Large deviations for one-dimensional random walk in a random environment - a survey

NINA GANTERT[1] and OFER ZEITOUNI[2]

24 November, 1998. Revised April 10, 1999

Abstract

Suppose that the integers are assigned i.i.d. random variables $\{\omega_x\}$ (taking values in the unit interval), which serve as an environment. This environment defines a random walk $\{X_n\}$ (called a RWRE) which, when at x, moves one step to the right with probability ω_x, and one step to the left with probability $1-\omega_x$. Solomon (1975) determined the almost-sure asymptotic speed v_α (=rate of escape) of a RWRE. Subsequent work provided limit theorems for the RWRE $\{X_n\}$. We review in this article recent results on the asymptotics of the probability of rare events of the form $\mathbf{P}(X_n/n \in A)$ or $\mathbf{P}_\omega(X_n/n \in A)$, in the situation where $\mathbf{P}(X_n/n \in A) \underset{n\to\infty}{\longrightarrow} 0$. It turns out that the RWRE exhibits a wide range of non-trivial behavior, and different regimes are possible.

KEY WORDS: Random walk in random environment, large deviations.

AMS (1991) subject classifications: 60J15,60F10,82C44,60J80.

---

[1]Dept. of Mathematics, TU Berlin, Strasse des 17. Juni 136, 10623 Berlin, GERMANY. Research partially supported by the Swiss National Science foundation under grant 8220–046518.

[2]Department of Electrical Engineering, Technion- Israel Institute of Technology, Haifa 32000, ISRAEL. Research partially supported by a US-Israel BSF grant.

1 Introduction

In this review, we consider large deviations from the law of large numbers for a nearest-neighbor random walk on $\mathbb{Z}$ with site-dependent transition probabilities.

Let $\Sigma = [0,1]^{\mathbb{Z}}$, let $\omega = (\omega_i)_{i\in\mathbb{Z}} \in \Sigma$ be a collection of random variables which serve as an environment. We write ρ for ρ_o. Here and throughout, we omit ω from the argument of functions if no confusion occurs. We denote by $\theta : \Sigma \to \Sigma$ the shift on Σ, given by $(\theta\omega)(i) = \omega(i+1)$.

For every fixed ω, let $X = (X_n)_{n\geq 0}$ be the Markov chain on $\mathbb{Z}$ starting at $X_0 = 0$, with transition probabilities

$$\mathbf{P}_\omega(X_{n+1} = y \mid X_n = x) = \begin{cases} \omega_x & \text{if } y = x+1 \\ 1-\omega_x & \text{if } y = x-1 \\ 0 & \text{otherwise} \end{cases}. \tag{1}$$

Let η be a stationary, ergodic probability measure on Σ. In the particular case that η is a product measure, we denote its marginal by α, and write $\eta = \alpha^{\mathbb{Z}}$. We denote by $\omega_{\min} = \omega_{\min}(\eta) := \min\{z : z \in \text{supp } \eta_0\}$ where η_0 denotes the marginal of η, $\omega_{\max} = \omega_{\max}(\eta) := \max\{z : z \in \text{supp } \eta_0\}$, $\rho_i = \rho_i(\omega) := (1-\omega_i)/\omega_i$, $i \in \mathbb{Z}$, and let $\rho_{\max} = \rho_{\max}(\eta) := (1-\omega_{\min})/\omega_{\min}$. We write, for any function f of the environment, $\langle f \rangle = \int f(\omega)\eta(d\omega)$.

Throughout this paper, we call a probability *quenched* if it is taken under $\mathbf{P}_\omega$, i.e. conditional on the environment. A probability is called *annealed* if it is taken according to $\mathbf{P}$, that is when the environment ω is averaged according to the measure η. We will sometimes use the notation $\mathbf{P}_\eta$ to emphasize the measure η used in taking expectations over ω.

The RWRE model, which seems to have been first considered in a particular case by Temkin [45], exhibits a number of phenomena not shared by classical random walk. Assume first that η is a product measure. It was established by Solomon [41] (see also [24] for a particular case) that X is $\mathbf{P}_\omega$-a.s. transient, for η-a.a. ω, iff $\langle \log\rho \rangle \neq 0$. In the transient case, for $\eta - a.a.\omega$, $\lim_n X_n = +\infty$ $\mathbf{P}_\omega$-a.s. if $\langle \log\rho \rangle < 0$ (and $\lim_n X_n = -\infty$ $\mathbf{P}_\omega$-a.s. if $\langle \log\rho \rangle > 0$). An easy proof of the transience may be obtained by noting that the function

$$f(x) = \begin{cases} \rho_0 + \cdots + \rho_0\rho_1\cdots\rho_{x-1}, & x > 0 \\ 0, & x = 0 \\ -\left(1 + \frac{1}{\rho_{-1}} + \cdots + \frac{1}{\rho_{-1}\cdots\rho_{x+1}}\right), & x < 0 \end{cases}$$

is $\mathbf{P}_\omega$-harmonic.

With $v_\alpha = \lim_n X_n/n$ denoting the (almost sure) speed of the RWRE, there are two speed regimes, namely,

(i) $v_\alpha = (1-\langle\rho\rangle)/(1+\langle\rho\rangle)$ when $\langle\rho\rangle < 1$ and $v_\alpha = (\langle\rho^{-1}\rangle - 1)/(\langle\rho^{-1}\rangle + 1)$ when $\langle\rho^{-1}\rangle < 1$.

(ii) $v_\alpha = 0$ when $\langle\rho\rangle^{-1} \le 1 \le \langle\rho^{-1}\rangle$.

Similar results hold if η is a stationary, ergodic probability measure on Σ. Assume $\langle\log\rho\rangle \le 0$: then, the RWRE (X_n) is (almost surely) either recurrent (if $\langle\log\rho\rangle = 0$) or $X_n \to +\infty$ (if $\langle\log\rho\rangle < 0$), see [28, Chap. IV, Theorem 2.3 and Corollary 2.4] or [2]. Further, let $T_k = \inf\{n : X_n = k\}$, $k = 0, \pm1, \pm2, \cdots$ and

$$\begin{aligned} \tau_k &= T_k - T_{k-1} \qquad k > 0 \\ \tau_k &= T_k - T_{k+1} \qquad k < 0\,, \end{aligned}$$

with the convention that $\infty - \infty = \infty$ in this definition. Let $Z_i^- := \rho_i + \rho_i\rho_{i-1} + \rho_i\rho_{i-1}\rho_{i-2} + \ldots$. Note that if the walk is transient to the right, i.e. if $\langle\log\rho\rangle < 0$, then $Z_i^- < \infty$, η-a.s.. It can be shown that if

$$v_\eta^{-1} := \int (1 + 2Z_0^-)\eta(d\omega) = \int (1 + Z_0^- + Z_{-1}^-)\eta(d\omega) < \infty\,, \tag{2}$$

then the random walk has the positive speed v_η, i.e. for η-a.e. ω, we have $X_n/n \to v_\eta$, $\mathbf{P}_\omega$-a.s., cf. [2]. In fact, we will see, cf. the proof of Lemma 1 below, that $\mathbf{E}_\omega(\tau_1) = 1 + 2Z_0^-$, and one could use this to rerun the argument of Solomon [41] yielding the speed of the RWRE. In particular, if η is a product measure, one recovers Solomon's formula for the speed: $\int(1+2Z_0^-)\eta(d\omega) < \infty$ if $\langle\rho\rangle < 1$, and in this case,

$$v_\eta = \frac{1-\langle\rho\rangle}{1+\langle\rho\rangle}\,.$$

In the i.i.d. case, i.e. when η is a product measure, this law of large numbers is supplemented in [22] by central limit type theorems. For instance, in regime (i) above the classical central limit theorem holds if $\langle\rho^2\rangle < 1$. In regime (ii), on the other hand, under some additional hypothesis, $n^{-s}X_n$ converges in law where $s \in (0,1)$ is the unique solution of $\langle\rho^s\rangle = 1$.

In the *recurrent* case the motion is extremely slow. Sinai [37] proved that in this case $(\log n)^{-2}X_n$ converges in law, and Kesten [21] identified explicitly the limiting law.

We will concentrate here on one topic, namely, large deviations from the law of large numbers. For more about the history of the model, connections with other fields and various limit laws not mentioned above (and, of course, more references!), we refer to [18] and [36]. Some recent asymptotic results, of a nature different from ours, may be found e.g. in [17] or in [1]. We also note that a direct relationship exists between results on RWRE and on certain branching processes in random environments (BPRE), see [24], [25] for an early account of this relationship and [4] for recent results and references.

Recall that a sequence of measures μ_n on $\mathbb{R}$ is said to satisfy the Large Deviation Principle (LDP) with rate function $I(\cdot)$ if for any measurable set G,

$$-\inf_{x\in G^o} I(x) \le \liminf_{n\to\infty} \frac{1}{n}\log \mu_n(G) \le \limsup_{n\to\infty} \frac{1}{n}\log \mu_n(G) \le -\inf_{x\in \bar{G}} I(x)\,,$$

where G^o and $\bar{G}$ denote the interior and closure of the set G. For a product measure $\eta = \alpha^{\mathbb{Z}}$, the study of large deviations for the law of X_n/n was initiated in the seminal paper [16], where a large deviation principle for the distributions of X_n/n under $\mathbf{P}_\omega$ was derived, for η -a.e. ω. The resulting rate function turns out to be deterministic, and in the case where $v_\alpha > 0$ and $\alpha(\omega_0 < 1/2) > 0$, it vanishes on the interval $[0, v_\alpha]$. This paper motivated refined estimates, i.e. estimates in the subexponential regime, where the rate function vanishes, both in the quenched and in the annealed case, cf. [9], [14], [35], [34]. Further, a large deviation principle for the distribution of X_n/n under the annealed mesure $\mathbf{P}$ was recently derived in [7].

Our goal in this review is to describe the recent results concerning large deviations for the law of X_n/n, in a unified framework, starting from the paper [16]. We present at the end of this review several open problems concerning RWRE in one dimension, and the only new result we prove here, Theorem 9, is intended to raise the interest of the readers in proving Conjecture 1. Maybe more importantly, we briefly mention at the end of the paper several questions of interest in extending the picture emerging in the one dimensional case to the multi-dimensional situation.

The approach of [16] to large deviation statements involves looking at the RWRE as a Markov chain in the space of environments, and the quenched Large Deviation Principle (LDP) is obtained by an appropriate contraction. More precisely, the rate function is the solution of a variational problem and is shown to be the Legendre transform of certain Lyapunov exponents. We follow here a different path, initiated in [9] and fully developed in [7]: namely, we derive a large deviation principle for X_n/n by first considering a large deviation principle for T_n/n. The fact that the walk is one dimensional and only moves to nearest neighbors, allows for a recursive decomposition of the hitting times T_n. This leads to rather simple proofs of the LDP's in the exponential regime, of Solomon's speed formulae, and of the annealed tail estimates in the subexponential regime. We will indicate some additional arguments, appearing in [14], for handling the quenched sub-exponential regime, and also briefly describe how the techniques of coarse graining are used in [35], [34] to obtain sharp tail asymptotics (including constants) in the subexponential regime.

It turns out that a full description of annealed large deviations, even in the i.i.d. case, forces us to first handle quenched large deviations for certain ergodic measures. This is why we start with a description of the (quenched) LDP results for a rather general class of ergodic measures η, contained in [7]. Specializing to product measures, we derive then annealed LDP's. These results form the content of Section 2. In Section 3, we turn our attention to annealed sub-exponential estimates, while Section 4 contains the corresponding quenched sub-exponential estimates. The following summary shows what is known in terms of subexponential asymptotics for a product measure $\eta = \alpha^{\mathbb{Z}}$.

Case I: $\langle \rho^s \rangle = 1$, some $s > 1$: *the support of* α *contains values* $> 1/2$ *and values* $< 1/2$.

Case 0: $\rho_{\max} = 1, \alpha(\{1/2\}) > 0$: α *is concentrated on* $[1/2, 1[$ *with a positive weight on* $1/2$.

$$\boxed{v \in (0, v_\alpha)}$$

	Annealed	Quenched
Case I	$[9, Thm.1.1]: \ \frac{1}{\log n} \log \mathbf{P}\left(\frac{X_n}{n} \le v\right) \to s - 1$	$[14, Thm.1]: \ \forall \delta > 0, \ \frac{1}{n^{1-\frac{1}{s}\pm\delta}} \log \mathbf{P}_\omega\left(\frac{X_n}{n} \le v\right) \longrightarrow \begin{cases} 0 \\ -\infty \end{cases}$ (Random fluctuations for $n^{\frac{1}{s}-1} \log \mathbf{P}_\omega(\frac{X_n}{n} \le v)$?)
Case 0	$[9, Thm.1.2], [35, Thm.2]:$ $\frac{1}{n^{1/3}} \log \mathbf{P}\left(\frac{X_n}{n} \le v\right) \to -\frac{3}{2}(1 - v/v_\alpha)^{1/3} \lvert \frac{\pi}{2} \log \alpha(\{1/2\}) \rvert^{2/3}$	$[14, Thm.2], [34, Thm.1]:$ $\frac{(\log n)^2}{n} \log \mathbf{P}_\omega\left(\frac{X_n}{n} \le v\right) \to -\frac{1}{8}(1 - v/v_\alpha) \lvert \pi \log \alpha(\{1/2\}) \rvert^2$

Section 5 is devoted to extensions, comments and open problems.

Acknowledgments We would like to thank our collaborators Francis Comets, Amir Dembo, Yuval Peres, Agoston Pisztora and Tobias Povel for their contributions, and Didier Piau for his remarks concerning the proof of Theorem 9. We would also like to thank the organizers of the school, Pal Révész and Balint Tòth, for the stimulating program and very enjoyable meeting.

2 LDP's - the exponential scale

2.1 Quenched LDP's

We present first quenched LDP's for the distribution of T_n/n or T_{-n}/n, respectively. LDP's in the annealed case follow. The LDP's for the distribution of X_n/n follow then by "renewal duality", see the end of Section 2.3. The linear pieces of the rate function of X_n/n near 0 in the transient case, first discovered by Greven and den Hollander, can be explained in terms of the linear pieces of the rate function for T_n/n or T_{-n}/n at infinity.

A crucial role is played by the function

$$\varphi(\lambda,\omega) := \mathbf{E}_\omega(e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty})\,. \tag{3}$$

A characterization of $\varphi(\lambda,\omega)$ in terms of continued fraction expansions is provided in Section 2.3, Lemma 1. Let next

$$\tau_\omega := \mathbf{E}_\omega(\tau_1|\tau_1<\infty) \tag{4}$$

(with the value $+\infty$ allowed), and define

$$I_\eta^{\tau,q}(u) = \sup_{\lambda\in\mathbb{R}}[\lambda u - \int \log\varphi(\lambda,\omega)\eta(d\omega)]\,. \tag{5}$$

Let $M_1(\Sigma)$, $M_1^s(\Sigma)$ and $M_1^e(\Sigma)$ be the spaces of probability measures, stationary probability measures and ergodic probability measures, on Σ. All spaces of probability measures in this paper are equipped with the topology induced from weak convergence. Let $K\subset(0,1)$ be some fixed compact subset of $(0,1)$. We denote by $M_1^e(\Sigma)^+ := \{\eta\in M_1^e(\Sigma) : \int\log\rho_0(\omega)\eta(d\omega)\le 0\}$ the set of ergodic measure on the environment making the walk recurrent or transient to the right, and, for any set $M\subset M_1(\Sigma)$, we let $M^K = M\cap\{\eta : \mathrm{supp}(\eta_0)\subseteq K\subset(0,1)\}$.

Theorem 1. *([7, Thm. 4]) Assume $\eta\in M_1^e(\Sigma)^K$. Then, for η-a.e. ω, the distributions of T_n/n under $\mathbf{P}_\omega$ satisfy a weak LDP with deterministic, convex rate function $I_\eta^{\tau,q}$. Further, $I_\eta^{\tau,q}(\cdot)$ is decreasing on $[1,\int\tau_\omega\eta(d\omega)]$ and increasing on $[\int\tau_\omega\eta(d\omega),\infty)$.*

For a formal definition of a LDP and weak LDP, we refer to [10, Section 1.2]. A discussion of different possible shapes of the rate function $I_\eta^{\tau,q}$, as well as graphs of such functions, are provided in [7].

Theorem 1 obviously implies also a LDP for T_{-n}/n, simply by symmetry (i.e., space reversal of the measure η). The logarithmic moment generating function of τ_{-1} can be expressed in terms of $\varphi(\lambda,\omega)$; in fact, this is needed in the proof of Theorem 1.

Proposition 1. *([7, Prop. 1]) Assume $\eta\in M_1^e(\Sigma)^K$. Then,*

$$\int\log\mathbf{E}_\omega(e^{\lambda\tau_{-1}}\mathbf{1}_{\tau_{-1}<\infty})\eta(d\omega) = \int\log\mathbf{E}_\omega(e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty})\eta(d\omega) + \int\log\rho_0(\omega)\eta(d\omega) \tag{6}$$

where both sides may be infinite for positive values of λ. Further, if $\eta\in M_1^e(\Sigma)^K$, then the distributions of T_{-n}/n under $\mathbf{P}_\omega$ satisfy, for η -a.e. ω, a weak LDP with deterministic rate function

$$I_\eta^{-\tau,q}(u) := I_\eta^{\tau,q}(u) - \int\log\rho_0(\omega)\eta(d\omega), \quad 1\le u<\infty. \tag{7}$$

We note that in both Theorem 1 and Proposition 1, the LDP's are weak due to possible positive probability mass at $+\infty$. The LDP of Theorem 1 can be strengthened to a full LDP if $\eta \in M_1^e(\Sigma)^{+,K}$.

We may now turn our attention to LDP's for the X_n process. Let, for $\eta \in M_1^e(\Sigma)^{+,K}$,

$$I_\eta^q(v) = \begin{cases} vI_\eta^{\tau,q}\left(\frac{1}{v}\right), & 0 \le v \le 1 \\ |v|I_\eta^{-\tau,q}\left(\frac{1}{|v|}\right), & -1 \le v \le 0\,, \end{cases} \tag{8}$$

where $I_\eta^{\tau,q}$ and $I_\eta^{-\tau,q}$ were defined in (5) and (7), and the value at $v = 0$ is taken as

$$I_\eta^q(0) = \lim_{v\to 0} vI_\eta^{\tau,q}(\frac{1}{v})\,.$$

Let the space reversal $\mathrm{Inv} : \Sigma \to \Sigma$ denote the map satisfying $(\mathrm{Inv}\,\omega)_i = 1 - \omega_{-i}$, and let $\eta^{\mathrm{Inv}} = \eta \circ \mathrm{Inv}^{-1}$. For $\eta \in M_1^e(\Sigma)^K \setminus M_1^e(\Sigma)^{+,K}$, define $I_\eta^q(v) = I_{\eta^{\mathrm{Inv}}}^q(-v)$, and note that $\eta^{\mathrm{Inv}} \in M_1^e(\Sigma)^{+,K}$ while $I_\eta^{\tau,q}(\cdot) = I_{\eta^{\mathrm{Inv}}}^{-\tau,q}(\cdot)$.

Theorem 2. *([7, Thm. 1]) Assume $\eta \in M_1^e(\Sigma)^K$. For η-a.e. ω, the distributions of X_n/n under $\mathbf{P}_\omega$ satisfy a large deviation principle with convex rate function I_η^q.*

This rate function was derived in [16] for the i.i.d. case, i.e. the case where η is a product measure. For some properties of the rate function $I_\eta^q(\cdot)$, see Section 2.5.

2.2 Annealed LDP's

We next turn our attention to the annealed situation in the i.i.d. case, namely with $\eta = \alpha^{\mathbb{Z}}$. Denote by $h(\cdot|\alpha^{\mathbb{Z}})$ the specific relative entropy with respect to $\alpha^{\mathbb{Z}} \in M_1(\Sigma)$. Recall that by [13], $\alpha^{\mathbb{Z}}$ satisfies the process level LDP, that is, denoting $R_n = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{\theta^j\omega} \in M_1(\Sigma)$, the distributions of the random variables R_n (with values in $M_1(\Sigma)$) under $\alpha^{\mathbb{Z}}$ satisfy the LDP with rate function $h(\cdot|\alpha^{\mathbb{Z}})$.

For $u \ge 1$, let

$$I_\alpha^{\tau,a}(u) = \inf_{\eta \in M_1^e(\Sigma)} [I_\eta^{\tau,q}(u) + h(\eta|\alpha^{\mathbb{Z}})]\,. \tag{9}$$

Theorem 3. *([7, Thm. 5]) Let $\eta = \alpha^{\mathbb{Z}}$. Then the distributions of T_n/n under $\mathbf{P}$ satisfy a (weak) LDP with convex rate function $I_\alpha^{\tau,a}$.*

The appearance of the quenched rate function for general ergodic η in the expression for the annealed rate function is the reason why we were forced to consider ergodic η in the quenched situation. It is possible to construct examples where (and formally it seems to be the rule that) the infimum in the variational problem defining the annealed rate function is achieved on ergodic measures which are not product measures.

As in the quenched case, annealed LDP's for T_n/n imply immediately annealed LDP's for X_n/n. Let

$$I^a_\alpha(v) = \begin{cases} vI^{\tau,a}_\alpha\left(\frac{1}{v}\right), & 0 \le v \le 1 \\ |v|I^{\tau,a}_{\alpha^{\mathrm{Inv}}}\left(\frac{1}{|v|}\right), & -1 \le v \le 0\,. \end{cases} \tag{10}$$

We have the following large deviation principle.

Theorem 4. *([7, Thm. 2]) Assume $\eta = \alpha^{\mathbb{Z}} \in M^e_1(\Sigma)^K$. Then, the distributions of X_n/n under* $\mathbf{P}$ *satisfy a LDP with convex rate function I^a_α.*

We note that the quenched rate function I^q_η and the annealed rate function I^a_α are related by the following variational formula:

$$I^a_\alpha(v) = \inf_{\eta \in M^e_1(\Sigma)} \left[I^q_\eta(v) + |v| h(\eta|\alpha^{\mathbb{Z}}) \right] . \tag{11}$$

where $vh(\eta|\alpha^{\mathbb{Z}}) = \infty$ if $h(\eta|\alpha^{\mathbb{Z}}) = \infty$. In particular, we always have $I^a_\alpha \le I^q_{\alpha^{\mathbb{Z}}}$. Further, $I^a_\alpha(v) = 0$ only if $I^q_{\alpha^{\mathbb{Z}}}(v) = 0$.

2.3 Properties of $\varphi(\lambda, \omega)$ and sketch of proofs of the quenched LDP's.

We begin by deriving a representation of $\varphi(\lambda, \omega)$:

Lemma 1. *For any $\lambda \in \mathrm{I\!R}$, we have that whenever $\varphi(\lambda, \omega) < \infty$ a.s. then*

$$\varphi(\lambda, \omega) = \frac{1|}{e^{-\lambda}(1 + \rho_0(\omega))} - \frac{\rho_0(\omega)|}{e^{-\lambda}(1 + \rho_{-1}(\omega))} - \frac{\rho_{-1}(\omega)}{\cdots} . \tag{12}$$

Proof of Lemma 1. Pathwise decomposition yields the following formula for τ_1:

$$\tau_1 = 1_{\{X_1 = 1\}} + 1_{\{X_1 = -1\}}(\tau_1' + \tau_1'' + 1) \tag{13}$$

where $\tau_1' + 1$ is the first hitting time of 0 after time 1 (possibly infinite) and $\tau_1' + \tau_1'' + 1$ is the first hitting time of +1 after time $\tau_1' + 1$. Note that, under $\mathbf{P}_\omega$, the law of τ_1' conditioned on the event

$X_1 = -1$ is $\mathbf{P}_{\theta^{-1}\omega}(\tau_1 \in \cdot)$ and, conditioned on the event $\tau_1' < \infty$, τ_1'' is independent of τ_1' and has law $\mathbf{P}_\omega(\tau_1 \in \cdot)$. Therefore, we have

$$\begin{aligned}
\varphi(\lambda,\omega) &= \mathbf{E}_\omega(e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty}) \\
&= \mathbf{P}_\omega(X_1 = 1)\ \mathbf{E}_\omega\Big(e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty}|X_1 = 1\Big) + \mathbf{P}_\omega(X_1 = -1)\ \mathbf{E}_\omega\Big(e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty}|X_1 = -1\Big) \\
&= \omega_0 e^\lambda + (1-\omega_0)\mathbf{E}_\omega(e^{\lambda(\tau_1\circ\theta^{-1})}\mathbf{1}_{\tau_1\circ\theta^{-1}<\infty})\ \mathbf{E}_\omega(e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty})e^\lambda \\
&= \omega_0 e^\lambda + (1-\omega_0)e^\lambda\varphi(\lambda,\theta^{-1}\omega)\varphi(\lambda,\omega)\,.
\end{aligned}$$

Hence, if $\varphi(\lambda,\omega) < \infty$ then $\varphi(\lambda,\theta^{-1}\omega) < \infty$, and

$$\varphi(\lambda,\omega) = \frac{1}{(1+\rho_0(\omega))e^{-\lambda} - \rho_0(\omega)\varphi(\lambda,\theta^{-1}\omega)}\,. \tag{14}$$

In the same way,

$$\varphi(\lambda,\theta^{-1}\omega) = \frac{1}{(1+\rho_{-1})e^{-\lambda} - \rho_{-1}\varphi(\lambda,\theta^{-2}\omega)}\,.$$

By iteration, we get the representation of φ as a continued fraction, i.e. (12). We refer to [16] for the convergence of the continued fraction; for a reference on continued fractions, see [19] or [46]. $\square$

Remark: In the same way, taking expectations in (13) and iterating yields $\mathbf{E}_\omega(\tau_1) = 1 + 2Z_0^-$, cf. (2).

Various analytical properties of the function $\varphi(\lambda,\omega)$ are derived in [7]. In particular, the following smoothness properties are proved there:

Lemma 2. *Let $\eta \in M_1^e(\Sigma)^{+,K}$. Then*

(i) For $1 < u < \mathbf{E}(\tau_1) \le \infty$, there exists a unique $\lambda_0 = \lambda_0(u,\eta)$ such that $\lambda_0 < 0$ and

$$u = \int \frac{d}{d\lambda}\log\varphi(\lambda,\omega)\Big|_{\lambda=\lambda_0}\eta(d\omega)\,. \tag{15}$$

For u as above

$$\inf_{\eta\in M_1^e(\Sigma)^{+,K}} \lambda_0(u,\eta) > -\infty\,. \tag{16}$$

(ii) There is a deterministic $\infty > \lambda_{\mathrm{crit}} \ge 0$, depending only on η, such that for $\lambda < \lambda_{\mathrm{crit}}$, $\mathbf{E}_\omega(e^{\lambda\tau_1}) < \infty$ for η -a.e. ω, $\mathbf{E}_\omega(e^{\lambda\tau_1})$ has the form (12) and for $\lambda > \lambda_{\mathrm{crit}}$, $\mathbf{E}_\omega(e^{\lambda\tau_1}) = \infty$ for η -a.e. ω.

(iii) *Let* $u_{\text{crit}} = \infty$ *if* $\int \mathbf{E}_\omega(\tau_1 e^{\lambda_{\text{crit}}\tau_1})\ \eta(d\omega) = \infty$ *and* $u_{\text{crit}} := \int \frac{d}{d\lambda} \log\varphi(\lambda,\omega)\Big|_{\lambda=\lambda_{\text{crit}}} \eta(d\omega)$ *else. For* $\mathbf{E}(\tau_1) \le u < u_{\text{crit}}$*, there exists a unique* $\lambda_0 = \lambda_0(u,\eta)$ *such that* $\lambda_0 \ge 0$ *and*

$$u = \int \frac{d}{d\lambda} \log\varphi(\lambda,\omega)\Big|_{\lambda=\lambda_0} \eta(d\omega)\,. \tag{17}$$

(iv) *If* $\eta \in M_1^e(\Sigma)^{+,K}$ *is a product measure and* $\rho_{\max} < 1$*, then* $\lambda_{\text{crit}} = \bar\lambda := -\frac{1}{2}\log(4\omega_{\min}(1-\omega_{\min})) > 0$ *and* $\varphi(\lambda,\omega) = \mathbf{E}_\omega(e^{\lambda\tau_1}) < \infty$ *iff* $\lambda \le \lambda_{\text{crit}}$*. Further,* $u_{\text{crit}} := \int \frac{d}{d\lambda}\log\varphi(\lambda,\omega)\Big|_{\lambda=\lambda_{\text{crit}}} \eta(d\omega) < \infty$ *unless* η *is degenerate, i.e. unless* $\omega = const$ η*-a.s.*

(v) *If* $\eta \in M_1^e(\Sigma)^{+,K}$ *is a product measure and* $\rho_{\max} \ge 1$*, we have* $\lambda_{\text{crit}} = 0$*.*

Remarks: 1. Parts *(iv)* and *(v)* of Lemma 2 continue to hold true even if $\eta \in M_1^e(\Sigma)^{+,K}$ is not a product measure, provided it is locally equivalent to the product of its marginals, i.e. all the finite-dimensional marginals of η have the same zero sets as the corresponding product measures. 2. u_{crit} can be infinite in the general ergodic case, for instance if the environment is periodic: an example is $\eta = \frac{1}{2}\delta_{(\ldots,\omega_1,\omega_2,\omega_1,\omega_2,\ldots)} + \frac{1}{2}\delta_{(\ldots,\omega_2,\omega_1,\omega_2,\omega_1,\ldots)}$.

Equipped with Lemma 2, Theorem 1 for $\eta \in M_1^e(\Sigma)^{+,K}$ follows readily by standard large deviations techniques (see [7] for a detailed proof). Proposition 1 allows one to extend the proof to $\eta \in M_1^e(\Sigma)^K \setminus M_1^e(\Sigma)^{+,K}$, and hence to complete the proof of Theorem 1. We note that the key to the proof of Proposition 1, cf. [7], is again a hitting time decomposition. We give here the proof of Proposition 1 for the particular case $\lambda < 0$. The general case is more involved (one has to care about the integrability which allows us to take expectations in the last step of the present proof) and may be found in [7].

Proof of (6) for $\lambda < 0$**:** Let

$$\tilde\varphi(\lambda,\omega) = \mathbf{E}_\omega\left(e^{\lambda\tau_{-1}}\mathbf{1}_{\tau_{-1}<\infty}\right)\,. \tag{18}$$

Note that for $\lambda \le 0$, $\tilde\varphi(\lambda,\omega) \le 1$ while it is easy to check that because $\eta_0(1) = 0$, one has also that $\tilde\varphi(\lambda,\omega) > 0$. It is not hard to see that the same type of recursion as in (13) (using the indicators in the definition of $\tilde\varphi$!) leads to the formula

$$\tilde\varphi(\lambda,\omega) = (1-\omega_0)e^{\lambda} + \omega_0 e^{\lambda}\tilde\varphi(\lambda,\theta\omega)\tilde\varphi(\lambda,\omega)\,. \tag{19}$$

One obtains from this recursion that, η-a.s.,

$$\tilde\varphi(\lambda,\omega)\tilde\varphi(\lambda,\theta\omega) = \frac{e^{-\lambda}\tilde\varphi(\lambda,\omega)}{\omega_0} - \rho_0(\omega)\,,$$

and similarly from the equation before (14) that, η-a.s.,

$$\rho_0(\omega)\varphi(\lambda,\omega)\varphi(\lambda,\theta^{-1}\omega) = \frac{e^{-\lambda}\varphi(\lambda,\omega)}{\omega_0} - 1\,.$$

Hence, η-a.s,

$$\begin{aligned}\rho_0(\omega)\Big(1-\tilde{\varphi}(\lambda,\omega)\varphi(\lambda,\theta^{-1}\omega)\Big)\varphi(\lambda,\omega) &= \rho_0(\omega)\varphi(\lambda,\omega) - \tilde{\varphi}(\lambda,\omega)\frac{e^{-\lambda}\varphi(\lambda,\omega)}{\omega_0} + \tilde{\varphi}(\lambda,\omega)\\ &= \Big(1-\varphi(\lambda,\omega)\tilde{\varphi}(\lambda,\theta\omega)\Big)\tilde{\varphi}(\lambda,\omega)\,.\end{aligned}$$

Therefore, η-a.s,

$$\log\rho_0(\omega) + \log\varphi(\lambda,\omega) - \log\tilde{\varphi}(\lambda,\omega) = \log(1-\tilde{\varphi}(\lambda,\theta\omega)\varphi(\lambda,\omega)) - \log(1-\tilde{\varphi}(\lambda,\omega)\varphi(\lambda,\theta^{-1}\omega))\,.$$

Integration with respect to η (due to the stationarity of η, the integral of the r.h.s. vanishes!) yields (6). $\square$

We conclude this section by noting that given Theorem 1, the proof of the quenched LDP forX_n/n in Theorem 2 is standard by renewal duality: we have

$$\mathbf{P}_\omega\left(\frac{X_n}{n}\sim v\right) \approx \mathbf{P}_\omega\left(T_{nv}\sim n\right) = \mathbf{P}_\omega\left(\frac{T_{nv}}{nv}\sim\frac{1}{v}\right) \tag{20}$$

hence

$$\frac{1}{n}\log\mathbf{P}_\omega\left(\frac{X_n}{n}\sim v\right) \approx v\frac{1}{nv}\log\mathbf{P}_\omega\left(\frac{T_{nv}}{nv}\sim\frac{1}{v}\right)\,, \tag{21}$$

leading to $I_\eta^q(v) = vI_\eta^{\tau,q}\left(\frac{1}{v}\right)$. This argument can be made precise, we refer to [7].

2.4 Annealed LDP's - sketch of proofs.

Introduce the notation $f(\lambda,\omega) := \log\mathbf{E}_\omega(e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty}) = \log\varphi(\lambda,\omega)$. In what follows, $\omega_{\min}$, $\rho_{\max}$, etc. are always defined in terms of α, whereas if $\alpha^{\mathbb{Z}}\in M_1^e(\Sigma)^{+,K}$ then λ_{crit} is defined as in Lemma 2, while if $\alpha^{\mathbb{Z}}\in M_1^e(\Sigma)^K\setminus M_1^e(\Sigma)^{+,K}$ then $\lambda_{\text{crit}} := \lambda_{\text{crit}}((\alpha^{\mathbb{Z}})^{\text{Inv}})$. Also, unless denoted otherwise, expectations are taken with respect to $\alpha^{\mathbb{Z}}$ or $\mathbf{P}_{\alpha^{\mathbb{Z}}}$. We recall that $M_1(\Sigma)$ is equipped with the topology of weak convergence, and define the compact set

$$\mathcal{D}_\alpha := \{\mu\in M_1^s(\Sigma)^K : \operatorname{supp}\mu_0\subseteq\operatorname{supp}\alpha\}.$$

The following lemma, whose proof can be found in [7], will be needed in the derivation of the annealed large deviation statements.

Lemma 3. *Assume $\alpha^{\mathbb{Z}} \in M_1^e(\Sigma)^K$ is non-degenerate. Then, the function $(\mu, \lambda) \to \int f(\lambda, \omega)\mu(d\omega)$ is continuous on $\mathcal{D}_\alpha \times (-\infty, \lambda_{\mathrm{crit}}]$.*

Sketch of proof of Theorem 3 We sketch the proof of an upper bound for $\frac{1}{n} \log \mathbf{P}\left(\frac{1}{n} \sum_{j=1}^n \tau_j \le u\right)$, where $1 < u < \infty$. We have, for $\lambda \le 0$,

$$\mathbf{P}\left(\frac{1}{n}\sum_{j=1}^n \tau_j \le u\right) \le \mathbf{E}\left(\exp\left(\lambda \sum_{j=1}^n \tau_j\right) \mathbf{1}_{\tau_j<\infty, j=1,\ldots,n}\right) e^{-\lambda n u}. \tag{22}$$

But,

$$\begin{aligned}
\mathbf{E}\left(\exp\left(\lambda \sum_{j=1}^n \tau_j\right) \mathbf{1}_{\tau_j<\infty, j=1,\ldots,n}\right) &= \mathbf{E}\left(\prod_{j=1}^n \mathbf{E}_\omega\left(e^{\lambda\tau_j} \mathbf{1}_{\tau_j<\infty}\right)\right) \\
&= \mathbf{E}\left(\exp\left(\sum_{j=1}^n \log \mathbf{E}_\omega\left(e^{\lambda\tau_j}\mathbf{1}_{\tau_j<\infty}\right)\right)\right) \\
&= \mathbf{E}\left(\exp\left(\sum_{j=0}^{n-1} f(\lambda, \theta^j\omega)\right)\right) \\
&= \mathbf{E}\left(\exp\left(n \int f(\lambda,\omega) R_n(d\omega)\right)\right)
\end{aligned}$$

where $R_n = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{\theta^j\omega} \in M_1(\Sigma)$ denotes the empirical field.

Since the distributions of R_n satisfy a LDP with rate function $h(\cdot|\alpha^{\mathbb{Z}})$, Lemma 3 ensures that we can apply Varadhan's lemma (see [10, Lemma 4.3.6]) to get

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbf{E}\left(\exp\left(n\int f(\lambda,\omega)R_n(d\omega)\right)\right) \le \sup_{\eta \in M_1^s(\Sigma)} \left[\int f(\lambda,\omega)\eta(d\omega) - h(\eta|\alpha^{\mathbb{Z}})\right]. \tag{23}$$

Going back to (22), this yields the upper bound

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}\left(\frac{1}{n}\sum_{j=1}^n \tau_j \le u\right) &\le \inf_{\lambda\le 0} \sup_{\eta\in M_1^s(\Sigma)} \left[\int f(\lambda,\omega)\eta(d\omega) - h(\eta|\alpha^{\mathbb{Z}}) - \lambda u\right] \\
&= -\sup_{\lambda\le 0} \inf_{\eta \in M_1^s(\Sigma)} \left[\lambda u - \int f(\lambda,\omega)\eta(d\omega) + h(\eta|\alpha^{\mathbb{Z}})\right]. \quad (24)
\end{aligned}$$

Since $\mu \to -\int f(\lambda,\omega)\mu(d\omega) + h(\mu|\alpha^{\mathbb{Z}})$ is lower semi-continuous and $M_1(\Sigma)$ is compact, the infimum in (24) is achieved for each λ, on measures with support of their marginal included in K, for otherwise $h(\eta|\alpha^{\mathbb{Z}}) = \infty$. Further, by (16), the supremum over λ can be taken over a compact set (recall that $\infty > u > 1$!). Hence, by the Minimax Theorem (see [10, Pg. 151] for Sion's version), using the fact that the expression in the r.h.s. of (24) is convex-concave, the min-max is equal to the max-min in (24). Further, since taking first the supremum in λ in the right hand side of (24) yields a lower semicontinuous function, an achieving $\bar{\eta}$ exists, and then, due to compactness, there exists actually an achieving pair $\bar{\lambda}, \bar{\eta}$. One then checks, using approximations of stationary measures by ergodic ones (such that $h(\cdot|\alpha^{\mathbb{Z}})$ converges along the approximating sequence), that

$$\begin{aligned} &\inf_{\eta \in M_1^s(\Sigma)^K} \sup_{\lambda \le 0} \left(\left[\lambda u - \int f(\lambda,\omega)\eta(d\omega) \right] + h(\eta|\alpha^{\mathbb{Z}}) \right) \\ = \quad &\inf_{\eta \in M_1^e(\Sigma)^K} \sup_{\lambda \le 0} \left(\left[\lambda u - \int f(\lambda,\omega)\eta(d\omega) \right] + h(\eta|\alpha^{\mathbb{Z}}) \right) \end{aligned} \tag{25}$$

Then, the r.h.s. of (23) equals

$$- \inf_{\eta \in M_1^e(\Sigma)^K} \sup_{\lambda \le 0} \left(\left[\lambda u - \int f(\lambda,\omega)\eta(d\omega) \right] + h(\eta|\alpha^{\mathbb{Z}}) \right) = - \inf_{\eta \in M_1^e(\Sigma)^K} \inf_{w \le u} \left[I_\eta^{\tau,q}(w) + h(\eta|\alpha^{\mathbb{Z}}) \right] \tag{26}$$

where we used

$$\sup_{\lambda \le 0} \left[\lambda u - \int f(\lambda,\omega)\eta(d\omega) \right] = \inf_{w \le u} I_\eta^{\tau,q}(w)\,. \tag{27}$$

Hence,

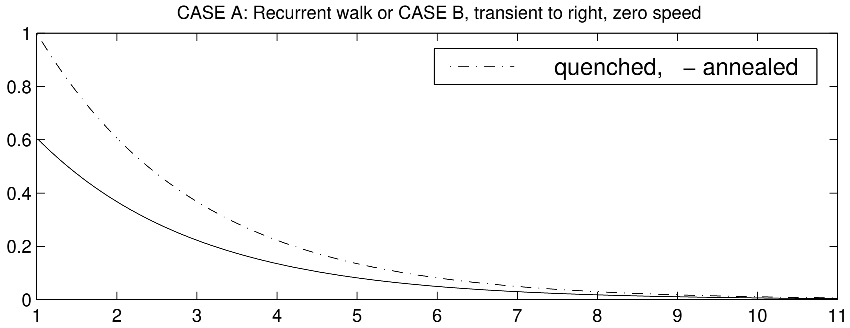
$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P} \left(\frac{1}{n} \sum_{j=1}^n \tau_j \le u \right) \le - \inf_{w \le u} \inf_{\eta \in M_1^e(\Sigma)^K} \left[I_\eta^{\tau,q}(w) + h(\eta|\alpha^{\mathbb{Z}}) \right] = - \inf_{w \le u} I_\alpha^{\tau,a}(w). \tag{28}$$

This completes the proof of the upper bound for the lower tail (the case $u = 1$ being handled directly by noting that $n^{-1} \sum_{j=1}^n \tau_j \le 1$ implies that $\tau_j = 1, j = 1, \ldots, n$). A similar argument holds for the upper tail $\frac{1}{n} \log \mathbf{P} \left(\frac{1}{n} \sum_{j=1}^n \tau_j \ge u \right)$, and then, once one checks that $I_\alpha^{\tau,a}(\cdot)$ is convex, the upper bound in Theorem 3 is established.

The lower bound in the LDP for T_n/n is proved by a standard change of measure, and we omit it here. Finally, the LDP for the distributions of X_n/n is derived by the standard duality from the hitting time LDP, exactly as in the quenched case.

Remark

For simplicity, we have restricted ourselves here to product measures. What we needed in the proof was the process level LDP for the empirical fields and an approximation property used in the proof of (25). In fact, these two requirements lead to mild assumptions on ergodic η under which Theorem 3 holds; we refer to [7] for details.



2.5 Properties of the rate functions

The various rate functions for the RWRE can have rather different shapes according to the law of the environment. A detailed discussion (and figures!) of the rate functions $I_\eta^{\tau,q}$, $I_\alpha^{\tau,a}$, I_η^q, I_α^a can be found in [7]. Since the rate functions for the positions are simple transformations of the rate functions for the hitting times, we discuss here only the latter. Further, we limit ourselves here to some intuitive arguments that explain the shape of the quenched rate functions. The analysis of the annealed rate function is technically more involved.

It turns out that the shape of the rate function $I_\eta^{\tau,q}$ shares many properties with that of Cramér's rate function for positive, i.i.d. random variables $Y_1, Y_2, \ldots$. More precisely, let $Y_1, Y_2, \ldots$ be positive, i.i.d. random variables, let $\Lambda(\lambda) := E\left(\exp(\lambda Y_1)\right) \le \infty$ and

$$\Lambda^*(y) := \sup_{\lambda \in \mathbb{R}} [\lambda y - \Lambda(\lambda)]$$

Then Λ^*, which is the rate function for the LDP of $n^{-1}\sum_{i=1}^n Y_i$, is convex, and we have the following (cf. [10]):

Case 1

Assume $E[Y_1] = \infty$. Then $\Lambda^*(y) > 0$ for all $y \ge 0$, Λ^* is decreasing, and $\lim_{y\to\infty} \Lambda^*(y) = 0$.

Case 2

Assume $m_0 := E[Y_1] < \infty$ and $E(\exp(\lambda Y_1)) = \infty$ for all $\lambda > 0$. Then Λ^* is decreasing for $0 \le y \le m_0$ and $\Lambda^*(y) = 0$ for $y \ge m_0$.

Case 3

Assume $m_0 := E[Y_1] < \infty$ and, for some $\lambda_{\text{crit}} > 0$, $E(\exp(\lambda Y_1)) < \infty$ iff $\lambda \le \lambda_{\text{crit}}$. Then Λ^* is decreasing for $y \le m_0$, increasing for $y \ge m_0$, $\Lambda^*(m_0) = 0$ and $\Lambda^*(y)/y \underset{y\to\infty}{\longrightarrow} \lambda_{\text{crit}}$.

Note that $\tau_1, \tau_2, \ldots$ are independent, but not identically distributed under $\mathbf{P}_\omega$. However, the **shape** of the rate function $I_\eta^{\tau,q}$ is the same as if they were i.i.d. Since a single picture is worth more than a thousand words, we include a plot of $I_\eta^{\tau,q}$ and $I_\alpha^{\tau,a}$ for the various possible cases.

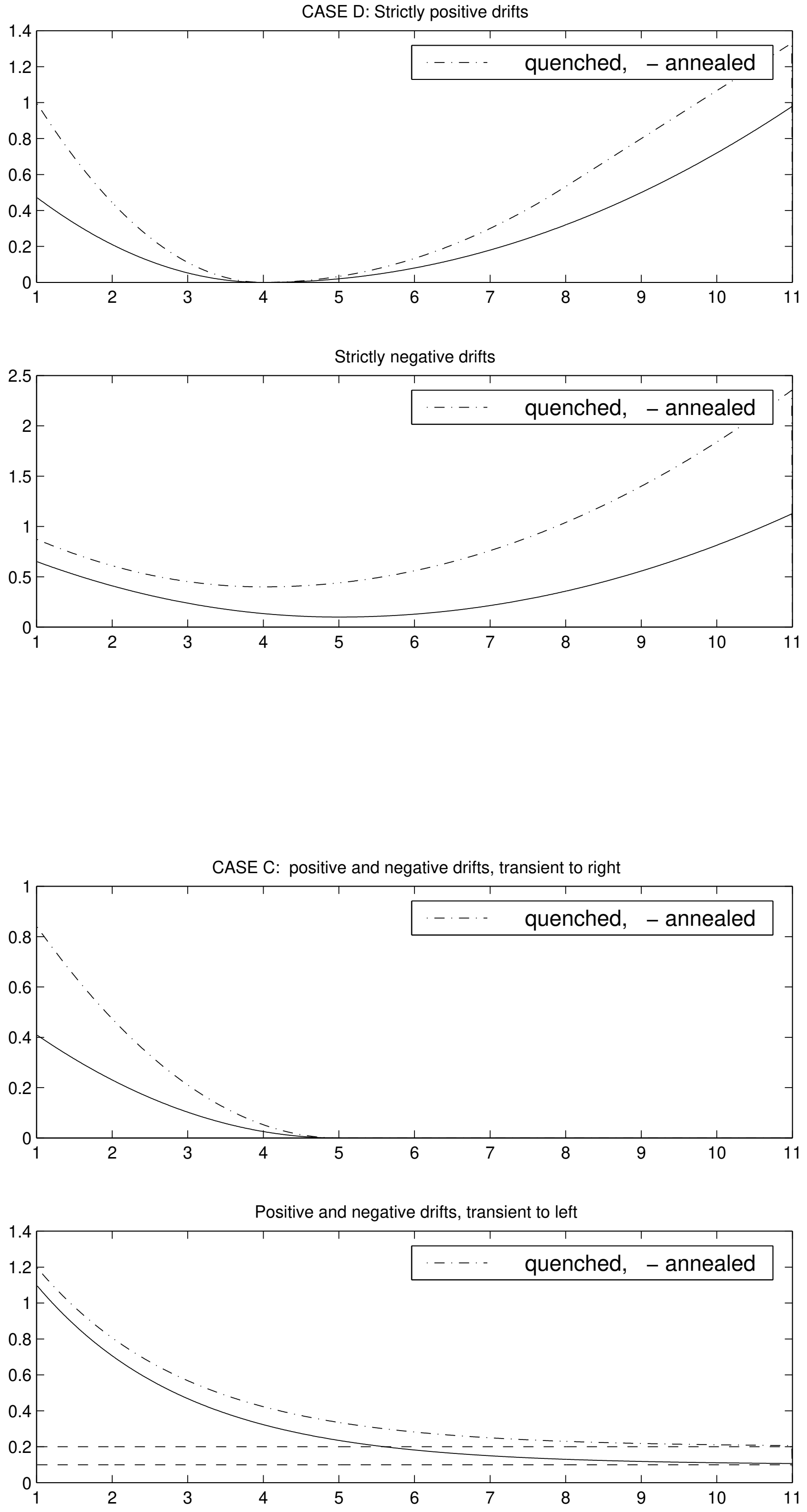


Figure 1: Shape of rate functions for hitting time, quenched and annealed

Case A

$\langle \log \rho_0 \rangle = 0$, i.e. (X_n) is recurrent. Then $\mathbf{E}_\omega(\tau_1) = \infty$ a.s. and $I_\eta^{\tau,q}$ has the same shape as Λ^* in Case 1 above.

Case B

$\langle \log \rho_0 \rangle < 0$ and $\mathbf{E}_\eta(\tau_1) = \infty$, i.e. (X_n) is transient to the right with zero speed. Again, $I_\eta^{\tau,q}$ has the same shape as Λ^* in Case 1 above.

Case C

Let η be a product measure, $\omega_{\min}(\eta) \leq 1/2$, $\omega_{\max}(\eta) \geq 1/2$, assume η is not concentrated on one point and $\langle \log \rho_0 \rangle < 0$, i.e. (X_n) is transient to the right with positive speed and "mixed drifts". Then, $\mathbf{E}(\tau_1) < \infty$ and, for η-a.a. ω, $\mathbf{E}_\omega(\exp(\lambda\tau_1)) = \infty$ for all $\lambda > 0$, cf. Lemma 2, and $I_\eta^{\tau,q}$ has the same shape as Λ^* in Case 2 above.

Case D

Let η be a product measure, $\rho_{\max}(\eta) < 1$ and assume η is not concentrated on one point, i.e. we have "all drifts to the right". This implies that (X_n) is transient to the right with positive speed. Then $\mathbf{E}(\tau_1) < \infty$ and there is $\lambda_{\text{crit}} > 0$ such that, for η-a.a. ω, $\mathbf{E}_\omega(\exp(\lambda\tau_1)) < \infty$ iff $\lambda \leq \lambda_{\text{crit}}$, cf. Lemma 2, and $I_\eta^{\tau,q}$ has the same shape as Λ^* in Case 3 above.

The case where (X_n) is transient to the left is slightly more complicated to describe (τ_1 can be infinite). We refer to [7] for details and to Figure 1 for a qualitative figure.

The rate function I_η^q is given by $I_\eta^q(v) = vI_\eta^{\tau,q}(1/v)$. In particular, the flat piece of $I_\eta^{\tau,q}(u)$ for u large in Case C leads to a flat piece of $I_\eta^q(v)$ for v between 0 and v_α and the linear piece of $I_\eta^{\tau,q}(u)$ for u large in Case D leads to a linear piece of $I_\eta^q(v)$ for v small.

In order to get $I_\eta^q(v)$ for $v < 0$, we have to consider $I_\eta^{-\tau,q}$ instead of $I_\eta^{\tau,q}$, cf. (8).

Remarks

1. In Cases C and D, we have the same behaviour if η is non-degenerate and locally equivalent to the product of its marginals. This rules out the case of a periodic environment. Our LDP covers the case of a periodic environment, but the shape of the rate function can be different, since we may have $\mathbf{E}_\omega(\exp(\lambda\tau_1)) < \infty$ for all $\lambda > 0$, cf. Lemma 2 and the remarks following it.

2. Returning to the i.i.d. case, since $Y_1, Y_2, \ldots$ are positive, the probability that the arithmetic mean is smaller than expected decays always exponentially - we have $\Lambda^*(y) > 0$ for $y < m_0$. Intuitively, in order to have $1/n \sum_{i=1}^n Y_i < m_0$, **all** the random variables $Y_1, Y_2, \ldots$ have to be small, whereas for $1/n \sum_{i=1}^n Y_i > m_0$, it suffices that **one** of the Y_i is very large. This explains, in terms of "extremal events" for the τ_i, why there can only be a flat piece of the rate function I_η^q between 0 and v_α: "speeding up" the RWRE (which corresponds to $1/n \sum_{i=1}^n \tau_i = T_n/n$ being

small!) has always an exponentially decaying probability, while the probability of "slowing down" (which corresponds to T_n/n being large!) can decay slower than exponentially. Already in the "toy example" of i.i.d random variables Y_i, one can see that various normalizations are possible, depending on the tail of the random variables Y_i, cf. [33]. This corresponds to the subexponential asymptotics for the RWRE.

3. Considering the rate function $I_\eta^{-\tau,q}$, we see that due to Proposition 1, I_η^q is symmetric in Case A, i.e. if $\langle \log \rho_0 \rangle = 0$, we have $I_\eta^q(-v) = I_\eta^q(v)$. This symmetry is not at all obvious, since τ_{-1} and τ_1 do not have, in general, the same distribution!

3 Annealed LDP's - subexponential speed

In this section we concentrate on the situation where $\eta = \alpha^{\mathbb{Z}}$, i.e. the environment consists of i.i.d. random variables. We further assume that $\langle \rho \rangle < 1$, hence the RWRE is transient to $+\infty$ with strictly positive speed v_α. As seen in Section 2, the rate functions I_η^q, I_η^a for the random variables X_n/n, both in the annealed and quenched situations, vanish on the interval $[0, v_\alpha]$. Our goal here is to describe, following [9] and [35], the appropriate large deviation results in this regime.

How can the walk deviate significantly from its almost-sure limiting speed v_α?
For ordinary RW, this is exponentially unlikely and given that such a deviation has occurred, it is most likely to arise from movement at an approximately constant different speed. For RWRE there are other possibilities- large deviations can arise from relatively short, atypical, segments in the environment ("traps").

The next two theorems characterize the subexponential slow-down probabilities $\mathbf{P}(n^{-1}X_n \in G)$ in the mixed-drift cases for any non-empty open $G \subset (0, v_\alpha)$ which is separated from v_α. A polynomial rate of decay is obtained when a negative local drift is possible, whereas for environments which allow only positive and zero drifts, the large-deviation slow-down probabilities decay like $\exp(-Cn^{1/3})$. These theorems (without the precise upper bound in Theorem 6) are taken from [9], while the derivation of the precise upper bound of Theorem 6 (and, thereby, the existence of the limit in Theorem 6!) is due to [35].

Theorem 5 (Positive and negative drifts) *([9, Thm. 1.1]) Suppose that $\langle \rho \rangle < 1$ and $\infty > \rho_{\max} > 1$. Then, there exists a unique $s > 1$ satisfying $\langle \rho^s \rangle = 1$ such that for any open $G \subset (0, v_\alpha)$ which is separated from v_α,*

$$\lim_{n\to\infty} \log \mathbf{P}(n^{-1}X_n \in G)/\log n = 1 - s\ .$$

Theorem 6 (Positive and zero drifts) *([9, Thm. 1.2], [35, Thm. 2]) Suppose that $\langle\rho\rangle < 1$, but $\rho_{\max} = 1$ and $\alpha(\{1/2\}) > 0$. Then, for any open $G \subset (0, v_\alpha)$ which is separated from v_α,*

$$\lim_{n\to\infty} n^{-1/3} \log \mathbf{P}(n^{-1}X_n \in G) = -\frac{3}{2} \inf_{v\in G} (1 - v/v_\alpha)^{1/3} |\frac{\pi}{2} \log \alpha(\{1/2\})|^{2/3} .$$

We sketch now the proof of Theorem 5. Since $\rho_{\max} > 1$, the existence and uniqueness of s are due to the strict convexity and monotonicity of the map $\lambda \mapsto \langle\rho^\lambda\rangle$. Maybe not surprisingly, hitting time decompositions are crucial throughout the proof.

We begin by describing the proof of the lower bounds. Essentially, the same proof works for all sub-exponential LDP's, both quenched and annealed, in different regimes. The strategy is to find a "trap", i.e. an interval where the random walk spends a lot of time, so that the probability of being slower than expected will not decay exponentially. In Theorem 6 and Theorem 8 below, this will be simply be an interval I consisting of fair coins, i.e. $\omega_i = 1/2$ for all $i \in I$. The main difference between the annealed and quenched setups is that the fluctuations in the environment are of different order. This can be seen by comparing the proofs of the lower bounds in Theorem 6 and Theorem 8 below. While in the annealed case, we have, with probability $\alpha(1/2)^{n^{1/3}}$, an interval consisting of fair coins of length $O(n^{1/3})$ at the origin, we have to use an almost sure limit law (here, the Erdös-Renyi law for longest runs) in the quenched case, to obtain a "fair" interval whose length is of order $\log n$. In the proof of the lower bound of Theorem 5, the fluctuations in the environment are given by the extreme values of the random variables $\{kR_k\}$ below.
For $y \in \mathbb{Z}$, let $T_y = \min\{n : X_n = y\}$. Let $\overline{X}_n$ denote the Markov chain (reflected at 0), initialized at zero, with the same ω-dependent transition kernel as in (1) but now with the value of ω_0 set to be $\omega_0 = 1$. Let

$$\overline{T}_k = \inf\{n : \overline{X}_n = k\} \tag{29}$$

and

$$R_k(m) = k^{-1} \sum_{i=m+1}^{m+k} \log \rho(i) \, , \text{ with } R_k = R_k(0). \tag{30}$$

The following tail estimates for $\overline{T}_k$ and for $L_y = \max\{y - X_n : n \geq T_y\}$, the longest excursion of the RWRE path to the left of y, are straightforward:

Lemma 4. *For all n, $k \geq 1$, $y \in \mathbb{Z}$, and any ω,*

$$\mathbf{P}_\omega(\overline{T}_k \geq n) \geq (1 - e^{-(k-1)R_{k-1}})^n \, , \; \mathbf{P}(L_y \geq k) \leq \frac{\langle\rho\rangle^k}{1 - \langle\rho\rangle} \, .$$

Since $G \subset (0, v_\alpha)$ is open and separated from v_α, it suffices to establish the lower bound for $G = (v - 2\gamma, v)$ for $0 < 2\gamma < v < v_\alpha$. The event $n^{-1}X_n \in (v - 2\gamma, v)$ contains the event

$$\left\{ \frac{(v-2\gamma)n}{v_\alpha} < T_{(v-\gamma)n} < n, \text{ the excursion distance } L_{(v-\gamma)n} < \gamma n, \text{ and } T_{vn} > n \right\},$$

namely, that the RWRE hits $(v-\gamma)n$ at about the expected time, from which point its longest excursion to the left is less than γn, but the RWRE does not arrive at position vn by time n. By Solomon's law of large numbers, $\mathbf{P}(T_{(v-\gamma)n} \in ((v-2\gamma)n/v_\alpha, n)) \to 1$ as $n \to \infty$. Set $\xi = 1 - (v-2\gamma)/v_\alpha > 0$. Since $T_{(v-\gamma)n}$ is independent of $\{\omega_x : x \geq (v-\gamma)n\}$, it follows by stationarity that

$$\mathbf{P}(T_{vn} > n | T_{(v-\gamma)n} \in ((v-2\gamma)n/v_\alpha, n)) \geq \mathbf{P}(\overline{T}_{\gamma n} > \xi n) .$$

Hence, by the exponential bound on $\mathbf{P}(L_{(v-\gamma)n} \geq \gamma n)$ (see Lemma 4), the lower bound holds as soon as we show that

$$\liminf_{n\to\infty} \log \mathbf{P}(\overline{T}_{\gamma n} > \xi n)/\log n \geq 1 - s . \tag{31}$$

To this end, let $y_\delta = \langle \rho^{s-\delta} \log \rho \rangle / \langle \rho^{s-\delta} \rangle$ for $\delta \geq 0$, and note that for every $\delta > 0$ small enough y_δ is finite and positive. Applying Cramér's theorem to the i.i.d. real-valued random variables $\{\log \rho(x)\}_{x\in\mathbb{Z}}$ gives

$$\mathbf{P}(R_{k-1} \geq y_\delta) \geq e^{-(s_\delta y_\delta + o(1))k} \text{ as } k \to \infty \tag{32}$$

where $s_\delta := s - \delta - (y_\delta)^{-1} \log\langle \rho^{s-\delta} \rangle$. Choose

$$k = k(n) = 1 + \frac{\log n}{y_\delta},$$

so that $e^{-(s_\delta y_\delta + o(1))k} = n^{-s_\delta + o(1)}$ as n and k tend to ∞ , and consider the event

$$\mathcal{A}_n = \{\omega : \max_{m=0,1,\ldots,[\gamma n/k]-1} R_{k-1}(mk, \omega) \geq y_\delta\} .$$

Since $\{R_{k-1}(mk)\}_{m\geq 0}$ are i.i.d. variables, we obtain from (32) that

$$\liminf_{n\to\infty} \frac{1}{\log n} \log \mathbf{P}(\mathcal{A}_n) \geq \liminf_{n\to\infty} \frac{1}{\log n} \log \left(\frac{\gamma n}{k(n)} n^{-s_\delta} \right) = 1 - s_\delta . \tag{33}$$

Decomposing the event $\mathcal{A}_n$ according to $m^* = \min\{m \geq 0 : R_{k-1}(mk) \geq y\}$ and ignoring the time which the chain X_n spends outside $[m^*k, m^*k + k)$, we get by stationarity that

$$\mathbf{P}(\overline{T}_{\gamma n} > \xi n \,|\, \mathcal{A}_n) \geq \inf_{\omega : R_{k-1}(\omega) \geq y_\delta} \mathbf{P}_\omega(\overline{T}_k > \xi n) .$$

By Lemma 4,

$$\inf_{\omega: R_{k-1}(\omega)\geq y_\delta} \mathbf{P}_\omega(\overline{T}_k > \xi n) \geq \inf_{z\geq y_\delta} (1-e^{-(k-1)z})^{(\xi n+1)} \geq (1-n^{-1})^{(\xi n+1)} . \tag{34}$$

Combining (33) and (34) and taking $\delta \downarrow 0$ (for which $s_\delta \to s$), we establish (31), thus completing the proof of the lower bound.

The upper bound on $\mathbf{P}(n^{-1}X_n \in G)$ hinges upon moment estimates on the hitting times T_k. To derive these observe that $T_k = \sum_{i=1}^k \tau_i$ is the sum of the identically distributed, (dependent) random variables τ_i, the law of each of which is identical to the law of τ_1. Let $C_\gamma = \mathbf{E}(\tau_1^\gamma)$ and note that by Minkowski's inequality for all $k \geq 1$

$$\mathbf{E}(T_k^\gamma) \leq C_\gamma k^\gamma . \tag{35}$$

The crucial observation is that $C_\gamma < \infty$ for all $\gamma < s$. Once this is established, standard estimates, due to Nagaev in [33], allow one to obtain the correct tail estimates for T_n/n, and the usual duality transfers these to tail estimates on X_n/n. We thus concentrate in the remainder of this sketch on the derivation of the bounds on $\mathbf{E}(\tau_1^\gamma)$. These bounds apply more generally in the context of Branching Processes in a Random Environment, and may be found in [9]. For the purpose of this review, we will prove a weaker result:

Lemma 5. *Assume $s > 2$. Then $\mathbf{E}(\tau_1) < \infty$ and $\mathbf{E}(\tau_1^2) < \infty$.*

Proof of Lemma 5: Let N_0 denote the number of excursions of the RWRE to the left of 0 before τ_1, and let, for $i = 1, 2, \ldots, N_0$, $\tau_0^{-1}(i)$ denote the length of the i-th excursion from -1 to 0. Note that given the environment, the random variables $\tau_0^{-1}(i)$ are i.i.d., and that their law depends on $\{\omega_j\}_{j=-\infty}^{-1}$ only while, because the RWRE is transient to $+\infty$, the law of N_0 depends on ω_0 only and is geometric with parameter ω_0, more precisely, we have $\mathbf{P}_\omega(N_0 = k) = \omega_0(1-\omega_0)^k$, $k = 0, 1, 2, \ldots$. In particular, under the measure $\mathbf{P}_\omega$, $\tau_0^{-1}(1), \tau_0^{-1}(2), \ldots$ are independent and independent of N_0. Using now the hitting time decomposition

$$\tau_1 = 1 + \sum_{i=1}^{N_0} (1 + \tau_0^{-1}(i)) \tag{36}$$

taking first expectations with $\mathbf{P}_\omega$ and then integrating over ω, one concludes that, whenever $\mathbf{E}(\tau_1) < \infty$,

$$\mathbf{E}(\tau_1) = 1 + \mathbf{E}(N_0)(1 + \mathbf{E}(\tau_0^{-1}(1))) = 1 + \langle\rho\rangle(1 + \mathbf{E}(\tau_1)) ,$$

establishing that $\mathbf{E}(\tau_1) = (1+\langle\rho\rangle)/(1-\langle\rho\rangle)$. One concludes that a necessary condition for $\mathbf{E}(\tau_1) < \infty$ is $s > 1$, while the sufficiency is established by a truncation argument: clearly, for any $1 < M < \infty$,

$$\tau_1 \wedge M \le 1 + \sum_{i=1}^{N_0} (1 + \tau_0^{-1}(i) \wedge M)\,,$$

and hence

$$\mathbf{E}(\tau_1 \wedge M) \le 1 + \mathbf{E}(N_0)(1 + \mathbf{E}(\tau_1 \wedge M))\,,$$

implying that $\mathbf{E}(\tau_1 \wedge M) \le (1+\langle\rho\rangle)(1-\langle\rho\rangle)$. Monotone convergence then yields that $\mathbf{E}(\tau_1) < \infty$.

To see the second moment bound, let τ_1, τ_1' denote two independent (given the environment) copies of τ_1. Using the decomposition (36) and

$$\tau_1' = 1 + \sum_{i=1}^{N_0'} (1 + \tau_0^{-1}(i)')\,,$$

one sees that if $\mathbf{E}(\tau_1\tau_1') < \infty$ then

$$\mathbf{E}(\tau_1\tau_1') = 1 + 2\mathbf{E}(N_0)(1+\mathbf{E}(\tau_1)) + \mathbf{E}(N_0N_0')(1 + 2\mathbf{E}(\tau_1) + \mathbf{E}(\tau_1\tau_1'))\,.$$

Since $\mathbf{E}(N_0N_0') = \langle\rho^2\rangle$, this establishes the necessity of $s > 2$, and, truncating and using monotone convergence, it also establishes that $\mathbf{E}(\tau_1\tau_1') < \infty$ as soon as $s > 2$. Using now (36) once more, one checks that whenever $\mathbf{E}(\tau_1^2) < \infty$ then

$$\mathbf{E}(\tau_1^2) = 1 + \mathbf{E}(N_0)(3 + 4\mathbf{E}(\tau_1) + \mathbf{E}(\tau_1^2)) + \mathbf{E}(N_0^2 - N_0)(\mathbf{E}(\tau_1\tau_1') + 2\mathbf{E}(\tau_1) + 1)\,,$$

and again, one concludes that for $s > 2$, $\mathbf{E}(\tau_1^2) < \infty$ with a truncation argument, using the fact that $\mathbf{E}(N_0) < 1$ and $\mathbf{E}(\tau_1\tau_1') < \infty$ in this case. $\square$

It is easy to check that one may generalize Lemma 5 by induction to other integer moments $i < s$. A somewhat more elegant argument, which also handles non integer moments of τ_1, can be found in [9, Lemma 2.4].

We conclude this section with comments on the proof of Theorem 6. As mentioned above, the proof of the lower bound follows the same track as in Theorem 5, this time with the contribution to large R_k coming essentially from blocks of "fair coins" of length $n^{1/3}$. The argument in [9] for the upper bound is a rather crude bootstrapping argument based on the upper bound in Theorem 5, and misses the correct constant in the exponent. The proof given in [35] which captures the correct constant in the upper bound is too technical to present it here. Very roughly, it involves a coarse graining of the environment into blocks of size $n^{1/3+\delta}$, some small δ, and classifying them as "biased" blocks (if the empirical measure of ω_i-s in the block has a significant proportion of $\omega_i > 1/2$) and "fair" blocks (if not). The biased blocks serve as effective barriers, in the sense that the random walk only rarely crosses such a block from right to left. Handling stretches of fair blocks is done by Chebycheff's inequality, and most of the effort is invested in proving that long stretches of fair blocks which are shorter than the maximal stretch do not contribute much to the tail asymptotics.

4 Quenched LDP's - subexponential speed

In this section we turn our attention to the quenched sub-exponential regime. Maybe surprisingly, it turns out that the annealed estimates are key to understanding the quenched asymptotics. The next theorems are the main results known. They quantify the fact that the annealed probabilities of large deviations are of bigger order than their quenched counterparts, due to the possibility of rare fluctuations in the environment which may slow down the RWRE.

Theorem 7 (Positive and negative drifts) *([14, Thm. 1])*
Suppose that $\langle\rho\rangle < 1$, $\rho_{\max} > 1$, and let $v \in (0, v_\alpha)$. Then, for η-a.a. ω, the following statements hold:

1. For any $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{n^{1-1/s-\delta}} \log \mathbf{P}_\omega\left(X_n < nv\right) = -\infty \,. \tag{37}$$

2. For any $\delta > 0$,

$$\liminf_{n\to\infty} \frac{1}{n^{1-1/s+\delta}} \log \mathbf{P}_\omega\left(X_n < nv\right) = 0 \,. \tag{38}$$

Furthermore,

$$\limsup_{n\to\infty} \frac{1}{n^{1-1/s}} \log \mathbf{P}_\omega\left(X_n < nv\right) = 0 \,. \tag{39}$$

One should compare the rate of decay obtained in Theorem 7 with the annealed polynomial rate of decay (see Theorem 5) $\mathbf{P}(X_n < nv) \simeq n^{1-s}$.

As in Theorem 6, tail estimates are different when the drift cannot be negative:

Theorem 8 (Positive and zero drifts) *([14, Thm. 2], [34, Thm. 1]) Suppose that* $\langle\rho\rangle < 1$, $\rho_{\max} = 1$, *and* $\alpha(\{1/2\}) > 0$. *Then, for* η*-a.a.* ω, *and for* $v \in (0, v_\alpha)$,

$$\lim_{n\to\infty} \frac{(\log n)^2}{n} \log \mathbf{P}_\omega(X_n < nv) = -|\pi \log \alpha(\{1/2\})|^2/8(1 - \frac{v}{v_\alpha}) . \tag{40}$$

Again, the rate in Theorem 8 should be compared with the annealed rate (c.f. Theorem 6) $\mathbf{P}(X_n < nv) \simeq \exp(-Cn^{1/3})$. Theorem 7 is contained in [14], as well as the order of decay in Theorem 8 without the sharp value of the r.h.s. The exact value of the r.h.s. in (40) (and, thereby, the existence of the limit in (40)!) are due to [34], where the authors sharpen the coarse graining approach described in the last section for the annealed case. We concentrate in the rest of this section on Theorem 7, and try to give a rough sketch of its proof as well as some intriguing questions and challenges that it poses.

We begin by noting that the lower bound in (38) follows the same outline as in the annealed case, and was actually predicted in [9]: The maximal value of all possible kR_k's (with different initial and final points) in the block $[0, nv]$ is $kR_k = \log n/s + Z_n$, where Z_n is a random variable whose length is of order 1 (but on appropriate subsequences may be arbitrarily large or small). Taking a (random) subsequence with Z_n large yields the improved lower bound (39).

The proof of the upper bound is based on dividing the interval $[0, nv]$ into blocks of size $n^{1/s+\delta}$. A typical such block is transient to the right, and the RWRE only rarely crosses such a block from right to left. The annealed estimates, together with the Borel-Cantelli lemma, can be used to give uniform estimates for the time needed to cross a block from left to right. They also allow one to estimate how rarely a "backtrack", i.e. a crossing from the right to the left, occurs. Then, each such "backtrack" can be treated as increasing by one the number of blocks that the RWRE has to cross. Taking that into account yields the quenched estimates.

Intuitively, since the random variables Z_n can be made arbitrarily small on appropriate subsequences, one expects the following conjecture to hold true:

Conjecture 1. *In the setting of Theorem 7, we have for* η*-a.a.* ω

$$\liminf_{n\to\infty} \frac{1}{n^{1-1/s}} \log \mathbf{P}_\omega(X_n < nv) = -\infty .$$

Together with (39), this conjecture says that there does not exist a refined LDP in the subexponential regime: the random variables $a_n := \log \mathbf{P}_\omega(X_n < nv)$ oscillate according to random subsequences given by the random environment.

The difficulty in verifying Conjecture 1 is that, unlike in the lower bound, it is not enough to establish that all "fair stretches" are shorter than usual: one has to show that even when all "fair stretches" are short, a combination of several such stretches does not contribute to the tail asymptotics. A case which we can analyze explicitly is the "two-coins case" where $\alpha_1 := \alpha(1) > 0$ and $\alpha(p) = 1 - \alpha_1$ for some $p < 1/2$. The RWRE in this environment is a simple random walk with drift to the left with randomly placed nodes, i.e. locations where the random walk is forced to go right. In this case, $s = -\log(1-\alpha_1)/\log\bar{\rho}$, where $\bar{\rho} := (1-p)/p > 1$. Here, $\langle\rho\rangle = (1-\alpha_1)\bar{\rho}$ and we will assume that $(1-\alpha_1)\bar{\rho} < 1$, implying that the RWRE has the positive speed $v_\alpha = (1-\langle\rho\rangle)/(1+\langle\rho\rangle)$. While the measure α does not satisfy our standard assumption that its support is included in the open interval $(0,1)$, subexponential asymptotics still occur in this case. Since Theorem 9 below has not appeared elsewhere, we present its proof with all details.

Theorem 9. *Assume we are in the two-coins case, i.e.* $\omega_0, \omega_1, \ldots$ *are i.i.d. ,* ω_0 *has value* 1*, with probability* α_1 *or value* p*, with probability* $1-\alpha_1$*. Assume* $(1-\alpha_1)\bar{\rho} < 1$*. Let* $u > 1/v_\alpha$*. Then, for* η*-a.a.* ω*,*

$$\liminf_{n\to\infty} \frac{1}{n^{1-1/s}} \log \mathbf{P}_\omega(T_n \geq nu) = -\infty\,, \tag{41}$$

$$\limsup_{n\to\infty} \frac{1}{n^{1-1/s}} \log \mathbf{P}_\omega(T_n \geq nu) = 0\,. \tag{42}$$

As a consequence, for $v < v_\alpha$*, we have for* η*-a.a.* ω

$$\liminf_{n\to\infty} \frac{1}{n^{1-1/s}} \log \mathbf{P}_\omega(X_n \leq nv) = -\infty\,, \tag{43}$$

$$\limsup_{n\to\infty} \frac{1}{n^{1-1/s}} \log \mathbf{P}_\omega(X_n \leq nv) = 0\,. \tag{44}$$

(In fact, statements slightly stronger than (41), (42) hold, see (53), (79)).

Proof

1. Proof of (42) (see also [14])

Since $\ell_0(\omega) = \inf\{i \geq 0 : \omega_i = 1\} < \infty$ a.s., we may and will assume w.l.o.g. that $\omega_0 = 1$, i.e. that we have a node at 0. Let $\ell_1(\omega), \ell_2(\omega), \ldots$ be the lengths of the successive intervals without nodes, i.e.

$$\ell_1(\omega) := \inf\{i \geq 1 : \omega_i = 1\} \tag{45}$$

$$\ell_k(\omega) := \inf\{i \geq 1 : \omega_{\ell_1+\ldots+\ell_{k-1}+i} = 1\}, \quad k \geq 1\,. \tag{46}$$

Let $N_l(n) := \max\{j : \ell_1 + \ldots + \ell_j \leq n\}$ and

$$\ell_{\max}(n) := \max_{1\leq j\leq N_l(n)} \ell_j(\omega)\,. \tag{47}$$

We will now give an estimate for the exit time of an interval without node. Let P_x be the distribution of the random walk (X_j), started at $x \in \{0, 1, \dots, \ell+1\}$, with $\omega_0 = 1$, $\omega_1 = \omega_2 = \dots = \omega_\ell = p$, let $\overline{T}_{\ell+1} := \inf\{j : X_j = \ell+1\}$ and $\overline{T}_0 := \inf\{j : X_j = 0\}$. Then

$$P_1\left(\overline{T}_0 < \overline{T}_{\ell+1}\right) = \frac{\bar\rho^{\ell+1} - \bar\rho}{\bar\rho^{\ell+1} - 1} \geq \frac{\bar\rho^{\ell} - 1}{\bar\rho^{\ell}}$$

and

$$P_1(\overline{T}_{\ell+1} \geq n) \geq P_1\left(\overline{T}_0 < \overline{T}_{\ell+1}\right)^n \geq \left(1 - \frac{1}{\bar\rho^{\ell}}\right)^n .$$

Using this estimate, we give a lower bound for $\mathbf{P}_\omega(T_n \geq nu)$ by simply picking the largest intervals without nodes: for each ω,

$$\mathbf{P}_\omega(T_n \geq nu) \geq P_0(\overline{T}_{\ell_{\max}(n)+1} \geq nu) \geq \left(1 - \frac{1}{\bar\rho^{\ell_{\max}(n)}}\right)^{\lfloor nu \rfloor + 1} \tag{48}$$

Theorem 2 in [8] implies that, with $\log_2 n = \log\log n$ and $\log_3 n = \log\log\log n$,

$$\mathbf{P}\left(\ell_{\max}(n) \geq \frac{\log n}{-\log(1-\alpha_1)} + \frac{\log_2 n}{-\log(1-\alpha_1)} \text{ for infinitely many } n\right) = 1 \tag{49}$$

and

$$\mathbf{P}\left(\ell_{\max}(n) \leq \frac{\log n}{-\log(1-\alpha_1)} - \frac{\log_3 n}{-\log(1-\alpha_1)} \text{ for infinitely many } n\right) = 1\,. \tag{50}$$

We now choose a (random) subsequence (n_k) such that

$$\ell_{\max}(n_k) \geq \frac{\log n_k}{-\log(1-\alpha_1)} + \frac{\log_2 n_k}{-\log(1-\alpha_1)} = \frac{\log n_k}{s \log \bar\rho} + \frac{\log_2 n_k}{s \log \bar\rho} \tag{51}$$

for all k. Then, due to (48), for each $\varepsilon > 0$

$$\begin{aligned}
\log \mathbf{P}_\omega(T_{n_k} \geq n_k u) &\geq \log\left(1 - \frac{1}{\bar\rho^{\ell_{\max}(n_k)}}\right)^{n_k u} \\
&= n_k u \frac{1}{\bar\rho^{\ell_{\max}(n_k)}} \bar\rho^{\ell_{\max}(n_k)} \log\left(1 - \frac{1}{\bar\rho^{\ell_{\max}(n_k)}}\right) \\
&\geq n_k u \exp\left(-\frac{\log n_k}{s} - \frac{1}{s}\log_2 n_k\right)(-1-\varepsilon) \qquad (52)
\end{aligned}$$

for k large enough. Hence, for η-a.a. ω,

$$\lim_{k\to\infty} \frac{1}{n_k^{1-1/s}} \log \mathbf{P}_\omega(T_{n_k} \geq n_k u) = 0$$

and (42) follows.

Remark

More precisely, (52) implies that, for η-a.a. ω, on the random subsequence $n_k \to \infty$ introduced in (51),

$$\limsup_{k\to\infty} \frac{(\log n_k)^{1/s}}{n_k^{1-1/s}} \log \mathbf{P}_\omega(T_{n_k} \geq n_k u) \geq -u\,. \tag{53}$$

2. Proof of (41)

We again may and will assume that $\omega_0 = 1$. Let $N(n) := \min\{j : \ell_1 + \ldots + \ell_j \geq n\}$. Denote

$$\ell_{\mathrm{MAX}}(n) := \max_{1\leq j\leq N(n)} \ell_j(\omega)\,. \tag{54}$$

and note that (49) and (50) still hold true if we replace $\ell_{\max}(n)$ with $\ell_{\mathrm{MAX}}(n)$. We first prove the following formula for the exit time of a random walk with reflection at 0.

Lemma 6. *Let $\omega_0 = 1$, $\omega_1 = \omega_2 = \ldots = \omega_{\ell-1} = p < \frac{1}{2}$, $\overline{T}_\ell := \inf\{j : X_j = \ell\}$ and $g(\ell) := E_0(\exp(\lambda\overline{T}_\ell))$. Let $q := 1-p$. For $\lambda < -\frac{1}{2}\log(4pq)$, define*

$$y_1 := y_1(\lambda) = \frac{1}{2p}\left(e^{-\lambda} + \sqrt{e^{-2\lambda} - 4pq}\right)\,, \tag{55}$$

$$y_2 := y_2(\lambda) = \frac{1}{2p}\left(e^{-\lambda} - \sqrt{e^{-2\lambda} - 4pq}\right)\,. \tag{56}$$

Fix

$$\lambda_c := \lambda_c(\ell) = \sup\{\lambda > 0 : e^{-2\lambda} > 4pq,\ y_2^\ell(e^\lambda y_1 - 1) > y_1^\ell(e^\lambda y_2 - 1)\}\,.$$

Then, for $\lambda < \lambda_c$, we have

$$g(\ell) = \frac{e^\lambda(y_1 - y_2)}{y_2^\ell(e^\lambda y_1 - 1) - y_1^\ell(e^\lambda y_2 - 1)}\,. \tag{57}$$

Proof Let $g_x(\ell) := E_x(\exp(\lambda\overline{T}_\ell))$, $0 \leq x \leq \ell$. We have $g(\ell) = g_0(\ell) = e^\lambda g_1(\ell)$, $g_\ell(\ell) = 1$ and $g_x(\ell) = e^\lambda p g_{x+1}(\ell) + e^\lambda q g_{x-1}(\ell)$, $1 \leq x \leq \ell - 1$. For $0 \leq x \leq \ell$, $g_x(\ell)$ has the form

$$g_x(\ell) = Ay_1^x + By_2^x \tag{58}$$

where y_1 and y_2 satisfy, for $1 \leq x \leq \ell - 1$, $y_{1,2}^x = e^\lambda p y_{1,2}^{x+1} + e^\lambda q y_{1,2}^{x-1}$, and are, therefore, given by (55) and (56). Substituting in (58) the boundary condition $g_\ell(\ell) = 1$ yields $B = (1 - Ay_1^\ell)/y_2^\ell$, hence

$$g_x(\ell) = Ay_1^x + y_2^{x-\ell} - Ay_2^x\left(\frac{y_1}{y_2}\right)^\ell\,. \tag{59}$$

Now, using the second boundary condition $g_0(\ell) = e^{\lambda} g_1(\ell)$, we compute

$$A = \frac{e^{\lambda} y_2 - 1}{y_2^{\ell} - y_1^{\ell} - e^{\lambda} y_1 y_2^{\ell} + e^{\lambda} y_1^{\ell} y_2} .$$

Finally, setting $x = 0$ in (59) gives

$$g_0(\ell) = A + y_2^{-\ell} - A \left(\frac{y_1}{y_2}\right)^{\ell}$$

and a little arithmetic yields (57), as long as $\lambda < \lambda_c$. $\square$

As usual, we start the proof of the upper bound (41) with Chebyshev's inequality. Fix $C > 0$ and let $\lambda_n = Cn^{-1/s}$. Define the (random) subsequence $(\tilde{n}_k)$ such that

$$\ell_{\mathrm{MAX}}(\tilde{n}_k) \leq \frac{\log \tilde{n}_k}{s \log \bar{\rho}} - \frac{\log_3 \tilde{n}_k}{s \log \bar{\rho}} , \tag{60}$$

which is possible due to (50). We claim that

$$\mathbf{P}_{\omega}(T_{\tilde{n}_k} \geq \tilde{n}_k u) \leq \mathbf{E}_{\omega}(e^{\lambda_{\tilde{n}_k} T_{\tilde{n}_k}}) e^{-\lambda_{\tilde{n}_k} \tilde{n}_k u} \leq \prod_{j=1}^{N(\tilde{n}_k)} g(\ell_j(\omega)) e^{-\lambda_{\tilde{n}_k} \tilde{n}_k u} \tag{61}$$

To verify (61), we have to show that, for all k large enough, $\lambda_{\tilde{n}_k} < \lambda_c(\ell_i)$ is satisfied for $i = 1, \ldots, \tilde{n}_k$. Because $(e^{\lambda} y_1 - 1) \geq y_1 - 1 > 0$, it suffices to show that

$$\left(\frac{y_1}{y_2}\right)^{\ell_{\mathrm{MAX}}(\tilde{n}_k)} \left(e^{\lambda_{\tilde{n}_k}} y_2 - 1\right) \underset{k \to \infty}{\longrightarrow} 0 \tag{62}$$

Note that, since $y_2(0) = 1$,

$$\frac{e^{\lambda_n} y_2 - 1}{\lambda_n} \underset{n \to \infty}{\longrightarrow} 1 + y_2'(0) = 1 + \frac{1}{1 - 2p} . \tag{63}$$

Since y_1 is decreasing and y_2 is increasing, we have

$$\frac{y_1(\lambda_n)}{y_2(\lambda_n)} \leq \frac{y_1(0)}{y_2(0)} = \frac{q}{p} = \bar{\rho} \tag{64}$$

Using (60) and (64), we have for some C_1 independent of k,

$$\left(\frac{y_1}{y_2}\right)^{\ell_{\mathrm{MAX}}(\tilde{n}_k)} \left(e^{\lambda_{\tilde{n}_k}} y_2 - 1\right) \leq \tilde{n}_k^{1/s} \lambda_{\tilde{n}_k} \frac{e^{\lambda_{\tilde{n}_k}} y_2 - 1}{\lambda_{\tilde{n}_k}} e^{-C_1 \log_3 \tilde{n}_k} \underset{k \to \infty}{\longrightarrow} 0 , \tag{65}$$

proving (62). Taking logarithms in (61) yields

$$\log \mathbf{P}_\omega(T_{\tilde{n}_k} \geq \tilde{n}_k u) \leq \sum_{j=1}^{N(\tilde{n}_k)} \log g(\ell_j(\omega)) - \lambda_{\tilde{n}_k} \tilde{n}_k u \tag{66}$$

Next we analyze the first term on the r.h.s. of (66). Note that, on the sequence $\{\tilde{n}_k\}$, i.e. for $\lambda = \lambda(\tilde{n}_k)$, $y_1 = y_1(\lambda(\tilde{n}_k)), y_2 = y_1(\lambda(\tilde{n}_k))$,

$$\begin{aligned} \log g(\ell) &= -\log\left(\frac{y_2^\ell(e^\lambda y_1 - 1)}{e^\lambda(y_1 - y_2)} - \frac{y_1^\ell(e^\lambda y_2 - 1)}{e^\lambda(y_1 - y_2)}\right) \\ &= -\log\frac{y_2^\ell(e^\lambda y_1 - 1)}{e^\lambda(y_1 - y_2)} - \log\left(1 - \frac{y_1^\ell(e^\lambda y_2 - 1)}{y_2^\ell(e^\lambda y_1 - 1)}\right) \end{aligned} \tag{67}$$

We consider, for n on the sequence $\{\tilde{n}_k\}$,

$$\frac{1}{n^{1-1/s}} \sum_{j=1}^{N(n)} \log g(\ell_j(\omega))$$

$$= -\frac{1}{n^{1-1/s}} \sum_{j=1}^{N(n)} \log\left(\frac{y_2^{\ell_j}(e^{\lambda_n} y_1 - 1)}{e^{\lambda_n}(y_1 - y_2)}\right) - \frac{1}{n^{1-1/s}} \sum_{j=1}^{N(n)} \log\left(1 - \frac{y_1^{\ell_j}(e^{\lambda_n} y_2 - 1)}{y_2^{\ell_j}(e^{\lambda_n} y_1 - 1)}\right) \tag{68}$$

The first term in (68) can be splitted again:

$$-\frac{1}{n^{1-1/s}} \sum_{j=1}^{N(n)} \log\left(\frac{y_2^{\ell_j}(e^{\lambda_n} y_1 - 1)}{e^{\lambda_n}(y_1 - y_2)}\right) = -\left(\frac{1}{n^{1-1/s}} \sum_{j=1}^{N(n)} \ell_j \log y_2\right) - \left(\frac{N(n)}{n^{1-1/s}} \log\frac{(e^{\lambda_n} y_1 - 1)}{e^{\lambda_n}(y_1 - y_2)}\right) \tag{69}$$

Note that, for η-a.a. ω,

$$\frac{N(n)}{n} \underset{n\to\infty}{\longrightarrow} \alpha_1 \tag{70}$$

due to the law of large numbers, and

$$\frac{1}{n} \sum_{j=1}^{N(n)} \ell_j(\omega) = \frac{N(n)}{n} \frac{1}{N(n)} \sum_{j=1}^{N(n)} \ell_j(\omega) \underset{n\to\infty}{\longrightarrow} 1\,. \tag{71}$$

Using the mean value theorem, there is $\xi_n \in (0, \lambda_n)$ such that

$$\log y_2(\lambda_n) = \lambda_n \frac{y_2'(\xi_n)}{y_2(\xi_n)}$$

Since $y_2(0) = 1$ and $y_2'(0) = (q-p)^{-1}$, we see that

$$n^{1/s} \log y_2(\lambda_n) = n^{1/s} \lambda_n \frac{y_2'(\xi_n)}{y_2(\xi_n)} = C \frac{y_2'(\xi_n)}{y_2(\xi_n)} \underset{n\to\infty}{\longrightarrow} \frac{C}{q-p}$$

which shows, together with (71), that the first term on the r.h.s. of (69) converges, for η-a.a. ω, to $\frac{-C}{q-p}$. For the second term on the r.h.s. of (69), we proceed similarly: Let

$$\varphi(\lambda) := \frac{e^{\lambda}y_1 - 1}{e^{\lambda}(y_1 - y_2)}\,.$$

There is $\xi_n \in (0, \lambda_n)$ such that

$$\log\varphi(\lambda_n) = \log\varphi(0) + \lambda_n \frac{\varphi'(\xi_n)}{\varphi(\xi_n)} = \lambda_n \frac{\varphi'(\xi_n)}{\varphi(\xi_n)}\,.$$

We have

$$\varphi'(\lambda) = \frac{e^{\lambda}y_1 + e^{\lambda}y_1'}{e^{\lambda}(y_1 - y_2)} - \frac{(e^{\lambda}y_1 - 1)\left(e^{\lambda}(y_1 - y_2) + e^{\lambda}(y_1' - y_2')\right)}{e^{2\lambda}(y_1 - y_2)^2}$$

Using $y_1(0) = q/p = \bar{\rho}$, $y_2(0) = 1$, $y_2'(0) = (q-p)^{-1}$, a little calculation yields

$$\lim_{\lambda\to 0} \varphi'(\lambda) = \frac{2qp}{(q-p)^2}\,.$$

Therefore, the second term on the r.h.s. of (69) equals

$$-\frac{N(n)}{n} n^{1/s} \lambda_n \frac{\varphi'(\xi_n)}{\varphi(\xi_n)} \underset{n\to\infty}{\longrightarrow} -\alpha_1 C \frac{2qp}{(q-p)^2} \quad \eta - a.a.\omega$$

Going back to (68), we have proved that

$$-\frac{1}{n^{1-1/s}} \sum_{j=1}^{N(n)} \log\left(\frac{y_2^{\ell_j}(e^{\lambda_n}y_1 - 1)}{e^{\lambda_n}(y_1 - y_2)}\right) \underset{n\to\infty}{\longrightarrow} -\frac{C}{q-p} - \alpha_1 C \frac{2qp}{(q-p)^2}\,, \quad \eta - a.a.\omega\,. \tag{72}$$

Considering the second term in (68), note that, for n on the sequence $\{\tilde{n}_k\}$,

$$-\sum_{j=1}^{N(n)} \log\left(1 - \frac{y_1^{\ell_j}(e^{\lambda_n}y_2 - 1)}{y_2^{\ell_j}(e^{\lambda_n}y_1 - 1)}\right) \le \sum_{j=1}^{N(n)} \frac{y_1^{\ell_j}(e^{\lambda_n}y_2 - 1)}{y_2^{\ell_j}(e^{\lambda_n}y_1 - 1)} \psi(\gamma_n) \tag{73}$$

where we set

$$\psi(x) := \frac{-\log(1-x)}{x}, \quad \gamma_n := \left(\frac{y_1}{y_2}\right)^{\ell_{\mathrm{MAX}}(n)} \frac{e^{\lambda_n}y_2 - 1}{e^{\lambda_n}y_1 - 1} \tag{74}$$

Note that $\psi(x) \downarrow 1$ for $x \to 0$, and $\gamma_n \to 0$ due to (65). Let θ^B be the shift up to the next node, i.e. $(\theta^B\omega)(i) = \omega(i + \ell_1)$. Denote by $R_n^B(\omega) := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{(\theta^B)^j\omega}$ the corresponding empirical field.

Note that due to our assumption $\omega_0 = 1$, θ^B is an ergodic transformation. The r.h.s. of (73) is dominated by

$$\frac{e^{\lambda}y_2 - 1}{e^{\lambda}y_1 - 1} N(n) \mathbf{E}_{R^B_{N(n)}(\omega)} \left(\left(\frac{y_1}{y_2} \right)^{\ell_1} \right) \psi(\gamma_n) \tag{75}$$

Since ℓ_1 possesses a geometric distribution,

$$\mathbf{E}\left(\bar{\rho}^{\ell_1}\right) = \sum_{j=1}^{\infty} \alpha_1 (1-\alpha_1)^{j-1} \bar{\rho}^j = \alpha_1 \bar{\rho} \frac{1}{1-(1-\alpha_1)\bar{\rho}} .$$

Therefore, due to (64) and the η-almost sure convergence of the empirical fields, we have for η-a.a. ω,

$$\limsup_{n\to\infty} \mathbf{E}_{R^B_{N(n)}(\omega)} \left(\left(\frac{y_1}{y_2} \right)^{\ell_1} \right) \le \mathbf{E}\left(\bar{\rho}^{\ell_1}\right) = \alpha_1 \bar{\rho} \frac{1}{1-(1-\alpha_1)\bar{\rho}} . \tag{76}$$

Taking into account (70), (73), (63), (76), the definition of λ_n, the convergence of $e^{\lambda_n} y_1$ to $\bar{\rho}$ and the convergence of $\psi(\gamma_{\tilde{n}_k}) \to 1$, we see that for η-a.a. ω,

$$\begin{aligned}
&\limsup_{k\to\infty} -\frac{1}{\tilde{n}_k^{1-1/s}} \sum_{j=1}^{N(\tilde{n}_k)} \log\left(1 - \frac{y_1^{\ell_j}(e^{\lambda_{\tilde{n}_k}} y_2 - 1)}{y_2^{\ell_j}(e^{\lambda_{\tilde{n}_k}} y_1 - 1)}\right) \\
&\le \limsup_{k\to\infty} \frac{N(\tilde{n}_k)}{\tilde{n}_k} \tilde{n}_k^{1/s} \lambda_{\tilde{n}_k} \frac{e^{\lambda_{\tilde{n}_k}} y_2 - 1}{\lambda_{\tilde{n}_k}} \frac{1}{e^{\lambda_{\tilde{n}_k}} y_1 - 1} E_{R^B_{N(\tilde{n}_k)}(\omega)} \left(\left(\frac{y_1}{y_2} \right)^{\ell_1} \right) \psi(\gamma_{\tilde{n}_k}) \\
&\le \alpha_1 C \frac{2pq}{(q-p)^2} \mathbf{E}\left(\bar{\rho}^{\ell_1}\right) \le \frac{2q^2 \alpha_1^2 C}{(q-p)^2 (1-(1-\alpha_1)\bar{\rho})} .
\end{aligned} \tag{77}$$

Putting together (66), (68), (72) and (77), we have proved that, for η-a.a. ω,

$$\limsup_{k\to\infty} \frac{1}{\tilde{n}_k^{1-1/s}} \log \mathbf{P}_\omega (T_{\tilde{n}_k} \ge \tilde{n}_k u)$$

$$\le -\frac{C}{q-p} - \alpha_1 C \frac{2qp}{(q-p)^2} + \alpha_1^2 C \frac{2q^2}{(q-p)^2} \frac{1}{1-(1-\alpha_1)\bar{\rho}} - Cu \tag{78}$$

A straightforward calculation shows that the last term equals

$$C \frac{1 + (1-\alpha_1) q/p}{1-(1-\alpha_1) q/p} - Cu = C\left(1/v_\alpha - u\right) = -C\left(u - 1/v_\alpha\right) .$$

(41) now follows since $C > 0$ was arbitrary. The proof of the statements involving X_n/n is straightforward from (41) and (42), we refer to [14]. $\square$

Remark We have in fact shown that, for η-a.a. ω, on the random subsequence $\tilde{n}_k \to \infty$ introduced in (60),

$$\lim_{k\to\infty} \frac{1}{(\tilde{n}_k)^{1-1/s}} \log \mathbf{P}_\omega(T_{\tilde{n}_k} \geq \tilde{n}_k u) = -\infty\,. \tag{79}$$

5 Concluding remarks

1. A setting which was not covered in the subexponential regime is that of $\rho_{\max} = 1$ while $\alpha(\{1/2\}) = 0$. It can be checked that in this case, the annealed large deviation probabilities can decay like $\exp(-Cn^\beta)$ for any $\beta \in (1/3, 1)$, with the value of β determined by the behavior of the measure $\alpha(\cdot)$ in the neighborhood of $1/2$. The same proof as in Theorem 8 then shows that the upper quenched estimates in Theorem 8 become $\exp(-dn/(\log n)^\gamma)$, with $\gamma = 1/\beta - 1$.

2. The assumption that the closed support of α is contained in the open interval $(0, 1)$ can often be dispensed if the support does not include both $\{0\}$ and $\{1\}$. For example, in the context of Theorems 5 and 7, it can be replaced by the weaker assumption of $\langle \rho^s \rangle = 1$ for some $s > 1$. One place where this assumption seems essential is in the derivation of the annealed exponential LDP's.

3. In general, one does not know how to solve the variational problem in (11), and hence we do not have explicit expressions for I^a_α (except in certain special cases, cf. [7]) or a precise understanding of the atypical environments leading to large deviations. Similarly, except when the results of [35] and [34] apply, it has not been proved what is the local environment which causes large fluctuations in the sub-exponential regime. The solution of this problem requires a more refined understanding of the upper bound than is currently available.

4. It is speculated in [7], but not proved, that $I^{\tau,a}_\alpha(u) \neq I^{\tau,q}_\alpha(u)$ for any u such that $I^{\tau,q}_\alpha$ is strictly convex at u.

5. As in the i.i.d. environment case described in Sections 3 and 4, one may look, in the general ergodic case, for refined asymptotics in the flat pieces of $I^{\tau,q}_\eta$ or I^q_η. When η is locally equivalent to the product of its marginals, it is believed to exhibit the same qualitative behavior as in the i.i.d. case, that is polynomial decay in the case $\omega_{\min} < 1/2 < \omega_{\max}$ and sub-exponential decay when $\omega_{\min} = 1/2$. Some explicit computations are possible in the Markov environment case, or in the "two-coins" case described before Theorem 9, cf. [15].

6. The multi-dimensional case presents many challenges. Because a precise hitting time decomposition is not available, explicit criteria for transience and recurrence, as well as formulae for

the speed in the transient case, are not known in general. Notable exceptions are situations where some symmetry is present, in which case one can prove CLT statements, cf. [26],[27], [3]. Under restrictions on the distribution of the environment, Kalikow [20] has proved transience of the RWRE. Very recently, Sznitman and Zerner [40] showed that in fact, under Kalikow's assumption, the RWRE has a non-vanishing speed. Sznitman succeeded in [39] in obtaining the CLT under the same assumption. The general case remains however open. As far as large deviations are concerned, some recent results have been obtained by [47] and [38]. Indeed, when the environment is i.i.d. and the convex hull of the support of α includes the origin, Zerner [47] has used a subadditive hitting time decomposition to show that a quenched LDP (in the exponential scale) holds true. Unfortunately, his results do not allow for the explicit evaluation of the rate function, nor for the evaluation of its zero set. Many questions still remain open, most notably what happens when $0 \notin \operatorname{conv} \operatorname{supp} \alpha_0$, what is the annealed rate function, and what is the relation of the latter to the quenched rate function. In a different direction, Sznitman [38], [39] has obtained, in the multidimensional case, some analogues of the results of Sections 3 and 4. Applications of RWRE models to model long range correlation in time series have been suggested by Marinari et al. in [31] and further discussed by Durrett in [11].

7. An analogous model for continuous time was described by Brox [5] and further developed by Tanaka [44], and, in the multidimensional case, by Mathieu [32]. Some results concerning large deviations for the continuous model were recently obtained by Taleb [43].

8. The RWRE model we have discussed here is a *nearest neighbor* model. There exists some work on non nearest neighbor models, which are much harder and exhibit some of the difficulties present in the multi-dimensional case. Specifically, let $L, R \geq 1$ be integers, and assume that $\mathbf{P}_\omega(X_{n+1} = x + i | X_n = x) = p_x(i)$, with $p_x(i) = 0$ for $i \notin [-L, R]$, and $p_x(\cdot)$ a sequence of i.i.d. random vectors. Key [23] (c.f. also [28]) provided a transience or recurrence criterion based on the evaluation of certain Lyapunov exponents. Other limit results are also available, see [29], [30] and the references therein. It should be noted that if $R = 1$ and $X_n \to \infty$ $\mathbf{P}$-a.s., the rate of growth of X_n/n can be evaluated by mimicking the argument used in the proof of Lemma 5. Indeed, with $\tau_1 = \min\{n : X_n = 1\}$, the same recursion reveals that, whenever $\mathbf{E}(\tau_1) < \infty$,

$$\mathbf{E}(\tau_1) = \frac{1 + \mathbf{E}\left(\frac{1-p_0(1)}{p_0(1)}\right)}{1 - \mathbf{E}\left(\frac{\sum_{i=1}^{L} i p_0(-i)}{p_0(1)}\right)}, \tag{80}$$

and further $\mathbf{E}(\tau_1) = \infty$ as soon as the r.h.s. in (80) is not strictly positive. As in the nearest neighbor case, one concludes that if $X_n \to \infty$, $\mathbf{P}$-a.s., it holds that $X_n/n \to 1/\mathbf{E}(\tau_1)$, $\mathbf{P}$-a.s.

References

[1] V. I. Afanasev, "On the maximum of a transient random walk in a random environment", *Theor. Prob. Appl.* **35** (1991), pp. 205–215.

[2] S. Alili, "Asymptotic behaviour for random walks in random environments", Preprint (1997).

[3] J. Bricmont, A. Kupiainen, "Random walks in asymmetric random environments", *Commun. Math. Phys.* **142** (1991) pp. 345-420.

[4] K. A. Borovkov and V. A. Vatutin, "Reduced critical branching processes in random environments", *Stoch. Proc. Appl.* **71** (1997), pp. 225–240.

[5] Th. Brox, "A one- dimensional diffusion process in a Wiener medium", *Annals of Probability* **14** (1986), pp. 1206 -1218.

[6] K. L. Chung, *Markov chains with stationary transition probabilities*, Berlin, Springer, 1960.

[7] F. Comets, N. Gantert and O. Zeitouni, "Quenched, annealed and functional large deviations for one dimensional random walk in random environment", submitted, 1998.

[8] P. Deheuvels, "On the Erdös-Rényi Theorem for random fields and sequences and its relationships with the theory of runs and spacings", *Zeitschrift für Wahrscheinlichkeitstheorie verw. Gebiete* **70** (1985) pp. 91-115.

[9] A. Dembo, Y. Peres and O. Zeitouni, "Tail estimates for one–dimensional random walk in random environment", *Commun. Math. Phys.* **181** (1996) pp. 667–683.

[10] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, second edition, Springer, New York (1998).

[11] R. Durrett, "Some multidimensional RWRE with subclassical limiting behavior", *Commun. Math. Phys.* **104** (1986) pp. 87-102.

[12] M. D. Donsker and S. R. S. Varadhan, "Asymptotic evaluation of certain Markov process expectations for large time III" *Comm. Pure Appl. Math.* **29** (1976) pp. 389-461.

[13] M. D. Donsker and S. R. S. Varadhan, "Asymptotic evaluation of certain Markov process expectations for large time IV" *Comm. Pure Appl. Math.* **36** (1983) pp. 183-212.

[14] N. Gantert and O. Zeitouni, "Quenched sub-exponential tail estimates for one-dimensional random walk in random environment". *Commun. Math. Phys.* **194** (1998) pp. 177-190.

[15] N. Gantert, "Subexponential tail asymptotics for a random walk with randomly placed nodes", submitted.

[16] A. Greven and F. den Hollander, "Large deviations for a random walk in random environment", *Ann. Probab.* **22** (1994) pp. 1381–1428.

[17] Y. Hu and Z. Shi, "The local time of simple random walk in random environment", *J. Theoretical Prob.* **11** (1998) pp. 765–793.

[18] B. D. Hughes, *Random walks and random environments*, Clarendon Press, Oxford (1996).

[19] W. B. Jones and W. J. Thron, *Continued fractions*, Encyclopedia of Mathematics and its Applications, vol. 11, Addison-Wesley, 1980.

[20] S. A. Kalikow, "Generalized random walk in a random environment". *Annals Probab.* **9** (1981), pp. 753–768.

[21] H. Kesten, "The limit distribution of Sinai's random walk in random environment", *Phys. A* 138 (1986) pp. 299-309.

[22] H. Kesten, M. V. Kozlov and F. Spitzer, "A limit law for random walk in a random environment", *Comp. Math.* **30** (1975) pp. 145–168.

[23] E. S. Key, "Recurrence and transience criteria for random walk in a random environment", *Annals Probab.* **12** (1984), pp. 529–560.

[24] M. V. Kozlov, "Random walk in a one-dimensional random medium", *Th. Prob. and Appl.* **18** (1973), pp. 387–388.

[25] M. V. Kozlov, "On the asymptotic behavior of the probability of non-extinction for critical branching processes in random environment", *Th. Prob. and Appl.* **21** (1976), pp. 791–804.

[26] S. M. Kozlov, "The averaging method and walks in inhomogeneous environments", *Russian Math. Surveys* **40** (1985), pp. 73–145.

[27] G. F. Lawler, "Weak convergence of a random walk in a random environment", *Comm. Math. Physics* **87** (1982), pp. 81–87.

[28] F. Ledrappier, "Quelques propriétés des exposants charactéristiques", Springer, LNM 1097, (1984), pp. 305 -396.

[29] A. V. Letchikov, "Localization of one-dimensional random walks in random environments", *Soviet Scientific Reviews, Section C: Mathematical Physics Reviews* **8** (1989), pp. 173-220.

[30] A. V. Letchikov, "A criterion for the applicability of the central limit theorem to one-dimensional random walks in random environments", *Theor. Prob. Appl.* **37** (1992), pp. 553–557.

[31] E. Marinari, G. Parisi, D. Ruelle, and P. Windey, "On the interpretation of $1/f$ noise", *Comm. Math. Physics* **89** (1983), pp. 1–12.

[32] P. Mathieu, "Zero White Noise Limit through Dirichlet forms, with application to diffusions in a random medium", *Probab. Theory Relat. Fields* **99** (1994) pp. 549-580.

[33] S. V. Nagaev, "Large deviations of sums of independent random variables", *Ann. Probab.* **7** (1979), pp. 745–789.

[34] A. Pisztora, T. Povel, "Large deviation principle for random walk in a quenched environment in the low speed regime", to appear, Annals Probab. (1998).

[35] A. Pisztora, T. Povel and O. Zeitouni, "Precise large deviations estimates for one dimensional random walk in random environment", *Probab. Theory Relat. Fields* **113** (1998), pp. 191-219.

[36] P. Révész, *Random walk in random and non-random environments*, World Scientific, Singapore (1990).

[37] Ya. G. Sinai, "The limiting behavior of a one dimensional random walk in a random medium", *Theory Probab. Appl.* **27** (1982), pp. 256-268.

[38] A.-S. Sznitman, "Slowdown and neutral pockets for a random walk in a random environment", to appear, PTRF (1999).

[39] A.-S. Sznitman, "Slowdown and central limit theorem for random walks in a random environment", preprint (1999).

[40] A.-S. Sznitman and M. P. W. Zerner, "A law of large numbers for random walks in random environment", preprint (1998).

[41] F. Solomon, "Random walks in random environment", *Ann. Probab.* **3** (1975), pp. 1–31.

[42] F. Spitzer, *Principles of random walk*, Springer, Berlin (1976).

[43] M. Taleb, "Large deviations for a Brownian motion in a drifted Brownian potential", *Preprint*, 1998.

[44] H. Tanaka, "Diffusion processes in random environments", *Proc. ICM* (S. D. Chatterij, ed., Birkhäuser, Basel) (1995) pp. 1047 -1054.

[45] D. E. Temkin, "One dimensional random walks in two-component chain", *Dokl. Akad. Nauk. USSR* **206** (1972), pp. 27–30.

[46] H. S. Wall, *Analytic theory of continued fractions*, Van Nostrand, 1948.

[47] M. P. W. Zerner, "Lyapounov exponents and quenched large deviations for multi-dimensional random walk in random environment", to appear, *Annals Probab.* (1999).