Recursions and tightness for the maximum of the discrete, two dimensional Gaussian Free Field

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Abstract

We consider the maximum of the discrete two dimensional Gaussian free field in a box, and prove the existence of a (dense) deterministic subsequence along which the maximum, centered at its mean, is tight; this still leaves open the conjecture that tightness holds without the need for subsequences. The method of proof relies on an argument developed by Dekking and Host for branching random walks with bounded increments and on comparison results specific to Gaussian fields.

1 Introduction and main result

We consider the discrete Gaussian Free Field (GFF) in a two-dimensional box of side $N+1$, with Dirichlet boundary conditions. That is, let $V_N = ([0,N] \cap \mathbb{Z})^2$, $V_N^o = ((0,N) \cap \mathbb{Z})^2$, and let $\{w_m\}_{m\geq 0}$ denote a simple random walk started in V_N and killed at $\tau = \min\{m : w_m \in \partial V_N\}$ (that is, killed upon hitting the boundary $\partial V_N = V_N \setminus V_N^o$). For $x, y \in V_N$, define $G_N(x,y) = E^x(\sum_{m=0}^{\tau} \mathbf{1}_{w_m=y})$, where E^x denotes expectation with respect to the random walk started at x. The GFF is the zero-mean Gaussian field $\{X_z^N\}_z$ indexed by $z \in V_N$ with covariance G_N.

Let $X_N^* = \max_{z \in V_N} X_z^N$. It was proved in [5] that $X_N^*/(\log N) \to c$ with $c = 2\sqrt{2/\pi}$, and the proof is closely related to the proof of the law of large numbers for the maximal displacement of a branching random walk (in $\mathbb{R}$). Based on the analogy with the maximum of independent Gaussian variables and the case of branching random walks, the following is a natural conjecture.

Conjecture 1 *The sequence of random variables $Y_N := X_N^* - EX_N^*$ is tight.*

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To the best of our knowledge, the sharpest result in this direction is due to [7], who shows that the variance of Y_N is $o(\log N)$; in the same paper, Chatterjee also analyzes related Gaussian fields, but in all these examples, does not prove tightness. We defer to Section 4 for some pointers to the relevant literature concerning the Gaussian free field and the origin of Conjecture 1.

The goal of this note is to prove a weak form of the conjecture. Namely, we will prove the following.

Theorem 1 *There is a deterministic sequence $\{N_k\}_{k\geq 1}$ such that the sequence of random variables $\{Y_{N_k}\}_{k\geq 1}$ is tight.*

More information on the sequence $\{N_k\}_{k\geq 1}$ is provided below in Section 3.

It is of course natural to try to improve the tightness from subsequences to the full sequence. As will be clear from the proof, for that it is enough to prove the existence of a constant C such that $EX^*_{2N} \leq EX^*_N + C$. This is weaker than, and implied by, the conjectured behavior of EX^*_N, which is

$$EX^*_N = c \log N - c_2 \log\log N + O(1), \tag{1}$$

for $c = 2\sqrt{2/\pi}$ and an appropriate c_2, see e.g. [6] and Remark 3.

Finally, although we deal here exclusively with the GFF, it should be clear from the proof that the analysis applies to a much wider class of models.

2 Preliminary considerations

Our approach is motivated by the proof of tightness of branching random walks (BRW) with independent increments, in the spirit of [9] (see also the argument in [3]). We will thus first introduce a branching-like structure in the GFF. Unfortunately, this structure is not directly suitable for analysis, and so we later modify it.

2.1 The basic branching structure

We consider $N = 2^n$ in what follows, write $Z_n = X^*_N$ and identify an integer $m = \sum_{\ell=0}^{n-1} m_i 2^i$ with its binary expansion $(m_{n-1}, m_{n-2}, \ldots, m_0)$. For $k \geq 1$, introduce the sets of k-diadic integers

$$A_k = \{m \in \{1,\ldots,N\} : m = (2l+1)N/2^k \quad \text{for some integer } l\}.$$

Note that if $m \in A_k$ then $m_i = 0$ for $i \leq n-k$ and $m_{n-k} = 1$. Then, define the σ-algebras

$$\mathcal{A}_k = \sigma(X^N_z : z = (x,y), x \,\text{or}\, y \in \cup_{i\leq k} A_i).$$

Finally, for every $z = (x,y) \in V^o_N$, write $z_i = (x_i, y_i)$ with x_i, y_i denoting as above the ith digit in the binary expansion of x, y. We introduce the random variables

$$\xi^{z_1,\ldots,z_k}_{z_{k+1},\ldots,z_n} = E[X^N_z | \mathcal{A}_k]. \tag{2}$$

We then have the decomposition

$$X_z^N = \xi^{z_1}_{z_2,\ldots,z_n} + X^{z_1}_{z_2,\ldots,z_n}, \tag{3}$$

where, by the Markov property of the GFF, the collections $\{X^{z_1}_\cdot\}_{z_1\in V_1}$ are i.i.d. copies of the GFF in the box $V_{N/2}$, and are independent of the collection of random variables $\{\xi^{z_1}_\cdot\}$. Iterating, we have the representation

$$X_z^N = \xi^{z_1}_{z_2,\ldots,z_n} + \xi^{z_1,z_2}_{z_2,\ldots,z_n} + \ldots + \xi^{z_1,z_2,\ldots,z_{n-1}}_{z_n}\,, \tag{4}$$

where all the summands in the right side of (4) are independent, and the ith summand is a sample from a GFF in $V_{N/2^i}$. Recall that $X_N^* = \max_{z\in V_N} X_z$.

We can now explain the relation with branching random walks: should the random variables in the right side of (4) not depend on the conditioning (that is, the superscript), (4) would correspond precisely to a branching random walk (on a four-ary tree). For such BRW, a functional recursion for the law of X_N^* can be written down, and used to prove tightness (see [4] and [1]). Unfortunately, no such simple functional recursions are available in the case (4). For this reason, we first modify the representation (4), and then adapt an argument of [9], originally presented in the context of BRW. To explain our goal, note that we have for (4) that

$$X_N^* = \max_{z\in V_1}((X_{N/2}^*)^z + D^{z,N}), \tag{5}$$

where the variables $\{(X^*_{N/2})^z\}_z$ are four i.i.d. copies of $X^*_{N/2}$, and the $D^{z,N}$ are complicated fields but $D^{z,N}_{z_2,\ldots,z_n} \geq \min_{z_2,\ldots,z_n} \xi^z_{z_2,\ldots,z_n}$. Unfortunately, the $D^{z,N}$ variables are far from being uniformly bounded, and this fact prevents the application of the argument from [9].

2.2 A modified recursion

We continue to take $N = 2^n$. Let $\Delta_N < N$ be a given sequence, to be determined below. We introduce the set $V_N^\Delta = \{z \in V_N^o : d(z, \partial V_N) \geq \Delta_N\}$. We define

$$\bar{X}_N^* = \max_{z\in V_N^\Delta} X_z^N.$$

Clearly, $\bar{X}_N^* \leq X_N^*$. In fact, writing $Z_n = X_N^*$ and $\bar{Z}_n = \bar{X}_N^*$, we have the basic inequality

$$Z_n \geq \max_{z\in V_1}(\bar{Z}^z_{n-1} + \bar{D}^z_n), \tag{6}$$

where the $\{\bar{Z}^z_{n-1}\}_{z\in V_1}$ are i.i.d. of the same law as $\bar{Z}_{n-1}$ and

$$\bar{D}^z_n = \min_{(z_2,\ldots,z_n)\in V^\Delta_{2^{n-1}}} \xi^z_{z_2,\ldots,z_n}$$

are identically distributed (but not independent!).

Let $u_n = E\bar{D}^z_n \leq 0$. The main result of this section is the following.

Proposition 1 *Assume that there exists a constant C independent of n such that*

$$EZ_{n-1} \leq E\bar{Z}_{n-1} + C, \tag{7}$$

$$u_n \geq -C, \tag{8}$$

$$EZ_n \leq EZ_{n-1} + C. \tag{9}$$

Then,

$$E|Z_{n-1} - Z'_{n-1}| \leq 10C, \tag{10}$$

where Z'_{n-1} is an independent copy of Z_{n-1}.

Proof By (6),

$$\begin{aligned}
EZ_n &\geq E \max_{z=(0,0),(0,1)} (\bar{Z}^z_{n-1} + \bar{D}^z_n) \\
&\geq E \max_{z=(0,0),(0,1)} (\bar{Z}^z_{n-1}) + 2u_n \\
&= E\frac{1}{2}(\bar{Z}^{(0,0)}_{n-1} + \bar{Z}^{(0,1)}_{n-1} + |\bar{Z}^{(0,0)}_{n-1} - \bar{Z}^{(0,1)}_{n-1}|) + 2u_n \\
&\geq E\bar{Z}^{(0,0)}_{n_1} + 2u_n + \frac{1}{2}E|\bar{Z}^{(0,0)}_{n-1} - \bar{Z}^{(0,1)}_{n-1}|.
\end{aligned}$$

Using (7), it follows that

$$C + EZ_n \geq EZ_{n-1} + 2u_n + \frac{1}{2}E|\bar{Z}^{(0,0)}_{n-1} - \bar{Z}^{(0,1)}_{n-1}|.$$

Applying (9) one then gets

$$2C - 2u_n \geq \frac{1}{2}E|\bar{Z}^{(0,0)}_{n-1} - \bar{Z}^{(0,1)}_{n-1}|.$$

Using again (7) and the fact that $Z_{n-1} \geq \bar{Z}_{n-1}$, one gets

$$2C - 2u_n \geq \frac{1}{2}E|Z^{(0,0)}_{n-1} - Z^{(0,1)}_{n-1}| - C.$$

Together with (8), this completes the proof. $\square$

3 Verification of assumptions and proof of Theorem 1

Fix $\epsilon \in (0, 1/2)$ and set $\Delta_n = \epsilon 2^n$. We recall the following consequence of the Sudakov–Fernique inequality [10].

Lemma 1 *Suppose G_α and g_α, $\alpha \in T$, are zero mean Gaussian fields, independent of each other. Then,*

$$E \sup_{\alpha \in T} (G_\alpha + g_\alpha) \geq E \sup_{\alpha \in T} G_\alpha \,.$$

Proof Write $\bar{G}_\alpha = G_\alpha + g_\alpha$. Note that

$$E(\bar{G}_\alpha - \bar{G}_\beta)^2 = E(G_\alpha - G_\beta)^2 + E(g_\alpha - g_\beta)^2 \geq E(G_\alpha - G_\beta)^2.$$

Now apply the Sudakov–Fernique inequality. $\square$
Lemma 1 immediately implies that for all n,

$$EZ_n \geq EZ_{n-1}. \tag{11}$$

We can now verify (9).

Lemma 2 *There exists a sequence $n_k \to \infty$ and a constant C such that*

$$EZ_{n_k} \leq EZ_{n_k-1} + C\,, \quad EZ_{n_k-1} \leq EZ_{n_k-2} + C\,. \tag{12}$$

Proof From [5] there exists a constant $c > 0$ so that $EZ_n/n \to c$. Fixing arbitrary K and defining $I_{n,K} = \{i \in [n, 2n] : EZ_{n+1} > EZ_n + K\}$, one has from (11) and the existence of the limit $EZ_n/n \to c$ that

$$\limsup_{n\to\infty} \frac{|I_{n,K}|}{2n} \leq \frac{c}{K}\,.$$

In particular, choosing $K = 3c$ it follows that for all n large, there exists an $n' \in [n, 2n]$ so that

$$EZ_{n'} \leq EZ_{n'-1} + K\,, \quad EZ_{n'-1} \leq EZ_{n'-2} + K,$$

as claimed. $\square$
We can now check (7).

Lemma 3 *With the sequence n_k as in Lemma 2, it holds that*

$$E\bar{Z}_{n_k-1} \geq EZ_{n_k-1} - C.$$

Proof Again with $N = 2^n$, let $\mathcal{F}_N = \sigma(X_z^N : d(z, \partial V_N) \leq N/4)$. We have, for $z \in (N/4, N/4) + V_{N/2}$, $X_z^N = E(X_z^N|\mathcal{F}_N) + G_z^{N/2}$, where $\{G_z^{N/2}\}_z$ is a copy of the GFF in $(N/4, N/4) + V_{N/2}$ and is independent of $\{E(X_z^N|\mathcal{F}_N)\}_z$. Then,

$$E\bar{Z}_n = E \max_{z\in V_N^\Delta} X_z^N \geq E \max_{z\in(N/4,N/4)+V_{N/2}} X_z^N \geq EZ_{n-1}$$

where the last inequality follows from Lemma 1. In particular, it follows from this and (12) that

$$E\bar{Z}_{n_k-1} \geq EZ_{n_k-2} \geq EZ_{n_k-1} - C,$$

as claimed. $\square$
Proof of Theorem 1 We still need to check (8). Toward that end, let $t = ((0,0), z_2, \ldots, z_n) \in V_{2^{n-1}}^\Delta$. We have the representation

$$\xi_t := \xi_{z_2,\ldots,z_n}^{(0,0)} = \sum_{y\in S_n} \pi_{n-1}(t, y) X_y^N\,,$$

where $S_n = \partial V_{2^{n-1}}$ and, with $\tau_{n-1} = \min\{m : w_m \in \partial V_{2^{n-1}}\}$, $\pi_{n-1}(t,y) = P^t(w_{\tau_{n-1}} = y)$. Setting $f_n(t,t',y) = [\pi_{n-1}(t,y) - \pi_{n-1}(t',y)]$, it follows that

$$E(\xi_t - \xi_{t'})^2 = \sum_{y,y' \in \partial V_{2^{n-1}}} f_n(t,t',y) f_n(t,t',y') G_N(y,y'), \tag{13}$$

where again $N = 2^n$. Standard estimates for simple random walk on V_N, see [15, Proposition 1.6.7 and Theorem 1.7.1], give that there exists a constant $C_1 = C_1(\epsilon)$ such that, uniformly for $t, t' \in V^{\Delta}_{2^{n-1}}$,

$$G_N(y,y') \leq C_1 \log\left(\frac{N}{|y-y'|+1}\right), \quad |f_n(t,t',y)| \leq C_1 \frac{|t-t'|}{N^2}. \tag{14}$$

It follows from (13) and (14) that

$$E(\xi_t - \xi_{t'})^2 \leq C_1^3 \left(\frac{|t-t'|}{N}\right)^2. \tag{15}$$

Using now Fernique's criterion, see [11] or [2, Theorem 4.1], with the uniform measure on $V_{N/2}$ as majorizing measure, we conclude that

$$E \sup_{t \in V^{\Delta}_{N/2}} |\xi_t| < C_2$$

for some constant $C_2 = C_2(\epsilon)$. This proves (8).

Combining Lemmas 2, 3 and Proposition 1 together with the last estimate, we conclude the existence of a constant C such that two independent copies of Z''_{n_k}, Z'_{n_k} satisfy $E|Z''_{n_k} - Z'_{n_k}| \leq 10C$, with the subsequence n_k provided by Lemma 2. This implies that the sequence $\{Z_{n_k}\}_k$ is tight, and completes the proof of Theorem 1. □

Remark 1 *The subsequence n_k provided in Lemma 2 can be taken with density arbitrary close to 1, as can be seen from the following modification of the proof. Fixing arbitrary K and ϵ and defining $I_{n,\epsilon,K} = \{i \in [n, n(1+\epsilon)] : EZ_{n+1} > EZ_n\}$, one has from (11) and the existence of the limit $EZ_n/n \to c$ that*

$$\limsup_{n\to\infty} \frac{|I_{n,\epsilon,K}|}{n\epsilon} \leq \frac{c}{K}.$$

It is of course of interest to see whether one can take $n_k = k$. Minor modifications of the proof of Theorem 1 would then yield Conjecture 1.

Remark 2 *Minor modifications of the proof of Theorem 1 also show that if there exists a constant C so that $EX^*_{2N} \leq EX^*_N + C$ for all integer N, then Conjecture 1 holds.*

Remark 3 *For Branching Random Walks, under suitable assumptions it was established in [1] that (1) holds. Running the argument above then immediately implies the tightness of the minimal (maximal) displacement, centered around its mean.*

4 Some bibliographical remarks

The Gaussian free field has been extensively studied in recent years, in both its continuous and discrete forms. For an accessible review, we refer to [16]. The fact that the GFF has a logarithmic decay of correlation invites a comparison with branching random walks, and through this analogy a form of Conjecture 1 is implicit in [6]. This conjecture is certainly "folklore", see e.g. open problem #4 in [7]. For some one-dimensional models (with logarithmic decay of correlation) where the structure of the maxima can be analysed, we refer to [12, 13]. The analogy with branching random walks has been reinforced by the study of the so called *thick points* of the GFF, both in the discrete form [8] and in the continuous form [14].

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