

Let

$$\overline{L}_n(x) := \frac{L^o_{t(n,x)}}{g(n)\psi(n)},\quad x\in[0,1]\,.$$

Note that $\overline{L}_n(x)\in M_+$, the space of non-negative Borel measures on $[0,1]$. Equip M_+ with the topology of weak convergence. Our main functional statement is the following:

Theorem 3 (Functional Moderate Deviations) *$\overline{L}_n(x)$ satisfies in M_+ a large deviation principle with speed $g(n)\psi(n)/g(\gamma_n)$ and rate function*

$$I(m) = \begin{cases} \int_0^1 \frac{1}{x}m(dx) &, \quad \frac{1}{x}\in L_1(m) \\ \infty &, \quad \textit{otherwise.} \end{cases}$$

As in the one-dimensional case, we can deduce convergence in distribution from our large deviation bounds, taking $\psi(n) \equiv 1$.

Theorem 4 (Functional Limit Law) *Let $t(n,x)$ be such that $g(t(n,x)) \sim xg(n)$, $x \in [0,1]$. The distributions of $\left(\frac{L^0_{t(n,x)}}{g(n)}\right)_{0\le x\le 1}$ converge weakly to $\mu \in M_1(M_+)$, the distribution of the process $(Z_x)_{0\le x\le 1}$ with increasing paths and independent increments given by*

$$P[Z_{x_2} - Z_{x_1} \in B] = \frac{x_1}{x_2}\delta_o(B) + \left(1 - \frac{x_1}{x_2}\right)\int_B \frac{1}{x_2} e^{-\frac{1}{x_2}u}\, du\,, \tag{7}$$

for any $0 \le x_1 < x_2 \le 1$, B Borel subset of $[0,\infty[$.

J. Bertoin kindly pointed out to us that in fact the process $(Z_x)_{0\le x\le 1}$ in Theorem 4 is a pure jump process which can be constructed from an inhomogeneous Poisson point process. Indeed, one may construct a Poisson point process $N(x,z)$ on $[0,1]\times \mathbb{R}_+$ with intensity $n(x,z)dxdz = x^{-2}\exp(-z/x)dxdz$ and define $Y_x = \int_0^\infty z d_z N(x,z)$. Obviously, $(Y_x)_{0\le x\le 1}$ possesses increasing paths and independent increments. Moreover, it is not hard to check, using the identity valid for any $\alpha, \beta > 0$,

$$\lim_{\epsilon\to 0}\left(\int_\epsilon^\infty \frac{e^{-\alpha z}}{z}dz - \int_\epsilon^\infty \frac{e^{-\beta z}}{z}dz\right) = \log\beta - \log\alpha\,,$$

that for any $\lambda \ge 0$,

$$E\Big(\exp\left(-\lambda(Y_{x+y} - Y_x)\right)\Big) = \frac{1+\lambda x}{1+\lambda(x+y)} = E\Big(\exp\left(-\lambda(Z_{x+y} - Z_x)\right)\Big)\,,$$

proving that the processes $(Z_x)_{0\le x\le 1}$ and $(Y_x)_{0\le x\le 1}$ have the same law.

We close this section by mentioning that the functional moderate deviations of Theorem 3 are strong enough to derive by standard arguments the following Strassen law of the iterated logarithm presented in [1, Theorem 5]. Obtaining such a derivation was actually the original motivation for this work. Since the arguments are standard, see [3, Theorem 1.4.1], we do not provide a proof.

Theorem 5 (E. Csáki, P. Révész and J. Rosen) *Let $t(n,x)$ be such that $g(t(n,x)) \sim xg(n)$, $x \in [0,1]$. The set $\left(\frac{L^0_{t(n,x)}}{g(n)\log_2 g(n)}\right)_{0\le x\le 1}$, n large enough, is relatively compact in M_+ with limit points K, where $K = \{m : I(m) \le 1\}$.*

2 Proofs

We begin by stating some simple bounds on $g(n)$.

Lemma 1 *We have*

$$\lim_{n\to\infty}\frac{g(n)}{g(ng(n))} = 1\,, \tag{8}$$

and

$$\lim_{n\to\infty}\frac{g(n)}{g(n/g(n))} = 1\,. \tag{9}$$

Proof of Lemma 1
We have

$$\begin{aligned} g(ng(n)) - g(n) &\leq \sum_{j=n}^{\lceil ng(n) \rceil} P[X_j = (0,0)] \\ &\leq C \sum_{j=n}^{\lceil ng(n) \rceil} \frac{1}{j} \leq C' \log g(n) , \end{aligned}$$

where C' is some (fixed, depending on C) constant. The limit (8) follows by dividing by $g(ng(n))$ and using the monotonicity of $g(\cdot)$. The proof of (9) is analogous. $\square$

Lemma 1 is needed for the following crucial estimate for the tail of the distribution of the excursion ρ_1. For a more precise statement, which we do not need here, see [6].

Proposition 1

$$P[\rho_1 > n] \leq \frac{1}{g(n)}$$

and

$$P[\rho_1 > n] \sim \frac{1}{g(n)}$$

i.e. $g(n)P[\rho_1 > n] \underset{n \to \infty}{\longrightarrow} 1$.

Proof of Proposition 1:

1. A last exit decomposition gives

$$\sum_{k=0}^{n} P[X_k = (0,0)]\, P[L^0_{n-k} = 0] = 1 \, .$$

   Since $P[L^0_{n-k} = 0] \geq P[L^0_n = 0], k = 0, 1, ..., n$, this implies $g(n)P[L^0_n = 0] \leq 1$, hence

$$P[\rho_1 > n] = P[L^0_n = 0] \leq \frac{1}{g(n)} \, .$$

2. In the same way,

$$1 \leq \sum_{j=0}^{k} P[X_j = (0,0)] P[L^0_{n-k} = 0] + \sum_{j=k+1}^{n} P[X_j = (0,0)]$$

   hence $1 \leq g(k)P[L^0_{n-k} = 0] + g(n) - g(k)$, so

$$g(k)P[L^0_{n-k} = 0] \geq 1 - (g(n) - g(k)) \, . \tag{10}$$

   Choose $k = k(n) = \left\lfloor n - \frac{n}{g(n)} \right\rfloor$, and note that, for some $C', C'' > 0$,

$$g(n) - g(k) = \sum_{j=k}^{n} P[X_j = (0,0)] \leq C \sum_{j=k}^{n} \frac{1}{j} \leq C'(\log n - \log k) \leq C'' \log(1 - \frac{1}{g(n)}) \underset{n \to \infty}{\longrightarrow} 0 \, .$$

   This, together with (9) of Lemma 1, yields the proposition. $\square$

Proof of Theorem 1

We begin with a quick proof of the lower bound in (4). Let $Y_1, Y_2, \dots$ be i.i.d. with the same distribution as ρ_1. Then

$$
\begin{aligned}
P[L_n^0 \geq \psi(n)g(n)y] &\geq P\left[\sum_{i=1}^{\lceil g(n)\psi(n)y\rceil} Y_i \leq n\right] \\
&\geq P\left[\max_{1\leq i\leq \lceil g(n)\psi(n)y\rceil} Y_i \leq \frac{n}{\lceil g(n)\psi(n)y\rceil}\right] \\
&= \left(1 - P\left[\rho_1 > \frac{n}{\lceil g(n)\psi(n)y\rceil}\right]\right)^{\lceil g(n)\psi(n)y\rceil}
\end{aligned}
$$

Now apply Proposition 1 and the fact that $g(\cdot)$ is slowly varying to get

$$
\liminf_{n\to\infty} \frac{g\left(\frac{n}{\psi(n)g(n)}\right)}{g(n)\psi(n)} \log P[L_n^0 \geq \psi(n)g(n)y] \geq -y \, .
$$

We next turn to the proof of the upper bound. We follow the standard strategy to apply Chebycheff's inequality and to optimize over the parameter. Due to Chebycheff's inequality,

$$
P[L_n^0 \geq g(n)\psi(n)y] \leq P\left[\sum_{i=1}^{\lfloor g(n)\psi(n)y\rfloor} Y_i \leq n\right] \leq E\left[e^{-\lambda_n Y_1}\right]^{\lfloor g(n)\psi(n)y\rfloor} e^{\lambda_n n} \tag{11}
$$

for each $\lambda_n > 0$. Recall $\gamma_n = \frac{n}{\psi(n)g(n)}$. Taking logarithms and dividing by $\frac{g(n)\psi(n)}{g(\gamma_n)}$, (11) yields

$$
\frac{g(\gamma_n)}{g(n)\psi(n)} \log P[L_n^0 \geq g(n)\psi(n)y] \leq g(\gamma_n) y \frac{\lfloor g(n)\psi(n)y\rfloor}{g(n)\psi(n)y} \log E[e^{-\lambda_n Y_1}] + \frac{g(\gamma_n)\lambda_n n}{\psi(n)g(n)} \tag{12}
$$

Next we show that for each $\delta > 0$, and $C_n > 0$ large enough, we have

$$
\log E[e^{-\lambda_n Y_1}] \leq \frac{1-\delta}{g(C_n)}(e^{-\lambda_n C_n} - 1). \tag{13}
$$

Indeed, observe that

$$
\begin{aligned}
\log E[e^{-\lambda_n Y_1}] &= \log E[e^{-\lambda_n \rho_1}] \leq E[e^{-\lambda_n \rho_1}] - 1 \\
&\leq e^{-\lambda_n C_n} P[\rho_1 \geq C_n] + P[\rho_1 < C_n] - 1 \\
&= P[\rho_1 \geq C_n](e^{-\lambda_n C_n} - 1) \leq \frac{1-\delta}{g(C_n)}(e^{-\lambda_n C_n} - 1)
\end{aligned}
$$

where we used Proposition 1 in the last inequality.

Substituting this estimate in (12), we get

$$
\frac{g(\gamma_n)}{\psi(n)g(n)} \log P[L_n^0 \geq g(n)\psi(n)y] \leq y(1-\delta)\frac{g(\gamma_n)}{g(C_n)}(e^{-\lambda_n C_n} - 1) + \frac{g(\gamma_n)\gamma_n}{C_n}\lambda_n C_n \, . \tag{14}
$$

Choose $C_n = K\gamma_n g(\gamma_n)$, $\lambda_n = \frac{K'}{C_n}$ with $K, K' > 0$. Then the r.h.s. of (14) is

$$
y(1-\delta)\frac{g(\gamma_n)}{g(K\gamma_n g(\gamma_n))}(e^{-K'} - 1) + \frac{g(\gamma_n)}{Kg(\gamma_n)}K' \, . \tag{15}
$$

Due to Lemma 1 and the fact that $g(\cdot)$ is slowly varying, $\frac{g(\gamma_n)}{g(K\gamma_n g(\gamma_n))} \underset{n\to\infty}{\longrightarrow} 1$. Hence (14) and (15) yield

$$\limsup_{n\to\infty} \frac{g(\gamma_n)}{\psi(n)g(n)} \log P[L_n^0 \geq g(n)\psi(n)y] \leq y(1-\delta)(e^{-K'}-1) + \frac{K'}{K}$$

and the upper bound follows by letting $\delta \to 0$, $K' \to \infty$, $\frac{K'}{K} \to 0$. □

Remark In particular, taking in the proof of the upper and the lower bound $\psi(n) \equiv 1$, we have

$$\frac{g\left(\frac{n}{g(n)}\right)}{g(n)} \log P\left[\frac{L_n^0}{g(n)} \geq y\right] \underset{n\to\infty}{\longrightarrow} -y\,.$$

Together with (9) in Lemma 1, this implies that for $y \geq 0$,

$$P\left[\frac{L_n^0}{g(n)} \geq y\right] \underset{n\to\infty}{\longrightarrow} e^{-y}\,,$$

as noted in (3).

Proof of Theorem 2
Note first that $P[L_n^0 \geq ny] = 0$ if $y > 1$. As in the proof of Theorem 1, we have

$$P\left[\sum_{i=1}^{\lceil ny\rceil} Y_i \leq n\right] \leq P[L_n^0 \geq ny] \leq P\left[\sum_{i=1}^{\lfloor ny\rfloor} Y_i \leq n\right]\,.$$

But

$$P\left[\sum_{i=1}^{\lfloor ny\rfloor} Y_i \leq n\right] \leq P\left[\frac{1}{\lfloor ny\rfloor}\sum_{i=1}^{\lfloor ny\rfloor} Y_i \leq \frac{1}{y}\right]$$

so we ask about large deviations of the arithmetic mean of a sequence of i.i.d. random variables. Cramér's theorem (see [2, Theorem 2.2.3]) implies that the distributions of $\frac{1}{\lfloor ny\rfloor}\sum_{i=1}^{\lfloor ny\rfloor} Y_i$ (or $\frac{1}{\lceil ny\rceil}\sum_{i=1}^{\lceil ny\rceil} Y_i$) satisfy a LDP with speed $\lfloor ny\rfloor$ (or $\lceil ny\rceil$) and rate function Λ^*. Note that $Y_1 \geq 0$, $E[Y_1] = \infty$ hence $\Lambda^*(y) \to 0$ for $y \to \infty$. Since we have

$$\frac{1}{n}\log P\left[\frac{1}{\lfloor ny\rfloor}\sum_{i=1}^{\lfloor ny\rfloor} Y_i \leq \frac{1}{y}\right] = \frac{\lfloor ny\rfloor}{n}\frac{1}{\lfloor ny\rfloor}\log P\left[\frac{1}{\lfloor ny\rfloor}\sum_{i=1}^{\lfloor ny\rfloor} Y_i \leq \frac{1}{y}\right]$$

and $\frac{\lfloor ny\rfloor}{n} \underset{n\to\infty}{\longrightarrow} y$, the claim follows. □

In order to prove Corollary 1, we need the following preliminary proposition.

Proposition 2 *Let $\psi(n) \to 0$, $\psi(n)g(n) \to \infty$. Then, for each $x > 0$, $\frac{1}{\psi(n)}P\left[\frac{L_n^0}{g(n)\psi(n)} \leq x\right] \underset{n\to\infty}{\longrightarrow} x$.*

Proof of Proposition 2

1. We have

$$P\left[L_n^0 \le g(n)\psi(n)x\right] \le P\left[\sum_{j=1}^{\lceil g(n)\psi(n)x\rceil} Y_j \ge n\right] \le P\left[\max_{1\le j\le \lceil g(n)\psi(n)x\rceil} Y_j \ge \frac{n}{\lceil g(n)\psi(n)x\rceil}\right]$$
$$= 1-\left(1-P\left[Y_1 \ge \frac{n}{\lceil g(n)\psi(n)x\rceil}\right]\right)^{\lceil g(n)\psi(n)x\rceil}$$
$$\le 1-\left(1-\frac{1}{g\left(\frac{n}{\lceil g(n)\psi(n)x\rceil}\right)}\right)^{\lceil g(n)\psi(n)x\rceil}$$

where we used Proposition 1 in the last inequality. Since $1-z \le -\log z$, the last term is

$$\le -\lceil g(n)\psi(n)x\rceil \log\left(1-\frac{1}{g\left(\frac{n}{\lceil g(n)\psi(n)x\rceil}\right)}\right).$$

Hence

$$\frac{1}{\psi(n)}P\left[\frac{L_n^0}{g(n)\psi(n)} \le x\right] \le -\frac{\lceil g(n)\psi(n)x\rceil}{g(n)\psi(n)} \log\left(1-\frac{1}{g\left(\frac{n}{\lceil g(n)\psi(n)x\rceil}\right)}\right)^{g(n)}. \tag{16}$$

Provided that

$$\frac{g(n)}{g\left(\frac{n}{g(n)\psi(n)}\right)} \underset{n\to\infty}{\longrightarrow} 1, \tag{17}$$

(16) implies that

$$\limsup_{n\to\infty} \frac{1}{\psi(n)}P\left[\frac{L_n^0}{g(n)\psi(n)} \le x\right] \le x. \tag{18}$$

But (17) holds true since

$$g(n) \ge g\left(\frac{n}{g(n)\psi(n)}\right) \ge g\left(\frac{n}{g(n)}\right)$$

and $\frac{g(n)}{g(n/g(n))} \underset{n\to\infty}{\longrightarrow} 1$ due to Lemma 1.

2.

$$P\left[L_n^0 \le g(n)\psi(n)x\right] \ge P\left[\sum_{j=1}^{\lfloor g(n)\psi(n)x\rfloor} Y_j \ge n\right]$$
$$\ge P\left[\max_{1\le j\le \lfloor g(n)\psi(n)x\rfloor} Y_j \ge n\right] = 1-(1-P\left[Y_1 \ge n\right])^{\lfloor g(n)\psi(n)x\rfloor}.$$

Now we use the inequality $1-z \ge -z\log z \quad (0<z<1)$ with $z=(1-P[Y_1\ge n])^{\lfloor g(n)\psi(n)x\rfloor}$ to get

$$P[L_n^0 \le g(n)\psi(n)x] \ge -\frac{\lfloor g(n)\psi(n)x\rfloor}{g(n)x}\log(1-P[Y_1\ge n])^{g(n)x}\cdot(1-P[Y_1\ge n])^{\lfloor g(n)\psi(n)x\rfloor}. \tag{19}$$

Proposition 1 implies that

$$(1-P[Y_1\ge n])^{g(n)x} \underset{n\to\infty}{\longrightarrow} e^{-x}$$

and therefore

$$(1 - P[Y_1 \geq n])^{\lfloor g(n)\psi(n)x \rfloor} \underset{n\to\infty}{\longrightarrow} 1\,.$$

We conclude from (19) that

$$\liminf_{n\to\infty} \frac{1}{\psi(n)} P\left[\frac{L_n^0}{g(n)\psi(n)} \geq x\right] \geq x\,.$$

□

Proof of Corollary 1

1. Let $d \in \mathbb{R}, J(d) > \frac{1}{c}$, choose $\delta > 0$ such that $J(d) - \delta > \frac{1}{c}$, and fix any $d' > d$. We show that

$$P\Big[\sup_{j=0,1,\cdots,n-\lfloor c \log g(n) \rfloor} \eta_{n,j} \geq d' \text{ for infinitely many } n\Big] = 0\,. \tag{20}$$

Let $\psi(n) = (\log g(n))^\gamma$ where $\gamma > 1$. Since we can take the sup in $\sup_{j=0,1,\cdots,n-\lfloor c\log g(n)\rfloor} \eta_{n,j}$ over those j with $X_j = (0,0)$ only, without changing the value, and since $\eta_{n,j}$ has the same distribution as $\eta_{n,0}$ for those j, we have

$$P\Big[\sup_{j=0,1,\cdots,n-\lfloor c \log g(n) \rfloor} \eta_{n,j} \geq d\Big] \leq P\Big[L_n^0 \geq g(n)\psi(n)\Big] + \psi(n)g(n)P\Big[\eta_{n,0} \geq d\Big]\,. \tag{21}$$

Now we have to estimate the terms on the r.h.s. of (21):

$$P\Big[L_n^0 \geq g(n)\psi(n)\Big] \leq e^{-\psi(n)(1-\delta)} \tag{22}$$

for n big enough, due to Theorem 1 and

$$P\Big[\eta_{n,0} \geq d\Big] \leq e^{-c\log g(n)(J(d)-\delta)} \tag{23}$$

for n big enough, due to Theorem 2.

Let $\lambda > 1$, $n_0 = 0$ and $n_k = \lceil g^{-1}(\lambda^k) \rceil$, $k = 1, 2, \cdots$. Then we see from (22) and (23), applying the Borel-Cantelli lemma, that

$$P\Big[\sup_{j=0,1,\cdots,n_k-\lfloor c \log g(n_k) \rfloor} \eta_{n_k,j} \geq d \text{ for infinitely many } k\Big] = 0\,.$$

In other words, we have proved (20) along the subsequence (n_k) with d replacing d'. Let $n_k \leq n \leq n_{k+1}$ and observe that, for $j = 0, 1, \cdots, n - \lfloor c \log g(n) \rfloor$,

$$\begin{aligned} \eta_{n,j} \leq \eta_{n_{k+1},j} \frac{\log g(n_{k+1})}{\log g(n)} &\leq \eta_{n_{k+1},j} \frac{\log g(n_{k+1})}{\log g(n_k)} \\ &\leq \eta_{n_{k+1},j} \frac{k+1}{k} \end{aligned}$$

For k big enough, $\eta_{n_{k+1},j} < d$ implies $\eta_{n,j} < d'$. This completes the proof of (20).

2. Let $d \in \mathbb{R}, J(d) < \frac{1}{c}$. Choose $\delta > 0$ and $\lambda > 1$ such that $\lambda(J(d)+\delta) < \frac{1}{c}$. We will construct a subsequence n_k such that

$$P[\sup_{0\leq j \leq n_k - \lfloor c\log g(n_k) \rfloor} \eta_{n_k,j} < d \text{ for infinitely many k}] = 0\,. \tag{24}$$

Fixing n, let $j_0^n := 0$, $j_m^n := \inf\{j : j > j_{m-1}^n + \lfloor c\log g(n)\rfloor, X_j = (0,0)\}$, $M^n := M^n(\omega) = \max\{m : j_m^n \leq n\}$ and $J^n := \{j_0^n, \ldots, j_{M^n-1}^n\}$. Then $(\eta_{n,j})_{j\in J^n}$ are i.i.d. with the same distribution as $\eta_{n,0}$. Let $\psi(n)$, to be determined below, satisfy the assumptions of Proposition 2. We have

$$P[\sup_{0\leq j\leq n-\lfloor c\log g(n)\rfloor} \eta_{n,j} < d] \leq P[M^n < \frac{\lfloor g(n)\psi(n)\rfloor}{\lfloor c\log g(n)\rfloor}] + P[\eta_{n,0} < d]^{\frac{\lfloor g(n)\psi(n)\rfloor}{\lfloor c\log g(n)\rfloor}}. \tag{25}$$

But, for each $\tilde{\delta} > 0$, and all n large enough,

$$P[M^n < \frac{\lfloor g(n)\psi(n)\rfloor}{\lfloor c\log g(n)\rfloor}] \leq P[L_n^0 < \lfloor g(n)\psi(n)\rfloor] \leq (1+\tilde{\delta})\psi(n) \tag{26}$$

for n large enough, where we used Proposition 2 in the last inequality. Turning now to the second term in (25), we first note that, by Theorem 2, for all n large enough,

$$P[\eta_{n,0} \geq d] \geq e^{-c\log g(n)(J(d)+\delta)} \geq e^{-\beta\log g(n)}$$

for n large enough, where $\beta := c(J(d)+\delta) < 1$. Hence

$$P[\eta_{n,0} < d]^{\frac{\lfloor g(n)\psi(n)\rfloor}{\lfloor c\log g(n)\rfloor}} \leq \left(1 - e^{-\beta\log g(n)}\right)^{\frac{\lfloor g(n)\psi(n)\rfloor}{\lfloor c\log g(n)\rfloor}} \leq e^{-\frac{(1-\delta)\psi(n)g(n)^{1-\beta}}{c\log g(n)}} \tag{27}$$

for n large enough. Considering (26) and (27), it remains to specify a subsequence (n_k) and a positive function $\psi(\cdot)$ such that $\psi(n) \underset{n\to\infty}{\longrightarrow} 0$, $\psi(n)g(n) \underset{n\to\infty}{\longrightarrow} \infty$ and

$$\sum_k \psi(n_k) < \infty \tag{28}$$

$$\sum_k e^{-\frac{(1-\delta)\psi(n_k)g(n_k)^{1-\beta}}{c\log g(n_k)}} < \infty \tag{29}$$

Then, (24) follows from (25), (26) and (27) together with the Borel-Cantelli lemma. We finish the proof by observing that (28) and (29) are satisfied for $n_k = g^{-1}(2^k)$ and $\psi(n) = \log g(n)/g(n)^\gamma$ where $0 < \gamma < 1-\beta$. □

Proof of Theorem 3

We begin by proving a finite distribution result, from which the required LDP will follow by standard projective limits arguments. Note first that for $0 = x_0 < x_1 < x_2 < \cdots < x_k \leq 1$, and $0 = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_k < \infty$, and with Y_i as in the proof of Theorem 1,

$$\begin{aligned}
&P[\overline{L}_n(x_1) \geq a_1, \overline{L}_n(x_2) \geq a_2, \cdots, \overline{L}_n(x_k) \geq a_k] \\
&\leq P\left[\sum_{i=1}^{\lfloor g(n)\psi(n)a_1\rfloor} Y_i \leq t(n,x_1), \cdots, \sum_{i=1}^{\lfloor g(n)\psi(n)a_k\rfloor} Y_i \leq t(n,x_k)\right] \\
&\leq P\Bigg[\sum_{i=1}^{\lfloor g(n)\psi(n)a_1\rfloor} Y_i \leq t(n,x_1), \sum_{i=\lfloor g(n)\psi(n)a_1\rfloor+1}^{\lfloor g(n)\psi(n)a_2\rfloor} Y_i \leq t(n,x_2), \cdots, \\
&\qquad \sum_{i=\lfloor g(n)\psi(n)a_{k-1}\rfloor+1}^{\lfloor g(n)\psi(n)a_k\rfloor} Y_i \leq t(n,x_k)\Bigg] \\
&= \prod_{j=1}^k P\left[\sum_{i=\lfloor g(n)\psi(n)a_{j-1}\rfloor+1}^{\lfloor g(n)\psi(n)a_j\rfloor} Y_i \leq t(n,x_j)\right].
\end{aligned}$$

Write $g(n)\psi(n) = g\Big(t(n,x_j)\Big)\overline{\psi}_j\Big(t(n,x_j)\Big)$, then for any $\delta > 0$ and n large enough,

$$
\begin{aligned}
& P\Big[\overline{L}_n(x_1) \geq a_1, \cdots, \overline{L}_n(x_k) \geq a_k\Big] \\
& \leq \prod_{j=1}^{k} P\left[\sum_{i=\lfloor g(t(n,x_j))\overline{\psi}_j(t(n,x_j))a_{j-1}\rfloor+1}^{\lfloor g(t(n,x_j))\overline{\psi}_j(t(n,x_j))a_j\rfloor} Y_i \leq t(n,x_j)\right] \\
& \leq \prod_{j=1}^{k} P\left[\sum_{i=1}^{\lfloor g(t(n,x_j))\overline{\psi}_j(t(n,x_j))(a_j-a_{j-1})\rfloor-1} Y_i \leq t(n,x_j)\right] \\
& \leq \prod_{j=1}^{k} \exp\left(-(a_j-a_{j-1})\frac{\overline{\psi}_j(t(n,x_j))g(t(n,x_j))}{g\left(\frac{t(n,x_j)}{\overline{\psi}_j(t(n,x_j))g(t(n,x_j))}\right)}(1-\delta)\right) \\
& = \prod_{j=1}^{k} \exp\left(-(a_j-a_{j-1})\frac{\psi(n)g(n)}{g\left(\frac{t(n,x_j)}{\psi(n)g(n)}\right)}(1-\delta)\right)
\end{aligned}
$$

where the last inequality holds for n large enough and follows from the proof of the upper bound in Theorem 1. Therefore, using the assumption (5),

$$
\limsup_{n\to\infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P[\overline{L}_n(x_1) \geq a_1, \cdots, \overline{L}_n(x_k) \geq a_k] \leq -\sum_{j=1}^{k}(a_j - a_{j-1})\frac{(1-\delta)}{x_j}\,.
$$

Taking now $\delta \to 0$ yields

$$
\limsup_{n\to\infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P[\overline{L}_n(x_1) \geq a_1, \cdots, \overline{L}_n(x_k) \geq a_k] \leq -\sum_{j=1}^{k}\frac{(a_j - a_{j-1})}{x_j}\,, \tag{30}
$$

proving a finite dimensional upper bound.

We next turn to a complementary lower bound. We first show that

$$
\liminf_{n\to\infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P[\overline{L}_n(x_1) \geq a_1, \cdots, \overline{L}_n(x_k) \geq a_k] \geq -\sum_{j=1}^{k}\frac{a_j - a_{j-1}}{x_j}\,. \tag{31}
$$

Indeed, assume w.l.o.g. $a_{j-1} < a_j, j = 1, 2, \cdots, k$. We have, setting $\varphi_{n,j} := \lceil g(n)\psi(n)a_j\rceil$,

$$
\begin{aligned}
& P\left[\frac{L^o_{t(n,x_j)}}{g(n)\psi(n)} \geq a_j, j = 1, 2, \cdots, k\right] \\
\geq\ \ & P\left[\sum_{i=1}^{\varphi_{n,1}} Y_i \leq t(n,x_1),\ \sum_{i=\varphi_{n,1}+1}^{\varphi_{n,2}} Y_i \leq t(n,x_2) - t(n,x_1),\ \cdots\right. \\
& \qquad \left.\sum_{i=\varphi_{n,k-1}+1}^{\varphi_{n,k}} Y_i \leq t(n,x_k) - t(n,x_{k-1})\right] \\
\geq\ \ & \prod_{j=1}^{k} P\left[\sum_{\varphi_{n,j-1}+1}^{\varphi_{n,j}} Y_i \leq t(n,x_j) - t(n,x_{j-1})\right]\,.
\end{aligned} \tag{32}
$$

Observe that for $j = 1, 2, \cdots, n$

$$
\begin{aligned}
& P\left[\sum_{i=\varphi_{n,j-1}+1}^{\varphi_{n,j}} Y_i \leq t(n,x_j) - t(n,x_{j-1})\right] \\
\geq \quad & P\left[\max_{\varphi_{n,j-1}+1\leq i\leq \varphi_{n,j}} Y_i \leq \frac{t(n,x_j)-t(n,x_{j-1})}{\varphi_{n,j}-\varphi_{n,j-1}-1}\right] \\
\geq \quad & P\left[\max_{\varphi_{n,j-1}+1\leq i\leq \varphi_{n,j}} Y_i \leq \frac{t(n,x_j)-t(n,x_{j-1})}{\varphi_{n,j}}\right] \\
\geq \quad & \left(1-\frac{1}{g\left(\frac{t(n,x_j)-t(n,x_{j-1})}{\varphi_{n,j}}\right)}\right)^{\varphi_{n,j}-\varphi_{n,j-1}-1} \qquad (33)
\end{aligned}
$$

where the last inequality is due to Proposition 1. Note that due to (5) and (6),

$$
\frac{g(\gamma_n)}{g\left(\frac{t(n,x_j)-t(n,x_{j-1})}{\lceil g(n)\psi(n)a_j\rceil}\right)} \underset{n\to\infty}{\longrightarrow} \frac{1}{x_j} \qquad (34)
$$

(31) now follows from (32), (33) and (34).

In the second step, we prove that, for $0 < \delta < \min\{a_j - a_{j-1}, j = 1, 2, \cdots k\}$ we have

$$
\liminf_{n\to\infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P[\overline{L}_n(x_1) \in (a_1-\delta, a_1+\delta), \ldots, \overline{L}_n(x_k) \in (a_k-\delta, a_k+\delta)] \geq -\sum_{j=1}^{k} \frac{a_j - a_{j-1}}{x_j}. \qquad (35)
$$

To prove (35), observe that

$$
\begin{aligned}
& P\left[\frac{L^o_{t(n,x_j)}}{\psi(n)g(n)} \in (a_j-\delta, a_j+\delta), j = 1, 2, \cdots, k\right] \\
\geq \quad & P\left[\frac{L^o_{t(n,x_j)}}{g(n)\psi(n)} \geq a_j - \delta,\, j = 1, 2, \cdots, k\right] - \sum_{\ell=1}^{k} P\left[\frac{L^o_{t(n,x_j)}}{g(n)\psi(n)} \geq a_j - \delta, j \neq \ell, \frac{L^o_{t(n,x_\ell)}}{g(n)\psi(n)} \geq a_\ell + \delta\right].
\end{aligned}
$$

Since

$$
\begin{aligned}
& \liminf_{n\to\infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P\left[\frac{L^o_{(n,x_j)}}{g(n)\psi(n)} \geq a_j - \delta, j = 1, 2, \cdots, k\right] \\
& \geq -\frac{a_1-\delta}{x_1} - \sum_{j=2}^{k} \frac{a_j - a_{j-1}}{x_j} \geq -\sum_{j=1}^{k} \frac{a_j - a_{j-1}}{x_j}
\end{aligned}
$$

due to the first step, it is enough to show that for $\ell = 1, 2, \cdots, k$ we have

$$
\limsup_{n\to\infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P\left[\frac{L^o_{t(n,x_j)}}{g(n)\psi(n)} \geq a_j - \delta, j \neq \ell, \frac{L^o_{t(n,x_\ell)}}{g(n)\psi(n)} \geq a_\ell + \delta\right] < -\sum_{j=1}^{k} \frac{a_j - a_{j-1}}{x_j}.
$$

But, using the upper bound (30), we have

$$
\liminf_{n\to\infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P\left[\frac{L^o_{t(n,x_j)}}{g(n)\psi(n)} \geq a_j - \delta, j \neq \ell, \frac{L^o_{t(n,x_\ell)}}{g(n)\psi(n)} \geq a_\ell + \delta\right]
$$

$$
\begin{aligned}
&\leq -\sum_{j=1}^{\ell-1} \frac{a_j - a_{j-1}}{x_j} - \frac{a_\ell + 2\delta - a_{\ell-1}}{x_\ell} - \frac{a_{\ell+1} - a_\ell - 2\delta}{x_{\ell+1}} - \sum_{j=\ell+2}^{k} \frac{a_j - a_{j-1}}{x_j} \\
&< -\sum_{j=1}^{k} \frac{a_j - a_{j-1}}{x_j}
\end{aligned}
$$

where we used $\frac{2\delta}{x_\ell} - \frac{2\delta}{x_{\ell+1}} > 0$ in the last inequality. This completes the proof of the lower bound.

It now follows from (30) and (35) that for $0 < x_1 < \cdots < x_k < 1$, the random vector $\{\overline{L}_n(x_j)\}_{j=1}^k$ satisfies in $\mathbb{R}^k$ the LDP with good rate function

$$
I_k(y_1, \cdots, y_k) = \sum_{j=1}^{k} \frac{(y_j - y_{j-1})}{x_j}.
$$

where $y_0 := 0$. By [2, Thm 4.6.1] (see Section 5.1 in [2] for a similar argument), we have that the random monotone function $\{\overline{L}_n(x)\}_{x\in[0,1]}$ satisfies the LDP in $M_+^\omega([0,1])$ (with $M_+^\omega([0,1])$ denoting $M_+([0,1])$ equipped with the topology of pointwise convergence) with good rate function

$$
I_\chi(m) = \sup_{0=x_0<x_1<\cdots<x_k<1} \sum_{i=1}^{k} \frac{m(x_i) - m(x_{i-1})}{x_i}.
$$

It then follows by monotone convergence that

$$
I_\chi(m) = I(m) = \int_0^1 \frac{m(dx)}{x}.
$$

Finally, note that the topology in $M_+^\omega([0,1])$ is stronger than the topology in $M_+([0,1])$, which concludes the proof of the theorem by an application of [2, Corollary 4.2.6]. $\square$

Proof of Theorem 4 Let $0 = a_0 < a_1 < \cdots < a_k \leq 1$ as before. Recall that with $\psi(n) \equiv 1$, (30) and (31) imply that

$$
P\left(\frac{L^0_{t(n,x_j)}}{g(n)} \geq a_j, j = 1, 2, \cdots, k\right) \underset{n\to\infty}{\longrightarrow} \exp\left(-\sum_{j=1}^{k} \frac{a_j - a_{j-1}}{x_j}\right).
$$

But sets of the form $A = \{f:\ f(x_j) \geq a_j, j = 1, 2, \cdots, k\}$ generate the Borel σ-field on M_+, hence in order to prove convergence of the finite-dimensional marginals of $\frac{L^0_{t(n,\cdot)}}{g(n)}$ to those of Z_x, we only have to check that

$$
P[Z_{x_j} \geq a_j, j = 1, 2, \cdots, k] = \exp\left(-\sum_{j=1}^{k} \frac{a_j - a_{j-1}}{x_j}\right),
$$

which follows from an explicit computation using (7). Tightness of the distributions of $\frac{L^0_{t(n,\cdot)}}{g(n)}$ is immediate from Prohorov's theorem. $\square$

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