LECTURE NOTES ON RANDOM WALKS IN RANDOM ENVIRONMENT

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Preface

These notes on random walks in random environments (RWRE) reflect what I hoped to cover in the 15 hours of the St Flour course on this topic, July 9–25, 2001. Of course, this turned out to be over optimistic. Departing even further from the actually delivered lectures, I have taken advantage of the year that elapsed to add some material (especially, related to multi-dimensional walks) and to correct numerous mistakes and omissions.

The manuscript consist roughly of two parts: the first deals with RWRE on $\mathbb{Z}$. The interest in the model began in the early 70's, and with the detailed analysis of RWRE asymptotics in the last decade, has now reached maturity (for an account of the history of the subject and many of the results through the early 90's, see [38]). I have tried to present different tools for the study of such walks, risking some repetition of results in a few cases, and deferring to the bibliographical notes a discussion of refinements and sharpening of the results. It is worthwhile to point out that RWRE's on $\mathbb{Z}$ have already been considered in previous St Flour courses (most notably by Ledrappier [51] and by Molchanov [54]), but the emphasis in this presentation is quite different.

The second part of the notes deals with $\mathbb{Z}^d$. This is currently an active research area, and one hopes that much progress will be made in the next few years. My goal here was to expose the audience to some tools which have proved useful, and to point out several directions where further progress could be made. In several places, I have tried to lay the groundwork for relaxing the often made assumption of i.i.d. environment.

When preparing the notes, and taking into account the time frame of these lectures, it became clear that there were topics that had to be left out. Even the uninitiated will quickly realize that the most glaring omission is the study of RWRE's by renormalization techniques. There are three reasons for this: first, it would take too long to properly expose it. Second, these methods have not yet reached the full scope of their applicability, and in view of very active current research efforts in this direction, any account written now risks being outdated very quickly. And third, an overview of the current status of these techniques can be found in [70] and [71]. Time constraints also did not allow me to discuss random walks on Galton-Watson trees, a topic that has seen much progress in recent years.

Parts of the material presented here is based on joint work, some still unpublished, with F. Comets, A. Dembo, N. Gantert, and Y. Peres. I thank them all, both for the many hours spent together on thinking about RWRE, and for their generosity. I would also like to thank my colleagues in Haifa who suffered through a first draft of these notes in the winter of 2000. In particular, comments from D. Ioffe, H. Kaspi, E. Mayer-Wolf, A. Roitershtein and M. Zerner are gratefully acknowledged. Similarly acknowledged are useful remarks from D. Cheliotis, A. Dembo, N. Gantert, A. Guionnet, H. Kesten, D. Piau, and S.R.S. Varadhan. Comments from participants at the St Flour summer school helped improve the presentation and strengthen numerous results. I am particularly grateful to P. Bougerol who allowed me to incorporate some of his suggestions in the final text of these notes, and D. Ocone and F. Rassoul-Agha for stimulating discussions. Last but not least, I am grateful to J. Picard for the smooth and gentle running of the summer school.

A typographical comment: for aesthetic reasons, I consistently use P_ω^o, E_ω^o, $\mathbb{P}^o$, $\mathbb{E}^o$, etc., when I mean $P_\omega^0, E_\omega^0, \mathbb{P}^0, \mathbb{E}^0$.

1 Introduction

The definition of a RWRE involves two components: first, the *environment*, which is randomly chosen but kept fixed throughout the time evolution, and second, the random walk, which, given the environment, is a time homogeneous Markov chain whose transition probabilities depend on the environment. We do not attempt here a historical review of RWRE's, or in greater generality of motion in homogeneous media, except for stating that we insist on the environment being static, i.e. time independent, and that in general the random walk (conditioned on the environment) is not necessarily reversible.

1.1 Model

We begin with a general setup, that will be specialized later to the cases of interest to us. Let (V, E) denote an (infinite, oriented) graph with countable vertex set V and edges set $E = \{(v, w)\}$ (we allow, but do not require, $(v, v) \in E$). For each $v \in V$, we define its *neighborhood* N_v by

$$N_v = \{w \in V : (v, w) \in E\},$$

throughout assuming that $|N_v| < \infty$, for all $v \in V$.

For each $v \in V$, let $M_1(N_v)$ denote the collection of probability measures on V with support N_v. Formally, an element of $M_1(N_v)$, called a *transition law at* v, is a measurable function $\omega_v : V \to [0,1]$ satisfying:

$$\begin{aligned} &\text{(a)} \quad \omega_v(w) \geq 0 \qquad \forall\, w \in V \\ &\text{(b)} \quad \omega_v(w) = 0 \qquad \forall\, w \notin N_v \\ &\text{(c)} \quad \sum_{w \in N_v} \omega_v(w) = 1 \end{aligned} \tag{1.1.1}$$

Note that if $v \in N_v$ then in (1.1.1c) we allow for $\omega_v(v) > 0$.

We equip $M_1(N_v)$ with the weak topology on probability measures, which makes it into a Polish space. Further, it induces a Polish structure on $\Omega = \prod_{v \in V} M_1(N_v)$. We let $\mathcal{F}$ denote the Borel σ-algebra on Ω (which is the same as the σ-algebra generated by cylinder functions). Given a probability measure P on $(\Omega, \mathcal{F})$, a *random environment* is an element ω of Ω distributed according to P.

We turn next to define the class of random walks of interest to us. For *each* $\omega \in \Omega$, we define the *random walk in the environment* ω as the time-homogeneous Markov chain $\{X_n\}$ taking values in V with transition probabilities

$$P_\omega(X_{n+1} = w | X_n = v) = \omega_v(w)\,.$$

We use P_ω^v to denote the law induced on $(V^{\mathbb{N}}, \mathcal{G})$ where $\mathcal{G}$ is the σ-algebra generated by cylinder functions and

$$P_\omega^v(X_0 = v) = 1\,.$$

In the sequel, we refer to $P_\omega^v(\cdot)$ as the *quenched* law of the random walk $\{X_n\}$. Note that for each $G \in \mathcal{G}$, the map

$$\omega \mapsto P_\omega^v(G)$$

is $\mathcal{F}$-measurable. Hence, we may define the measure $\mathbb{P}^v := P \otimes P_\omega^v$ on $(\Omega \times V^{\mathbb{N}}, \mathcal{F} \times \mathcal{G})$ from the relation

$$\mathbb{P}^v(F \times G) = \int_F P_\omega^v(G) P(d\omega), \quad F \in \mathcal{F}, G \in \mathcal{G}\,. \tag{1.1.2}$$

The marginal of $\mathbb{P}^v$ on $V^{\mathbb{N}}$, denoted also $\mathbb{P}^v$ whenever no confusion occurs, is called the *annealed law* of the random walk $\{X_n\}$; note that under $\mathbb{P}^v$, the random walk in random environment (RWRE) $\{X_n\}$ is *not* a Markov chain!

1.2 Examples

Throughout these notes, we only treat nearest neighbor RWRE's on $\mathbb{Z}^d$:

Nearest neighbor RWRE on $\mathbb{Z}$ Here, we take $V = \mathbb{Z}$ and $E = \cup_{z \in \mathbb{Z}} \{(z, z+1), (z, z)\}$. Then, $N_v = \{v-1, v, v+1\}$ and $M_1(N_v)$ can be

identified with the three dimensional simplex; We let $\omega_z^+ := \omega_z(z+1)$, $\omega_z^- := \omega_z(z-1)$, and $\omega_z^0 := \omega_z(z)$. One defines naturally the shift θ on Ω by $(\theta\omega)_z = \omega_{z+1}$. We always make the following assumption:
$(\omega, \mathcal{F}, P, \theta)$ is an ergodic system.
It is worthwhile commenting, already at this stage, that for each ω there exists a reversing measure that makes the RWRE reversible. More details are provided in Section 2.1.
Nearest neighbor RWRE on $\mathbb{Z}^d$ Here, $V = \mathbb{Z}^d$ and $E = \cup_{z\in\mathbb{Z}}\{\cup_{y\sim z}(z,y)\cup (z,z)\}$. For each $v \in V$, N_v contains $2d+1$ vertices, and $M_1(N_v)$ is identified with the $2d+1$-dimensional simplex. One may define the family of shifts $\{\theta^e\}_{|e|_1=1}$. As in the case of $d=1$, we always require P to be ergodic with respect to this family. We write throughout $\omega(x,e) := \omega_x(x+e)$. Unlike the case with $d=1$, the Markov chain defined by P_ω^v is, in general, not reversible.

Bibliographical notes: the preface section contains relevant bibliography on the RWRE model in $\mathbb{Z}^d$, $d \geq 1$. We mention here some other models of random walks in random media that can be adapted into the general framework presented above, but that will not be considered in these notes:
- *Non nearest neighbor walks: For $\mathbb{Z}^1$, see the recent thesis [7], that includes also a summary of earlier work and in particular of [44]. I am not aware of a systematic study of non nearest neighbor RWRE's on $\mathbb{Z}^d$, see however [80] for some results valid in that generality.*
- *Reversible random walks in random environments in $\mathbb{Z}^d$, $d > 1$: the prime example is the random conductance model, in which bonds on $\mathbb{Z}^d$ carry i.i.d. conductances and modulate the transition mechanism of the walk, see [15]. Other models in the same spirit, and their surprising behavior, are described in [6] and the references therein.*
- *Random walks on Galton-Watson trees: see [16, 52, 53, 60] for recent developments.*

2 RWRE – d=1

This chapter is devoted to the study of the one-dimensional model, where sharp results are available. As a warm-up to the high dimensional case, we sometimes present different proofs of the same statement.

Our exposition progresses from ergodic properties and law of large numbers (Section 2.1), to the study of central limit theorems (Section 2.2), large deviations (Section 2.3), subexponential tail estimates (2.4), and subdiffusive behaviour and aging (Section 2.5). Each section contains a (non-exhaustive!) list pointing to the literature.

2.1 Ergodic theorems

In this section, we are interested in questions concerning transience, recurrence and laws of large numbers, in the most general nearest neighbour one-dimensional setup. Define $\rho_z = \omega_z^- / \omega_z^+$.

Assumption 2.1.1

(A1) *P is stationary and ergodic.*
(A2) *$E_P(\log \rho_0)$ is well defined (with $+\infty$ or $-\infty$ as possible values).*
(A3) $P(\omega_0^+ + \omega_0^- > 0) = 1$.

Theorem 2.1.2 *Assume Assumption 2.1.1. Then,*

$$\begin{array}{llll} \text{(a)} & E_P(\log \rho_0) < 0 & \Rightarrow \lim_{n\to\infty} X_n = +\infty, & \mathbb{P}^o \ \textit{a.s.} \\ \text{(b)} & E_P(\log \rho_0) > 0 & \Rightarrow \lim_{n\to\infty} X_n = -\infty, & \mathbb{P}^o \ \textit{a.s.} \\ \text{(c)} & E_P(\log \rho_0) = 0 & \Rightarrow -\infty = \liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n = \infty, & \mathbb{P}^o \ \textit{a.s.} \end{array}$$

(Note that (a), (b) above imply that X_n is $\mathbb{P}^o$-a.s. transient, whereas (c) implies it is recurrent).

Proof. Due to **(A2)** and **(A3)**, if $P(\omega_0^+ = 0) > 0$ then $P(\omega_0^- = 0) = 0$, and then $E_P(\log \rho_0) = \infty$. To see that (b) holds in this case, let $n_0 = \min\{z > 0 : \omega_z^+ = 0\}$ and $n_i = \max\{z < n_{i-1} : \omega_z^+ = 0\}$. Then, $P(n_0 < \infty) = 1$, hence $\limsup_{n\to\infty} X_n < \infty$, $\mathbb{P}^o$-a.s. Note that by ergodicity, $P(n_i > -\infty) = 1$, and further $P_\omega^{n_i}(X_n \text{ does not hit } \ n_{i+1}) = 0$, as can be seen either from a coupling with (biased) random walk or from (2.1.4) below. This completes the proof of (b) when $P(\omega_0^+ = 0) > 0$.

A similar argument applies to proving (a) in case $P(\omega_0^- = 0) > 0$. Thus, using **(A3)**, we assume in the sequel that $P(\min(\omega_0^+, \omega_0^-) = 0) = 0$. We begin by deriving some formulae related to one-dimensional random walks in a *fixed* environment. These can be derived concisely by using the link between nearest-neighbor random walks on z and electrical networks, see appendix A.

Fix an environment ω with $|\log \rho_z| < \infty$ for each $z \in \mathbb{Z}$. For $z \in [-m_-, m_+]$, define

$$\mathcal{V}_{m_-,m_+,\omega}(z) := P_\omega^z(\{X_n\} \text{ hits } -m_- \text{ before hitting } m_+)\,.$$

Note that due to the assumption $|\log \rho_z| < \infty$, for each z, it holds that $\mathcal{V}_{m_-,m_+,\omega}(z)$ is well defined as

$$P_\omega^z(\{X_m\} \text{ never hits } [-m_-, m_+]^c) = 0\,.$$

The Markov property implies that $\mathcal{V}_{m_-,m_+,\omega}(\cdot)$ is harmonic, that is it satisfies

$$\begin{cases} (\omega_z^+ + \omega_z^-)\mathcal{V}_{m_-,m_+,\omega}(z) = \omega_z^- \mathcal{V}_{m_-,m_+,\omega}(z-1) \\ \qquad\qquad\qquad\qquad + \omega_z^+ \mathcal{V}_{m_-,m_+,\omega}(z+1), \quad z \in (-m_-, m_+)\,, \\ \mathcal{V}_{m_-,m_+,\omega}(-m_-) = 1, \quad \mathcal{V}_{m_-,m_+,\omega}(m_+) = 0\,. \end{cases} \tag{2.1.3}$$

Solving (2.1.3), we find

$$\mathcal{V}_{m_-,m_+,\omega}(z) = \frac{\sum_{i=z+1}^{m_+} \prod_{j=z+1}^{i-1} \rho_j}{\sum_{i=z+1}^{m_+} \prod_{j=z+1}^{i-1} \rho_j + \sum_{i=-m_-+1}^{z} \left(\prod_{j=i}^{z} \rho_j^{-1} \right)}\,. \tag{2.1.4}$$

(Note that the solution to (2.1.3) is unique due to the maximum principle, hence it is enough to verify that the function in (2.1.4) satisfies (2.1.3).)

Define $S(\omega) = \sum_{n=1}^{\infty} \rho_1 \cdots \rho_n$, $F(\omega) = \sum_{n=0}^{\infty} \rho_0^{-1} \ldots \rho_{-n}^{-1}$. Further, define the events

$$\mathcal{S}_+ = \{S(\omega) < \infty\}, \quad \mathcal{F}_+ = \{F(\omega) < \infty\}\,.$$

Then:

- on $T_+ := \{\mathcal{S}_+ \cap \mathcal{F}_+^c\}$, it holds that

$$\lim_{m\to\infty} \Big[1 - \mathcal{V}_{1,m,\omega}(0)\Big] > 0, \lim_{k\to\infty} \lim_{m\to\infty} \Big[1 - \mathcal{V}_{k,m,\omega}(0)\Big] = 1.$$

  Hence, for $\omega \in T_+$,

$$P_\omega^o(\lim_{n\to\infty} X_n = \infty) = 1\,.$$

- Similarly, for $\omega \in T_- := \{\mathcal{S}_+^c \cap \mathcal{F}_+\}$,

$$P_\omega^o(\lim_{n\to\infty} X_n = -\infty) = 1\,.$$

- Finally, if $\omega \in R := \{\mathcal{S}_+^c \cap \mathcal{F}_+^c\}$ then, for any fixed k,

$$1 - \lim_{m\to\infty} \mathcal{V}_{k,m,\omega}(0) = \lim_{m\to\infty} \mathcal{V}_{m,k,\omega}(0) = 0\,,$$

  and hence, for $\omega \in R$,

$$P_\omega^o\Big(-\infty = \liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n = \infty\Big) = 1\,.$$

We observe next that both $\mathcal{S}_+$ and $\mathcal{F}_+$ are invariant events, hence $P(\mathcal{S}_+) \in \{0,1\}$, $P(\mathcal{F}_+) \in \{0,1\}$ by the ergodicity of P. Next, $P(\mathcal{S}_+) = 1 \Rightarrow P(\mathcal{F}_+) = 0$ by the shift-invariance of P. Thus, it is enough to prove that $P(\mathcal{S}_+) = 1$ if and only if $E_P(\log \rho_0) < 0$ and $P(\mathcal{F}_+) = 1$ if and only if $E_P(\log \rho_0) > 0$. We prove the first claim only, the second one possessing a similar proof.

Assume first $c := E_P(\log \rho_0) < 0$. Then, by the ergodic theorem, there exists an $n_0(\omega)$ with $P(n_0(\omega) < \infty) = 1$ such that $\frac{1}{n}\sum_{i=1}^n \log \rho_i \le c/2 < 0$ for all $n > n_0(\omega)$. But then, for some $C_1(\omega) < \infty$ P-a.s.,

$$\sum_{n=1}^{\infty} \rho_1 \cdots \rho_n \le C_1(\omega) + \sum_{k=n_0(\omega)+1}^{\infty} e^{kc/2} < \infty, \quad P\text{-.a.s}$$

implying $P(\mathcal{S}_+) = 1$. Conversely, for $\omega \in \mathcal{S}_+$, $\lim_{n\to\infty} \sum_{k=1}^n \log \rho_k = -\infty$. But, $\{Y_i = -\log \rho_i\}$ are stationary, and the claim follows from the following well known:

Lemma 2.1.5 (Kesten[41]) *For any real valued, stationary sequence $\{Y_i\}$, fix $Z_n = \sum_{i=1}^n Y_i$. Then, one has with probability 1 that the event $\{Z_n \to_{n\to\infty} \infty\}$ implies $\{\liminf_{n\to\infty} Z_n/n > 0\}$.*

Indeed, Kesten's lemma implies that on $\mathcal{S}_+$,

$$E_P(\log \rho_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \log \rho_k < 0, \quad P-\text{a.s.}, \tag{2.1.6}$$

and $P(\mathcal{S}_+) = 1$ thus implies $E_P(\log \rho_0) < 0$ and completes the proof of Theorem 2.1.2. $\square$

Remarks: 1. P. Bougerol has kindly indicated to me the followig proof of the implication $P(\mathcal{S}_+) = 1 \Rightarrow E_P(\log \rho_0) < 0$, which bypasses the use of Kesten's lemma: define the function $f(\omega) = \log S(\omega)$. $P(\mathcal{S}_+) = 1$ implies that $f(\omega)$ is well defined. Since $S(\omega) = \rho_1 + \rho_1 S(\theta\omega)$, it holds that $f(\omega) > \log \rho_1 + f(\theta\omega)$, and we conclude by **(A2)** that $(f(\theta\omega) - f(\omega))_+$ is P-integrable and hence $E_P[f(\omega) - f(\theta\omega)] = 0$ (this is Mañe's lemma, apply the ergodic theorem to see it!). Using again $f(\omega) > \log \rho_1 + f(\theta\omega)$, one concludes that $0 > E_P \log \rho_1 = E_P \log \rho_0$, as claimed.

2. If P is i.i.d., Theorem 2.1.2 remains valid when the left hand side of conditions (a), (b), (c) is replaced, respectively, by

$$(\text{a}'): \quad \sum_{n=1}^{\infty} n^{-1} P\left(\prod_{j=1}^n \rho_j > 1\right) < \infty.$$

$$(\text{b}'): \quad \sum_{n=1}^{\infty} n^{-1} P\left(\prod_{j=1}^n \rho_j < 1\right) < \infty.$$

$$(\text{c}'): \quad \sum_{n=1}^{\infty} n^{-1} P\left(\prod_{j=1}^n \rho_j < 1\right) = \sum_{n=1}^{\infty} n^{-1} P\left(\prod_{j=1}^n \rho_j > 1\right) = \infty.$$

This is useful in particular when $E_P(\log \rho_0)$ is not well defined. See [68] for details.

Having developed transience and recurrence criteria, we turn to the law of large numbers. We first note that one cannot apply directly ergodic theorems to the sequence X_n/n: The sequence $\{X_n - X_{n-1}\}$ is not even stationary! We will exhibit two approaches to the LLN: The first is based on a hitting times decomposition. The second approach is based on the point of view of the "environment viewed from the particle".

LLN-version I: hitting time decompositions

Introduce the following notations:

$$\overline{S} = \sum_{i=1}^{\infty} \frac{1}{\omega^+_{(-i)}} \prod_{j=0}^{i-1} \rho_{(-j)} + \frac{1}{\omega^+_0} \tag{2.1.7}$$

$$\overline{F} = \sum_{i=1}^{\infty} \frac{1}{\omega^-_i} \prod_{j=0}^{i-1} \rho^{-1}_j + \frac{1}{\omega^-_0} \tag{2.1.8}$$

Theorem 2.1.9 *Assume Assumption 2.1.1. Then,*

$$\begin{array}{llll} \text{(a)} & E_P(\overline{S}) < \infty & \Rightarrow \lim\limits_{n\to\infty} \dfrac{X_n}{n} = \dfrac{1}{E_P(\overline{S})}, & \mathbb{P}^o \ \textit{a.s.} \\ \text{(b)} & E_P(\overline{F}) < \infty & \Rightarrow \lim\limits_{n\to\infty} \dfrac{X_n}{n} = -\dfrac{1}{E_P(\overline{F})}, & \mathbb{P}^o \ \textit{a.s.} \\ \text{(c)} & E_P(\overline{S}) = \infty & \textit{and } E_P(\overline{F}) = \infty \Rightarrow \lim\limits_{n\to\infty} \dfrac{X_n}{n} = 0, & \mathbb{P}^o \ \textit{a.s.} \end{array}$$

Remark: In the case that P is i.i.d., (a)–(c) of Theorem 2.1.9 become

$$\begin{array}{llll} \text{(a}'\text{)} & E_P(\rho_0) < 1 & \Rightarrow \lim\limits_{n\to\infty} \dfrac{X_n}{n} = \dfrac{1 - E_P(\rho_0)}{E_P\left(\frac{1}{\omega^+_0}\right)}, & \mathbb{P}^o \text{ a.s.} \\ \text{(b}'\text{)} & E_P(\rho_0^{-1}) < 1 & \Rightarrow \lim\limits_{n\to\infty} \dfrac{X_n}{n} = -\dfrac{1 - E_P\left(\frac{1}{\rho_0}\right)}{E_P\left(\frac{1}{\omega^-_0}\right)}, & \mathbb{P}^o \text{ a.s.} \\ \text{(c}'\text{)} & \dfrac{1}{E_P(\rho_0)} \le 1 \le E_P(\rho_0^{-1}) & \Rightarrow \lim\limits_{n\to\infty} \dfrac{X_n}{n} = 0, & \mathbb{P}^o \text{ a.s.} \end{array}$$

since $E_P \log \rho_0 \le \log E_P \rho_0$ with a strict inequality whenever P is non-degenerate, it follows that one can find examples where $X_n \to \infty$ $\mathbb{P}^o$-a.s. but $X_n/n \to 0$, $\mathbb{P}^o$-a.s. This does not contradict Kesten's lemma (Lemma 2.1.5) because $\{X_n - X_{n-1}\}$ is not in general a stationary sequence under $\mathbb{P}^o$.

Proof of Theorem 2.1.9 We introduce hitting times which will serve us later too. Let $T_0 = 0$, and

$$T_n = \min\{k : X_k = n\}$$

with the usual convention that the minimum over an empty set is $+\infty$. Set $\tau_0 = 0$ and

$$\tau_n = T_n - T_{n-1}, \quad n \geq 1\,.$$

Similarly, set

$$T_{-n} = \min\{k : X_k = -n\}$$

and

$$\tau_{-n} = T_{-n} - T_{-n+1}, \quad n \geq 1\,,$$

the convention being that $\tau_{\pm n} = \infty$ if $T_{\pm n} = \infty$. We have the following lemma:

Lemma 2.1.10 *If* $\limsup_{n\to\infty} X_n = +\infty$, $\mathbb{P}^o$*-a.s., then* $\{\tau_i\}_{i\geq 1}$ *is a stationary and ergodic sequence. If further* P *is strongly mixing, then* $\{\tau_i\}_{i\geq 1}$ *is also strongly mixing.*

Proof of Lemma 2.1.10 The stationarity of $\{\tau_i\}_{i\geq 1}$ follows from the stationarity of the environment. To see the ergodicity, let $\Xi = [0,1]^{\mathbb{N}}$, let U_Ξ denote the measure on Ξ making all coordinates $\{\xi_i\}$ independent and of uniform law on $[0,1]$, and note that $\{X_n\}$ may be constructed by writing

$$X_{n+1} = X_n + \mathbf{1}_{\{\omega^+_{X_n} > \xi_{n+1}\}} - \mathbf{1}_{\{\xi_{n+1} \in [\omega^+_{X_n}, \omega^+_{X_n} + \omega^-_{X_n})\}}\,.$$

Suppose $A = A(\omega, \xi) = A(\tau)$ is an event, measurable w.r.t. $\mathcal{G}_n = \sigma\{\tau_i, i \geq 1\}$, which is invariant with respect to the shift $(\theta\tau)_i = \tau_{i+1}$ (we write in the sequel $\theta A = A(\theta\tau)$). We need only show that $P \otimes U_\Xi(A) \in \{0,1\}$. Note however that $\theta^k A$, conditioned on $\sigma\{\omega_i, i \in \mathbb{Z}\}$, is independent of $\xi_1, \ldots, \xi_k$. Thus, since U_Ξ is an i.i.d. law and hence the tail sigma-field of $\{\xi_i\}$ is trival, it follows that $A = \theta^k A$ is, under the above conditioning, independent of $\sigma\{\xi_i, i \geq 1\}$. Thus A depends only on ω. But the shift θ on the sequence $\{\tau_i\}$ induces the usual shift θ on Ω. Thus, $\theta A = A(\theta\omega) = A(\omega)$ and hence $P(A) \in \{0,1\}$.

To prove the strong mixing properties (which we do not actually need in the sequel), consider sets $A_1 \cdots A_k, B_1 \cdots B_j \subset \mathbb{Z}$, and let

$$\mathcal{A} = \bigcap_{i=1}^{k} \{\tau_i \in A_i\}, \qquad \mathcal{B}^m = \bigcap_{i=1}^{j} \{\tau_{m+i} \in B_i\}, \,.$$

Clearly, $\mathbb{P}^o(\mathcal{B}^m) = \mathbb{P}^o(\mathcal{B}^0)$, and thus we need to prove that whenever $\limsup_{n\to\infty} X_n = \infty$ $\mathbb{P}^o$-a.s., then

$$\lim_{m\to\infty} \mathbb{P}^o(\mathcal{A} \cap \mathcal{B}^m) = \mathbb{P}^o(\mathcal{A})\mathbb{P}^o(\mathcal{B}^0)\,.$$

Toward this end, let

$$B_i^K = B_i \cap [0, K] \quad \text{and} \quad \mathcal{B}^{m,K} = \bigcap_{i=1}^{j} \{\tau_{m+i} \in B_i^K\} \,.$$

Fix $\varepsilon > 0$ and then $K = K(\varepsilon)$ large enough such that

$$\mathbb{P}^o(\mathcal{B}^m \setminus \mathcal{B}^{m,K}) = \mathbb{P}^o(\mathcal{B}^0 \setminus \mathcal{B}^{0,K}) \leq \varepsilon$$

which is possible since $\limsup_{n\to\infty} X_n = \infty$, $\mathbb{P}^o$-a.s. Note that, for $m > k$,

$$P_\omega^o(\mathcal{A} \cap \mathcal{B}^{K,m}) = P_\omega^o(\mathcal{A}) P_\omega^o(\mathcal{B}^{K,m})$$

and that $P_\omega^o(\mathcal{A})$ is measurable with respect to $\sigma(\omega_i, i \leq k-1)$. On the other hand, since $|X_{n+1} - X_n| = 1$, on the event $\{\tau_{m+i} \leq K, \, 1 \leq i \leq j\}$, it holds that $X_n \geq m - K$ for $T_m \leq n \leq T_{m+j+1}$. Thus, for $m > K + k$, $P_\omega^o(\mathcal{B}^{K,m})$ is measurable with respect to $\sigma(\omega_i, i \geq m - K)$. It follows from the strong mixing of P that

$$\begin{aligned}
\lim_{m\to\infty} \mathbb{P}^o(\mathcal{A} \cap \mathcal{B}^{K,m}) &= \lim_{m\to\infty} E_P(P_\omega^o(\mathcal{A}) P_\omega^o(\mathcal{B}^{K,m})) \\
&= E_P(P_\omega^o(\mathcal{A})) \cdot E_P(P_\omega^o(\mathcal{B}^{K,0})) \\
&= \mathbb{P}^o(\mathcal{A}) \mathbb{P}^o(\mathcal{B}^{K,0}) \,. \qquad (2.1.11)
\end{aligned}$$

On the other hand,

$$\mathbb{P}^o(\mathcal{A} \cap \mathcal{B}^m) - \varepsilon \leq \mathbb{P}^o(\mathcal{A} \cap \mathcal{B}^{K,m}) \leq \mathbb{P}^o(\mathcal{A} \cap \mathcal{B}^m)$$

while

$$\mathbb{P}^o(\mathcal{B}^m) - \varepsilon \leq \mathbb{P}^o(\mathcal{B}^{K,m}) \leq \mathbb{P}^o(\mathcal{B}^m)$$

and one concludes from (2.1.11) that

$$\left| \lim_{m\to\infty} \mathbb{P}^o(\mathcal{A} \cap \mathcal{B}^m) - \mathbb{P}^o(\mathcal{A}) \mathbb{P}^o(\mathcal{B}^0) \right| \leq \varepsilon$$

which implies the claim since ε is arbitrary. □

Remark: Note that an attempt to mimick this argument in $\mathbb{Z}^d$, $d \geq 1$, with T_i denoting the hitting times of hyperplanes at distance i from the origin, fails because of the extra information contained in the hitting location.

Our strategy consists now of applying the ergodic theorem to the sequence $\{\tau_i\}$. As a first step, we have the

Lemma 2.1.12 *Assume Assumption 2.1.1. Then,*

$$\begin{aligned}
&\text{(a)} \quad E_{\mathbb{P}^o}(\tau_1) = E_P(\overline{S}) \,, \\
&\text{(b)} \quad E_{\mathbb{P}^o}(\tau_{-1}) = E_P(\overline{F}) \,.
\end{aligned}$$

Proof. We prove only (a), the proof of (b) being similar. Decompose, with $X_0 = 0$,

$$\tau_1 = \mathbf{1}_{X_1=1} + \mathbf{1}_{X_1=0}(1+\tau_1') + \mathbf{1}_{X_1=-1}(1+\tau_0''+\tau_1'') \,. \tag{2.1.13}$$

Here, (τ_1') is the first hitting time of 1 after time 1 (possibly infinite), $(1+\tau_0'')$ is the first hitting time of 0 after time 1, and $1+\tau_0''+\tau_1''$ is the first hitting time of 1 after time $1+\tau_0''$.

Under P^o_ω, the law of τ_1' conditioned on the event $\{X_1 = 0\}$ is identical to the law of τ_1, the law of τ_0'' conditioned on the event $\{X_1 = -1\}$ is $P^o_{\theta^{-1}\omega}(\tau_1 \in \cdot)$, while conditioned on the event $\{X_1 = -1\} \cap \{\tau_0'' < \infty\}$, τ_1'' also has law identical to that of τ_1.

Consider first the case $E_{\mathbb{P}^o}(\tau_1) < \infty$. Then, both $E^o_\omega(\tau_1) < \infty$ and $E_{\theta^{-1}\omega}(\tau_1) < \infty$, P-a.s. Taking expectations in (2.1.13), one gets then

$$E^o_\omega(\tau_1) = 1 + (1-\omega_0^+)E^o_\omega(\tau_1) + \omega_0^- E^o_{\theta^{-1}\omega}(\tau_1) \,.$$

Hence,

$$E^o_\omega(\tau_1) = \frac{1}{\omega_0^+} + \rho_0 E^o_{\theta^{-1}\omega}(\tau_1) \,.$$

Iterating this equation, we get

$$E^o_\omega(\tau_1) = \frac{1}{\omega_0^+} + \frac{\rho_0}{\omega^+_{(-1)}} + \frac{\rho_0\rho_{(-1)}}{\omega^+_{(-2)}} + \cdots + \frac{\prod_{i=0}^{-(m-1)} \rho_i}{\omega^+_{(-m)}} + \left(\prod_{i=0}^{-m+1} \rho_i\right) E^o_{\theta^{-m}\omega}(\tau_1) \,. \tag{2.1.14}$$

Omitting the last term, taking expectations on both sides, and then taking $m \to \infty$ using dominated convergence, we get

$$E_{\mathbb{P}^o}(\tau_1) \geq E_P(\overline{S}) \,. \tag{2.1.15}$$

To see the reverse inequality, note that by (2.1.13), for any $M < \infty$,

$$E^o_\omega(\tau_1 \mathbf{1}_{\tau_1<M}) \leq 1 + (1-\omega_0^+)E^o_\omega(\tau_1\mathbf{1}_{\tau_1<M}) + \omega_0^- E^o_{\theta^{-1}\omega}(\tau_1\mathbf{1}_{\tau_1<M}) \,.$$

Iterating, we get that

$$E^o_\omega(\tau_1\mathbf{1}_{\tau_1<M}) \leq \overline{S} + M \prod_{i=0}^{-m+1} \rho_i \,.$$

Taking expectations, we get that

$$E_{\mathbb{P}^o}(\tau_1\mathbf{1}_{\tau_1<M}) \leq E_P(\overline{S}) + M E_P\left(\prod_{i=0}^{-m+1} \rho_i\right) \,.$$

Assuming $E_P(\overline{S}) < \infty$ and hence $E_P\left(\prod_{i=0}^{-m+1} \rho_i\right) \underset{m\to\infty}{\longrightarrow} 0$, we get that

$$E_{\mathbb{P}^o}(\tau_1 \mathbf{1}_{\tau_1 < M}) \leq E_P(\overline{S})\,.$$

Taking $M \to \infty$ and using monotone convergence we conclude, using also (2.1.15), that

$$E_{\mathbb{P}^o}(\tau_1 \mathbf{1}_{\tau_1 < \infty}) = E_P(\overline{S})\,,$$

completing the proof that $E_{\mathbb{P}^o}(\tau_1) < \infty \Rightarrow E_{\mathbb{P}^o}(\tau_1) = E_P(\overline{S})$.

It thus remains to show that $E_{\mathbb{P}^o}(\tau_1) = \infty \Rightarrow E_P(\overline{S}) = \infty$. Note next that if $E_P(\log \rho_0) \leq 0$, we have by Theorem 2.1.2 that

$$E_{\mathbb{P}^o}(\tau_1 \mathbf{1}_{\tau_1 < \infty}) = E_{\mathbb{P}^o}(\tau_1)$$

hence $E_{\mathbb{P}^o}(\tau_1) = \infty$ implies $E_P(\overline{S}) = \infty$. On the other hand, if $E_P(\log \rho_0) > 0$ then $\prod_{i-1}^{0} \rho_{(-j)} \to_{j\to\infty} \infty$, P-a.s. by the ergodic theorem and hence also $E_P(\overline{S}) = \infty$. This concludes the proof of Lemma 2.1.12. $\square$

Remark: In fact, a similar proof shows that in the uniformly elliptic case, $E^o_\omega(\tau_1) = \overline{S}$, for *every* environment ω.

An application of Lemmas 2.1.10 and 2.1.12 yields that in case (a)

$$\frac{T_n}{n} = \frac{\sum_{i=1}^n \tau_i}{n} \longrightarrow_{n\to\infty} E_{\mathbb{P}^o}(\tau_1) < \infty, \quad \mathbb{P}^o\text{-a.s.}\,. \tag{2.1.16}$$

On the other hand, we have the following:

Lemma 2.1.17 *Assume $T_n/n \to \alpha$, for some constant $\alpha < \infty$. Then,*

$$\frac{X_n}{n} \underset{n\to\infty}{\longrightarrow} \frac{1}{\alpha}.$$

Proof of Lemma 2.1.17 Let k_n be the unique (random) integers such that

$$T_{k_n} \leq n < T_{k_n+1}\,.$$

Note that $X_n < k_n + 1$ while $X_n \geq k_n - (n - T_{k_n})$. Hence,

$$\frac{k_n}{n} - \left(1 - \frac{T_{k_n}}{n}\right) \leq \frac{X_n}{n} \leq \frac{k_n + 1}{n}\,.$$

But, $\lim_{n\to\infty} k_n/n = \lim_{n\to\infty} n/T_n$ (due to the existence of the second limit and the definition of k_n). Thus,

$$\frac{1}{\alpha} \geq \limsup_{n\to\infty} \frac{X_n}{n} \geq \liminf_{n\to\infty} \frac{X_n}{n} \geq \frac{1}{\alpha}\,. \qquad \square$$

Lemma 2.1.17 and (2.1.16) complete the proof of Theorem 2.1.9 in case (a). Case (b) is similar, while case (c) is a minor modification of the above argument and is left out. $\square$

Bibliographical notes: The proof of Theorem 2.1.2 is essentially from [68], except that the use of Kesten's lemma is borrowed from [1]. See also [51] for an "ergodic" approach. The rest of the section is an adaptation of the argument in [68], which requires a strongly mixing assumption. The proof of ergodicity in Lemma 2.1.10 was suggested to me by P. Bougerol. F. Rassoul-Agha has kindly shown me a different proof of this fact.

Transience and recurrence results for non nearest-neighbour RWRE on $\mathbb{Z}$, in terms of certain Lyapunov exponents of products of random matrices, are developed in [44], see also [51] and [7]. This is further developed in [3], [40], where transience and recurrence criteria for RWRE on graphs of the form $\mathbb{Z} \times G$, G finite, are derived.

LLN-version II: auxiliary Markov chains

We use the evaluation of the LLN as an excuse for introducing the machinery of the "environment viewed from the particle". The first step consists of introducing an auxiliary Markov chain.

Starting from the RWRE X_n, define $\overline{\omega}(n) = \theta^{X_n}\omega$. The sequence $\{\overline{\omega}_n\}$ is a process with paths in $\Omega^{\mathbb{N}}$. What is maybe more useful is that it is in fact a *Markov* process. More precisely:

Lemma 2.1.18 *The process $\{\overline{\omega}(n)\}$ is a Markov process under either P_ω^0 or $\mathbb{P}^0$, with state space Ω and transition kernel*

$$M(\omega, d\omega') = \omega_0^+ \delta_{\theta\omega=\omega'} + \omega_0^- \delta_{\theta^{-1}\omega=\omega'} + \omega_0^0 \delta_{\omega=\omega'} \,.$$

Proof. For bounded functions $f_i : \Omega \to \mathbb{R}$,

$$\begin{aligned}
E_\omega^o\left(\prod_{i=1}^n f_i(\overline{\omega}(i))\right) &= E_\omega^o\left(\prod_{i=1}^n f_i(\theta^{X_i}\omega)\right) \\
&= E_\omega^o\left(\prod_{i=1}^{n-1} f_i(\theta^{X_i}\omega) E_\omega^{X_{n-1}}(f_n(\theta^{X_n}\omega))\right) \\
&= E_\omega^o\left(\prod_{i=1}^{n-1} f_i(\theta^{X_i}\omega)\left[\omega_{X_{n-1}}^+ f_n(\theta\cdot\theta^{X_{n-1}}\omega)\right.\right. \\
&\qquad \left.\left. +\omega_{X_{n-1}}^- f_n(\theta^{-1}\cdot\theta^{X_{n-1}}\omega) + \omega_{X_{n-1}}^o f_n(\theta^{X_{n-1}}\omega)\right]\right) \\
&= E_\omega^o\left(\prod_{i=1}^{n-1} f_i(\theta^{X_i}\omega) M f_n(\overline{\omega}(n-1))\right) \qquad (2.1.19)
\end{aligned}$$

where

$$Mf(\omega) = \int f(\omega')M(\omega, d\omega') \,,$$

which proves the Markov property of $\{\overline{\omega}(n)\}$ under P_ω^o. Integrating both sides of (2.1.19) with respect to P yields the Markov property under $\mathbb{P}^o$. □

Our next step is to construct an invariant measure for the transition kernel M. In most of this section we will assume that $E_P \log \rho_0 < 0$, implying, by Theorem 2.1.2, that $T_1 < \infty$, $\mathbb{P}^o$-a.s. Whenever $E_{\mathbb{P}^o}(T_1) < \infty$, define the measures

$$Q(B) = E_{\mathbb{P}^o}\left(\sum_{i=0}^{T_1-1} \mathbf{1}_{\{\overline{\omega}(i)\in B\}}\right), \quad \overline{Q}(B) = \frac{Q(B)}{Q(\Omega)} = \frac{Q(B)}{E_{\mathbb{P}^o}T_1}.$$

Using Lemma 2.1.12, one checks that under Assumption 2.1.1 and if $E_P(\bar{S}) < \infty$ then $E_{\mathbb{P}^o}T_1 < \infty$, and $\overline{Q}(\cdot)$ in this case is a probability measure.

Lemma 2.1.20 *Assume Assumption 2.1.1 and $E_P(\bar{S}) < \infty$. Then, $Q(\cdot)$ is invariant under the Markov kernel M, that is*

$$Q(B) = \iint \mathbf{1}_{\omega' \in B} M(\omega, d\omega') Q(d\omega).$$

Proof. We have

$$\begin{aligned}
&\iint \mathbf{1}_{\omega' \in B} M(\omega, d\omega') Q(d\omega) \\
&\quad = \sum_{k=0}^{\infty} E_{\mathbb{P}^o}\Big(T_1 > k;\ \mathbf{1}_{\overline{\omega}(k+1)\in B}\Big) \\
&\quad = \sum_{k=0}^{\infty} E_{\mathbb{P}^o}\Big(T_1 = k+1;\ \mathbf{1}_{\overline{\omega}(k+1)\in B}\Big) + \sum_{k=0}^{\infty} E_{\mathbb{P}^o}\Big(T_1 > k+1;\ \mathbf{1}_{\overline{\omega}(k+1)\in B}\Big) \\
&\quad = \mathbb{P}^o(T_1 < \infty; \overline{\omega}(T_1) \in B) + \sum_{k=1}^{\infty} \mathbb{P}^o(T_1 > k; \overline{\omega}(k) \in B).
\end{aligned}$$

But $\mathbb{P}^o(T_1 < \infty) = 1$ while $\mathbb{P}^o(\overline{\omega}(T_1) \in B) = P(\theta\omega \in B) = P(\omega \in B)$, hence

$$= \sum_{k=0}^{\infty} \mathbb{P}^o(T_1 > k; \overline{\omega}(k) \in B) = Q(B). \qquad \square$$

Define next

$$\Lambda(\omega) = \frac{1}{\omega_0^+}\left[1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} \rho_j\right].$$

It is not hard to check, by the shift invariance of P, that the condition $E_P(\Lambda(\omega)) < \infty$ is equivalent to $E_P(\overline{S}) < \infty$, c.f. Section 2.1. We next claim the

Lemma 2.1.21 *Under the assumptions of Lemma 2.1.20, it holds that*

$$\frac{dQ}{dP} = \Lambda(\omega).$$

Proof. Note first that by Jensen's inequality, $E_P(\Lambda) < \infty$ implies that $E_P(\log \rho_0) < 0$ and hence $X_n \to_{n\to\infty} \infty$, $\mathbb{P}^o$-a.s., by Theorem 2.1.2. Let $f : \Omega \to \mathbb{R}$ be measurable. Then,

$$\int f dQ = E_{\mathbb{P}^o}\left(\sum_{i=0}^{T_1-1} f(\overline{\omega}_i)\right) = E_{\mathbb{P}^o}\left(\sum_{i\leq 0} f(\theta^i\omega)N_i\right)$$

where $N_i = \{\#k \in [0,T_1) : X_k = i\}$ (note the difference in the role the index i plays in the two sums!). Using the shift invariance of P, we get

$$\begin{aligned}\int f dQ &= \sum_{i\leq 0} E_P\Big(f(\theta^i\omega)E^o_\omega N_i\Big)\\ &= \sum_{i\leq 0} E_P\Big(f(\omega)E^o_{\theta^{-i}\omega} N_i\Big) = E_P\left(f(\omega)\left(\sum_{i\leq 0} E^o_{\theta^{-i}\omega}N_i\right)\right).\end{aligned}$$

Hence,

$$\frac{dQ}{dP} = \sum_{i\leq 0} E^o_{\theta^{-i}\omega} N_i\,, \tag{2.1.22}$$

and the right hand side converges, P-a.s.

In order to prove both the convergence in (2.1.22) and the lemma, we turn to evaluate $E^o_\omega N_i$. Define, for $i \leq 0$,

$$\begin{aligned}\eta_{i,0} &= \min\{k \leq T_1 :\ X_k = i\}\\ \theta_{i,0} &= \min\{\eta_{i,0} < k \leq T_1 :\ X_{k-1} = i, X_k = i-1\}\end{aligned}$$

and, for $j \geq 1$,

$$\begin{aligned}\eta_{i,j} &= \min\{\theta_{i,j-1} < k \leq T_1 :\ X_k = i\}\\ \theta_{i,j} &= \min\{\eta_{i,j} < k \leq T_1 :\ X_{k-1} = i, X_k = i-1\}\end{aligned}$$

(with the usual convention that the minimum over an empty set is $+\infty$). We refer to the time interval $(\theta_{i,j-1}, \eta_{i,j})$ as the j-th excursion from $i-1$ to i. For any $j \geq 0$, any $i \leq 0$, define

$$\begin{aligned}U_{i,j} &= \{\#\ell \geq 0 :\ \theta_{i+1,j} < \theta_{i,\ell} < \eta_{i+1,j+1}\}\\ Z_{i,j} &= \{\#k \geq 0 :\ X_{k-1} = i, X_k = i, \theta_{i+1,j} < k < \eta_{i+1,j+1}\}\,.\end{aligned}$$

Note that $U_{i,j}$ is the number of steps from i to $i-1$ during the $j+1$-th excursion from i to $i+1$, whereas $Z_{i,j}$ is the number of steps from i to i during the same excursion. The Markov property implies that

$$P^o_\omega\Big(U_{i,\ell}=k_\ell, Z_{i,\ell}=m_\ell\,,\ell=1,\ldots,L\Big|\{U_{i',j}\}_{i'>i},\eta_{i+1,L+1}<\infty\Big)$$
$$=\prod_{\ell=1}^{L}\left[\left(\frac{\omega_i^-}{\omega_i^-+\omega_i^+}\right)^{k_\ell}\left(\frac{\omega_i^+}{\omega_i^-+\omega_i^+}\right)\left(\frac{\omega_i^0}{\omega_i^0+\omega_i^+}\right)^{m_\ell}\left(\frac{\omega_i^+}{\omega_i^0+\omega_i^+}\right)\right]. \tag{2.1.23}$$

Defining $U_i=\sum_j U_{i,j}$, $Z_i=\sum_j Z_{i,j}$, and noting that $\mathbb{P}^o(\{U_i<\infty\}\cap\{Z_i<\infty\})=1$ because $X_n\to\infty$, $\mathbb{P}^o$-a.s., (2.1.23) implies that $\{U_i\}$ is under P^o_ω an (inhomogeneous) branching process with geometric offspring distribution of parameter $\frac{\omega_i^-}{\omega_i^-+\omega_i^+}$. Further,

$$\begin{aligned}E^o_\omega(U_i|U_{i+1},\cdots,U_0)&=\rho_i U_{i+1}\\ E^o_\omega(Z_i|U_{i+1},\cdots,U_0)&=\frac{\omega_i^0}{\omega_i^+}U_{i+1}\end{aligned} \tag{2.1.24}$$

and using the relation $N_i=U_i+U_{i+1}+Z_i$, $\mathbb{P}^o$-a.s., we get

$$E^o_\omega(N_i|U_{i+1},\ldots,U_0)=E^o_\omega\Big(U_i+U_{i+1}+Z_i|U_{i+1},\cdots,U_0\Big)=\frac{1}{\omega_i^+}E^o_\omega U_{i+1}\,.$$

Iterating (2.1.24), one gets

$$E^o_\omega N_i=\frac{1}{\omega_i^+}\rho_0\cdots\rho_{i+1}\,.$$

Hence, using (2.1.22), and the assumption,

$$\frac{dQ}{dP}=\frac{1}{\omega_0^+}\left[1+\sum_{i=1}^{\infty}\prod_{j=1}^{i}\rho_j\right]<\infty, P\text{-a.s.}$$

which completes the proof of the Lemma. □

Remark: Note that $dQ/dP>0$, P-a.s., and hence under the assumption $E_P(\bar S)<\infty$ it holds that $Q\sim P$. This fact is true in greater generality, see the discussion in [70] and in Section 3.3 below.

Corollary 2.1.25 *Under the law induced by $\overline{Q}\otimes P^o_\omega$, the sequence $\{\overline{\omega}(n)\}$ is stationary and ergodic.*

Proof. The stationarity follows from the stationarity of $\overline{Q}$. Let $\overline{\theta}$ denote the shift on $\overline{\Omega}=\Omega^{\mathbb{N}}$, that is, for $\overline{\omega}\in\overline{\Omega}, \overline{\theta}\overline{\omega}(n)=\overline{\omega}(n+1)$. Denote by $\overline{P}_\omega$ the law of the sequence $\{\overline{\omega}(n)\}$ with $\overline{\omega}(0)=\omega$, that is, for any measurable sets $B_i\subset\Omega$,

$$\overline{P}_\omega\Big(\overline{\omega}(i)\in B_i, i=1,\ldots,\ell\Big)$$
$$=\int_{B_1}\cdots\int_{B_\ell}M(\omega,d\omega^1)M(\omega^1,d\omega^2)\cdots M(\omega^{\ell-1},d\omega^\ell)$$

and set $\overline{\mathcal{Q}} = \overline{Q} \otimes \overline{P}_\omega$ (as usual, we also use $\overline{\mathcal{Q}}$ to denote the corresponding marginal induced on $\overline{\Omega}$).

We need to show that for any invariant A, that is $A \in \overline{\Omega}$ such that $\overline{\theta} A = A$, $\overline{\mathcal{Q}}(A) \in \{0,1\}$. Set $\varphi(\omega) = \overline{P}_\omega(A)$, we claim that $\{\varphi(\overline{\omega}(n))\}$ is a martingale with respect to the filtration $\mathcal{G}_n = \sigma(\overline{\omega}(0), \ldots, \overline{\omega}(n))$: indeed,

$$\varphi(\overline{\omega}(n)) = \overline{P}_{\overline{\omega}(n)}(A) = E_{\overline{\mathcal{Q}}}\Big(\mathbf{1}_{\theta^n A} | \mathcal{G}_n\Big) = E_{\overline{\mathcal{Q}}}\Big(\mathbf{1}_A | \mathcal{G}_n\Big),$$

where the second equality is due to the Markov property and the third due to the invariance of A. Hence, by the martingale convergence theorem,

$$\varphi(\overline{\omega}(n)) \underset{n\to\infty}{\longrightarrow} \mathbf{1}_A, \quad \overline{\mathcal{Q}}\text{-a.s.} \tag{2.1.26}$$

Further, $Q(\varphi(\omega) \notin \{0,1\}) = 0$ because otherwise there exists an interval $[a,b]$ with $\{0\}, \{1\} \notin [a,b]$ and $Q(\varphi(\omega) \in [a,b]) > 0$, while

$$\frac{1}{n} \sum_0^{n-1} \mathbf{1}_{\{\varphi(\overline{\omega}(n)) \in [a,b]\}} \to E_{\overline{\mathcal{Q}}}\Big(\mathbf{1}_{\{\varphi(\overline{\omega}(0)) \in [a,b]\}} | \mathcal{I}\Big), \tag{2.1.27}$$

where $\mathcal{I}$ is the invariant σ-field.

Taking expectations in (2.1.27) and using (2.1.26), one concludes that

$$0 = \overline{\mathcal{Q}}\Big(\varphi(\overline{\omega}(0)) \in [a,b]\Big) = \overline{Q}\Big(\varphi(\omega) \in [a,b]\Big),$$

a contradiction. Thus for some measurable $B \subset \Omega$, $\varphi(\omega) = \mathbf{1}_B\,, \overline{Q} -$ a.s.. Further, the Markov property and invariance of A yield that $M\mathbf{1}_B = \mathbf{1}_B$, $\overline{Q}$-a.s. and hence P-a.s. But then,

$$\mathbf{1}_B = M\mathbf{1}_B \geq \omega_0^+ \mathbf{1}_{\theta B}, P\text{-a.s.}\,.$$

Since $E\Lambda(\omega) < \infty$ implies $P(\omega_0^+ = 0) = 0$, it follows that $\mathbf{1}_B \geq \mathbf{1}_{\theta B}$, P-a.s., and then $E_P(\mathbf{1}_B) = E_P(\mathbf{1}_{\theta B})$ implies that $\mathbf{1}_B = \mathbf{1}_{\theta B}$, P-a.s. But then, by ergodicity of P, $P(B) \in \{0,1\}$, and hence $\overline{Q}(B) \in \{0,1\}$. Since $\overline{\mathcal{Q}}(A) = E_{\overline{Q}} \varphi(\omega) = \overline{Q}(B)$, the conclusion follows. $\square$

We are now ready to give the:

Proof of Theorem 2.1.9 - Environment version We begin with case (a), noting that the proof of case (b) is identical by the transformation $\omega_i \mapsto \hat{\omega}_{-i}$, where $\hat{\omega}_i^+ = \omega_i^-, \hat{\omega}_i^- = \omega_i^+$. Set $d(x,\omega) = E_\omega^x(X_1 - x)$. Then

$$\begin{aligned} X_n = \sum_{i=1}^n (X_i - X_{i-1}) &= \sum_{i=1}^n \Big(X_i - X_{i-1} - d(X_{i-1},\omega)\Big) + \sum_{i=1}^n d(X_{i-1},\omega) \\ &:= M_n + \sum_{i=1}^n d(X_{i-1},\omega)\,. \end{aligned} \tag{2.1.28}$$

But, under P^o_ω, M_n is a martingale, with $|M_{n+1} - M_n| \le 2$; Hence, with $\mathcal{G}_n = \sigma(M_1, \cdots M_n)$,

$$\begin{aligned} E^o_\omega(e^{\lambda M_n}) &= E^o_\omega\Big(e^{\lambda M_{n-1}} E^o_\omega(e^{\lambda(M_n - M_{n-1})}|\mathcal{G}_{n-1})\Big) \\ &\le E^o_\omega\Big(e^{\lambda M_{n-1}} e^{2\lambda^2}\Big) \end{aligned}$$

and hence, iterating, $E^o_\omega(e^{\lambda M_n}) \le e^{2n\lambda^2}$ (this is a version of Azuma's inequality, see [20, Corollary 2.4.7]). Chebycheff's inequality then implies

$$\frac{M_n}{n} \to 0, \quad \mathbb{P}^o\text{-a.s.}$$

(and even with exponential rate). Next, note that

$$\sum_{i=1}^n d(X_{i-1}, \omega) = \sum_{i=1}^n d(0, \overline{\omega}(i-1))\,.$$

The ergodicity of $\{\overline{\omega}(i)\}$ under $\overline{Q} \otimes P^o_\omega$ implies that

$$\frac{1}{n}\sum_{i=1}^n d(0, \overline{\omega}(i-1)) \longrightarrow E_{\overline{Q}}(d(0, \overline{\omega}(0))),\ \overline{Q} \otimes P^o_\omega\text{-a.s.}\,. \tag{2.1.29}$$

But,

$$\begin{aligned} &E_{\overline{Q}}(d(0, \overline{\omega}(0))) \\ &\quad = \frac{E_P\Big[\Lambda(\omega)(\omega_0^+ - \omega_0^-)\Big]}{E_P(\Lambda(\omega))} \\ &\quad = \frac{1 + E_P\Big(\omega_1^-\Big[\frac{1}{\omega_1^+} + \sum_{i=2}^\infty \prod_{j=2}^i \rho_j\Big] - \omega_0^-\Big[\frac{1}{\omega_0^+} + \sum_{i=1}^\infty \prod_{j=1}^i \rho_j\Big]\Big)}{E_P(\Lambda(\omega))} \\ &\quad = \frac{1}{E_P(\Lambda(\omega))} = \frac{1}{E_P(\overline{S}(\omega))}\,. \end{aligned}$$

Finally, since $E_P(\Lambda(\omega)) < \infty$, (2.1.29) holds also $\mathbb{P}^o$-a.s., completing the proof of the theorem in cases (a),(b).

Case (c) is handled by appealing to Lemma 2.1.12. Suppose $\limsup X_n = +\infty, \mathbb{P}^o -$ a.s.. Then, $\tau_1 < \infty$, $\mathbb{P}^o$-a.s.. Define $\tau_i^K = \min(\tau_i, K)$. Note that under P^o_ω, the random variables $\{\tau_i^K\}$ are independent and bounded, and hence, with $G_n^K = n^{-1}\sum_{i=1}^n \tau_i^K$, we have

$$|G_n^K - E^o_\omega G_n^K| \to_{n\to\infty} 0\,, \quad P^o_\omega - \text{a.s.}$$

But $f(\omega) := E^o_\omega \tau_1^K$ is a bounded, measurable, local function on Ω, and $E^o_\omega G_n^K = n^{-1}\sum_{i=1}^n f(\theta^i\omega)$. Hence, by the ergodic theorem, $E^o_\omega G_n^K \to_{n\to\infty}$

$E_{\mathbb{P}^o}\tau_1^K, P-$ a.s.. Since, by Lemma 2.1.12 we have $E_{\mathbb{P}^o}\tau_1^K \to_{K\to\infty} \infty$, we conclude that

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \tau_i \geq \lim_{K\to\infty} E_{\mathbb{P}^o}\tau_1^K = \infty\,, \mathbb{P}^o - \text{a.s.}$$

This immediately implies $\limsup_{n\to\infty} X_n/n \leq 0, \mathbb{P}^o -$ a.s.. The reverse inequality is proved by considering the sequence $\{\tau_{-i}\}$, yielding part (c) of the Theorem. □

Remark: Exactly as in Lemma 2.1.17, it is not hard to check that under Assumption 2.1.1, it holds that

$$\lim_{n\to\infty}\frac{T_n}{n} = E_P(\overline{S})\,, \quad \mathbb{P}^o - \text{a.s.} \tag{2.1.30}$$

Bibliographical notes: The construction presented here goes back at least to [46]. Our presentation is heavily influenced by [1] and [70].

2.2 CLT for ergodic environments

In this section, we continue to look at the environment from the point of view of the particle. Our main goal is to prove the following:

Theorem 2.2.1 *Assume 2.1.1. Further, assume that for some* $\varepsilon > 0$,

$$E_Q(\overline{S}^{2+\varepsilon}(\omega) + \overline{S}(\theta^{-1}\omega)^{2+\varepsilon}) < \infty\,, \tag{2.2.2}$$

and that

$$\sum_{n\geq 1}\sqrt{E_P\Big(E_P\Big(v_P\overline{S}(\omega)-1\,\Big|\,\sigma(\omega_i, i\leq -n)\Big)^2\Big)} < \infty, \tag{2.2.3}$$

where $v_P := 1/E_P(\overline{S}(\omega))$. *Then, with*

$$\sigma^2_{P,1} := v_P^2 E_Q\Big(\omega_0^+(\overline{S}(\omega)-1)^2 + \omega_0^-(\overline{S}(\theta^{-1}\omega)+1)^2 + \omega_0^0\Big)\,,$$

and

$$\sigma^2_{P,2} := E_P(v_P\overline{S}(\omega)-1)^2 + 2\sum_{n=1}^\infty E_P\left(\left(v_P\overline{S}(\omega)-1\right)\left(v_P\overline{S}(\theta^n\omega)-1\right)\right)\,,$$

we have that

$$\mathbb{P}^o\left(\frac{X_n - nv_P}{\sigma_P\sqrt{n}} > x\right) \longrightarrow_{n\to\infty} \Phi(-x)\,,$$

where

$$\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{\theta^2}{2}}\,d\theta\,,$$

and $\sigma_P^2 = \sigma^2_{P,1} + v_P\sigma^2_{P,2}$.

Proof. The basic idea in the proof is to construct an appropriate martingale, and then use the Martingale CLT and the CLT for stationary ergodic sequences. We thus begin with recalling the version of these CLT's most useful to us.

Lemma 2.2.4 ([27], pg. 417) *Suppose $(Z_n, \mathcal{F}_n)_{n\geq 0}$ is a martingale difference sequence, and let $V_n = \sum_{1\leq k\leq n} E(Z_k^2|\mathcal{F}_{k-1})$. Assume that*

$$\text{(a)} \quad \frac{V_n}{n} \to_{n\to\infty} \sigma^2, \text{ in probability.}$$
$$\text{(b)} \quad \frac{1}{n}\sum_{m\leq n} E\Big(Z_m^2 \mathbf{1}_{\{|Z_m|>\varepsilon\sqrt{n}\}}\Big) \to_{n\to\infty} 0\,.$$

Then, $\sum_{i=1}^n Z_i/\sigma\sqrt{n}$ converges in distribution to a standard Gaussian random variable.

Lemma 2.2.5 ([27], p. 419) *Suppose $\{Z_n\}_{n\in\mathbb{Z}}$ is a stationary, zero mean, ergodic sequence, and set $\mathcal{F}_n = \sigma(Z_i, i\leq n)$. Assume that*

$$\sum_{n\geq 0}\sqrt{E\left(E\left(Z_0|\mathcal{F}_{-n}\right)\right)^2} < \infty\,. \tag{2.2.6}$$

Then, $\left\{\sum_{i=1}^{nt} Z_i/\sigma\sqrt{n}\right\}_{t\in[0,1]}$ converges in distribution to a standard Brownian motion, where

$$\sigma^2 = EZ_0^2 + 2\sum_{n=1}^{\infty} E(Z_0 Z_n)\,.$$

We next recall that by Theorem 2.1.9,

$$\frac{X_n}{n} \to v_P, \quad \mathbb{P}^o\text{-a.s.},$$

where $v_P := 1/E_P(\overline{S})$. One is tempted to use the martingale M_n appearing in the environment proof of Theorem 2.1.9 (see (2.1.28)), however this strategy is not so successful because of the difficulties associated with separating the fluctuations in M_n and $\sum_{i=1}^n d(X_{i-1},\omega)$. Instead, write

$$f(x,n,\omega) = x - v_P n + h(x,\omega), \quad x\in\mathbb{Z}\,.$$

We want to make $f(X_n,n,\omega)$ into a martingale w.r.t. $\mathcal{F}_n := \sigma(X_1,\ldots,X_n)$ and the law P_ω^o. This is automatic if we can ensure that

$$E_\omega^{X_n} f(X_{n+1}, n+1,\omega) = f(X_n,n,\omega), \quad P_\omega^o\text{-a.s.} \tag{2.2.7}$$

Developing this equality and defining $\Delta(x,\omega) = h(x+1,\omega) - h(x,\omega)$, we get that (2.2.7) holds true if a bounded solution to the equation

$$\Delta(x,\omega) = -\left[\frac{\omega_x^+ - \omega_x^- - v_P}{\omega_x^+}\right] + \frac{\omega_x^-}{\omega_x^+}\Delta(x-1,\omega)$$

exists. One may verify that $\Delta(x,\omega) = -1 + v_P\overline{S}(\theta^x\omega)$ is such a solution.

Fixing $h(0,\omega) = 0$, and defining $\overline{M}_0 = 0$ and $\overline{M}_n = f(X_n, n, \omega)$, one concludes that $\overline{M}_n$ is a martingale, and further

$$\begin{aligned}
&E_\omega^o\Big((\overline{M}_{k+1} - \overline{M}_k)^2|\mathcal{F}_k\Big) \\
&\quad = \omega_{X_k}^+ v_P^2(\overline{S}(\theta^{X_k}\omega) - 1)^2 + \omega_{X_k}^- v_P^2(\overline{S}(\theta^{X_k-1}\omega) + 1)^2 + \omega_{X_k}^0 v_P^2 \\
&\quad = v_P^2\Big[\overline{\omega}(k)_0^+(\overline{S}(\overline{\omega}_k) - 1)^2 + \overline{\omega}(k)_0^-(\overline{S}(\theta^{-1}\overline{\omega}_k) + 1)^2 + \overline{\omega}(k)_0^0\Big] .
\end{aligned}$$

Hence,

$$\frac{V_n}{n} = \frac{1}{n}\sum_{k=1}^{n} E_\omega^o\Big((\overline{M}_{k+1} - \overline{M}_k)^2|\mathcal{F}_k\Big) \underset{n\to\infty}{\longrightarrow} \sigma_{P,1}^2\,, \quad \mathbb{P}^o\text{-a.s.},$$

using the machinery developed in Section 2.1. The integrability condition (2.2.2) is enough to apply the Martingale CLT (Lemma 2.2.4), and one concludes that for any $\delta > 0$,

$$P\left(\left|P_\omega^o\left(\frac{\overline{M}_n}{\sigma_{P,1}\sqrt{n}} \geq x\right) - \Phi(-x)\right| > \delta\right) \to_{n\to\infty} 0\,. \qquad (2.2.8)$$

Note that since both $P_\omega^o(\overline{M}_n \geq x\sigma_{P,1}\sqrt{n})$ and $\Phi(x)$ are monotone in x, and that $\Phi(\cdot)$ is continuous, the convergence in (2.2.8) actually is uniform on $\mathbb{R}$. Further, note that

$$h(X_n,\omega) = \sum_{j=1}^{X_n-1}\Delta(j,\omega) = \sum_{j=1}^{nv_P}\Delta(j,\omega) + R_n := Z_n + R_n\,.$$

Note that, for every $\delta > 0$ and some $\delta_n \to 0$,

$$\begin{aligned}
&\mathbb{P}^o\left(\frac{|R_n|}{\sqrt{n}} \geq \delta\right) \leq \mathbb{P}^o\Big(|X_n - nv_P| \geq \delta_n n\Big) \\
&+P\left(\max_{j_-,j_+\in(-n\delta_n,n\delta_n)}\Big|\sum_{i=j_-}^{j_+}\frac{\Delta(i,\omega)}{\sqrt{n}}\Big| \geq \delta\right) := P_{1,n}(\delta_n)+P_{2,n}(\delta,\delta_n) \underset{n\to\infty}{\longrightarrow} 0,
\end{aligned} \qquad (2.2.9)$$

where the convergence of the first term is due (choosing an appropriate $\delta_n \to_{n\to\infty} 0$ slowly enough) to Theorem 2.1.9 and that of the second one due to $E_P\Delta(i,\omega) = 0$ and the stationary invariance principle (Lemma 2.2.5), which can be applied, for any $\delta_n \to_{n\to\infty} 0$, due to (2.2.3).

Another application of Lemma 2.2.5 yields that

$$\lim_{n\to\infty} P(Z_n \geq z\sqrt{nv_P}\sigma_{P,2}) = \Phi(-z)\,. \tag{2.2.10}$$

Writing $X_n - nv_P = \overline{M}_n - Z_n - R_n$, and using that $R_n/\sqrt{n} \to_{n\to\infty} 0$ in $\mathbb{P}^o$-probability, one concludes that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}^o(\frac{X_n - nv_P}{\sqrt{n}} > x) &= \lim_{n\to\infty} E_P(P^o_\omega(\overline{M}_n/\sqrt{n} > x + Z_n/\sqrt{n})) \\ &= \lim_{n\to\infty} E_P(\Phi\left(-\frac{x + Z_n/\sqrt{n}}{\sigma_{P,1}}\right))\,, \end{aligned} \tag{2.2.11}$$

where the second equality is due to the uniform convergence in (2.2.8). Combining (2.2.11) with (2.2.10) yields the claim. □

Remark: The alert reader will have noted that under assumptions (2.1.1) and (2.2.2), and a mild mixing assumption on P which ensures that for any $\delta > 0$ and $\delta_n \to 0$, $P_{2,n}(\delta, \delta_n) \to_{n\to\infty} 0$, c.f. (2.2.9),

$$P^o_\omega\Big(\frac{X_n - v_P n - Z_n}{\sqrt{n}\sigma_{P,1}} > x\Big) \to_{n\to\infty} \Phi(-x)\,,$$

in probability. That is, using a random centering one also has a *quenched* CLT in probability. This can be strengthened to an almost sure quenched CLT (the argument will be included in the forthcoming thesis of J. Peterson).

Exercise 2.2.12 *Check that the integrability conditions (2.2.2) and (2.2.3) allow for the application of Lemmas 2.2.4 and 2.2.5 in the course of the proof of Theorem 2.2.1.*

Exercise 2.2.13 *Check that in the case of P being a product measure, the assumption (2.2.2) in Theorem 2.2.1 can be dropped.*

Bibliographical notes: The presentation here follows the ideas of [46], as developed in [54]. The latter provides an explicit derivation of the CLT in case $P(\omega_0^0 = 0) = 1$, but it seems that in his derivation only the quenched CLT is derived and the random centering then is missing. A different approach to the CLT is presented in [1], using the hitting times $\{\tau_i\}$; It is well suited to yield the quenched CLT, and under strong assumptions on P which ensure that the random quenched centering vanishes P-a.s., also the annealed CLT. Note however that the case of P being a product measure is not covered in the hypotheses of [1]. See [7] for some further discussion and extensions.

There are situations where limit laws which are not of the CLT type can be exhibited. The proof of such results uses hitting time decompositions, and techniques as discussed in Section 2.4. We refer to Section 2.5 and its bibliographical notes for an example of such a situation and additional information.

2.3 Large deviations

Having settled the issue of the LLN, the next logical step (even if not following the historical development) is the evaluation of the probabilities of large deviations. As already noted in the evaluation of the CLT in Section 2.2, there can be serious differences between quenched and annealed probabilities of deviations. In order to address this, we make the following definitions; throughout this section, $\mathcal{X}$ denotes a completely regular topological space.

Definition 2.3.1 *A function $I : \mathcal{X} \to [0, \infty]$ is a rate function if it is lower semicontinuous. It is a good rate function if its level sets are compact.*

Definition 2.3.2 *A sequence of $\mathcal{X}$ valued random variables $\{Z_n\}$ satisfies the quenched Large Deviations Principle (LDP) with speed n and deterministic rate function I if for any Borel set A,*

$$-I(A^o) \le \liminf_{n\to\infty} \frac{1}{n} \log P^o_\omega(Z_n \in A) \le \limsup_{n\to\infty} \frac{1}{n} \log P^o_\omega(Z_n \in A) \le -I(\overline{A}) \quad P\text{-a.s.} \tag{2.3.3}$$

where A^o denotes the interior of A, $\overline{A}$ the closure of A, and for any Borel set F,

$$I(F) = \inf_{x\in F} I(x) . \tag{2.3.4}$$

Definition 2.3.5 *A sequence of $\mathcal{X}$ valued random variables $\{Z_n\}$ satisfies the annealed LDP with speed n and rate function I if, for any Borel set A,*

$$-I(A^o) \le \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}^o(Z_n \in A) \le \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}^o(Z_n \in A) \le -I(\overline{A}) . \tag{2.3.6}$$

Finally, we note the

Definition 2.3.7 *A LDP is called weak if the upper bound in (2.3.3) or (2.3.6), holds only with $\overline{A}$ compact.*

For background on the LDP we refer to [20]. It is well known, c.f. [20, Lemma 4.1.4] that if the LDP holds then the rate function is uniquely defined. The following easy lemma is intuitively clear: annealed deviation probabilities allow for atypical fluctuations of the environment and hence are not smaller than corresponding quenched deviation probabilities:

Lemma 2.3.8 *Let $\{A_n\}$ be a sequence of events, subsets of $\Omega \times \mathbb{Z}^N$. Then,*

$$c := \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}^o(A_n) \ge \limsup_{n\to\infty} \frac{1}{n} \log P^o_\omega(A_n), \quad P-a.s. \tag{2.3.9}$$

Further,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}^o(A_n) \geq \liminf_{n\to\infty} \frac{1}{n} \log P^o_\omega(A_n)\,, \quad P-a.s. \tag{2.3.10}$$

In particular, if a sequence of $\mathcal{X}$ valued random variables $\{Z_n\}$ satisfies annealed and quenched LDP's with rate functions $I_a(\cdot)$, $I_q(\cdot)$, respectively, then,

$$I_a(x) \leq I_q(x)\ , \forall x \in \mathcal{X}.$$

Proof. Assume first $c < 0$. Fix $\delta > 0$ and let $B_n^\delta = \{\omega : P^o_\omega(A_n) \geq \exp((c+\delta)n)\}$. Then, by the definition of c, see (2.3.9), and Markov's bound, for n large enough,

$$P(B_n^\delta) \leq e^{-\delta n/2}\,.$$

Hence, $\omega \in B_n^\delta$ occurs only finitely many times, P-a.s., implying that for P-almost all ω there exists an $n_0(\omega)$ such that for all $n \geq n_0(\omega)$, $P^o_\omega(A_n) < \exp((c+\delta)n)$. Hence,

$$\limsup_{n\to\infty} \frac{1}{n} \log P^o_\omega(A_n) \leq c + \delta\,, \quad P-\text{a.s.}$$

(2.3.9) follows by the arbitrariness of $\delta > 0$. Next, set $\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}^o(A_n) := c_1 \leq c$. Define $\{n_k\}$ such that

$$\lim_{k\to\infty} \frac{1}{n_k} \log \mathbb{P}^o(A_{n_k}) = c_1\,.$$

Apply now the first part of the lemma to conclude that

$$c_1 \geq \limsup_{k\to\infty} \frac{1}{n_k} \log P^o_\omega(A_{n_k}) \geq \liminf_{k\to\infty} \frac{1}{n_k} \log P^o_\omega(A_{n_k}) \geq \liminf_{n\to\infty} \frac{1}{n} \log P^o_\omega(A_n)$$
$$P-\text{a.s.}$$

The case $c = 0$ is the same, except that (2.3.9) is trivial. This completes the proof. $\square$

Quenched LDP's

The LDP in the quenched setting makes use in its proof of the hitting times $\{\tau_i\}$. Introduce, for any $\lambda \in \mathbb{R}$,

$$\varphi(\lambda,\omega) = E^o_\omega(e^{\lambda\tau_1}\,\mathbf{1}_{\{\tau_1<\infty\}})\,, \quad f(\lambda,\omega) = \log\varphi(\lambda,\omega)$$
$$G(\lambda,P,u) = \lambda u - E_P\Big(f(\lambda,\omega)\Big)\,.$$

We need throughout the following modification of Assumption 2.1.1.

Assumption 2.3.11

(B1) *P is stationary and ergodic,*

(B2) *There exists an* $\varepsilon > 0$ *such that* $P(\omega_0^+ \notin (0,\varepsilon))P(\omega_0^- \notin (0,\varepsilon)) = 1$,
(B3) $P(\omega_0^+ + \omega_0^- > 0) = 1$, $P(\omega_0^0 > 0, \omega_0^+\omega_0^- = 0) = 0$, *and* $P(\omega_0^+ = 0)P(\omega_0^- = 0) = 0$.

Note that we allow for the possibility of having one sided transitions (e.g., moves to the right only) of the RWRE. This allows one to deal with the case where "random nodes" are present.

Define

$$\begin{aligned}
\rho_{\min} &:= \inf[\rho : \ P(\rho_0 < \rho) > 0]\,, \\
\rho_{\max} &:= \sup[\rho : \ P(\rho_0 > \rho) > 0]\,, \\
\omega^0_{\max} &:= \sup[\alpha : P(\omega_0^0 > \alpha) > 0]\,.
\end{aligned}$$

With P_N denoting the restriction of P to the first N coordinates $\{\omega_i\}_{i=0}^{N-1}$, we say that P is locally equivalent to the product of its marginals if for any N finite, $P_N \sim \otimes^N P_1$.

Finally, we say that a measure P is *extremal* if it is locally equivalent to the product of its marginals and in addition it satisfies the following condition:

(C5) Either $\rho_{\min} \le 1$ and $\rho_{\max} \ge 1$, or if $\rho_{\min} > 1$ then for all $\delta > 0$, $P(\rho_0 < \rho_{\min} + \delta,\, \omega_0^0 > \omega^0_{\max} - \delta) > 0$, or if $\rho_{\max} < 1$ then for all $\delta > 0$, $P(\rho_0 > \rho_{\max} - \delta,\, \omega_0^0 > \omega^0_{\max} - \delta) > 0$.

Note that **(C5)**, which is used only in the proof of the annealed LDP, can be read off the support of P_0 and represents an assumption concerning the inclusion of "extremal environments" in the support of P. The introduction of this assumption is not essential and can be avoided at the cost of a slightly more cumbersome proof, see the remarks at the end of this chapter.

For a fixed $\varepsilon > 0$, we denote by $M_1^{e,\varepsilon}$ the set of probability measures satisfying Assumption 2.3.11 with parameter ε in **(B2)**. Define also the maps $F : \Omega \mapsto \Omega$ by $(F\omega)_k^+ = \omega_k^-$, $(F\omega)_k^- = \omega_k^+$, and $(\mathrm{Inv}\,\omega)_k = (F\omega)_{-k}$. We now have:

Theorem 2.3.12 *Assume Assumption 2.3.11.*
a) The random variables $\{T_n/n\}$ *satisfy the weak quenched LDP with speed* n *and convex rate function*

$$I_P^{\tau,q}(u) = \sup_{\lambda\in\mathbb{R}} G(\lambda, P, u)\,.$$

b) Assume further that $E_P \log\rho_0 \le 0$. *Then, the random variables* X_n/n *satisfy the quenched LDP with speed* n *and good convex rate function*

$$I_P^q(v) = \begin{cases} v\, I_P^{\tau,q}\left(\frac{1}{v}\right) & ,\ 0 < v \le 1 \\ |v| \left(I_P^{\tau,q}\left(\frac{1}{|v|}\right) - E_P(\log\rho_0)\right) & ,\ -1 \le v < 0 \end{cases}$$

and

$$I_P^q(0) = \lim_{v \downarrow 0} v I_P^{\tau,q}\left(\frac{1}{v}\right).$$

c) Finally, if $E_P \log \rho_0 > 0$, define $P^{\mathrm{Inv}} := P \circ \mathrm{Inv}^{-1}$. Then, $E_{P^{\mathrm{Inv}}}(\log \rho_0) < 0$, and the LDP for (X_n/n) holds with good convex rate function

$$I_P^q(v) = I_{P^{\mathrm{Inv}}}^q(-v).$$

Proof. It should come as no surprise that we begin with the LDP for T_n/n. We divide the proof of Theorem 2.3.12 into the following steps:

Step I: $E_P \log \rho_0 \leq 0$, quenched LDP for T_n/n with convex rate function $I_P^{\tau,q}(\cdot)$:

(I.1) upper bound, lower tail: $P_\omega^o(T_n \leq nu)$
(I.2) upper bound, upper tail: $P_\omega^o(T_n \geq nu)$
(I.3) lower bound

Step II: $E_P \log \rho_0 > 0$, quenched LDP for T_n/n with convex rate function $I_P^{\tau,q}(\cdot) + E_P(\log \rho_0)$,
Step III: quenched LDP for X_n/n with convex rate function $I_P^q(\cdot)$.

As a preliminary step we have the following technical lemma, whose proof is deferred:

Lemma 2.3.13 *Assume $P \in M_1^{e,\varepsilon}$ and $E_P(\log \rho_0) \leq 0$; Then*
(a) The convex function $I_P^{\tau,q}(\cdot) : \mathbb{R} \mapsto [0,\infty]$ is nonincreasing on $[1, E_P(\overline{S})]$, nondecreasing on $[E_P(\overline{S}), \infty)$. Further, if $E_P(\overline{S}) < \infty$ then $I_P^{\tau,q}(E_P(\overline{S})) = 0$.
(b) For any $1 < u < E_P(\overline{S})$, there exists a unique $\lambda_0 = \lambda_0(u,P)$ such that $\lambda_0 < 0$ and

$$u = \int \frac{d}{d\lambda} \log \varphi(\lambda,\omega)\Big|_{\lambda=\lambda_0} P(d\omega). \tag{2.3.14}$$

Further,

$$\inf_{P \in M_1^{e,\varepsilon}} \lambda_0(u,P) > -\infty. \tag{2.3.15}$$

(c) There is a deterministic $\lambda_{\mathrm{crit}} := \lambda_{\mathrm{crit}}(P) \in [0,\infty]$ such that

$$\varphi(\lambda,\omega) \begin{cases} < \infty, & \lambda < \lambda_{\mathrm{crit}}, \quad P\text{ - a.s.} \\ = \infty, & \lambda > \lambda_{\mathrm{crit}}, \quad P\text{ - a.s.} \end{cases}$$

with $\lambda_{\mathrm{crit}} < \infty$ if $P(\omega_0^+ \omega_0^- = 0) = 0$. In the latter case, $E_\omega^o(e^{\lambda_{\mathrm{crit}} \tau_1}) < e^{-\lambda_{\mathrm{crit}}}/\varepsilon$, P-a.s., and with

$$u_{\mathrm{crit}} = \begin{cases} \infty, & E_P\left[\frac{E_\omega^o(\tau_1 e^{\lambda_{\mathrm{crit}} \tau_1})}{E_\omega^o(e^{\lambda_{\mathrm{crit}} \tau_1})}\right] = \infty \\ E_P\left(\frac{d}{d\lambda} \log \varphi(\lambda,\omega)\Big|_{\lambda=\lambda_{\mathrm{crit}}}\right), & E_P\left[\frac{E_\omega^o(\tau_1 e^{\lambda_{\mathrm{crit}} \tau_1})}{E_\omega^o(e^{\lambda_{\mathrm{crit}} \tau_1})}\right] < \infty, \end{cases}$$

and $E_P(\overline{S}) \le u < u_{\mathrm{crit}}$, there exists a unique $\lambda_0 := \lambda_0(u,P)$ such that $\lambda_0 \ge 0$ and (2.3.14) holds.
(d) Assume P is extremal and further assume that $\rho_{\max} < 1$. Then,

$$\lambda_{\mathrm{crit}} = \overline{\lambda} := -\log\left(\omega^0_{\max} + \frac{2(1-\omega^0_{\max})\sqrt{\rho_{\max}}}{1+\rho_{\max}}\right) .$$

Further, define

$$\hat{\omega}^+ = (1-\omega^0_{\max})/(1+\rho_{\max}) \, , \; \hat{\omega}^- = \rho_{\max}\hat{\omega}^+ \, , \; \hat{\omega}^0 = \omega^0_{\max} \, ,$$

and let $\tilde{\omega}^{\min}$ denote the deterministic environment with $\tilde{\omega}^{\min}_k = \hat{\omega}$. Then, for any $\lambda \le \lambda_{\mathrm{crit}}$, and any ω such that $\omega^0_i \le \omega^0_{\max}$, $\rho_i \le \rho_{\max}$,

$$\varphi(\lambda,\omega) \le \varphi(\lambda,\tilde{\omega}^{\min}) < \infty \, . \tag{2.3.16}$$

Step I.1: Obviously, it is enough to deal with $u \le E_P(\overline{S})$. Indeed, for $u > E_P(\overline{S})$ we have by (2.1.30) that $\mathbb{P}^o(T_n \le nu) \underset{n\to\infty}{\longrightarrow} 1$, and there is nothing to prove. Next, by Chebycheff's inequality, for all $\lambda \le 0$,

$$\begin{aligned} P^o_\omega\left(\frac{T_n}{n} \le u\right) &\le e^{-\lambda nu} \, E^o_\omega\left(e^{\lambda \sum_{i=1}^n \tau_i}\right) = e^{-\lambda nu} \prod_{i=1}^n E^o_{\theta^i\omega}\left(e^{\lambda\tau_1}\right) \\ &= e^{-\lambda nu} \prod_{i=1}^n \varphi(\lambda,\theta^i\omega) \, , \quad P\text{ - a.s.} \end{aligned} \tag{2.3.17}$$

where the first equality is due to the Markov property and the second due to $\tau_i < \infty$, $\mathbb{P}^o$ - a.s. (the null set in (2.3.17) does not depend on λ). An application of the ergodic theorem yields that

$$\frac{1}{n}\,\log\prod_1^n \varphi(\lambda,\theta^i\omega) \longrightarrow E_P\Big(f(\lambda,\omega)\Big) \, , \quad P\text{ - a.s.}$$

first for all λ rational and then for all λ by monotonicity. Thus,

$$\limsup_{n\to\infty}\frac{1}{n}\,\log P^o_\omega\left(\frac{T_n}{n} \le u\right) \le -\sup_{\lambda\le 0} G(\lambda,P,u) \, , \quad P\text{ - a.s.}$$

Note that if $E_P(\overline{S}) = \infty$ then clearly $E_P[\log E^o_\omega(e^{\lambda\tau_1})] = \infty$ by Jensen's inequality for $\lambda > 0$, and then $\sup_{\lambda\le 0} G(\lambda,P,u) = I^{\tau,q}_P(u)$. If $E_P(\overline{S}) < \infty$ then, because $u < E_P(\overline{S})$, it holds that for any $\lambda > 0$,

$$\lambda u - E_P f(\lambda,\omega) \le \lambda E_P(\overline{S}) - E_P f(\lambda,\omega) \le 0 \, ,$$

where Jensen's inequality was used in the last step. Since $G(0,P,u) = 0$, it follows that also in this case $\sup_{\lambda\le 0} G(\lambda,P,u) = I^{\tau,q}_P(u)$. Hence,

$$\limsup_{n\to\infty} \frac{1}{n} \log P^o_\omega\Big(\frac{T_n}{n} \le u\Big) \le -I_P^{\tau,q}(u) = -\inf_{w\le u} I_P^{\tau,q}(w)\,,$$

where the last inequality is due to part a) of Lemma 2.3.13, completing Step I.1.

Step I.2: is similar, using this time $\lambda \ge 0$.

Step I.3: The proof of the lower bound is based on a change of measure argument. We present it here in full detail for $u < u_{\text{crit}}$. Fix $\lambda_0 = \lambda_0(u, P)$ as in Lemma 2.3.13, and set a probability measure $Q^o_{\omega,n}$ such that

$$\frac{dQ^o_{\omega,n}}{dP^o_\omega} = \frac{1}{Z_{n,\omega}} \exp\Big(\lambda_0 T_n\Big)\,, \quad Z_{n,\omega} = \mathbb{E}^o_\omega \,\Big(\exp\Big(\lambda_0 T_n\Big)\Big)\;,$$

and let $\overline{Q}^o_{\omega,n}$ denote the induced law on $\{\tau_1\,, \ldots\,, \tau_n\}$. Due to the Markov property, $\overline{Q}^o_{\omega,n}$ is a product measure, whose first n marginals do not depend on n, hence we will write $\overline{Q}^o_\omega$ instead of $\overline{Q}^o_{\omega,n}$ when integrating over events depending only on $\{\tau_i\}_{i<n}$. But, for any $\delta > 0$,

$$\begin{aligned}
&P^o_\omega\Big(\frac{T_n}{n} \in (u-\delta,\, u+\delta)\Big) \\
&\ge \exp\Big(-nu\lambda_0 - n\delta|\lambda_0| + \sum_{i=1}^n \log\varphi\Big(\lambda_0,\, \theta^i\omega\Big)\Big)\overline{Q}^o_\omega\,\Big(\Big|\frac{T_n}{n} - u\Big| < \delta\Big)\,.
\end{aligned} \tag{2.3.18}$$

By the ergodic theorem and the fact that $u < u_{\text{crit}}$, it holds that

$$E_{\overline{Q}^o_\omega}(T_n/n) \to_{n\to\infty} E_P\Big(E_{\overline{Q}^o_\omega}(\tau_1)\Big) = u\,, P\text{ - a.s.} \tag{2.3.19}$$

where we used again (2.3.14). On the other hand, again because $\lambda_0 < \lambda_{\text{crit}}$ it holds that there exists an $\eta > 0$ such that

$$E_P\Big(E_{\overline{Q}^o_\omega}\,\Big(e^{\eta\tau_1}\Big)\Big) < \infty\,,$$

implying that

$$\overline{Q}^o_\omega\Big(\Big|\frac{T_n}{n} - u\Big| \ge \delta\Big) \underset{n\to\infty}{\longrightarrow} 0\,, P\text{ - a.s.} \tag{2.3.20}$$

Combining (2.3.20) with (2.3.18), we get

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \log P^o_\omega\Big(\frac{T_n}{n} \in (u-\delta, u+\delta)\Big) \\
&\quad\ge -u\lambda_0 - \delta|\lambda_0| + \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \log\varphi(\lambda_0, \theta^i\omega) \\
&\quad= -u\lambda_0 - \delta|\lambda_0| + E_P(\log\varphi(\lambda_0, \omega)) \\
&\quad= -G(\lambda_0, P, u) - \delta|\lambda_0| = -I_P^{\tau,q}(u) - \delta|\lambda_0|\,,\ P\text{ - a.s.}
\end{aligned}$$

where the first equality is due to the ergodic theorem and the last one to Lemma 2.3.13. This completes Step I.3 when $u < u_{\rm crit}$, since $\delta > 0$ is arbitrary. For $u > u_{\rm crit}$, the proof is similar, except that one needs to truncate the variables $\{\tau_i\}$, we refer to [12, Theorem 4] for details. Step I is complete, except for the:

Proof of Lemma 2.3.13 We consider in what follows only the case $P(\omega_0^+\omega_0^- = 0) = 0$, the modifications in the case where random nodes are allowed are left to the reader.
a) The convexity of $I_P^{\tau,q}(\cdot)$ is immediate from its definition as a supremum of affine functions.

As in the course of the proof of Step I, recall that

$$\sup_{\lambda\in\mathbb{R}} G(\lambda,u,P) = \begin{cases} \sup_{\lambda\leq 0} G(\lambda,u,P)\,, & u < E_P(\overline{S}) \\ \sup_{\lambda\geq 0} G(\lambda,u,P)\,, & u > E_P(\overline{S}) \\ 0\,, & u = E_P(\overline{S})\,. \end{cases}$$

The stated monotonicity properties are then immediate.

b)+c) Recall the path decomposition (2.1.13). Exponentiating and taking expectations using $\tau_1 < \infty$, $\mathbb{P}^o$ - a.s., we have that if $\varphi(\lambda,\omega) < \infty$ then

$$\varphi(\lambda,\omega) = \omega_0^+\, e^\lambda + \omega_0^0\, e^\lambda\, \varphi(\lambda,\omega) + \omega_0^-\, e^\lambda \varphi(\lambda,\omega)\, \varphi(\lambda,\theta^{-1}\omega)\,. \qquad (2.3.21)$$

Thus $\varphi(\lambda,\omega) < \infty$ implies $\varphi(\lambda,\theta^{-1}\omega) < \infty$, yielding that $\mathbf{1}_{\varphi(\lambda,\omega)<\infty}$ is constant P - a.s., and hence for all λ rational, $P(\varphi(\lambda,\omega)<\infty) \in \{0,1\}$. This, and the monotonicity of $\varphi(\lambda,\omega)$ in λ, immediately yields the existence of a deterministic $\lambda_{\rm crit}$. (We note in passing that (2.3.21) gives, by iterating, a representation of $\varphi(\lambda,\omega)$ as a continued fraction, but we do not need this now.) We also conclude from (2.3.21) that for $\lambda < \lambda_{\rm crit}$ it holds that $\varphi(\lambda,\omega) \leq e^{-\lambda}/\varepsilon$, P-a.s., which implies by monotone convergence that $\varphi(\lambda_{\rm crit},\omega) < \infty$, P-a.s.

Next, for $\lambda < 0$ we have that

$$g(\lambda) := \int \frac{E^o_\omega(\tau_1\, e^{\lambda\tau_1})}{E^o_\omega\,(e^{\lambda\tau_1})}\, P(d\omega) = \int \frac{d}{d\lambda}\, \log\varphi(\lambda,\omega) P(d\omega)\,.$$

Further, $g(0) = E_{\mathbb{P}^o}(\tau_1)$, whereas $g(\cdot) \geq 1$ is strictly monotone increasing, satisfying $g(\lambda) \underset{\lambda\to-\infty}{\longrightarrow} 1$. This implies (2.3.14). Finally, to see (2.3.15), note that

$$\begin{aligned} 1 \leq \frac{E^o_\omega(\tau_1\, e^{\lambda\tau_1})}{E^o_\omega(e^{\lambda\tau_1})} &\leq \frac{\omega_0^+ e^\lambda + E_\omega(\tau_1\, e^{\lambda\tau_1}\, \mathbf{1}_{\tau_1\geq 2})}{\omega_0^+ e^\lambda} \\ &\leq 1 + \frac{c\, e^{3\lambda/2}}{\omega_0^+ e^\lambda} \leq 1 + \frac{c}{\varepsilon}\, e^{\lambda/2} \underset{\lambda\to-\infty}{\longrightarrow} 1\,, \end{aligned}$$

where c is a constant independent of ω or λ. Hence, $g(\lambda) \underset{\lambda\to-\infty}{\longrightarrow} 1$ uniformly in $M_1^{e,\varepsilon}$.

d) Assume that P is extremal. The first inequality in (2.3.16) follows from a simple coupling argument: let $\bar{\varphi}(\lambda) := E^o_{\hat{\omega}^{\min}}[e^{\lambda\tau_1}]$. By the recursions (2.3.21), it holds that if $\bar{\varphi}(\lambda) < \infty$ then as long as $\lambda \leq \overline{\lambda}$ it holds that

$$\bar{\varphi}(\lambda) = \frac{(1-\omega^0_{\max}e^\lambda) - \sqrt{(1-\omega^0_{\max}e^\lambda)^2 - 4\hat{\omega}^+\hat{\omega}^- e^{2\lambda}}}{2\hat{\omega}^- e^\lambda} .$$

Thus, we have to show that if $\lambda > \lambda_{\mathrm{crit}}$ then $E^o_\omega(e^{\lambda\tau_1}) = \infty$, P-a.s. Since $E^o_{\hat{\omega}^{\min}}(e^{\lambda\tau_1}) = \infty$, we may find an M large enough such that $E^o_{\hat{\omega}^{\min}}(e^{\lambda\tau_1}\mathbf{1}_{\tau_1<M})$ $> 1/\epsilon+1$. Since the last expression is local, i.e. depends only on $\{\omega_i\}_{i=0}^{-M+1}$, it follows (from the assumption of local equivalence to the product of marginals) that with P positive probability, $E^o_\omega(e^{\lambda\tau_1}) > 1/\epsilon$, and hence by part (c) actually $E^0_\omega(e^{\lambda\tau_1}) = \infty$ with P positive probability, and hence with P probability 1. $\square$

Remark: Before proceeding, we note that a direct consequence of Lemma 2.3.13 is that if $E_P(\log\rho_0) \leq 0$, then for $u < E_P(\overline{S})$,

$$I^{\tau,q}_P(u) = \lambda_0 u - E_P(f(\lambda_0,\omega)) = \sup_{\lambda\in\mathbb{R}} G(\lambda,P,u) > G(0,P,u) = 0$$

since the function $G(\cdot,P,u)$ is strictly concave.

Step II: Recall the transformation $\mathrm{Inv} : \Omega \mapsto \Omega$ and the law $P^{\mathrm{Inv}} = P \circ \mathrm{Inv}^{-1}$. Proving the LDP for T_n/n when $E_P(\log\rho_0) > 0$ is the same, by space reversal, as proving the quenched LDP for T_{-n}/n under the law P^{Inv} on the environment. Note that in this case, $E_{P^{\mathrm{Inv}}}(\log\rho_0) < 0$, and further, $P \in M_1^{e,\varepsilon}$ implies that $P^{\mathrm{Inv}} \in M_1^{e,\varepsilon}$. Thus, Step II will be completed if we can prove a quenched LDP for T_{-n}/n for $P \in M_1^{e,\varepsilon}$ satisfying $E_P \log\rho_0 < 0$. We turn to this task now.

Note that if $P(\omega_0^- = 0) > 0$ then $P^o_\omega(T_{-n} < \infty) = 0$ for some $n = n(\omega)$ large enough, and the LDP for T_{-n}/n is trivial. We thus assume throughout that $\omega_0^- \geq \varepsilon$, P-a.s. As a first step in the derivation of the LDP, we compute logarithmic moment generating functions. Define, for any $\lambda \in \mathbb{R}$,

$$\overline{\varphi}(\lambda,\omega) = E^o_\omega(e^{\lambda\tau_{-1}}\,\mathbf{1}_{\{\tau_{-1}<\infty\}})\,, \quad \overline{f}(\lambda,\omega) = \log\overline{\varphi}(\lambda,\omega)\,.$$

Lemma 2.3.22 *Assume $P \in M_1^{e,\varepsilon}$ and further assume that $\min(\omega_0^+,\omega_0^-) >$ ε, P-a.s. Then,*

$$E_P(\overline{f}(\lambda,\omega)) = E_P(f(\lambda,\omega)) + E_P\log\rho_0\,. \tag{2.3.23}$$

Proof of Lemma 2.3.22: Define the map $I_n : \Omega \mapsto \Omega$ by

$$(I_n\omega)_k = \begin{cases} \omega_k\,, & k \notin [0,n] \\ (F\omega)_{n-k}\,, & k \in [0,n]\,. \end{cases}$$

Introduce

$$\varphi_n(\lambda,\omega) = E^o_\omega(e^{\lambda\tau_1}\,;\,\tau_1 < T_{-(n+1)})\,,\ \overline{\varphi}_n(\lambda,\omega) = E^o_\omega(e^{\lambda\tau_{-1}}\,;\,\tau_{-1} < T_{(n+1)})\,.$$

We will show below that

$$G_n(\lambda,\omega) := \varphi_n(\lambda,\theta^n\omega)\overline{\varphi}_{n-1}(\lambda,\omega) = \varphi_{n-1}(\lambda,\theta^n\omega)\overline{\varphi}_n(\lambda,\omega) := F_n(\lambda,\omega)\,. \tag{2.3.24}$$

Because $\min(\omega_0^+,\omega_0^-) > \varepsilon$, P-a.s., the function $\log\varphi_n(\lambda,\omega)$ and $\log\overline{\varphi}_n(\lambda,\omega)$ are P-integrable for each n. Taking logarithms in (2.3.24), we find that $E_P(\log\varphi_n(\lambda,\omega)) - E_P(\log\overline{\varphi}_n(\lambda,\omega))$ does not depend on n. On the other hand, both terms are monotone in n, hence by monotone convergence either both sides of (2.3.24) are $+\infty$ or both are finite, in which case

$$E_P(\log\varphi(\lambda,\omega)) - E_P(\log\overline{\varphi}(\lambda,\omega)) = E_P\left(\log\left(\frac{\varphi_0(\lambda,\omega)}{\overline{\varphi}_0(\lambda,\omega)}\right)\right) = -E_P(\log\rho_0)\,,$$

yielding (2.3.23).

We thus turn to the proof of (2.3.24). It is straight forward to check, by space inversion, that $F_n(\lambda, I_n\omega) = G_n(\lambda,\omega)$. Thus, the proof of (2.3.24) will be complete once we show that $F_n(\lambda, I_n\omega) = F_n(\lambda,\omega)$. Toward this end, note that by the Markov property,

$$\begin{aligned}\overline{\varphi}_n(\lambda,\omega) &= E^o_\omega(e^{\lambda T_{-1}}\,;\,T_{-1} < T_{n+1})\\ &= E^o_\omega(e^{\lambda T_{-1}}\,;\,T_{-1} < T_n)\\ &\quad + E^o_\omega(e^{\lambda T_n}\,;\,T_n < T_{-1})E^n_\omega(e^{\lambda T_{-1}}\,;\,T_{-1} < T_{n+1})\,.\end{aligned}$$

Hence, defining

$$\begin{aligned}B_n(\lambda,\omega) &:= E^o_\omega(e^{\lambda T_{-1}}\,;\,T_{-1} < T_n)\,,\\ C_n(\lambda,\omega) &:= E^o_\omega(e^{\lambda T_n}\,;\,T_n < T_{-1})\,,\end{aligned}$$

one has, using again space reversal and the Markov property in the second equality,

$$\begin{aligned}F_n(\lambda,\omega) &= E^n_\omega(e^{\lambda T_{n+1}}\,;\,T_{n+1} < T_0)E^o_\omega(e^{\lambda T_{-1}}\,;\,T_{-1} < T_n)\\ &+ E^n_\omega(e^{\lambda T_{n+1}}\,;\,T_{n+1} < T_0)E^n_\omega(e^{\lambda T_{-1}}\,;\,T_{-1} < T_{n+1})E^o_\omega(e^{\lambda T_n}\,;\,T_n < T_{-1})\\ &= B_n(\lambda,\omega)B_n(\lambda,I_n\omega)\\ &+ E^n_\omega(e^{\lambda T_{n+1}}\,;\,T_{n+1} < T_0)E^n_\omega(e^{\lambda T_0}\,;\,T_0 < T_{n+1})E^o_\omega(e^{\lambda T_{-1}}\,;\,T_{-1} < T_{n+1})\\ &\quad E^o_\omega(e^{\lambda T_n}\,;\,T_n < T_{-1})\\ &= B_n(\lambda,\omega)B_n(\lambda,I_n\omega) + C_n(\lambda,\omega)C_n(\lambda,I_n\omega)F_n(\lambda,\omega)\,,\end{aligned}$$

implying the invariance of $F_n(\lambda,\omega)$ under the action of I_n on Ω, except possibly at λ where $C_n(\lambda,\omega)C_n(\lambda,I_n\omega) = 1$. The latter λ is then handled by continuity. This completes the proof of Lemma 2.3.22 $\square$

Step II now is completed by following the same route as in the proof of Step I, using Lemma 2.3.22 to transfer the analytic results of Lemma 2.3.13 to this setup. The details, which are straightforward and are given in [12], are omitted here. $\square$

Remarks: 1. Note that the conclusion of Lemma 2.3.22 extends immediately, by the ergodic decomposition, to stationary measures $P \in M_1^{s,\varepsilon}$.
2. Lemma 2.3.22 is the key to the large deviations principle, and deserves some discussion. First, by taking $\lambda \uparrow 0$, one sees that if $E_P(\log \rho_0) \le 0$ then $E_P[\log P_\omega^o(\tau_{-1} < \infty)] = E_P(\log \rho_0)$. Next, let $\bar\tau_{-1}, \bar\tau_{-2}, \bar\tau_{-3}, ..., \bar\tau_{-N}$ have the distribution of $\tau_{-1},\ \tau_{-2},\ \tau_{-3}, ...\tau_{-N}$ under P_ω^o conditioned on $T_{-N} < \infty$. In fact the law of $\{\bar\tau_{-i}\}_{i=1}^N$ does not depend on N. This can be seen by a discrete h-transform: the distributions of $X_0^{T_{-N}} := (X_0, \dots, X_{T_{-N}})$ under P_ω^o, conditioned on $T_{-N} < \infty$, $N = 1, 2, ...$ form a consistent family whose extension is again a Markov chain. To see this, let $\tilde P_{\omega,N}^o := P_\omega^o(\cdot | T_{-N} < \infty)$, restricted to $X_0^{T_{-N}}$. Denoting $x_1^n := (x_1, ..., x_n)$, compute (with $x_i > -N$),

$$
\begin{aligned}
\tilde P_{\omega,N}^o&(X_{n+1} = x_n + 1 | X_1^n = x_1^n)\\
&= \frac{\tilde P_{\omega,N}^o(X_{n+1} = x_n + 1, X_1^n = x_1^n)}{\tilde P_{\omega,N}^o(X_1^n = x_1^n)}\\
&= \frac{P_\omega^o(X_{n+1} = x_n + 1, X_1^n = x_1^n, T_{-N} < \infty)}{P_\omega^o(X_1^n = x_1^n, T_{-N} < \infty)}\\
&= \frac{P_\omega^o(X_{n+1} = x_n + 1, X_1^n = x_1^n) P_{\theta^{x_n+1}\omega}^o(T_{-N-x_n-1} < \infty)}{P_\omega^o(X_1^n = x_1^n) P_{\theta^{x_n}\omega}^o(T_{-N-x_n} < \infty)}\\
&= P_\omega^o(X_{n+1} = x_n + 1 | X_1^n = x_1^n) P_{\theta^{x_n+1}\omega}(T_{-1} < \infty)\\
&\qquad = \omega_{x_n}^+ P_{\theta^{x_n+1}\omega}^o(T_{-1} < \infty)\,,
\end{aligned}
$$

where we used the Markov property in the third and in the fourth equality. The last term depends neither on N nor on x_1^{n-1}. Therefore, the extension of $(\tilde P_{\omega,N})_{N\ge1}$ is the distribution of the Markov chain with transition probabilities $\tilde\omega_i^+ = \omega_i^+ P_{\theta^{i+1}\omega}(T_{-1} < \infty), \tilde\omega_i^0 = \omega_i^0,\ i \in \mathbb{Z}$. In particular, $\bar\tau_{-1}, \bar\tau_{-2}, \bar\tau_{-3}, ...$ are independent under P_ω^o and, with a slight abuse of notations, form a stationary sequence under $\mathbb{P}^o$. Note now that if we set

$$
\overline{\phi}(\lambda, \omega) := E_\omega^o(e^{\lambda \bar\tau_{-1}}) = \frac{\overline{\varphi}(\lambda, \omega)}{P_\omega^o(T_{-1} < \infty)} \tag{2.3.25}
$$

then Lemma 2.3.22 tells us that $E_P\overline{\phi}(\lambda, \omega) = E_P\varphi(\lambda, \omega)$. In particular, $\mathbb{E}_P^o(\bar\tau_1) = \mathbb{E}_P^o(\tau_1) = E_P(\overline{S})$ if $E_P(\log \rho_0) \le 0$ and, repeating the arguments leading to the LDP of T_n/n, we find that the sequence of random variables T_{-n}/n, *conditioned on* $T_{-n} < \infty$, satisfy a quenched LDP under P_ω^o with the same rate function as T_n/n!

Step III: By space reversal, it is enough to prove the result for $E_P(\log \rho_0) \le 0$. Further, as in Step II, it will be enough to consider the case where $\min(\omega_0^+, \omega_0^-) \ge \varepsilon$, P-a.s. Since $I_P^{\tau,q}(\cdot)$ is convex, and since $x \mapsto xf(1/x)$ is convex if $f(\cdot)$ is convex, it follows that $I_P^q(\cdot)$ is convex on $(0, 1]$ and on $[-1, 0)$ separately. If $\lambda_{\mathrm{crit}}(P) = 0$ then $I_P^q(0) = 0$ and the convexity on $[-1, 1]$ follows. In the general case, note that I_P^q is continuous at 0, and

$(I_P^q)'(0^-) = -(I_P^q)'(0^+) + E_P(\log \rho_0)$. Note that for $\lambda \le \lambda_{\text{crit}}$, by the Markov property,

$$E_\omega^o(e^{\lambda T_M}\mathbf{1}_{\tau_{-1}<\tau_M}) = E_\omega^o(e^{\lambda \tau_{-1}}\mathbf{1}_{\tau_{-1}<\tau_M})\varphi(\lambda, \theta^{-1}\omega)E_\omega^o(e^{\lambda T_M})\,,$$

and hence,

$$1 \ge E_\omega^o(e^{\lambda \tau_{-1}}\mathbf{1}_{\tau_{-1}<\tau_M})\varphi(\lambda, \theta^{-1}\omega)\,,$$

leading (by taking $M \to \infty$) to the conclusion that

$$\varphi(\lambda, \omega)\bar{\varphi}(\lambda, \theta^{-1}\omega) \le 1\,.$$

Taking logarithms and P-expectations, we conclude that

$$E_P f(\lambda, \omega) + E_P \bar{f}(\lambda, \omega) \le 0.$$

Combined with Lemma 2.3.22, we deduce that for all $\lambda \le \lambda_{\text{crit}}$, $2E_P(f(\lambda, \omega)) \le -E_P(\log \rho_0)$. Hence,

$$(I_P^q)'(0^+) = -E_P\left(\log E_\omega(e^{\lambda_{\text{crit}}\tau_1})\right) = -E_P\left(f(\lambda_{\text{crit}}, \omega)\right) \ge \frac{1}{2}E_P(\log \rho_0)\,,$$

implying that $(I_P^q)'(0^-) \le (I_P^q)'(0^+)$, and hence that $I_P^q(\cdot)$ is convex on $[-1, 1]$.

Using the monotonicity of $I_P^{\tau,q}(\cdot)$, c.f. the remark following the proof of Step I, it follows easily that $I_P^q(\cdot)$ is non increasing on $[-1, v_P]$ and non decreasing on $[v_P, 1]$.

Let $v > v_P$. We have

$$P_\omega^o\left(\frac{X_n}{n} \ge v\right) \le P_\omega^o\Big(T_{\lfloor nv \rfloor} \le n\Big) = P_\omega^o\left(\frac{T_{\lfloor nv \rfloor}}{\lfloor nv \rfloor} \le \frac{n}{\lfloor nv \rfloor}\right)\,.$$

Step I and the monotonicity of $I_P^{\tau,q}(\cdot)$ now imply

$$\limsup_{n\to\infty} \frac{1}{n}\,\log P_\omega^o\left(\frac{X_n}{n} \ge v\right) \le -vI_P^{\tau,q}\left(\frac{1}{v}\right)\,,$$

which yields the required upper bound by the monotonicity of $I_P^q(\cdot)$. The same argument applies to yield the desired upper bound on $P_\omega^o\left(\frac{X_n}{n} \le v\right)$ for $v < 0$, by considering the hitting times $T_{\lceil nv \rceil}$.

In the same way, for any $0 < \eta < \delta/2$,

$$P_\omega^o\left((v+\delta) \ge \frac{X_n}{n} \ge (v-\delta)\right) \ge P_\omega^o\Big((1-\eta)n \le T_{\lceil nv \rceil} \le n\Big),$$

hence, from Step I it follows that for $v \ge 0$,

$$\liminf_{n\to\infty} \frac{1}{n}\,\log P_\omega^o\left(\frac{X_n}{n} \in (v-\delta, v+\delta)\right) \ge -vI_P^{\tau,q}\left(\frac{1-\eta}{v}\right)\,,\; P-\text{a.s.}\,,$$

and the lower bound is obtained by letting $\eta \to 0$. The same argument also yields the lower bounds for $v < 0$, using this time the function $I_P^{-\tau,q}(\cdot)$.

Next, we turn to evaluate an upper bound on $P_\omega^o(X_n/n \le v), 0 \le v < v_P$, with $v_P \ge 0$. Starting with $v = 0$, let $\eta, \delta > 0$, with $\delta < v_P$. Then,

$$\begin{aligned} P_\omega^o\left(X_n \le 0\right) &\le P_\omega^o\Big(T_{[n\delta]} \ge n\Big) + P_\omega^o\Big(T_{[n\delta]} < n, \frac{X_n}{n} \le 0\Big) \qquad (2.3.26)\\ &\le P_\omega^o\Big(T_{[n\delta]} \ge n\Big) + \sum_{1/\eta \le k,l;(k+l)\eta \le 1/\delta} P_\omega^o\Big(\frac{T_{[n\delta]}}{n\delta} \in [k\eta,(k+1)\eta)\Big) \times \\ &\quad P^o_{\theta^{[n\delta]}\omega}\Big(\frac{T_{-[n\delta]}}{n\delta} \in [l\eta,(l+1)\eta)\Big) \sum_{-2n\delta\eta \le m-n(1-(k+l)\delta\eta)\le 0} P_\omega^o\left(X_m \le 0\right)\,, \end{aligned}$$

by the strong Markov property. Define the random variable

$$a = \limsup_{n\to\infty} \frac{1}{n} \sup_{m:-2n\delta\eta \le m-n \le 0} \log P_\omega^o\left(X_m \le 0\right)\,,$$

and note, using the inequality

$$P_\omega^o(X_n \le 0) \ge P_\omega^o(X_m \le 0) \inf_{i \le 0} P^o_{\theta^i\omega}[X_{n-m} = -(n-m)]$$

with a worst-environment estimate, that

$$a - C\delta\eta \le \limsup_{n\to\infty} \frac{1}{n} \log P_\omega^o\left[X_n \le 0\right] \le a \qquad (2.3.27)$$

with $C = -2\log\varepsilon > 0$. The first two probabilities in the right-hand side of (2.3.26) will be estimated using Step I. By convexity, the rate functions $I_P^{\tau,q}$ and $I_P^{-\tau,q} := I_P^{\tau,q} - E_P(\log\rho_0)$ are continuous, so that the oscillation

$$w(\delta;\eta) = \\ \max\{|I_P^{\tau,q}(u) - I_P^{\tau,q}(u')| + |I_P^{-\tau,q}(u) - I_P^{-\tau,q}(u')|; u,u' \in [1,1/\delta], |u-u'| \le \eta\}$$

tends to 0 with η, for all fixed δ. From the proof of Step II, it is not difficult to see that the third term in the right-hand side of (2.3.26) can be estimated similarly (it does not cause problems to consider $P^o_{\theta^{[n\delta]}\omega}$ instead of P^o_ω):

$$\limsup_{n\to\infty} \frac{1}{n} \log P^o_{\theta^{[n\delta]}\omega}\Big(\frac{T_{-[n\delta]}}{n\delta} \in [l\eta,(l+1)\eta)\Big) \le -\delta\left(I_P^{-\tau,q}(l\eta) - w(\delta;\eta)\right) P\text{–a.s.}$$

Finally, we get from (2.3.27) and (2.3.26)

$$\begin{aligned} a \le C\delta\eta + \max\{-I_P^q(\delta), &\max_{1/\eta \le k,l;(k+l)\eta \le 1/\delta} \\ &[-\delta\eta(kI_P^q(1/k\eta) + lI_P^q(-1/l\eta)) + 2\delta w(\delta;\eta) + (1-(k+l+2)\delta\eta)a]\}\,. \end{aligned}$$

By convexity and since $\delta \le v_P$, it holds $kI_P^q(1/k\eta) + lI_P^q(-1/l\eta) \ge (k+l)I_P^q(0) \ge (k+l)I_P^q(\delta)$, and therefore $a' := a + I_P^q(\delta)$ is such that

$$a' \le C\delta\eta + \Big(\max_{1/\eta \le k,l;(k+l)\eta \le 1/\delta} [2\delta w(\delta;\eta) + 2\delta\eta I_P^q(\delta) + (1-(k+l+2)\delta\eta)a']\Big)^+ .$$

Computing the maximum for positive a', we derive that $2a' \le C\eta + 2(w(\delta;\eta) + \eta I_P^q(\delta))$. Letting now $\eta \to 0$ and $\delta \to 0$, we conclude that

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\omega^o\Big(X_n \le 0\Big) \le -I_P^q(0) \quad , \quad P-\text{a.s.} \tag{2.3.28}$$

In fact, the same proof actually shows that

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\omega^o\Big(\exists \ell \ge n : X_\ell \le 0\Big) \le -I_P^q(0) \quad , \quad P-\text{a.s.} \tag{2.3.29}$$

For an arbitrary $v \in [0, v_P)$, we write

$$\begin{aligned} &P_\omega^o\Big(\frac{X_n}{n} \le v\Big) \le P_\omega^o(\exists \ell \ge n : X_\ell \le nv) \le P_\omega^o\Big(T_{[nv]} \ge n\Big) \\ &+ \sum_{k: v/\eta \le k < 1/\eta} P_\omega^o\Big(\frac{T_{[nv]}}{n} \in [k\epsilon, (k+1)\epsilon)\Big) P_{\theta^{[nv]}\omega}^o\Big(\exists \ell \ge n - n(k+1)\eta : X_\ell \le 0\Big) \end{aligned} \tag{2.3.30}$$

where the two first probabilities in the right-hand side can be estimated using Step I, and concerning the last one we note that from (2.3.29) one has that

$$\frac{1}{n - n(k+1)\eta} \log P_{\theta^{[nv]}\omega}^o\Big(\exists \ell \ge n - n(k+1)\eta : X_\ell \le 0\Big) \to_{n\to\infty} -I_P^q(0) \,,$$

in probability, and hence a.s. along a random subsequence. Therefore,

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{n} \log P_\omega^o\Big(\exists \ell \ge n : X_\ell \le nv\Big) \\ &\le \limsup_{\eta\to 0} \Big(-I_P^q(v) \vee \max_{v/\eta \le k \le 1/\eta} [-k\eta I_P^q(v/k\eta) - (1-k\eta)I_P^q(0)]\Big) \\ &= -I_P^q(v) \,, \end{aligned} \tag{2.3.31}$$

by convexity. But, due to Kingman's sub-additive ergodic theorem, the left hand side of the last expression converges P-a.s., resulting with

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\omega^o\Big(X_n \le nv\Big) \le -I_P^q(v) \,, \quad P-a.s..$$

The upper bound for general subsets of $[0,1]$ follows by noting the convexity of $I_P^q(\cdot)$. $\square$

Remarks 1. If P is extremal, a simpler proof of (2.3.28) can be given. Indeed, note that $I_P^q(0) = \lambda_{\text{crit}}(0)$, and, by extremality,

$$\begin{aligned} P_\omega^o(X_n \le 0) &\le P_\omega(X_m \le 0,\ \text{some}\, m \ge 0) \le P_{\tilde{\omega}^{\min}}(X_m \le 0,\ \text{some}\, m \ge 0) \\ &\le \sum_{m=n}^{\infty} P_{\tilde{\omega}^{\min}}(X_m \le 0)\,. \end{aligned}$$

A simple computation reveals that $P_{\tilde{\omega}^{\min}}(X_m \le 0) \le C_\lambda e^{-\lambda m}$ for any $\lambda < \lambda_{\text{crit}}$, yielding that

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\omega^o(X_n \le 0) \le -I_P^q(0)\,, \quad P - a.s. \tag{2.3.32}$$

2. A lot of information is available concerning the shape of the rate function $I_P^q(\cdot)$, and in particular concerning the existence of pieces where the rate function is not strictly convex. We refer to the discussion in [12] for details.

Annealed LDP's

The LDP in the annealed setting also makes use of the hitting times T_n and T_{-n}. For technical reasons, we need to make stronger hypotheses on the environment. To state these, define the empirical process

$$R_n(\omega) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\theta^i \omega}\,.$$

R_n takes values in the space $M_1(\Omega)$ of probability measures on Ω, which we equip with the topology of weak convergence. We also need to introduce the specific relative entropy

$$h(\cdot|P) : M_1(\Omega) \mapsto [0,\infty],\ h(Q|P) := \begin{cases} \lim_{N\to\infty} \frac{1}{N} H(Q_N|P_N)\,, & Q \text{ stationary} \\ \infty\,, & \text{otherwise}\,, \end{cases}$$

where Q_N, P_N denote the restriction of Q, P to the first N coordinates $\{\omega_i\}_{i=0}^{N-1}$ and $H(\cdot|\cdot)$ denotes the relative entropy:

$$H(\mu|\nu) = \begin{cases} \int \log\left(\frac{d\mu}{d\nu}(x)\right) \mu(dx)\,, & \mu \ll \nu \\ \infty\,, & \text{otherwise} \end{cases}\,.$$

Assumption 2.3.33

(C1) *P is stationary and ergodic*
(C2) *There exists an $\varepsilon > 0$ such that $\min(\omega_0^+, \omega_0^-) > \varepsilon\,,\ P - a.s.$,*
(C3) *$\{R_n\}$ satisfies under P the process level LDP in $M_1(\Omega)$ with good rate function $h(\cdot|P)$,*

(C4) *P is locally equivalent to the product of its marginals and, for any stationary measure $\eta \in M_1(\Omega)$ there is a sequence $\{\eta^n\}$ of stationary, ergodic measures with $\eta^n \underset{n\to\infty}{\longrightarrow} \eta$ weakly and $h(\eta^n|P) \to h(\eta|P)$.*

(C5) *P is extremal.*

We note that product measures and Markov processes with bounded transition kernels satisfy **(C1)**–**(C4)** of Assumption 2.3.33, see [28, Lemma 4.8] and [24].
Define now

$$I_P^{\tau,a}(u) = \inf_{\eta\in M_1^{e,\varepsilon}(\Omega)} \left[I_\eta^{\tau,q}(u) + h(\eta|P) \right] \,,\; I_P^a(v) = \inf_{\eta\in M_1^{e,\varepsilon}(\Omega)} \left[I_\eta^q(v) + |v| h(\eta|P) \right] \,.$$

We now have the annealed analog of Theorem 2.3.12:

Theorem 2.3.34 *Assume Assumption 2.3.33. Then, the random variables $\{T_n/n\}$ satisfy the weak annealed LDP with speed n and rate function $I_P^{\tau,a}(\cdot)$. Further, the random variables X_n/n satisfy the annealed LDP with speed n and good rate function $I_P^a(\cdot)$.*

Proof. Throughout, $M_1^{s,\varepsilon} = M_1^{s,\varepsilon,P}$ denotes the set of stationary probability measures $\eta \in M_1(\Omega)$ satisfying $\operatorname{supp}\eta_0 \subset \operatorname{supp} P_0$. If $E_P(\log\rho_0) \le 0$ then $\lambda_{\text{crit}} = \lambda_{\text{crit}}(P)$ is as in Lemma 2.3.13, whereas if $E_P(\log\rho_0) > 0$ then $\lambda_{\text{crit}} = \lambda_{\text{crit}}(P^{\text{Inv}})$.

The following lemma, whose proof is deferred, is key to the transfer of quenched LDP's to annealed LDP's:

Lemma 2.3.35 *Assume P satisfies Assumption 2.3.33. Then, the function $(\mu,\lambda) \mapsto \int f(\lambda,\omega)\mu(d\omega)$ is continuous on $M_1^{s,\varepsilon,P} \times (-\infty, \lambda_{\text{crit}}]$.*

Steps I.1 + I.2: weak annealed LDP upper bound for T_n/n: We have, for $\lambda \le 0$,

$$\begin{aligned}
\mathbb{P}^o\left(T_n/n \le u\right) &\le e^{-\lambda n u}\mathbb{E}^o\left(\exp\left(\lambda\sum_{j=1}^n \tau_j\right)\mathbf{1}_{\tau_j<\infty, j=1,\ldots,n}\right) \\
&= e^{-\lambda n u} E_P\left(\prod_{j=1}^n E_\omega^o\left(e^{\lambda\tau_j}\mathbf{1}_{\tau_j<\infty}\right)\right) = e^{-\lambda n u} E_P\left(\exp\left(\sum_{j=0}^{n-1} f(\lambda,\theta^j\omega)\right)\right) \\
&= e^{-\lambda n u} E_P\left(\exp\left(n\int f(\lambda,\omega)R_n(d\omega)\right)\right) \,. \qquad (2.3.36)
\end{aligned}$$

By Assumption 2.3.33, $\{R_n\}$ satisfies a LDP with rate function $h(\cdot|P)$. Lemma 2.3.35 ensures that we can apply Varadhan's lemma (see [20, Lemma 4.3.6]) to get

$$\limsup_{n\to\infty} \frac{1}{n} \log E_P \left(\exp\left(n \int f(\lambda,\omega) R_n(d\omega)\right)\right)$$
$$\leq \sup_{\eta\in M_1^{s,\varepsilon}} \left[\int f(\lambda,\omega)\eta(d\omega) - h(\eta|P)\right] . \quad (2.3.37)$$

Going back to (2.3.36), this yields the upper bound

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}^o\left(T_n/n \leq u\right) \leq \inf_{\lambda\leq 0} \sup_{\eta\in M_1^{s,\varepsilon}} \left[\int f(\lambda,\omega)\eta(d\omega) - h(\eta|P) - \lambda u\right]$$
$$= -\sup_{\lambda\leq 0} \inf_{\eta\in M_1^{s,\varepsilon}} \left[G(\lambda,\eta,u) + h(\eta|P)\right] . \quad (2.3.38)$$

Since $\eta \to -\int f(\lambda,\omega)\eta(d\omega) + h(\eta|P)$ is lower semi-continuous (for $\lambda \leq 0$) and $M_1(\Omega)$ is compact, the infimum in (2.3.38) is achieved for each λ, on measures in $M_1^{s,\varepsilon}$, for otherwise $h(\eta|P) = \infty$. Further, by (2.3.15), the supremum over λ can be taken over a compact set (recall that $\infty > u > 1$!). By the Minimax theorem (see [65, Theorem 4.2] for this version), the min-max is equal to the max-min in (2.3.38). Further, since taking first the supremum in λ in the right hand side of (2.3.38) yields a lower semicontinuous function, an achieving $\bar\eta$ exists, and then, due to compactness, there exists actually an achieving pair $\bar\lambda, \bar\eta$. We will show below that the infimum may be taken over stationary, ergodic measures only, that is

$$\inf_{\eta\in M_1^{s,\varepsilon}} \sup_{\lambda\leq 0} \left(G(\lambda,\eta,u) + h(\eta|P)\right) = \inf_{\eta\in M_1^{e,\varepsilon}} \sup_{\lambda\leq 0} \left(G(\lambda,\eta,u) + h(\eta|P)\right) . \quad (2.3.39)$$

Then,

$$\text{R.H.S. of } (2.3.38) = -\inf_{\eta\in M_1^{e,\varepsilon}} \sup_{\lambda\leq 0} \left(G(\lambda,\eta,u) + h(\eta|P)\right)$$
$$= -\inf_{\eta\in M_1^{e,\varepsilon}} \inf_{w\leq u} \left(I_\eta^{\tau,q}(w) + h(\eta|P)\right) . \quad (2.3.40)$$

The second equality in (2.3.40) is obtained as follows: set $M_u = \{\eta \in M_1^{e,\varepsilon} : E_\eta(E_\omega^o(\tau_1|\tau_1 < \infty)) > u\}$, $M_u^- = \{\eta \in M_1^{e,\varepsilon} : E_\eta(E_\omega^o(\tau_1|\tau_1 < \infty)) \leq u\}$. For $\eta \in M_u$,

$$\inf_{w\leq u} I_\eta^{\tau,q}(w) = I_\eta^{\tau,q}(u) = \sup_{\lambda\in\mathbb{R}} G(\lambda,\eta,u) = \sup_{\lambda\leq 0} G(\lambda,\eta,u) .$$

Further, recall that $I_\eta^{\tau,q}(\cdot)$ is convex with minimum value $\max(0, E_\eta(\log\rho_0))$ achieved at $E_\eta(E_\omega^o(\tau_1|\tau_1 < \infty))$. Then, for $\eta \in M_u^-$,

$$\inf_{w\leq u} I_\eta^{\tau,q}(u) = \max(0, E_\eta(\log\rho_0))$$

whereas Jensen's inequality implies that for such η,

$$\sup_{\lambda\leq 0} G(\lambda,\eta,u) = G(0,\eta,u) = \max(0, E_\eta(\log\rho_0)) ,$$

completing the proof of (2.3.40). Hence,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}^o\left(T_n/n \le u\right) \le -\inf_{w\le u} \inf_{\eta\in M_1^{e,\varepsilon}} \left(I_\eta^{\tau,q}(w) + h(\eta|P)\right)$$
$$= -\inf_{w\le u} I_P^{\tau,a}(w). \tag{2.3.41}$$

Turning to the proof of (2.3.39), we have, due to **(C4)** in Assumption 2.3.33, a sequence of stationary, ergodic measures with $\eta^n \to \bar\eta$ and $h(\eta^n|P) \to h(\bar\eta|P)$. Let λ_n be the maximizers in (2.3.39) corresponding to η^n. We have

$$\inf_{\eta\in M_1^{e,\varepsilon}} \sup_{\lambda\le 0} \left(\left[\lambda u - \int f(\lambda,\omega)\eta(d\omega)\right] + h(\eta|P)\right) \le \left[\lambda_n u - \int f(\lambda_n,\omega)\eta^n(d\omega)\right]$$
$$+h(\eta^n|P)\,. \tag{2.3.42}$$

W.l.o.g. we can assume, by taking a subsequence, that $\lambda_n \to \lambda^* \le 0$. Using the joint continuity in Lemma 2.3.35, we have, for $\epsilon' > 0$ and $n \ge N_0(\epsilon')$,

$$\lambda_n u - \int f(\lambda_n,\omega)\eta^n(d\omega) + h(\eta^n|P)$$
$$\le \left[\lambda^* u - \int f(\lambda^*,\omega)\bar\eta(d\omega)\right] + h(\bar\eta|P) + \epsilon'$$
$$\le \inf_{\eta\in M_1^{s,\varepsilon}} \sup_{\lambda\le 0} \left(\left[\lambda u - \int f(\lambda,\omega)\eta(d\omega)\right] + h(\eta|P)\right) + \epsilon'\,.$$

But this shows the equality in (2.3.39), since the reverse inequality there is trivial.

The upper bound for the upper tail, that is for $\frac{1}{n}\log P[\infty > \frac{1}{n}\sum_{j=1}^n \tau_j \ge u]$, where $1 < u < \infty$, is achieved similarly. We detail the argument since there is a small gap in the proof presented in [12]. First, exactly as in (2.3.38), one has

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}^o\left(T_n/n \ge u\right) \le \inf_{0\le\lambda\le\lambda_{\mathrm{crit}}} \sup_{\eta\in M_1^{s,\varepsilon}} \left[-G(\lambda,\eta,u) - h(\eta|P)\right]$$
$$= -\sup_{0\le\lambda\le\lambda_{\mathrm{crit}}} \inf_{\eta\in M_1^{s,\varepsilon}} \left[G(\lambda,\eta,u) + h(\eta|P)\right]\,. \tag{2.3.43}$$

One may now apply the min-max theorem to deduce that the right hand side of (2.3.43) equals

$$\inf_{\eta\in M_1^{s,\varepsilon}} \sup_{0\le\lambda\le\lambda_{\mathrm{crit}}} \left[G(\lambda,\eta,u) + h(\eta|P)\right] = \inf_{\eta\in M_1^{e,\varepsilon}} \sup_{0\le\lambda\le\lambda_{\mathrm{crit}}} \left[G(\lambda,\eta,u) + h(\eta|P)\right]\,,$$

where the second equality is proved by the same argument as in (2.3.39). Here a new difficulty arises: the supremum is taken over $\lambda \in [0, \lambda_{\mathrm{crit}}(P)]$, but in general $\lambda_{\rm crit}(\eta) \geq \lambda_{\rm crit}(P)$ and hence the identification of the last expression with a variational problem involving $I_\eta^{\tau,q}(\cdot)$ is not immediate. To bypass this obstacle, we note, first by replacing η with $(1-n^{-1})\eta + n^{-1}P$ and then using again **(C4)** to approximate with an ergodic measure, that the last expression equals

$$\inf_{\{\eta \in M_1^{e,\varepsilon}, \lambda_{\rm crit}(\eta)=\lambda_{\rm crit}(P)\}} \sup_{0\leq\lambda\leq\lambda_{\rm crit}} [G(\lambda,\eta,u) + h(\eta|P)] \,.$$

From here, one proceeds as in the case of the lower tail, concluding that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}^o\left(T_n/n \geq u\right) \\
&\leq - \inf_{\{\eta \in M_1^{e,\varepsilon}, \lambda_{\rm crit}(\eta)=\lambda_{\rm crit}(P)\}} \sup_{0\leq\lambda\leq\lambda_{\rm crit}} [G(\lambda,\eta,u) + h(\eta|P)] \\
&= - \inf_{\{\eta \in M_1^{e,\varepsilon}, \lambda_{\rm crit}(\eta)=\lambda_{\rm crit}(P)\}} \inf_{w\geq u} I_\eta^{\tau,q}(w) + h(\eta|P) \leq - \inf_{\eta\in M_1^{e,\varepsilon}} \inf_{w\geq u} I_\eta^{\tau,q}(w) + h(\eta|P) \,.
\end{aligned}$$

This will then complete the proof of the (weak) upper bound, as soon as we prove the convexity of $I_P^{\tau,a}$. But, the function

$$\begin{aligned}
&\sup_{\lambda\in\mathbb{R}} \inf_{\eta\in M_1^{s,\varepsilon}} [G(\lambda,\eta,u) + h(\eta|P)] \\
&\qquad = \sup_{\lambda\in\mathbb{R}} \left[\lambda u + \inf_{\eta\in M_1^{s,\varepsilon}} \left(-\int f(\lambda,\omega)\eta(d\omega) + h(\eta|P)\right)\right] \,, \qquad (2.3.44)
\end{aligned}$$

being a supremum over affine functions in u, is clearly convex in u, while one shows, exactly as in (2.3.39), that

$$\inf_{\eta\in M_1^{s,\varepsilon}} \sup_{\lambda\in\mathbb{R}} [G(\lambda,\eta,u) + h(\eta|P)] = \inf_{\eta\in M_1^{e,\varepsilon}} \sup_{\lambda\in\mathbb{R}} [G(\lambda,\eta,u) + h(\eta|P)] \qquad (2.3.45)$$

and therefore

$$\inf_{\eta\in M_1^{s,\varepsilon}} \sup_{\lambda\in\mathbb{R}} [G(\lambda,\eta,u) + h(\eta|P)] = \inf_{\eta\in M_1^{e,\varepsilon}} \left[I_\eta^{\tau,q}(u) + h(\eta|P)\right] = I_P^{\tau,a}(u) \,.$$

Recalling that, as we saw above, supremum and infimum in (2.3.44) can be exchanged, this completes the proof of the upper bounds for the annealed LDP's for T_n/n.

Step I.3: Annealed lower bounds for T_n/n: We will use the following standard argument.

Lemma 2.3.46 *Let P be a probability distribution, $(\mathcal{F}_n)$ be an increasing sequence of σ-fields and A_n be $\mathcal{F}_n$-measurable sets, $n = 1,2,3,\ldots$. Let (Q_n) be a sequence of probability distributions such that $Q_n[A_n] \to 1$ and*

$$\limsup_{n\to\infty} \frac{1}{n} H(Q_n|P)\Big|_{\mathcal{F}_n} \leq h$$

where $H(\cdot|P)\Big|_{\mathcal{F}_n}$ denotes the relative entropy w.r.t. P on the σ-field $\mathcal{F}_n$ and h is a positive number. Then we have

$$\liminf_{n\to\infty} \frac{1}{n} \log P[A_n] \geq -h\,.$$

Proof of Lemma 2.3.46. From the basic entropy inequality ([23], p. 423),

$$Q_n[A_n] \leq \frac{\log 2 + H(Q_n|P)\Big|_{\mathcal{F}_n}}{\log(1+1/P[A_n])}\quad,\qquad A_n\in\mathcal{F}_n,$$

we have $-Q_n[A_n]\log P[A_n] \leq \log 2 + H(Q_n|P)\Big|_{\mathcal{F}_n}$. Dividing by n and taking limits we obtain the desired result. $\square$

We prove the lower bound for the lower tail only, the upper tail being handled by the same truncation as in the quenched case, see [12] for details. For $\eta\in M_1^{e,\varepsilon}$ satisfying $E_\eta(\log\rho_0)\leq 0$, define $\overline{Q}^o_\omega$ as in Step I.3 of Theorem 2.3.12, and let $\overline{Q}^o_\eta = \eta(d\omega)\otimes\overline{Q}^o_\omega$. Let $A_n=\{|n^{-1}T_n-u|<\delta\}$. We know already that $\overline{Q}^o_\omega[A_n^c]\underset{n\to\infty}{\longrightarrow}0\,,\ \eta-\text{a.s.}\,,$ and this implies $\overline{Q}^o_\eta[A_n^c]\underset{n\to\infty}{\longrightarrow}0\,.$ Let $\mathcal{F}_n := \sigma(\{\tau_i\}_{i=1}^n,\{\omega_j\}_{j=-\infty}^n)$, $\mathcal{F}_n^\omega=\sigma(\{\omega_j\}_{j=-\infty}^n)$. Note that

$$\overline{Q}^o_\eta|_{\mathcal{F}_n} = \eta|_{\mathcal{F}_n^\omega}(d\omega)\otimes\overline{Q}^o_\omega|_{\mathcal{F}_n}\,.$$

Hence,

$$H(\overline{Q}^o_\eta|\mathbb{P}^o)\Big|_{\mathcal{F}_n} = H(\eta|P)\Big|_{\mathcal{F}_n^\omega} + \int H(\overline{Q}^o_\omega|P^o_\omega)\Big|_{\mathcal{F}_n}\eta(d\omega)\,. \tag{2.3.47}$$

Considering the second term in (2.3.47), we have

$$\begin{aligned}
&\frac{1}{n}\int H(\overline{Q}^o_\omega|P^o_\omega)\Big|_{\mathcal{F}_n}\eta(d\omega)\\
&\quad= -\frac{1}{n}\int \log Z_{n,\omega}\eta(d\omega) + \lambda_0(u,\eta)\int\frac{T_n}{n}d\overline{Q}^o_\omega\eta(d\omega)\\
&\quad= -\frac{1}{n}\int\sum_{j=1}^n \log\varphi(\lambda_0(u,\eta),\theta^{j-1}\omega)\eta(d\omega) + \lambda_0(u,\eta)\int\frac{T_n}{n}d\overline{Q}^o_\omega\eta(d\omega)
\end{aligned}$$

and we see, as in the proof of the lower bound of Theorem 2.3.12, that

$$\frac{1}{n}\int H(\overline{Q}^o_\omega|P^o_\omega)\Big|_{\mathcal{F}_n}\eta(d\omega)\underset{n\to\infty}{\longrightarrow}\lambda_0(u,\eta)u - E_\eta f(\lambda_0(u,\eta),\omega)\leq I_\eta^{\tau,q}(u)\,.$$

Considering the first term in (2.3.47), we know that

$$\limsup_{n\to\infty}\ \frac{1}{n}\,H(\eta|P)\Big|_{\mathcal{F}_n^\omega} = h(\eta|P)\,.$$

Hence,

$$\limsup_{n\to\infty}\frac{1}{n}\,H(\overline{Q}^o_\eta|\mathbb{P}^o)\Big|_{\mathcal{F}_n}\leq I_\eta^{\tau,q}(u)+h(\eta|P)\,,$$

and we can now apply Lemma 2.3.46 to conclude that for any $\eta \in M_1^{e,\varepsilon}$ satisfying $E_\eta(\log \rho_0) \le 0$ one has,

$$\liminf_{n\to\infty} E_P(A_n) \ge -\left(I_\eta^{\tau,q}(u) + h(\eta|P)\right) .$$

As in the quenched case, one handles $\eta \in M_1^{e,\varepsilon}$ satisfying $E_\eta(\log \rho_0) > 0$ by repeating the above argument with the required (obvious) modifications, replacing $\overline{Q}_\omega^o$ by $\overline{Q}_\omega^o(\cdot|T_n < \infty)$. This completes the proof of Step I. $\square$

Proof of Lemma 2.3.35: For $\kappa > 1$, decompose $\varphi(\lambda, \omega)$ as follows:

$$\begin{aligned} E_\omega^o(e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty}) &= E_\omega^o(e^{\lambda\tau_1}; \tau_1 < \kappa) + E_\omega^o(e^{\lambda\tau_1}; \infty > \tau_1 \ge \kappa) \\ &:= \varphi_1^\kappa(\lambda,\omega) + \varphi_2^\kappa(\lambda,\omega) , \end{aligned} \tag{2.3.48}$$

where $(\lambda, \omega) \to \log \varphi_1^\kappa(\lambda, \omega)$ is bounded and continuous. We also have

$$0 \le \log\left(1 + \frac{\varphi_2^\kappa(\lambda,\omega)}{\varphi_1^\kappa(\lambda,\omega)}\right) \le \log\left(1 + \frac{\varphi_2^\kappa(\lambda_{\mathrm{crit}},\omega)}{\varepsilon e^\lambda}\right) .$$

Hence, the required continuity of the function $(\mu, \lambda) \to \int f(\lambda, \omega)\mu(d\omega)$ will follow from (2.3.48) as soon as we show that for any fixed constant $C_1 < 1$,

$$\lim_{\kappa\to\infty} \sup_{\mu\in M_1^{s,\varepsilon,P}} \int \log\left(1 + \frac{\varphi_2^\kappa(\lambda_{\mathrm{crit}},\omega)}{C_1}\right)\mu(d\omega) = 0 . \tag{2.3.49}$$

If $\rho_{\min} < 1$ and $\rho_{\max} > 1$ then one can easily check, by a coupling argument using **(C4)**, that $\lambda_{\mathrm{crit}} = 0$ (for a detailed proof see [12, Lemma 4]). Then, for each $\epsilon' > 0$ there exists a $\kappa_\mu = \kappa(\epsilon', \mu)$ large enough such that,

$$E_\mu\left(\log\left(1 + \frac{P_\omega^o(\infty > \tau_1 > \kappa_\mu)}{P_\omega^o(\tau_1 < \infty)}\right)\right) < \epsilon' .$$

Further, in this situation, for stationary, ergodic μ,

$$\int f(0,\omega)\mu(d\omega) = \left(-\int \log \rho_0(\omega)\mu(d\omega)\right) \wedge 0 . \tag{2.3.50}$$

In particular, $\mu \mapsto \int f(0, \omega)\mu(d\omega)$, being linear, is uniformly continuous on the compact set $M_1^{s,\varepsilon}$. Therefore, using (2.3.48), one sees that for each such μ one can construct a neighborhood B_μ of μ such that, for each $\nu \in B_\mu \cap M_1^{s,\varepsilon}$,

$$E_\nu\left(\log\left(1 + \frac{P_\omega^o(\infty > \tau_1 > \kappa_\mu + 1)}{P_\omega^o(\tau_1 < \infty)}\right)\right) < \epsilon' .$$

By compactness, it follows that there exists an $\kappa = \kappa(\epsilon')$ large enough such that, for all $\mu \in M_1^{s,\varepsilon}$,

$$E_\mu\left(\log\left(1 + \frac{P_\omega^o(\infty > \tau_1 > \kappa)}{P_\omega^o(\tau_1 < \infty)}\right)\right) < \epsilon' .$$

Using the inequality $\log(1+cx) \le c\log(1+x)$, valid for $x \ge 0$, $c \ge 1$, one finds that for κ large enough,

$$\sup_{\mu \in M_1^{s,\varepsilon}} \int \log\left(1+\frac{\varphi_2^{\kappa}(0,\omega)}{C_1}\right)\mu(d\omega) \le \epsilon'/C_1\,,$$

proving (2.3.49) under the condition $\rho_{\min} < 1, \rho_{\max} > 1$.

We next handle the case $\rho_{\max} < 1$. We now complete the proof of Lemma 2.3.35 in the case $\rho_{\min} > 1$. We have $f(\lambda,\omega) \ge \lambda + \log\omega_0^+ \ge \lambda + \log\varepsilon$. We show that $(\lambda,\omega) \mapsto \varphi(\lambda,\omega)$ is continuous as long as $\omega_i \le \omega^{\max}$, $\rho_i \le \rho_{\max}$ and $\lambda \le \lambda_{\text{crit}}$, which is enough to complete the proof. Write, for $\lambda \le \lambda_{\text{crit}}$,

$$E_\omega(e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty}) = E_\omega(e^{\lambda\tau_1};\tau_1<\kappa) + E_\omega(e^{\lambda\tau_1};\infty>\tau_1\ge\kappa) \qquad (2.3.51)$$

and observe that the first term in the right hand side of (2.3.51) is continuous as a function of ω and the second term goes to 0 for $\kappa\to\infty$, uniformly in ω. More precisely, due to (2.3.16), for all ω considered here,

$$E_\omega[e^{\lambda\tau_1};\infty>\tau_1\ge\kappa] \le E_{\tilde\omega^{\min}}(e^{\lambda_{\text{crit}}\tau_1};\tau_1\ge\kappa) \to_{\kappa\to\infty} 0\,. \qquad (2.3.52)$$

Finally, in the case $\rho_{\min} > 1$, the conclusion follows from the duality formula (2.3.23) and Remark 1 that follows its proof, by reducing the claim to the case $\rho_{\max} < 1$. $\square$

Step II: The proof is identical to Step I, and is omitted.

Step III: The proof of all statements, except for the convexity of I_P^a, and the upper bound on $\mathbb{P}^o(X_n \le nv)$, follow the argument in the quenched case. The latter proofs can be found in [12]. $\square$

Remarks: 1. We note that under the conditions of Theorem 2.3.34, if $E_P\log\rho_0 \le 0$ then both $I_P^a(v) \ne 0$ and $I_P^q(v) \ne 0$ for $v \notin [0,v_P]$. Indeed, since $h(\eta|P) \ne 0$ unless $\eta = P$, it holds that $I_P^a(v) = 0$ only if $I_P^q(v) = 0$. If $E_P\log\rho_0 = 0$, $v_P = 0$ and then for any $v \ne 0$, $I_P^q(v) = |v|I_P^{\tau,q}(1/|v|) > 0$ by the remark following the proof of Lemma 2.3.13. On the other hand, if $E_P\log\rho_0 < 0$, the same argument applies for $v > v_P$ while for $v < 0$ we have that $I_P^q(v) \ge -|v|E_P\log\rho_0 > 0$.

2. The condition **(C5)** can be avoided altogether. This is not hard to see if one is interested only in the LDP for T_n/n. Indeed, **(C5)** was used mainly in describing a worst case environment in the course of the proof of Lemma 2.3.35, see also part (d) of Lemma 2.3.13. When it is dropped, the following lemma, whose proof we provide below, replaces Lemma 2.3.35 when deriving the annealed LDP for T_n/n:

Lemma 2.3.53 *Assume P satisfies Assumption 2.3.33 except for* **(C5)***. Then, $\lambda_{\text{crit}}(P)$ depends only on* $\text{supp}(P_0)$*, and the map $(\mu,\lambda) \mapsto E_\mu(f(\lambda,\omega))$ is continuous on $M_1^{s,\varepsilon,P}\times(-\infty,0]\cup[0,\lambda_{\text{crit}})$.*

Given Lemma 2.3.53, we omit **(C5)** and replace **(C4)** in Assumption 2.3.33 by

(C4') *P is locally equivalent to the product of its marginals and, for any stationary measure $\eta \in M_1(\Omega)$ with $h(\eta|P) < \infty$ there is a sequence $\{\eta^n\}$ of stationary, ergodic measures, locally equivalent to the product of P's marginals, with* $\mathrm{supp}((\eta^n)_0) = \mathrm{supp}(P_0)$, *$\eta^n \underset{n\to\infty}{\longrightarrow} \eta$ weakly and $h(\eta^n|P) \to h(\eta|P)$.*

One now checks (we omit the details) that all approximations carried out in the proof of the upper bound of the upper tail of T_n/n can still be done, yielding the annealed LDP for T_n/n. To transfer this LDP to an annealed LDP for X_n/n does require a new argument, we refer to [17] for details.

We conclude our discussion of large deviation principles with the:

Proof of Lemma 2.3.53: Set $\Xi = \mathrm{supp}\,(P_0)$ and define $\bar\lambda := \inf_{\omega\in\Xi^{\mathbb{Z}}} \lambda_{\mathrm{crit}}(\omega)$ where

$$\lambda_{\mathrm{crit}}(\omega) := \sup\{\lambda \in \mathbb{R} : E^o_\omega(e^{\lambda\tau_1}\mathbf{1}_{\{\tau_1<\infty\}})\}\,.$$

By definition, $\lambda_{\mathrm{crit}}(P) \geq \bar\lambda$. On the other hand, if $\lambda > \bar\lambda$ then there exists a $\bar\omega \in \Xi^{\mathbb{Z}}$ with $E^o_{\bar\omega}(e^{\lambda\tau_1}\mathbf{1}_{\{\tau_1<\infty\}}) = \infty$. Fix $K = e^{-\lambda}/\varepsilon$, and using monotone convergence, choose an M large enough such that

$$\varphi_{M,\bar\omega}(\lambda) := E^o_{\bar\omega}(e^{\lambda\tau_1}\mathbf{1}_{\{\tau_1<M\}}) > K + 1.$$

Since $\varphi_{M,\bar\omega}$ depends only on $\{\omega_i, i \in (-M, 0)\}$, it holds that with positive P-probability, $E^o_\omega(e^{\lambda\tau_1}\mathbf{1}_{\{\tau_1<M\}}) \geq K + 1$. But, if $\lambda \leq \lambda_{\mathrm{crit}}(P)$ it follows from the recursions (2.3.21) that $\varphi(\lambda, \omega) < K$, P-a.s., a contradiction unless $\bar\lambda = \lambda_{\mathrm{crit}}(P)$. In particular, $\lambda_{\mathrm{crit}}(P)$ depends only on $\mathrm{supp}(P_0)$. Note that the characterization of $\lambda_{\mathrm{crit}}(P)$ as $\bar\lambda$ implies that for any $\mu \in M_1^{s,\varepsilon,P}$ it holds that $\lambda_{\mathrm{crit}}(\mu) \geq \lambda_{\mathrm{crit}}(P)$.

Next, as in the course of the proof of Lemma 2.3.35, see (2.3.49), it is enough to show that for any $\lambda < \lambda_{\mathrm{crit}}(P)$,

$$\lim_{\kappa\to\infty} \sup_{\mu\in M_1^{s,\varepsilon,P}} \int \varphi_2^\kappa(\lambda,\omega)\mu(d\omega) = 0\,. \tag{2.3.54}$$

But, since $\varphi(\lambda, \omega) \leq e^{-\lambda}/\epsilon$ μ-a.s. for all $\mu \in M_1^{s,\varepsilon,P}$ (use again the recursions (2.3.21) and that $\lambda_{\mathrm{crit}}(\mu) \geq \lambda_{\mathrm{crit}}(P)$), it holds that

$$\int \varphi_2^\kappa(\lambda,\omega)\mu(d\omega) \leq e^{(\lambda-\lambda_{\mathrm{crit}})M}\frac{e^{-\lambda}}{\varepsilon}\,,$$

yielding immediately (2.3.54). □

Remark 2.3.55 *A better version of the derivation of the LDP, that avoids some of the coupling arguments and hence holds under weaker conditions (in particular, without the extremality assumption* **C5***) can now be found in [14].*

Bibliographical notes: The first quenched LDP result is due to Greven and Den Hollander, [35], who proved it for i.i.d. environments using the method of the environment viewed from the particle. Our derivation here follows the hitting times approach developed in [12], except that the proof of Lemma 2.3.22 *follows the article [59]. Extensions of the LDP's in this chapter to more general models allowing for (non geometric) holding times is presented in [17], where the derivation avoids completely coupling arguments and thus bypasses altogether the need for* **(C5)** *in deriving the annealed LDP for* X_n/n.

The "process level LDP" for R_n *was first proved in [24] in the context of Markov chains with law* P *satisfying appropriate regularity conditions. It was extended to various ergodic situation in [56] and [57], see also [11]. We refer to [28] and [20, Chapter 6] for further information. Our presentation of the annealed LDP follows here [12], where additional information on the shape of the rate functions etc. can be found. Note that [12] treats the case* $P(\omega_0^0 = 0) = 1$. *In the exposition here, we corrected and simplified some of the arguments in [12], following [17], where a RWRE with general (i.e., not necessarily geometric) holding times is considered. Finally, a completely different approach to the derivation of the LDP, both annealed and quenched, is described in [80].*

2.4 The subexponential regime

We saw in Section 2.3 that, at least when P satisfies Assumption 2.3.33 and $E_P \log \rho_0 \le 0$, we have that for any δ small enough, any $v \in [0, v_P]$,

$$\lim_{n\to\infty} \frac{1}{n} \log P_\omega^o \left(\frac{X_n}{n} \in (v-\delta, v+\delta) \right) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}^o \left(\frac{X_n}{n} \in (v-\delta, v+\delta) \right) = 0\,. \quad (2.4.1)$$

Our goal in this section is to obtain more precise information on the rate of convergence in (2.4.1). Surprisingly, it turns out that it is better to consider first the annealed case.

Throughout this section, we impose the following assumption on the law P. Together with **(C4)** there, it implies Assumption 2.3.33.

Assumption 2.4.2

(D1) *There exists an* $\varepsilon > 0$ *such that* $\min(\omega_0^+, \omega_0^-) > \varepsilon$, *$P$-a.s.*
(D2) $\rho_{\min} < 1$, $\rho_{\max} > 1$, *and* $E_P \log \rho_0 \le 0$.
(D3) P *is* α*-mixing with* $\alpha(n) = \exp(-n(\log n)^{1+\eta})$ *for some* $\eta > 0$*; that is, for any* ℓ*-separated measurable bounded by* 1 *functions* f_1, f_2,

$$E_P\Big(f_1(\omega)(f_2(\omega) - E_P f_2(\omega))\Big) \le \alpha(\ell)\,.$$

(functions f_i *are* ℓ *separated if* f_i *is measurable on* $\sigma(\omega_j, j \in I_i)$ *with* I_i *intervals satisfying* $\mathrm{dist}(I_i, I_k) > \ell$ *for any* $i \ne k$*).*

It is known that **(D3)** implies **(C1)** and **(C3)** of Assumption 2.3.33, see [10]. In particular, letting $\overline{R}_k := k^{-1} \sum_{i=0}^{k-1} \log \rho_i$, it implies that $\overline{R}_k$ satisfies the LDP with good rate function $J(\cdot)$. We add the following assumption on $J(\cdot)$:

(D4) $J(0) > 0$.

Condition **(D4)** implies that $E_P(\log \rho_0) < 0$. Define next $s := \min_{y\geq 0} \frac{1}{y} J(y)$. Note that the condition $E_P(\overline{S}) < \infty$ and the existence of a LDP for $\overline{R}_k$ with good rate function $J(\cdot)$ are enough to imply, by Varadhan's lemma, that $0 \geq \sup_y (y - J(y))$, and in particular that $s \geq 1$. (In the case where P is a product measure, we can identify s as satisfying $E_P(\rho_0^s) = 1$, and then $E_P(\overline{S}) < \infty$, which is equivalent to $E_P(\rho_0) < 1$, implies that $s > 1$.)

Annealed subexponential estimates

Theorem 2.4.3 *Assume P satisfies Assumption 2.4.2, and $v_P > 0$. Then, for any $v \in (0, v_P)$ and any $\delta > 0$ small enough,*

$$\lim_{n\to\infty} \frac{\log \mathbb{P}^o\left(\frac{X_n}{n} \in (v-\delta, v+\delta)\right)}{\log n} = 1 - s\,.$$

Proof. We begin by proving the lower bound. Fix $0 < v-\delta < v-4\eta < v < v_P$; let

$$L_k = \max\Big\{n \geq T_k :\ (k - X_n)\Big\}$$

denote the largest excursion of $\{X_\cdot\}$ to the left of k after hitting it. Observe that the event $\{n^{-1}X_n \in (v-\delta, v+\delta)\}$ contains the event

$$A := \left\{\frac{(v-4\eta)n}{v_P} < T_{(v-2\eta)n} < n\,,\ L_{(v-2\eta)n} < \frac{\eta n}{2}\,,\ T_{vn} > n\right\}\,, \tag{2.4.4}$$

namely, the RWRE hits $(v-2\eta)n$ at about the expected time, from which point its longest excursion to the left is less than $\eta n/2$, but the RWRE does not arrive at position vn by time n.

Next, note that by (2.1.4),

$$P^o_\omega\Big(L_{(v-2\eta)n} \geq \eta n/2\Big) \leq \sum_{i=0}^{\infty} \prod_{j=-(\eta n/2-1)}^{i} \rho_{(v-2\eta)n+j}\,. \tag{2.4.5}$$

Hence, using the LDP for $\overline{R}_k$, we have for all n large enough

$$\begin{aligned}\mathbb{P}^o\Big(L_{(v-2\eta)n} \geq \eta n/2\Big) &\leq \sum_{i=0}^{\infty} E\Big(e^{(\eta n/2+i)\overline{R}_{\eta n/2+i}}\Big)\\ &\leq e^{-\eta n \sup_y (y-J(y))/4} \leq e^{-\delta_1 n}\end{aligned} \tag{2.4.6}$$

for some $\delta_1 > 0$. Thus, for all n large enough,

$$\begin{aligned}
\mathbb{P}^o(A) &\geq \mathbb{P}^o\left(\frac{(v-4\eta)n}{v_P} < T_{(v-2\eta)n} < n\,,\, T_{vn} > n\right) - e^{-\delta_1 n} \\
&\geq \mathbb{E}^o\left(P^o_\omega\left(\frac{(v-4\eta)n}{v_P} < T_{(v-2\eta)n} < n\right)\right. \\
&\qquad \left. P_\omega^{(v-\eta)n}\left(T_{vn} > \frac{4\eta n}{v_P}, L_{(v-\eta)n < \eta n/2}\right)\right) - e^{-\delta_1 n} \\
&\geq B\cdot C - \alpha(\eta n/2) - 2e^{-\delta_1 n}\,,
\end{aligned}$$

where

$$\begin{aligned}
B &= \mathbb{P}^o\left(\frac{(v-4\eta)n}{v_P} < T_{(v-2\eta)n} < n\right) \\
C &= \mathbb{P}^o\left(T_{\eta n} > \frac{4\eta n}{v_P}\right)\,.
\end{aligned}$$

and $\alpha(\cdot)$ is as in **(D3)**.

Next, note that $B \to_{n\to\infty} 1$ by (2.1.16). We will prove below that for any $\delta' > 0$,

$$C \geq n^{1-s-2\delta'} \tag{2.4.7}$$

and this implies that for all n large, $\mathbb{P}^o(A) \geq n^{1-s-4\delta'}$, which yields the required lower bound (recall δ' is arbitrary!) as soon as we prove (2.4.7).

Turning to the proof of (2.4.7), fix y such that $\frac{J(y)}{y} \leq s + \frac{\delta'}{4}, K = [n^{\frac{\delta'}{4}}]$, $k = [\frac{1}{y}\log n]$, and set $\overline{m}_K = [\eta n/K]$. Now, using **(D3)**,

$$\begin{aligned}
&P\left(\bigcap_{j=1}^{\overline{m}_K}\{\overline{R}_k(\theta^{jK}\omega) \leq y\}\right) \\
&\quad \leq \big(P(\overline{R}_k(\omega) \leq y)\big)^{\overline{m}_K} + \overline{m}_K\alpha(K-k) \\
&\quad = \big(1 - P(\overline{R}_k(\omega) > y)\big)^{\overline{m}_K} + \overline{m}_K\alpha(K-k) \\
&\quad \leq \left(1 - e^{-k(J(y)+\frac{\delta' y}{4})}\right)^{\overline{m}_K} + \overline{m}_K\alpha(K-k) \leq 1 - n^{1-s-\delta'}\,,
\end{aligned}$$

for all n large enough. Hence,

$$P\Big(\exists j \in \{1,\cdots,\overline{m}_K\}:\ \overline{R}_k(\theta^{j_K}\omega) > y\Big) \geq n^{1-s-\delta'}\,. \tag{2.4.8}$$

On the other hand, let ω and $j \leq \overline{m}_K$ be such that $\overline{R}_k(\theta^{jK}\omega) > y$. Then, using (2.1.6) in the second inequality, for such ω,

$$\begin{aligned}
P^o_\omega\left(T_{\eta n} > \frac{4\eta n}{v_P}\right) &\geq P^{jK}_\omega\left(T_k > \frac{4\eta n}{v_P}\right) \geq (1-e^{-ky})^{\frac{4\eta n}{v_P}} \\
&\geq \left(1 - \frac{1}{n}\right)^{\frac{6\eta n}{v_P}} \geq e^{-\frac{8\eta}{v_P}}\,.
\end{aligned} \tag{2.4.9}$$

Combining (2.4.8) and (2.4.9), we conclude that

$$C \geq n^{1-s-\delta'} \cdot e^{-\frac{8\eta}{v_P}},$$

as claimed.

We next turn to the proof of the upper bounds. We may and will assume that $s > 1$, for otherwise there is nothing to prove. We first note that, for some $\delta'' := \delta''(\delta) > 0$,

$$\begin{aligned} \mathbb{P}^o\left(\frac{X_n}{n} \in (v-\delta, v+\delta)\right) &\leq \mathbb{P}^o\left(\frac{X_n}{n} < v+\delta\right) \\ &\leq \mathbb{P}^o(T_{n(v+2\delta)} > n) + \mathbb{P}^o(L_0 > n\delta) \\ &\leq \mathbb{P}^o(T_{n(v+2\delta)} > n) + e^{-\delta'' n} \end{aligned} \tag{2.4.10}$$

where the stationarity of P was used in the second inequality, and (2.4.6) in the third. Thus, the required upper bound follows once we show that for any $v < v_P$, any $\delta' > 0$,

$$\mathbb{P}^o(T_{nv} > n) \leq n^{1-s+\delta'} \tag{2.4.11}$$

for all n large enough.

Set $a := \sup_y(y - J(y))$. Because $s > 1$ and $J(0) > 0$, it holds that $a < 0$. Fix $A > -s/a$, and set $k = k(n) = A \log n$. Next, define the process $\{Y_n\}$ in $\mathbb{Z}^{\mathbb{N}}$ and the hitting times $\tilde{T}_{ik} = \min(n \geq 0 : Y_n = ik), i = 0, 1, \cdots$ such that the only change between the processes $\{X_n\}$ and $\{Y_n\}$ is that the process $\{Y_n\}_{n \geq \tilde{T}_{ik}}$ is reflected at position $(i-1)k$ (with a slight abuse of notations, we continue to use P^o_ω, $\mathbb{P}^o$ to denote the law of $\{Y_n\}$ as well as that of $\{X_n\}$). Set $m_k = [vn/k] + 1$, and $\tilde{\tau}^{(i)}_k = \tilde{T}_{ik} - \tilde{T}_{(i-1)k}, i = 1, \cdots, m_k$. Note that the $\tilde{\tau}^{(i)}_k$ are identically distributed, each stochastically dominated by T_k. Hence, $\mathbb{E}^o\tilde{T}_{ik} \leq \mathbb{E}^o T_{ik}$. On the other hand, fixing $\lambda \in (1/s, 1)$, we will see below (cf. Lemma 2.4.15) that $\mathbb{E}^o(T_k^{1/\lambda}) \leq ck^{1/\lambda}$ for some $c := c(\lambda)$, yielding, by Hölder's inequality, that

$$\mathbb{E}^o T_k \leq \mathbb{E}^o(\tilde{T}_k) + \mathbb{P}^o(L_0 \geq k)^{1-\lambda}\mathbb{E}^o(T_k^{1/\lambda})^\lambda \leq \mathbb{E}^o(\tilde{T}_k) + ck\mathbb{P}^o(L_0 \geq k)^{1-\lambda}.$$

Thus, using (2.4.6) and the fact that $\mathbb{E}^o(T_k)/k = v_P$, we conclude that $\lim_{k\to\infty} \mathbb{E}^o\, T_k/\mathbb{E}^o\, \tilde{T}_k = 1$, implying that $\mathbb{E}^o\tilde{T}_k/k \to_{k\to\infty} 1/v_P$.

Next, note that on the event $\{L_{ik} < k \text{ for } i = 0, \cdots, m_k\}$, the processes $\{X_n\}$ and $\{Y_n\}$ coincide for $n < T_{m_k k}$. Hence

$$\mathbb{P}^o(T_{nv} > n) \leq \mathbb{P}^o(\tilde{T}_{m_k k} > n) + m_k \mathbb{P}^o(L_0 > k). \tag{2.4.12}$$

But, as in (2.4.6), for k large enough

$$\mathbb{P}^o(L_0 > k) \leq E_P(e^{k(\overline{R}_k + \delta)}) \leq e^{\log n(Aa+\delta')} \leq n^{-s+\delta''},$$

where $\delta'' := \delta''(\delta) \to_{\delta\to 0} 0$. Since $m_k < n$, the second term in (2.4.12) is of the right order, and the upper bound follows as soon as we prove that, for n large enough

$$\mathbb{P}^o(\tilde{T}_{m_k k} > n) \le n^{1-s+\delta'} . \tag{2.4.13}$$

To see (2.4.13), note that $\tilde{T}_{m_k k} = \sum_{i=1}^{m_k} \tilde{\tau}_k$, with $\mathbb{E}^o(\tilde{\tau}_k^{(i)})/k = \mathbb{E}^o(\tilde{T}_k)/k \to 1/v_P$. Hence, for some $\eta > 0$, using that $km_k \le v < v_P$,

$$\begin{aligned}\mathbb{P}^o(\tilde{T}_{m_k k} > n) &\le \mathbb{P}^o\left(\sum_{i=1}^{m_k}\Big(\tilde{\tau}_k^{(i)} - \mathbb{E}^o(\tilde{\tau}_k^{(i)})\Big) > 4\eta n\right)\\ &\le 4\mathbb{P}^o\left(\sum_{i=1}^{\lceil m_k/4\rceil}\Big(\tilde{\tau}_k^{(4i)} - \mathbb{E}^o(\tilde{T}_k)\Big) > \eta n\right) .\end{aligned}$$

Note that the quenched law of $\tilde{\tau}_k^{(4i)}$ depends on $\{\omega_j, j \in I_i\}$ where $I_i = \{4i-k, 4i-k+1, \cdots, 4i+k\}$. Let $\{\overline{\tau}_k^{(i)}\}$ be i.i.d. random variables such that for any Borel set G, $P(\overline{\tau}_k^{(i)} \in G) = \mathbb{P}^o(\tilde{\tau}_k^{(i)} \in G)$. Then, by iterating the definition of $\alpha(\cdot)$, one has that

$$\begin{aligned}\mathbb{P}^o&\left(\sum_{i=1}^{\lceil m_k/4\rceil}\Big(\tilde{\tau}_k^{(4i)} - \mathbb{E}^o(\tilde{T}_k)\Big) > \eta n\right) \le\\ &\qquad P\left(\sum_{i=1}^{\lceil m_k/4\rceil}\Big(\overline{\tau}_k^{(4i)} - \mathbb{E}^o(\tilde{T}_k)\Big) > \eta n\right) + B_n \end{aligned}\tag{2.4.14}$$

where

$$B_n = \frac{n^2 m_k \alpha(2k)}{4} + \frac{m_k \mathbb{P}^o(\tilde{\tau}^{(1)} > n^2)}{4} .$$

The following estimate, whose proof is deferred, is crucial to the proof of (2.4.13):

Lemma 2.4.15 *For each $\kappa < s$, there exists a constant $c(\kappa) < \infty$ such that*

$$\mathbb{E}^o(T_k)^\kappa \le c(\kappa) k^\kappa . \tag{2.4.16}$$

One gets from the definition of $\alpha(2k)$ and Lemma 2.4.15 that

$$B_n \le o(n^{1-s}) . \tag{2.4.17}$$

Further, by Markov's inequality, for any $\kappa < \kappa' < s$,

$$P(\overline{\tau}_k^{(4i)} - E\overline{\tau}_k^{(4i)} > \eta n) \le \frac{1}{(\eta n)^{\kappa'}} E|\overline{\tau}_k^{(4i)} - E\overline{\tau}_k^{(4i)}|^{\kappa'} \le n^{-\kappa}$$

where n is large enough and we used Lemma 2.4.15 and the fact that $E((\overline{\tau}_k^{(4i)})^{\kappa'}) = \mathbb{E}^o((\tilde{\tau}_k^{(4i)})^{\kappa'}) \le \mathbb{E}^o(T_k^{\kappa'})$. Hence, (see [55, (1.3),(1.7a)]),

$$P\left(\sum_{i=1}^{\lceil m_k/4\rceil}\left(\overline{\tau}_k^{(4i)}-E\overline{\tau}_k^{(4i)}\right)>\eta n\right)$$
$$\leq \lceil\frac{m_k}{4}\rceil P(\overline{\tau}_k^{(4)}-E\overline{\tau}_k^{(4)}>\eta n)+\frac{1}{2}\,n^{1-\kappa}\leq n^{1-\kappa}\,.$$

Since $\kappa<s$ is arbitrary, this completes the proof, modulo the

Proof of Lemma 2.4.15 Note first that by Minkowski's inequality, for any $k\geq 1$,

$$\mathbb{E}^o(T_k^\kappa)=\mathbb{E}^o\left(\sum_{i=1}^k \tau_i\right)^\kappa\leq k^\kappa\mathbb{E}^o\tau_1^\kappa\,.$$

Hence, it will be enough to prove that

$$\mathbb{E}^o(\tau_1^\kappa)<\infty\,. \tag{2.4.18}$$

To prove (2.4.18), we build upon the techniques developed in the course of proving Lemma 2.1.21. Indeed, recall the random variables $U_{i,j}, Z_{i,j}$ and N_i defined there, and note that since $\tau_1=\sum_{i=-\infty}^o N_i$, it is enough to estimate

$$\mathbb{E}^o\left(\sum_{i=-\infty}^0 N_i\right)^\kappa=\mathbb{E}^o\left(\sum_{i=-\infty}^0 U_i+U_{i+1}+Z_i\right)^\kappa\leq C_\varepsilon\mathbb{E}^o\left(\sum_{i=-\infty}^0 U_i\right)^\kappa\,. \tag{2.4.19}$$

An important step in the evaluation of the RHS in (2.4.19) involves the computation of moments of U_i. To present the idea, consider first the case $\kappa>2$, and write

$$U_i=\sum_{j=1}^{U_{i+1}} G_j$$

where, under P_ω^o, the G_j are i.i.d. geometric random variables, independent of $\{U_{i+1},\cdots U_0\}$, of parameter $\frac{\omega_i^-}{\omega_i^-+\omega_i^+}$. Hence,

$$\begin{aligned}E_\omega^o(U_i^2)&=E_\omega^o\left(\sum_{j=1}^{U_i+1}(G_j-E_\omega^oG_j)+\sum_{j=1}^{U_{i+1}}E_\omega^oG_j\right)^2 \\ &\leq c_\delta E_\omega^o\left(\sum_{j=1}^{U_i+1}(G_j-E_\omega^oG_j)\right)^2+(1+\delta)(E_\omega^oG_j)^2\cdot E_\omega^o(U_{i+1}^2)\\ &\leq c'_\delta E_\omega^o(U_{i+1})\cdot E_\omega^o(G_j^2)+(1+\delta)\rho_i^2E_\omega^o(U_{i+1}^2)\,.\end{aligned}\tag{2.4.20}$$

Here, c_δ, c'_δ are constants which depend on δ only. Since $E_\omega^o(G_j^2)$ is uniformly (in ω) bounded, and $E_\omega^o(U_{i+1})=\rho_i$, we get

$$E^o_\omega(U_i^2) \le c''_\delta \rho_i E^o_\omega U_{i+1} + (1+\delta)\rho_i^2 E^o_\omega(U_{i+1}^2)\,.$$

Iterating and using (cf. (2.1.24)) that $E^o_\omega U_{i+1} = \prod_{j=i+1}^0 \rho_j$, we conclude the existence of a constant c'''_δ such that

$$E^o_\omega(U_i^2) \le c_\delta''' \left(\sum_{j=0}^{|i|} \left(\prod_{k=-j}^0 \rho_k + \prod_{k=-j}^0 \left(\rho_k^2(1+\delta)\right)\right)\right)\,,$$

and hence

$$\mathbb{E}^o(U_i^2) \le c_\delta''' \sum_{j=0}^{|i|} \left(E_P \prod_{k=-j}^0 \rho_k + E_P \prod_{k=-j}^0 \Big(\rho_k^2(1+\delta)\Big)\right)\,. \tag{2.4.21}$$

Note that, by Varadhan's lemma (see [20, Theorem 4.3.1]), for any constant β,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}^o \left(\prod_{k=-n}^0 \rho_k^\beta \right) = \sup_y \Big(\beta y - J(y)\Big) = \sup_y y \left(\beta - \frac{J(y)}{y}\right) := \beta'(\beta)\,, \tag{2.4.22}$$

and $\beta'(\beta) < 0$ as soon as $\beta < s$. Hence, substituting in (2.4.21), and choosing δ such that $\log(1+\delta) < \beta'(\beta)/4$, we obtain that for some constant c''''_δ,

$$\mathbb{E}^o(U_i^2) \le c_\delta'''' e^{-i\beta'(2)/2},$$

implying that

$$\sqrt{\mathbb{E}^o \left(\sum_{i=-\infty}^0 N_i \right)^2} \le C_\varepsilon^{\frac{1}{2}} \sqrt{\sum_{i=-\infty}^0 \mathbb{E}^o(U_i^2)} < \infty\,.$$

A similar argument holds for any integer $\kappa < s$: mimicking the steps leading to (2.4.21), we get that

$$E^o_\omega(U_i^\kappa) \le c_\delta''' \left(\sum_{j=0}^{|i|} \left(\prod_{k=-j}^0 \rho_k^{\kappa/2} + \prod_{k=-j}^0 \rho_k^\kappa \right)\right)\,,$$

and using (2.4.22) and an induction on lower (integer) moments, we get that $\mathbb{E}^o \left(\sum_{i=-\infty}^0 N_i\right)^\kappa < \infty$ for all $\kappa < s$ integer. Finally, to handle $\lfloor s \rfloor < \kappa < s$, we replace (2.4.20) by

$$\begin{aligned} E^o_\omega(U_i^\kappa) &\le c'_\delta E^o_\omega(U_{i+1}^{\kappa/2 \vee 1}) E^o_\omega(G_j^\kappa) + (1+\delta)\rho_i^\kappa E^o_\omega(U_{i+1}^\kappa) \\ &\le c_\delta''(E^o_\omega U_i^{\lceil \kappa/2 \rceil})^{\frac{\kappa/2}{\lceil \kappa/2 \rceil}} + (1+\delta)\rho_i^\kappa E^o_\omega(U_{i+1}^\kappa)\,, \end{aligned}$$

and one proceeds as before. □

Quenched subexponential estimates

Theorem 2.4.23 *Assume P satisfies Assumption 2.4.2, and $v_P > 0$. Then, for any $v \in (0, v_P)$, any $\eta > 0$, and any $\delta > 0$ small enough,*

$$\liminf_{n\to\infty} \frac{1}{n^{1-1/s+\eta}} \log P_\omega^o\left(\frac{X_n}{n} \in (v-\delta, v+\delta)\right) = 0\,, \quad P-a.s. \tag{2.4.24}$$

Further,

$$\limsup_{n\to\infty} \frac{1}{n^{1-1/s-\eta}} \log P_\omega^o\left(\frac{X_n}{n} < v\right) = -\infty\,, \quad P-a.s.. \tag{2.4.25}$$

Proof. Starting with the lower bound, we have, using (2.4.4) and (2.4.5), that for some $\delta_1(\omega) > 0$,

$$\begin{aligned} P_\omega^o(\frac{X_n}{n} \in (v-\delta, v+\delta)) \geq P_\omega^o&\left(\frac{(v-4\eta)n}{v_P} < T_{(v-2\eta)n} < n\right) \\ &P_\omega^{(v-\eta)n}\left(T_{vn} > \frac{4\eta}{v_P}n, L_{(v-\eta)n<\eta n}\right) - e^{-\delta_1(\omega)n}\,. \end{aligned}$$

By (2.1.16), $P_\omega^o(\frac{X_n}{n} \in (v-\delta, v+\delta)) \to_{n\to\infty} 1$, P-a.s. On the other hand, as in the proof of (2.4.8), fix y such that $J(y)/y \leq s + \delta'/4$, $k = \lfloor (1-\delta')/ys \rfloor$, and $K = n^{\delta'/4}$. Then, one checks as in the annealed case that

$$P\Big(\forall j \in \{1, \cdots, \overline{m}_K\} : \overline{R}_k(\theta^{jK}\omega) \leq y\Big) \leq \frac{1}{n^2}\,,$$

and one concludes by the Borel-Cantelli lemma that there exists an $n_0(\omega)$ such that for all $n_0(\omega)$, there exists a $j \in \{1, \ldots, \overline{m}_K\}$ such that $\overline{R}_k(\theta^{jK}\omega) > y$. The lower bound (2.4.24) now follows as in the proof of (2.4.9).

Turning to the proof of the upper bound (2.4.25), as in the annealed setup it is straightforward to reduce the proof to proving

$$\lim_{n\to\infty} \frac{1}{n^{1-1/s-\delta}} \log P_\omega^o(T_n > n/v) = -\infty\,. \tag{2.4.26}$$

We provide now a short sketch of the proof of (2.4.26) before getting our hands dirty in the actual computations. Divide the interval $[0, nv]$ into blocks of size roughly $k = k_n := n^{1/s+\delta}$. By using the annealed bounds of Theorem 2.4.3, one knows that $P(T_k > k/v) \sim k^{1-s}$. Hence, taking appropriate subsequences, one applies a Borel-Cantelli argument to control uniformly the probability $P_\omega^{ik}(T_{(i+1)k} > k/v)$, c.f. Lemma 2.4.28.

The next step involves a decoupling argument. Define

$$\overline{T}_{(i+1)k} = \inf\{t > T_{ik} : X_t = (i+1)k \text{ or } X_t = (i-1)k\}\,. \tag{2.4.27}$$

Then one shows that for all relevant blocks, that is $i = \pm 1, \pm 2, \ldots, \pm n/k$, $P_\omega^{ik}(\overline{T}_{(i+1)k} \neq T_{(i+1)k})$ is small enough. Therefore, we can consider the random variables $\overline{T}_{(i+1)k} - T_{ik}$ instead of $T_{(i+1)k} - T_{ik}$, which have the advantage that their dependence on the environment is well localized. This allows us to obtain a uniform bound on the tails of $\overline{T}_{(i+1)k} - T_{ik}$, *for all relevant* i, see (2.4.30).

The final step involves estimating how many of the k-blocks will be traversed from right to left before the RWRE hits the point nv. This is done by constructing a simple random walk (SRW) S_t whose probability of jump to the left dominates $P_\omega^{ik}(T_{(i+1)} \neq \overline{T}_{(i+1)k})$ for all relevant i. The analysis of this SRW will allow us to claim (c.f. Lemma 2.1.17) that the number of visits to a k-block after entering its right neighbor is negligible. Thus, the original question on the tail of T_n is replaced by a question on the sum of (dominated by i.i.d.) random variables, which is resolved by means of the tail estimates obtained in the second step.

A slight complication is presented by the need to work with subsequences in order to apply the Borel-Cantelli lemma at various places. Going from subsequences to the original n sequence is achieved by means of monotonicity arguments. Indeed, by monotonicity, note that it is enough to prove the result when, for arbitrary δ small enough, n is replaced by the subsequence $n_j = \lfloor j^{2/\delta} \rfloor$, since $n_{j+1}/n_j \to_{j\to\infty} 1$.

Turning to the actual proof, fix $C_n = n^\delta$, $k = k_j = \dfrac{C_{n_j} n_j^{1/s}}{1-\varepsilon}$ for some $1 > \varepsilon > 0$, $b_n = C_n^{-\delta}$, and $I_j = \left\{-\left[\frac{n_j}{k_j}\right] - 1, \cdots, \left[\frac{n_j}{k_j}\right] + 1\right\}$. Finally, fix $v' < v$ and $\overline{T}_{(i+1)k}$ as in (2.4.27). (We will always use $\overline{T}_{(i+1)k}$ in conjunction with the RWRE started at ik!). We now claim the:

Lemma 2.4.28 *For P – a.e. ω, there exists a $J_0(\omega)$ such that for all $j > J_0(\omega)$, and all $i \in I_j$,*

$$P_\omega^{ik}\left(\frac{T_{(i+1)k_j}}{k_j} > \frac{1}{v'}\right) \le b_{n_j}\ . \tag{2.4.29}$$

Further, for all $j > J_0(\omega)$, and each $i \in I_j$, *and for $x \ge 1$,*

$$P_\omega^{(ik)}\left(\frac{\overline{T}_{(i+1)k_j}}{k_j} > \frac{x}{v'}\right) \le (2b_{n_j})^{[x/2]\vee 1}\ . \tag{2.4.30}$$

Proof of Lemma 2.4.28. By Chebycheff's bound,

$$\begin{aligned} P\left(P_\omega^{ik}\Big(\frac{T_{(i+1)k_j}}{k_j} > \frac{1}{v'}\Big) > b_{n_j}\right) &\le \frac{1}{b_{n_j}}\,\mathbb{P}^{ik}\Big(\frac{T_{(i+1)k_j}}{k_j} > \frac{1}{v'}\Big) \\ &\le \frac{1}{b_{n_j}}\,k_j^{1-s+o(1)}\ , \end{aligned}$$

where the last inequality follows from Theorem 2.4.3. Hence,

$$P\left(P_\omega^{ik}\Big(\frac{T_{(i+1)k_j}}{k_j}>\frac{1}{v'}\Big)>b_{n_j}\text{ for some }i\in I_j\right)\leq 3\left[\frac{n_j}{k_j}\right]\cdot\frac{1}{b_{n_j}}\cdot k_j^{1-s+o(1)}$$
$$\leq\frac{3}{n_j^{\delta(s-o(1)-\delta)}}\leq\frac{4}{j^{2(s-o(1)-\delta)}}$$

and (2.4.29) follows from the Borel-Cantelli lemma. (2.4.30) follows by iterating this inequality and using the Markov property. □

Recall that $a=\sup_y(y-J(y))<0$ and let $0<\theta<-\frac{a}{1-\varepsilon/4}$, $d_n^\theta=e^{-\theta n^{1/s}C_n}$. We now have:

Lemma 2.4.31 *For P – a.e. ω, there is a $J_1(\omega)$ s.t. for all $j\geq J_1(\omega)$, all $i\in I_j$,*
$$P_\omega^{ik}\Big(\overline{T}_{(i+1)k_j}\neq T_{(i+1)k_j}\Big)\leq d_{n_j}^\theta\ .$$

Proof of Lemma 2.4.31. Again, we use the Chebycheff bound:

$$\begin{aligned}&P\Big(P_\omega^{ik}\Big(\overline{T}_{(i+1)k_j}\neq T_{(i+1)k_j}\Big)>d_{n_j}^\theta\,,\text{ some }\quad i\in I_j\Big)\\&\leq\frac{1}{d_{n_j}^\theta}\cdot\frac{3n_j}{k_j}\,\mathbb{P}^o\Big(\overline{T}_{k_j}\neq T_{k_j}\Big)\\&\leq\frac{1}{d_{n_j}^\theta}\cdot\frac{3n_j}{k_j}\cdot\exp\left(-k_ja(1-\varepsilon/2)\right)\\&\leq 3\,n_j^{1-\frac{1}{s}-\delta}\exp\left(n_j^{\frac{1}{s}+\delta}\left(\frac{a}{1-\varepsilon/4}+\theta\right)\right)\ ,\end{aligned}$$

where the second inequality follows again from (2.1.4) and the LDP for $\overline{R}_k$. The conclusion follows from the Borel-Cantelli lemma. □

We need one more preliminary computation related to the bounds in (2.4.30). Let $\{Z_{k_j}^{(i)}\}$, $i=1,2,\ldots$ denote a sequence of i.i.d. positive random variables, with

$$P\left(\frac{Z_{k_j}^{(i)}}{k_j}<\mu'\right)=0\,,\quad P\left(\frac{Z_{k_j}^{(i)}}{k_j}>\mu'x\right)=\Big(2b_{n_j}\Big)^{[x/2]\vee 1}\,,\quad x\geq 1\,.$$

Note now that for any $\lambda>0$, and any $\varepsilon>0$,

$$\begin{aligned}E\left(\exp\left(\lambda\frac{Z_{k_j}^{(i)}}{k_j}\right)\right)&=\int_0^\infty P\Big(\frac{Z_{k_j}^{(i)}}{k_j}>\frac{\log u}{\lambda}\Big)\,du\\&\leq e^{\lambda\mu'(1+\varepsilon)}+\int_{e^{\lambda\mu'(1+\varepsilon)}}^\infty(2b_{n_j})^{\left[\frac{\log u}{2\lambda\mu'(1+\varepsilon)}\right]\vee 1}du\\&=e^{\lambda\mu'(1+\varepsilon)}+g_j\,.\end{aligned}\tag{2.4.32}$$

where $g_j \to_{j\to\infty} 0$.

In order to control the number of repetitions of visits to k_j–blocks, we introduce an auxiliary random walk. Let S_t, $t = 0, 1, \ldots,$ denote a simple random walk with $S_0 = 0$ and

$$P\Big(S_{t+1} = S_t + 1 \Big| S_t\Big) = 1 - P\Big(S_{t+1} = S_t - 1 \Big| S_t\Big) = 1 - d_n^\theta \,.$$

Set $M_{n_j} = \dfrac{1}{C_{n_j}} n_j^{1-\frac{1}{s}}$.

Lemma 2.4.33 *For θ as in Lemma 2.4.31, and n large enough,*

$$P\Big(\inf\{t: S_t = \Big[\frac{n_j}{k_j}\Big]\} > M_{n_j}\Big) \le \exp\left(-\frac{\theta\varepsilon}{2} n_j\right) \,.$$

Proof of Lemma 2.4.33.

$$\begin{aligned} P\left(\inf\Big\{t: S_t = \Big[\frac{n_j}{k_j}\Big]\Big\} > M_{n_j}\right) &\le P\left(\frac{S_{[M_{n_j}]}}{M_{n_j}} < \frac{n_j}{k_j\, M_{n_j}}\right) \\ &= P\left(\frac{S_{[M_{n_j}]}}{M_{n_j}} < 1-\varepsilon\right) \le 2\, e^{-M_{n_j} h_{n_j}(1-\varepsilon)} \,, \end{aligned}$$

where the last inequality is a consequence of Chebycheff's inequality and the fact that $d_n^\theta < \varepsilon$. Here,

$$h_n(1-x) = (1-x)\log\left(\frac{1-x}{1-d_n^\theta}\right) + x \log\frac{x}{d_n^\theta} \,.$$

Using $h_n(1-x) \ge -\frac{2}{e} - x\log d_n^\theta$, we get

$$P\left(\frac{S_{[M_{n_j}]}}{M_{n_j}} < 1-\varepsilon\right) \le 2\, e^{2M_{n_j}/e}\, e^{+\varepsilon M_{n_j}\log d_{n_j}^\theta} \le e^{-\frac{\varepsilon}{2}\theta n_j} \,. \qquad \square$$

We are now ready to prove (2.4.26). Note that, for all $j > J_0(\omega)$, and all $i \in I_j$, we may, due to (2.4.30), construct $\{Z_{k_j}^{(i)}\}$ and $\{\overline{T}_{(i+1)k_j}\}$ on the same probability space such that for all $i \in I_j$, $P_\omega^{ik}(Z_{k_j}^{(i)} \ge \overline{T}_{(i+1)k_j}) = 1$. Fix $1/v_P > 1/v' > 1/v$ and $\varepsilon > 0$ small enough. Recalling that under the law P_ω^o, the random variables $\overline{T}_{k_j}^{(i)} := \overline{T}_{(i+1)k_j} - T_{ik_j}$ are independent, we obtain, with $\{S_t\}$ defined before Lemma 2.4.33, and j large enough,

$$
\begin{aligned}
P^o_\omega(T_{n_j} > n_j/v) &\le P\Big(\inf\Big\{t:\, S_t = \Big[\frac{n_j}{k_j}\Big]\Big\} > M_{n_j}\Big) + P\Big(\sum_{i=1}^{M_{n_j}} Z^{(i)}_{k_j} > n_j/v\Big) \\
&\le e^{-\theta\varepsilon n_j/2} + P\Big(\frac{1}{M_{n_j}}\sum_{i=1}^{M_{n_j}} \frac{Z^{(i)}_{k_j}}{k_j} > 1/v(1-\varepsilon)\Big) \\
&\le e^{-\theta\varepsilon n_j/2} + \Big[E\Big(\exp\left(\lambda \frac{Z^{(i)}_{k_j}}{k^{(i)}_j}\right)\Big)\cdot e^{-\lambda(1-\varepsilon)/v}\Big]^{M_{n_j}} \\
&\le e^{-\theta\varepsilon n_j/2} + \Big(e^{\lambda(1/v'+2\varepsilon/v-1/v)} + g_j e^{-\lambda(1-\varepsilon)/v}\Big)^{M_{n_j}} \\
&\le e^{-\theta\varepsilon n_j/2} + \Big(e^{-\lambda\varepsilon/v}\Big)^{M_{n_j}}\,,
\end{aligned}
$$

where Lemma 2.4.33 was used in the second inequality and (2.4.32) in the fourth. Since $\lambda > 0$ is arbitrary, (2.4.26) follows. □

Remarks: 1. A study of the proof of the annealed estimates shows that the strong mixing condition **(D3)** can be replaced by the sightly milder one that $\alpha(n) = \exp(-Cn)$ for some C large enough such that (2.4.17) holds, *if* one also assumes the existence of a LDP for $\overline{R}_k$. In this form, the assumption is satisfied for many Markov chains satisfying a Doeblin condition.

2. It is worthwhile noting that the transfer of the annealed estimates to the quenched setting required very few assumptions on the environment, besides the existence of a LDP for $\overline{R}_k$. This technique, as we will see, is not limited to the one-dimensional setup, and works well in situations where a drift is present.

3. One may study by similar techniques also the case where $E_P(\overline{S}) < \infty$ but $\rho_{\max} = 1$ with $\alpha := P(\rho_{\max} = 1) > 0$. The rate of decay is then quite different: at least when the environment is i.i.d., the annealed rate of decay in Theorem 2.4.3 is exponential with exponent $n^{1/3}$, see [19], whereas the quenched one has exponent $n/(\log n)^2$, see [31], and it seems both proofs extend to the mixing setup. By adapting the method of enlargement of obstacles to this setup, one actually can show more in the i.i.d. environment case: it holds then that,

$$
\lim_{\delta\to 0}\lim_{n\to\infty} \frac{1}{n^{1/3}}\log \mathbb{P}^o\left(\frac{X_n}{n} \in (v-\delta, v+\delta)\right) = -\frac{3}{2}\Big|\frac{\pi\log\alpha}{2}\Big|^{2/3}, \qquad (2.4.34)
$$

and

$$
\lim_{\delta\to 0}\lim_{n\to\infty} \frac{(\log n)^2}{n}\log P^o\left(\frac{X_n}{n} \in (v-\delta, v+\delta)\right) = -\frac{|\pi\log\alpha|^2}{8}\Big(1-\frac{v}{v_P}\Big)\,, \qquad (2.4.35)
$$

see [61] and [62]. (Note that the lower bounds in (2.4.34) and (2.4.35) are not hard to obtain, by constructing “neutral” traps. The difficulty lies in matching the constants in the upper bound to the ones in the lower bound.) The

technique of enlargement of obstacles in this context is based on considerably refining the classification of blocks used above when going from annealed to quenched estimates, by introducing the notion of "good" and "bad" blocks (and double blocks...)
4. One can check, at least in the i.i.d. environment case, that when $\rho_{\max} = 1$ with $\alpha = 0$ then intermediate decay rates, between Theorems 2.4.3, 2.4.23 and (2.4.34), (2.4.35) can be achieved. We do not elaborate further here.
5. Again in the case of i.i.d. environment and the setup of Theorem 2.4.23, one can show, c.f. [31], that

$$\limsup_{n\to\infty} \frac{1}{n^{1-1/s}} \log P_\omega^o\left(\frac{X_n}{n} \in (v-\delta, v+\delta)\right) = 0\,, \quad P-a.s. \tag{2.4.36}$$

This is due to fluctuations in the length of the "significant" trap where the walk may stay for large time. Based on the study of these fluctuations, it is reasonable to conjecture that

$$\liminf_{n\to\infty} \frac{1}{n^{1-1/s}} \log P_\omega^o\left(\frac{X_n}{n} \in (v-\delta, v+\delta)\right) = -\infty\,, \quad P-a.s.,$$

explaining the need for δ in the statement of Theorem 2.4.23. This conjecture has been verified only in the case where $P(\rho_{\min} = 0) > 0$, i.e. in the presence of "reflecting nodes", c.f. [30, 29].
Bibliographical notes: The derivation in this section is based on [19] and [31]. Other relevant references, giving additional information not described here, are described in the remarks at the end of the section, so we only mention them here without repeating the description given there: [30, 61, 62].

2.5 Sinai's model: non standard limit laws and aging properties

Throughout this section, define $\overline{R}_k = k^{-1}\sum_{i=1}^{k-1} \log \rho_i(\text{sign } i)$. We assume the following

Assumption 2.5.1

(E1) *Assumption 2.1.1 holds.*
(E2) $E_P \log \rho_0 = 0$, *and there exists an* $\varepsilon > 0$ *such that* $E_P|\log\rho_0|^{2+\varepsilon} < \infty$.
(E3) *P is strongly mixing, and the functional invariance principle holds for* $\sqrt{k}\,\overline{R}_k/\sigma_P$; *that is,* $\{\sqrt{k}\,\overline{R}_{[kt]}/\sigma_P\}_{t\in\mathbb{R}}$ *converges weakly to a Brownian motion for some* $\sigma_P > 0$ *(sufficient conditions for such convergence are as in Lemma 2.2.4).*

(In the i.i.d. case, note that $\sigma_P^2 = E_P(\log\rho_0)^2$). Define

$$W^n(t) = \frac{1}{\log n}\sum_{i=0}^{\lfloor(\log n)^2 t\rfloor} \log\rho_i \cdot (\text{sign } t)$$

with $t \in \mathbb{R}$. By Assumption 2.5.1, $\{W^n(t)\}_{t\in\mathbb{R}}$ converges weakly to $\{\sigma_P B_t\}$, where $\{B_t\}$ is a two sided Brownian motion.

Next, we call a triple (a,b,c) with $a<b<c$ a valley of the path $\{W^n(\cdot)\}$ if

$$\begin{aligned} W^n(b) &= \min_{a\le t\le c} W^n(t)\,,\\ W^n(a) &= \max_{a\le t\le b} W^n(t)\,,\\ W^n(c) &= \max_{b\le t\le c} W^n(t)\,. \end{aligned}$$

The *depth* of the valley is defined as

$$d_{(a,b,c)} = \min(W^n(a)-W^n(b), W^n(c)-W^n(b))\,.$$

If (a,b,c) is a valley, and $a<d<e<b$ are such that

$$W^n(e)-W^n(d) = \max_{a\le x<y\le b} W^n(y)-W^n(x)$$

then (a,d,e) and (e,b,c) are again valleys, which are obtained from (a,b,c) by a *left refinement*. One defines similarly a *right refinement*. Define

$$\begin{aligned} c_0^n &= \min\{t\ge 0: \quad W^n(t)\ge 1\}\\ a_0^n &= \max\{t\le 0: \quad W^n(t)\ge 1\}\\ W^n(b_0^n) &= \min_{a_0^n\le t\le c_0^n} W^n(t)\,. \end{aligned}$$

(b_0^n is not uniquely defined, however, due to Assumption 2.5.1, with P-probability approaching 1 as $n\to\infty$, all candidates for b_0^n are within distance converging to 0 as $n\to\infty$; we define b_0^n then as the smallest one in absolute value.)

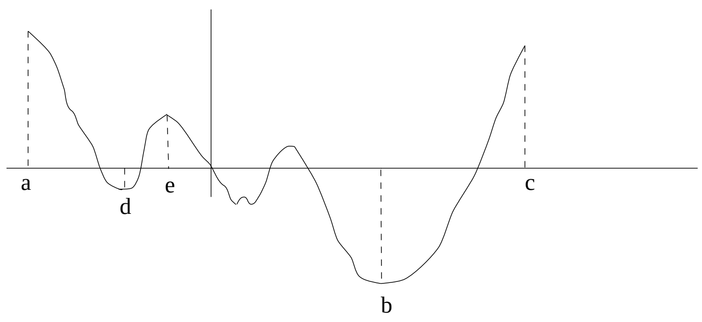


Fig. 2.5.1. Left refinement of (a,b,c)

One may now apply a (finite) sequence of refinements to find the *smallest* valley $(\overline{a}^n, \overline{b}^n, \overline{c}^n)$ with $\overline{a}^n < 0 < \overline{c}^n$, while $d_{(\overline{a}^n,\overline{b}^n,\overline{c}^n)} \geq 1$. We define similarly the smallest valley $(\overline{a}^n_\delta, \overline{b}^n_\delta, \overline{c}^n_\delta)$ such that $d_{(\overline{a}^n_\delta,\overline{b}^n_\delta,\overline{c}^n_\delta)} \geq 1+\delta$. Let

$$A_n^{J,\delta} = \left\{ \begin{array}{c} \omega \in \Omega : \overline{b}^n = \overline{b}^n_\delta, \text{ any refinement } (a,b,c) \text{ of } (\overline{a}^n_\delta, \overline{b}^n_\delta, \overline{c}^n_\delta) \text{ with} \\ b \neq \overline{b}^n \text{ has depth } < 1-\delta\,, |\overline{a}^n_\delta| + |\overline{c}^n_\delta| \leq J\,, \\ \min_{t\in[\overline{a}^n,\overline{c}^n]\setminus[\overline{b}^n-\delta,\overline{b}^n+\delta]} W^n(t) - W^n(\overline{b}^n) > \delta^3 \end{array} \right\}$$

then it is easy to check by the properties of Brownian motion that

$$\lim_{\delta\to 0} \lim_{J\to\infty} \lim_{n\to\infty} P(A_n^{J,\delta}) = 1\,. \tag{2.5.2}$$

The following is the main result of this section:

Theorem 2.5.3 *Assume $P(\min(\omega_0^-, \omega_0^+) < \varepsilon) = 0$ and Assumption 2.5.1. For any $\eta > 0$,*

$$\mathbb{P}^o\left(\left|\frac{X_n}{(\log n)^2} - \overline{b}^n\right| > \eta\right) \underset{n\to\infty}{\to} 0\,.$$

Proof. Fix $\delta < \eta/2, J$ and n large enough with $\omega \in A_n^{J,\delta}$. For simplicity of notations, assume in the sequel that ω is such that $\overline{b}^n > 0$. Write $a^n = \overline{a}^n(\log n)^2, b^n = \overline{b}^n(\log n)^2, c^n = \overline{c}^n(\log n)^2$, with similar notations for $a^n_\delta, b^n_\delta, c^n_\delta$. Define

$$\overline{T}_{b,n} = \min\{t \geq 0 : X_t = b^n \text{ or } X_t = a^n_\delta\}\,.$$

By (2.1.4),

$$P^o_\omega\Big(X_{\overline{T}_{b,n}} = a^n_\delta\Big) \leq \frac{1}{1 + \frac{\exp\{(\log n)(W^n(\overline{a}^n_\delta) - W^n(\overline{b}^n))\}}{Jn(\log n)^2}} \leq \frac{J(\log n)^2}{n^\delta}\,. \tag{2.5.4}$$

On the other hand, let $\tilde{T}_{b,n}$ have the law of $\overline{T}_{b,n}$ except that the walk $\{X_\cdot\}$ is reflected at a^n_δ, and define similarly $\tilde{\tau}_1$. Using the same recursions as in (2.1.14), we have that

$$E^o_\omega(\tilde{\tau}_1) = \frac{1}{\omega_0^+} + \frac{\rho_0}{\omega^+_{(-1)}} + \cdots + \frac{\prod_{i=0}^{a^n_\delta+2} \rho_{-i}}{\omega^+_{a^n_\delta-1}} + \prod_{i=0}^{a^n_\delta+1} \rho_{-i}\,.$$

Hence, with $\tilde{\omega}_i = \omega_i$ for $i \neq a^n_\delta$ and $\tilde{\omega}^+_{a^n_\delta} = 1$, for all n large enough,

$$\begin{aligned} E^o_\omega(\overline{T}_{b,n}) \leq E^o_\omega(\tilde{T}_{b,n}) &= \sum_{i=1}^{b^n} \sum_{j=0}^{i-1-a^n_\delta} \frac{\prod_{k=1}^j \rho_{i-k}}{\omega^+_{(i-j-1)}} \\ &\leq \frac{1}{\varepsilon} \sum_{i=1}^{b^n} \sum_{j=0}^{i-1-a^n_\delta} e^{(\log n)(W^n(i)-W^n(i-j))} \leq \frac{2J^2}{\varepsilon} e^{\log n(1-\delta)} \leq n^{1-\frac{\delta}{2}}\,. \end{aligned}$$

We thus conclude that

$$P_\omega^o\Big(\overline{T}_{b,n} < n, \quad X_{\overline{T}_{b,n}} = b^n\Big) \underset{n\to\infty}{\longrightarrow} 1$$

implying that

$$P_\omega^o\Big(T_{b^n} < n\Big) \underset{n\to\infty}{\to} 1\,. \tag{2.5.5}$$

Next note that another application of (2.1.4) yields

$$\begin{aligned} &P_\omega^{b^n-1}(X_\cdot \text{ hits } b^n \text{ before } a_\delta^n) \ge 1 - n^{-(1+\frac{\delta}{2})} \\ &P_\omega^{b^n+1}(X_\cdot \text{ hits } b^n \text{ before } c_\delta^n) \ge 1 - n^{-(1+\frac{\delta}{2})}\,. \end{aligned} \tag{2.5.6}$$

On the same probability space, construct a RWRE $\{\tilde{X}_t\}$ with the same transition mechanism as $\{X_t\}$ except that it is reflected at a_δ^n, i.e. replace ω by $\tilde{\omega}$. Then, using (2.5.6),

$$\begin{aligned} P_\omega^o\left(\left|\frac{X_n}{(\log n)^2} - \overline{b}^n\right| > \delta\right) &\le P_\omega^o\Big(T_{b^n} > n\Big) + \max_{t\le n} P_\omega^{b^n}\left(\left|\frac{X_t}{(\log n)^2} - \overline{b}^n\right| > \delta\right) \\ &\le P_\omega^o\Big(T_{b^n} > n\Big) + \Big[1 - (1 - n^{-(1+\frac{\delta}{2})})^n\Big] \\ &\quad + \max_{t\le n} P_\omega^{b^n}\left(\left|\frac{\tilde{X}_t}{(\log n)^2} - \overline{b}^n\right| > \delta\right)\,. \end{aligned}$$

Hence, in view of (2.5.2) and (2.5.5), the theorem holds as soon as we show that

$$\sup_{\omega\in A_n^{J,\delta}} \max_{t\le n} P_\omega^{b^n}\left(\left|\frac{\tilde{X}_t}{(\log n)^2} - \overline{b}^n\right| > \delta\right) \underset{n\to\infty}{\longrightarrow} 0\,. \tag{2.5.7}$$

To see (2.5.7), define

$$f(z) = \frac{\prod_{a_\delta^n+1\le i<z} \omega_i^+}{\prod_{a_\delta^n+1\le i<z} \omega_{i+1}^-}, \quad \overline{f}(z) = \frac{f(z)}{f(b^n)}$$

(as usual, the product over an empty set of indices is taken as 1. $\overline{f}(\cdot)$ corresponds to the invariant measure for the resistor network corresponding to $\tilde{X}_\cdot$). Next, define the operator

$$(Ag)(z) = \overline{\omega}_{z-1}^+ g(z-1) + \overline{\omega}_{z+1}^- g(z+1) + \overline{\omega}_z^0 g(z) \tag{2.5.8}$$

where $\overline{\omega}_z = \omega_z$ for $z > a_\delta^n$, $\overline{\omega}_{a_\delta^n}^+ = 1, \overline{\omega}_{a_\delta^n-1}^+ = 0$. Note that $A\overline{f} = \overline{f}$, and further that

$$P_\omega^{b^n}(\tilde{X}_t = z) = A^t \mathbf{1}_{b^n}(z)\,.$$

Since $\overline{f}(z) \ge \mathbf{1}_{b^n}(z)$ and A is a positive operator, we conclude that

$$P_\omega^{b^n}(\tilde{X}_t = z) \le \overline{f}(z)\,.$$

But, for z with $|z/(\log n)^2 - \overline{b}^n| > \delta$, it holds that $\overline{f}(z) \le e^{-\delta^3 \log n}$, and hence

$$P_\omega^{b^n}(\tilde{X}_t = z) \le n^{-\delta^3} .$$

Thus, for $\omega \in A_n^{J,\delta}$, using the fact that the second inequality in (2.5.6) still applies for $\tilde{X}$,

$$\max_{t\le n} P_\omega^{b^n}\left(\left|\frac{\tilde{X}_t}{(\log n)^2} - \overline{b}^n\right| > \delta\right) \le J(\log n)^2 n^{-\delta^3} + 1 - \left(1 - n^{-(1+\delta/2)}\right)^n ,$$

yielding (2.5.7) and completing the proof of the theorem. □

We next turn to a somewhat more detailed study of the random variable $\overline{b}^n$. By replacing 1 with t in the definition of $\overline{b}^n$, one obtains a process $\{\overline{b}^n(t)\}_{t\ge0}$. Further, due to Assumption 2.5.1, the process $\{\overline{b}^n(t/\sigma_P)\}_{t\ge0}$ converges weakly to a process $\{\overline{b}(t)\}_{t\ge0}$, defined in terms of the Brownian motion $\{B_t\}_{t\ge0}$; Indeed, $\overline{b}(t)$ is the location of the bottom of the smallest valley of $\{B_t\}_{t\ge0}$, which surrounds 0 and has depth t. Throughout this section we denote by $\mathcal{Q}$ the law of the Brownian motion $B_\cdot$. Our next goal is to characterize the process $\{\overline{b}(t)\}_{t\ge0}$. Toward this end, define

$$\begin{aligned}
m_+(t) &= \min\{B_s : 0 \le s \le t\}\,, m_-(t) = \min\{B_{-s} : 0 \le s \le t\}\\
T_+(a) &= \inf\{s \ge 0 : B_s - m_+(s) = a\}\,,\\
T_-(a) &= \inf\{s \ge 0 : B_{-s} - m_-(s) = a\}\\
s_\pm(a) &= \inf\{s \ge 0 : m_\pm(T_\pm(a)) = B_{\pm s}\}\,,\\
M_\pm(a) &= \sup\{B_{\pm\eta} : \quad 0 \le \eta \le s_\pm(a)\}\,.
\end{aligned}$$

Next, define $W_\pm(a) = B_{s_\pm(a)}$. It is not hard to check that the pairs $(M_+(\cdot), W_+(\cdot))$ and $(M_-(\cdot), W_-(\cdot))$ form independent Markov processes. Define finally

$$H_\pm(a) = (W_\pm(a) + a) \vee M_\pm(a)\,.$$

We now have the

Theorem 2.5.9 *For each $a > 0$, $\mathcal{Q}(\overline{b}(a) \in \{s_+(a), -s_-(a)\}) = 1$. Further, $\overline{b}(a) = s_+(a)$ iff $H_+(a) < H_-(a)$.*

Proof. Note that $\mathcal{Q}(H_+(a) = H_-(a)) = 0$. That $\overline{b}(a) \in \{s_+(a), -s_-(a)\}$ is a direct consequence of the definitions, i.e. assuming $\overline{b}(a) > 0$ and $\overline{b}(a) \ne s_+(a)$ it is easy to show that one may refine from the right the valley defining $\overline{b}(a)$, contradicting minimality. We begin by showing, after Kesten [42], that $\overline{b}(a) = s_+(a)$ iff either

$$W_-(a) > W_+(a), \quad M_+(a) < (W_-(a) + a) \vee M_-(a) \tag{2.5.10}$$

or

$$W_-(a) < W_+(a), \quad M_-(a) > (W_+(a) + a) \vee M_+(a)\,. \tag{2.5.11}$$

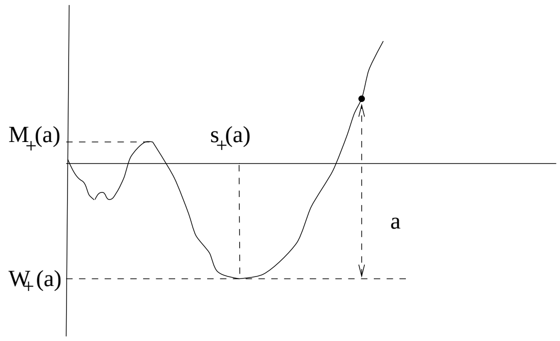


Fig. 2.5.2. The random variables $(M_+(a), W_+(a), s_+(a))$

Indeed, assume $\overline{b}(a) = s_+(a)$, and $W_-(a) > W_+(a)$. Let $(\alpha, \overline{b}(a), \gamma)$ denote the minimal valley defining $\overline{b}(a)$. If $-s_-(a) \leq \alpha$, then

$$\begin{aligned} M_-(a) &= \max\{B_{-s} : \; s \in (0, s_-(a))\} \geq B_\alpha \\ &= \max\{B_s : -\alpha \leq s \leq \overline{b}(a)\} \geq M_+(a) \end{aligned} \tag{2.5.12}$$

implying (2.5.10). On the other hand, if $-s_-(a) > \alpha$, refine $(\alpha, \overline{b}(a), \gamma)$ on the left (find α', β' with $\alpha < \alpha' < \beta' < \overline{b}(a)$), such that

$$B_{\beta'} - B_{\alpha'} = \max_{\alpha < x < y < \overline{b}(\alpha)} (B_y - B_x) \geq M_+(a) - W_-(a)$$

and thus minimality of $(\alpha, \overline{b}(a), \gamma)$ implies that $M_+(a) - W_-(a) < a$, implying (2.5.10).

We thus showed that if $\overline{b}(a) = s_+(a)$ and $W_-(a) > W_+(a)$ then (2.5.10) holds. On the other hand, if (2.5.10) holds, we show that $\overline{b}(a) = s_+(a)$ by considering the cases $\alpha \leq -s_-(a)$ and $-s_-(a) < \alpha$ separately. In the former case, necessarily $\gamma > s_+$, for otherwise $M_-(\alpha) \leq B_\gamma \leq M_+(a) \leq W_-(a) + a$ which together with $\overline{b}(a) = -s_-(a)$ would imply that the depth of $(\alpha, \overline{b}(a), \gamma)$ is smaller than a. Thus, under (2.5.10) if $\alpha \leq -s_-(a)$ then $\gamma > s_+$, and in this case $\overline{b}(a) = s_+(a)$ since $B_{s_+(a)} < B_{-s_-(a)}$. Finally, if $\alpha > -s_-(a)$ then $\overline{b}(a) \neq -s_-(a)$ and hence $\overline{b}(a) = s_+(a)$.

Hence, we showed that if $W_-(a) > W_+(a)$ then (2.5.10) is equivalent to $\overline{b}(a) = s_+(a)$. Interchanging the positive and negative axis, we conclude that if $W_-(a) < W_+(a)$, then $\overline{b}(a) = -s_-(a)$ iff $M_+(a) < (W_+(a)+1) \vee M_+(a)$. This completes the proof that $\overline{b}(a) = s_+(a)$ is equivalent to (2.5.10) or (2.5.11).

To complete the proof of the theorem, assume first $W_-(a) > W_+(a)$. Then, $\overline{b}(a) = s_+(a)$ iff (2.5.10) holds, i.e. $M_+(a) < (W_-(a) + a) \vee M_-(a) = H_-(a)$. But $H_-(a) \geq W_-(a) + a \geq W_+(a) + a$, and hence $M_+(a) < H_-(a)$ is equivalent to $M_+(a) \vee (W_+(a) + a) < H_-(a)$, i.e. $H_+(a) < H_-(a)$. The case $W_+(a) < W_-(a)$ is handled similarly by using (2.5.11). $\square$

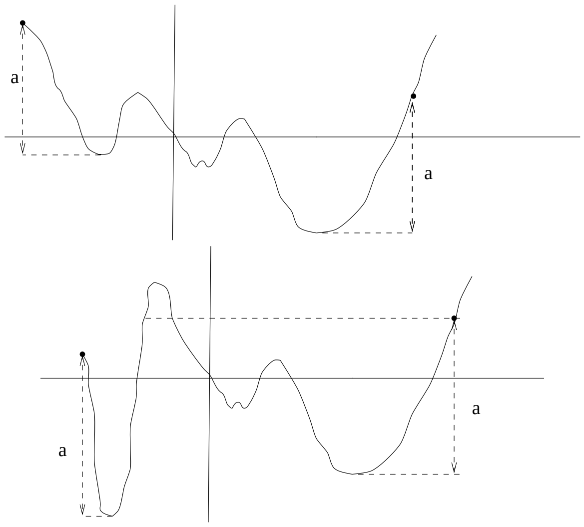


Fig. 2.5.3. $\overline{b}(a) = s_+(a)$

One may use the representation in Theorem 2.5.9 in order to evaluate explicitly the law of $\overline{b}(a)$ (note that $\overline{b}(a) \stackrel{\mathcal{L}}{=} a^2\overline{b}(1)$ by Brownian scaling). This is done in [42], and we do not repeat the construction here. Our goal is to use Theorem 2.5.9 to show that Sinai's model exhibits *aging* properties. More precisely, we claim that

Theorem 2.5.13 *Assume* $P(\min(\omega_0, \omega_0^+) < \varepsilon) = 0$ *and Assumption 2.5.1. Then, for* $h > 1$,

$$\lim_{\eta \to 0} \lim_{n \to \infty} \mathbb{P}^o\left(\frac{|X_{n^h} - X_n|}{(\log n)^2} < \eta\right) = \frac{1}{h^2}\left[\frac{5}{3} - \frac{2}{3}e^{-(h-1)}\right]. \tag{2.5.14}$$

Proof. Applying Theorem 2.5.3, the limit in the left hand side of (2.5.14) equals

$$\mathcal{Q}\Big(\overline{b}(h) = \overline{b}(1)\Big) = 2\mathcal{Q}\Big(\overline{b}(h) = \overline{b}(1) = s_+(1) = s_+(h)\Big).$$

Note that

$$\mathcal{Q}(s_+(h) = s_+(1)) = \mathcal{Q}\left(\begin{array}{c}\text{Brownian motion, started at height 1,}\\ \text{hits } h \text{ before hitting } 0\end{array}\right) = \frac{1}{h}.$$

Hence, using that on $s_+(1) = s_+(h)$ one has $W_+(1) = W_+(h), M_+(1) = M_+(h)$, and using that the event $\{s_+(h) = s_+(1)\}$ depends only on increments of the path of the Brownian motion after time $T_+(1)$, one gets

$$\mathcal{Q}\Big(\overline{b}(h)=\overline{b}(1)\Big)=\frac{2}{h}\mathcal{Q}\Big(H_+(1)<H_-(1),(W_+(1)+h)\vee M_+(1)<H_-(h)\Big)\,. \tag{2.5.15}$$

Next, let

$$\begin{aligned}\tau_0&=\min\{t>s_-(1):B_{-t}=W_-(1)+1\}\\ \tau_h&=\min\{t>\tau_0:\ B_{-t}=W_-(1)+h\ \text{ or }\ B_t=W_-(1)\}\,.\end{aligned}$$

Note that $\tau_h-\tau_0$ has the same law as that of the hitting time of $\{0,h\}$ by a Brownian motion Z_t with $Z_0=1$. (Here, $Z_t=B_{-(\tau_0+t)}-W_-(1)!$). Further, letting $I_h=\mathbf{1}_{\{B_{\tau_h}=W_-(1)\}}(=\mathbf{1}_{\{Z_{\tau_h-\tau_0}=0\}})$, it holds that

$$\begin{aligned}W_-(h)&=W_-(1)+I_h\tilde{W}_-(h)\\ M_-(h)&=\begin{cases}M_-(1), & I_h=0\\ M_-(1)\vee(\overline{M}_-(h)+W_-(1)+1)\vee(\tilde{M}_-(h)+W_-(1)), & I_h=1\end{cases}\end{aligned}$$

where $(\tilde{W}_-(h),\tilde{M}_-(h))$ are independent of $(W_-(1),M_-(1))$ and possess the same law as $(W_-(h),\ M_-(h))$, while $\overline{M}_-(h)$ is independent of both $(W_-(1),M_-(1))$ and $(\tilde{W}_-(h),\tilde{M}_-(h))$ and has the law of the maximum of a Brownian motion, started at 0, killed at hitting -1 and conditioned not to hit $h-1$. (See figure 2.5.4 for a graphical description of these random variables.)

Set now

$$\hat{M}_-(h)=\begin{cases}h, & I_h=0\\ 1+\overline{M}_-(h), & I_h=1\,,\end{cases}$$

$\tilde{H}_-(h)=(\tilde{W}_-(h)+h)\vee\tilde{M}_-(h)$ and $\Gamma(h)=\max(\tilde{H}_-(h),\hat{M}_-(h))$. Note that $\tilde{H}_-(h)$ has the same law as $H_-(h)$ but is independent of $\overline{M}_-(h)$. Further, it is easy to check that $(W_-(h)+h)\vee M_-(h)=(W_-(1)+\Gamma_h)\vee M_-(1)$ (note that either $M_-(h)=M_-(1)$ or $M_-(h)>M_-(1)$ but in the latter case, $M_-(h)\le W_-(1)+\Gamma(h)$.) We have the following lemma, whose proof is deferred:

Lemma 2.5.16 *The law of $\Gamma(h)$ is $\frac{1}{h}\delta_h+\frac{h-1}{h}U[1,h]$, where $U[1,h]$ denotes the uniform law on $[1,h]$.*

Substituting in (2.5.15), we get that

$$\mathcal{Q}(\overline{b}(h)=\overline{b}(1))=E_{\mathcal{Q}}\left(\mathcal{Q}(\overline{b}(h)=\overline{b}(1)|\Gamma(h))\right)=\frac{2}{h^2}\left[\int_1^h Q(t)dt+Q(h)\right] \tag{2.5.17}$$

where

$$Q(t)=\mathcal{Q}\left(H_+(1)<H_-(1),H_+(h)<H_-(t)\,|\,s_+(h)=s_+(1),s_-(1)=s_-(t)\right)\,.$$

In order to evaluate the integral in (2.5.17), we need to evaluate the joint law of $(H_+(1),H_+(t))$ (the joint law of $(H_-(1),H_-(t))$ being identical). Since

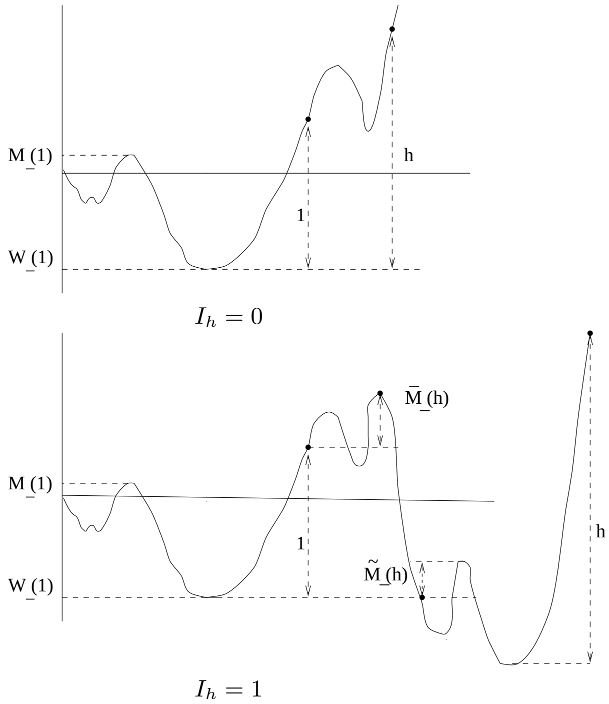


Fig. 2.5.4. Definition of auxiliary variables

$0 \le H_+(1) \le 1$ and $H_+(1) \le H_+(t) \le H_+(1) + t - 1$, the support of the law of $(H_+(1), H_+(t))$ is the domain A defined by $0 \le x \le 1$, $x \le y \le x + t - 1$. Note that for $(z, w) \in A$,

$$\begin{aligned}
&\mathcal{Q}(H_+(1) \le z, H_+(t) \le w \,|\, s_+(1) = s_+(t)) \\
&= \mathcal{Q}(M_+(1) \le z \wedge w, W_+(1) \le -[(1-z) \vee (t-w)]) \\
&= \mathcal{Q}\Big(M_+(1) \le z, W_+(1) \le -(t-w)\Big)\,.
\end{aligned}$$

We now have the following well known lemma. For completeness, the proof is given at the end of this section:

Lemma 2.5.18 *For $z + y \ge 1$, $0 \le z \le 1$, $y \ge 0$,*

$$\mathcal{Q}(M_+(1) \le z, W_+(1) \le -y) = z e^{-(z+y-1)}\,.$$

Lemma 2.5.18 implies that, for $(z, w) \in A$, $t > 1$,

$$\mathcal{Q}(H_+(1) \le z, H_+(t) \le w \,|\, s_+(1) = s_+(t)) = z e^{-(z+t-w-1)}\,. \qquad (2.5.19)$$

Denote by B_1 the segment $\{0 \le x = y \le 1\}$ and by B_2 the segment $\{t - 1 \le y = x + t - 1 \le t\}$. We conclude, after some tedious computations, that the conditional law of $(H_+(1), H_+(t))$:

– possesses the density $f(z,w)=(1-z)e^{-z}e^{w-(t-1)}$, $(z,w)\in A\backslash(B_1\cup B_2)$
– possesses the density $\underline{f}(z,y)=(1-z)e^{-(t-1)}$, $z=w\in B_1$
– possesses the density $\overline{f}(z,z+t-1)=z$, $w=z+t-1\in B_2$.

Substituting in the expression for $Q(t)$, we find that

$$Q(t)=\frac{5}{12}e^{-(h-t)}+\frac{1}{12}e^{-(h+t-2)}\,.$$

Substituting in (2.5.19), the theorem follows. □

Proof of Lemma 2.5.16: Note that $\mathcal{Q}(I_h=0)=1/h$, and in this case $\Gamma_h=h$. Thus, we only need to consider the case where $I_h=1$ and show that under this conditioning, $\max(H_-(h),1+\overline{M}_-(h))$ possesses the law $U[1,h]$. Note that by standard properties of Brownian motion,

$$\mathcal{Q}(\hat{M}_-(h)\le\xi|I_h=1)=\frac{\frac{\xi-1}{\xi}}{\frac{h-1}{h}}\,.$$

We show below that the law of $\tilde{H}_-(h)$, which is identical to the law of $H_-(h)$, is uniform on $[0,h]$. Thus, using independence, for $\xi\in[1,h]$,

$$\mathcal{Q}(\Gamma_h<\xi|I_h=1)=\frac{h(\xi-1)\xi}{\xi(h-1)h}=\frac{\xi-1}{h-1}\,,$$

i.e. the law of Γ_h conditioned on $I_h=1$ is indeed $U[1,h]$.

It thus only remains to evaluate the law of $H_-(h)$. By Brownian scaling, the law of $H_-(h)$ is identical to the law of $hH_+(1)$, so we only need show that the law of $H_+(1)$ is uniform on $[0,1]$. This in fact is a direct consequence of Lemma 2.5.18. □

Proof of Lemma 2.5.18: Let $\mathcal{Q}^x$ denote the law of a Brownian motion $\{Z_t\}$ starting at time 0 at x. The Markov property now yields

$$\begin{aligned}\mathcal{Q}(M_+(1)\le z,W_+(1)\le -y)&=\mathcal{Q}^o(\{Z_\cdot\}\text{ hits } z-1 \text{ before hitting } z)\\&\qquad\mathcal{Q}^{z-1}(M_+(1)\le z,W_+(1)\le -y)\\&=z\mathcal{Q}^o(M_+(1)\le 1,W_+(1)\le -y-z+1)\\&=z\mathcal{Q}^o(W_+(1)\le -(y+z-1))\,. \qquad (2.5.20)\end{aligned}$$

For $x\ge 0$, let $f(x):=\mathcal{Q}(W_+(1)\le -x)$. The Markov property now implies

$$f(x+\epsilon)=f(x)\mathcal{Q}^{-x}(W_+(1)\le -(x+\epsilon))=f(x)f(\epsilon)\,.$$

Since $f(0)=1$ and $f(\epsilon)=1-\epsilon+o(\epsilon)$, it follows that $f(x)=e^{-x}$. Substituting in (2.5.20), the lemma follows. □

Bibliographical notes: Theorem 2.5.3 is due to [67]. The proof here follows the approach of Golosov [32], who dealt with a RWRE reflected at 0, *i.e. with state space* $\mathbb{Z}_+$. *In the same paper, Golosov evaluates the analogue of*

Theorem 2.5.9 in this reflected setup, and in [33] he provides sharp (pathwise) localization results. These are extended to the case of a walk on $\mathbb{Z}$ in [34]. The statement of Theorem 2.5.9 and the proof here follow the article [42], where an explicit characterization of the law of $\overline{b}(1)$ is provided. The same characterization appears also in [34]. The aging properties of RWRE (Theorem 2.5.13) were first derived heuristically in [25], to which we refer for additional aging properties and discussion. The derivation here is based on [18]. The right hand side of formula (2.5.14) appears also in [34], in a slightly different context. We mention that results of iterated logarithm types, and results concerning most visited sites for Sinai's RWRE, can be found in [36], [37]. See [66] for a recent review. Finally, extensions of the results in this section and a theorem concerning the dichotomy between Sinai's regime and the classical CLT for ergodic environments can be found in [7].

Limit laws for transient RWRE in an i.i.d. environment appear in [43]. One distinguishes between CLT limit laws and stable laws: recall the parameter s introduced in Section 2.4. The main result of [43] is that if $s > 2$, a CLT holds true (see Section 2.2 for other approaches), whereas for $s \in (0,2)$ a Stable(s) limit law holds true. Note that this is valid even when $s < 1$, i.e. when $v_P = 0$! It is an interesting open problem to extend the results concerning stable limit laws to non i.i.d. environments. Some results in this direction are forthcoming in the Technion thesis of A. Roitershtein.

3 RWRE – $d > 1$

3.1 Ergodic Theorems

In this section we present some of the general results known concerning $0-1$ laws and laws of large numbers for nearest neighbour RWRE in $\mathbb{Z}^d$. Even if considerable progress was achieved in recent years, the situation here is, unfortunately, much less satisfying than for $d = 1$.

A standing assumption throughout this section is the following:

Assumption 3.1.1

(A1) *P is stationary and ergodic, and satisfies a ϕ-mixing condition: there exists a function $\phi(l) \underset{l\to\infty}{\to} 0$ such that any two l-separated events A, B with $P(A) > 0$,*

$$\left|\frac{P(A\cap B)}{P(A)} - P(B)\right| \leq \phi(l)\,.$$

(A2) *P is uniformly elliptic with no holding times: there exists an $\varepsilon > 0$ such that*

$$P(\omega(0,e) \geq \varepsilon) = 1, \qquad \forall e \in \{\pm e_i\}_{i=1}^d \ ,$$

and $P(\omega(0,0) > 0) = 0$.

(Events A, B are *l-separated* if the shortest lattice path connecting A and B is of length l or more.)

Remark: I have recently learnt that Assumption **(A1)** implies, in fact, that P is finitely dependent, c.f. [5]. On the other hand, the basic structure of what appears in the rest of this section remains unchanged if P is *mixing on cones*, see [13], and thus I have kept the proof in its original form.

Fix $\ell \in \mathbb{R}^d \setminus \{0\}$, and consider the events

$$A_{\pm\ell} = \{\lim_{n\to\infty} X_n \cdot \ell = \pm\infty\}\,.$$

We have the

Theorem 3.1.2 *Assume Assumption 3.1.1. Then*

$$\mathbb{P}^o(A_\ell \cup A_{-\ell}) \in \{0,1\}\,.$$

Proof. We begin by constructing an extension of our probability space: recall that the RWRE was defined by means of the law $\mathbb{P}^o = P \otimes P^o_\omega$ on $(\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{F} \times \mathcal{G})$. Set $W = \{0\} \cup \{\pm e_i\}_{i=1}^d$ and $\mathcal{W}$ the cylinder σ-algebra on $W^{\mathbb{N}}$. We now define the measure

$$\overline{\mathbb{P}}^o = P \otimes Q_\varepsilon \otimes \overline{P}^o_{\omega,\varepsilon}$$

on

$$\Big(\Omega \times W^{\mathbb{N}} \times (\mathbb{Z}^d)^{\mathbb{N}}, \quad \mathcal{F} \times \mathcal{W} \times \mathcal{G}\Big)$$

in the following way: Q_ε is a product measure, such that with $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$ denoting an element of $W^{\mathbb{N}}$, $Q_\varepsilon(\varepsilon_1 = \pm e_i) = \varepsilon$, $i = 1, \cdots, d$, $Q_\varepsilon(\varepsilon_1 = 0) = 1 - 2\varepsilon d$. For each fixed ω, ε, $\overline{P}^o_{\omega,\varepsilon}$ is the law of the Markov chain $\{X_n\}$ with state space $\mathbb{Z}^d$, such that $X_0 = 0$ and, for each $e \in W$, $e \neq 0$,

$$\overline{P}^o_{\omega,\varepsilon}(X_{n+1} = z + e | X_n = z) = \mathbf{1}_{\{\varepsilon_{n+1}=e\}} + \frac{\mathbf{1}_{\{\varepsilon_{n+1}=0\}}}{1-2d\varepsilon}[\omega(z, z+e) - \varepsilon]\,.$$

It is not hard to check that the law of $\{X_n\}$ under $\overline{\mathbb{P}}^o$ coincides with its law under $\mathbb{P}^o$, while its law under $Q_\varepsilon \otimes \overline{P}^o_{\omega,\varepsilon}$ coincides with its law under P^o_ω.

We will prove the theorem for $\ell = (1, 0 \ldots 0)$, the general case being similar but requiring more cumbersome notations. Note that for any $u < v$, the walk cannot visit infinitely often the strip $u \le z \cdot \ell \le v$ without crossing the line $z \cdot \ell = v$. More precisely, with

$$T_v = \inf\{n \ge 0 : \quad X_n \cdot \ell \ge v\}, \tag{3.1.3}$$

we have

$$\mathbb{P}^o\Big(\#\{n > 0 : \ X_n \cdot \ell \ge u\} = \infty, T_v = \infty\Big) = 0\,. \tag{3.1.4}$$

Indeed, note that for any z with $u \le z \cdot \ell \le v$, and any ω,

$$P_\omega^z(X_{v-u}\cdot\ell \ge v) = Q_\varepsilon \otimes P_{\omega,\varepsilon}^z(X_{v-u}\cdot\ell\ge v) \ge \varepsilon^{v-u}\,,$$

yielding (3.1.4) by the strong Markov property.

Assume next that $\mathbb{P}^o(A_\ell) > 0$. Set $D = \inf\{n \ge 0 :\ X_n\cdot\ell < X_0\cdot\ell\}$. Clearly, $\mathbb{P}^o(D=\infty) > 0$, because if $\mathbb{P}^o(D=\infty)=0$ then $\mathbb{P}^z(D<\infty)=1$ $\forall z\in\mathbb{Z}^d$, and thus P-a.s., for all $z\in\mathbb{Z}^d$, $P_\omega^z(D<\infty)=1$. This implies by the Markov property that

$$\liminf_{n\to\infty} X_n\cdot\ell \le 0, \quad \mathbb{P}^o\text{-a.s.}\,,$$

contradicting $\mathbb{P}^o(A_\ell) > 0$.

Define $\mathcal{O}_\ell$ to be the event that $X_n\cdot\ell$ changes its sign infinitely often. We next show that whenever $\mathbb{P}^o(A_\ell)>0$, then $\mathbb{P}^o(\mathcal{O}_\ell)=0$. Set $M=\sup_n X_n\cdot\ell$, fix $v>0$ and note by (3.1.4) that

$$\mathbb{P}^o(\mathcal{O}_\ell\cap\{M<v\}) = 0\,. \tag{3.1.5}$$

We next prove that if $\mathbb{P}^o(A_\ell)>0$ then $\mathbb{P}^o(\mathcal{O}_\ell\cap\{M=\infty\})=0$, by first noting that

$$\mathbb{P}^o(\mathcal{O}_\ell\cap\{M=\infty\}) = \overline{\mathbb{P}}^o(\mathcal{O}_\ell\cap\{M=\infty\})\,.$$

Then, set $\mathcal{G}_n = \sigma((\varepsilon_i, X_i),\, i\le n)$, fix $L>0$ and, setting $S_0=0$, define recursively $\mathcal{G}_n$ stopping times as follows:

$$\begin{aligned} R_k &= \inf\Big\{n\ge S_k :\ X_n\cdot\ell < 0\}\,,\\ S_{k+1} &= \inf\{n\ge R_k : X_{n-L}\cdot\ell \\ &\qquad \ge \max\{X_m\cdot\ell :\ m\le n-L\},\, \varepsilon_{n-1}=\varepsilon_{n-2}=\ldots=\varepsilon_{n-L}=e_1\Big\}\,. \end{aligned}$$

On $\mathcal{O}_\ell\cap\{M=\infty\}$, all these stopping times are finite. Now, at each time S_k-L the walk enters a half space it never visited before, and then *due to the action of the ε sequence alone*, it proceeds L steps in the direction e_1. Formally, "events in the σ-algebra $\mathcal{G}_{S_k}$ are L-separated from $\sigma(\omega_z :\ z\cdot\ell \ge X_{S_k}\cdot\ell)$". Note that, using $\mathbb{P}^o(A_\ell)>0$ in the second inequality,

$$\overline{\mathbb{P}}^o(R_0<\infty) = \overline{\mathbb{P}}^o(D<\infty) < 1\,,$$

whereas, using θ to denote both time and space shifts as needed from the context,

$$\begin{aligned} &\overline{\mathbb{P}}^o(R_1<\infty) \le \overline{\mathbb{P}}^o(R_0<\infty, R_0\circ\theta_{X_{S_1}}<\infty)\\ &\quad= \sum_{z\in\mathbb{Z}^d}\overline{\mathbb{P}}^o(R_0<\infty, R_0\circ\theta_z<\infty, X_{S_1}=z)\\ &\quad= \sum_{z\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}} E_{P\otimes Q_\varepsilon}\Big(\overline{P}^o_{\omega,\varepsilon}(R_0<\infty, X_{S_1}=z, S_1=n)\cdot\overline{P}^o_{\theta^z\omega,\theta^n\varepsilon}(R_0<\infty)\Big)\,. \end{aligned}$$

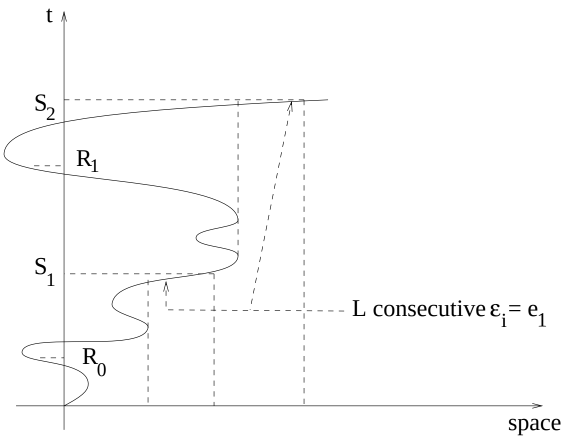


Fig. 3.1.1. Definition of the hitting times (S_k, R_k)

Note that $\overline{P}^o_{\theta^z\omega,\theta^n\varepsilon}(R_0<\infty)$ is measurable on $\sigma(\omega_x:\ x\cdot\ell\geq z\cdot\ell)\times\sigma(\varepsilon_i, i>n)$, whereas $\overline{P}^o_{\omega,\varepsilon}(R_0<\infty, X_{S_1}=z, S_1=n)$ is measurable on $\sigma(\omega_x: x\cdot\ell\leq z\cdot\ell-L)\times\sigma(\varepsilon_i, i\leq n)$. Hence, by the ϕ-mixing property of P and the product structure of Q_ε,

$$\begin{aligned}\overline{\mathbb{P}}^o(R_1<\infty)\leq \sum_{z\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}}\Big[&E_{P\otimes Q_\varepsilon}\Big(\overline{P}^o_{\omega,\varepsilon}(R_0<\infty, X_{S_1}=z, S_1=n)\Big)\\ &\cdot E_{P\otimes Q_\varepsilon}\Big(\overline{P}^o_{\omega,\varepsilon}(R_0<\infty)\Big)\Big]\\ +\phi(L)\sum_{z\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}}&E_{P\otimes Q_\varepsilon}\Big(\overline{P}^o_{\omega,\varepsilon}(R_0<\infty, X_{S_1}=z, S_1=n)\Big)\\ \leq (\overline{\mathbb{P}}^o(R_0<\infty))^2+&\phi(L)\overline{\mathbb{P}}^o(R_0<\infty)\leq(\overline{\mathbb{P}}^o(D<\infty)+\phi(L))^2\,.\end{aligned}$$

Repeating this procedure, we conclude that $\overline{P}^o(\mathcal{O}_\ell\cap\{M=\infty\})\leq\overline{\mathbb{P}}^o(R_k<\infty)\leq(\mathbb{P}^o(D<\infty)+\phi(L))^{k+1}$. Since k is arbitrary and $\phi(L)\underset{L\to\infty}{\to}0$, we conclude that $\overline{\mathbb{P}}^o(\mathcal{O}_\ell\cap\{M=\infty\})=0$, yielding with the above that $\mathbb{P}^o(\mathcal{O}_\ell)=0$ as soon as $\mathbb{P}^o(A_\ell)>0$. In a similar manner one proves that $\mathbb{P}^o(A_{-\ell})>0$ also implies $\mathbb{P}^o(\mathcal{O}_\ell)=0$.

Assume now $1>\mathbb{P}^o(A_\ell\cup A_{-\ell})$. Then one can find a v such that $\mathbb{P}^o(X_n\cdot\ell\in[-v,v]$ infinitely often$)>0$. Therefore, $\mathbb{P}^o(\mathcal{O}_\ell)>0$, implying by the above $\mathbb{P}^o(A_\ell)=\mathbb{P}^o(A_{-\ell})=0$. $\square$

Remark: It should be obvious that one does not need the full strength of **(A1)** in Assumption 3.1.1, and weaker forms of mixing suffice. For an example of how this can be relaxed, see [13].

Bibliographical notes: The 0-1 law described in this section is due to Kalikow [39], who handled the i.i.d. setup. Our proof borrows from [83], which, still *in the i.i.d. case, relaxes the uniform ellipticity assumption $\mathcal{A}2$. In that paper, they show that a stronger 0-1 law holds if P is a product measure and $d = 2$, namely they show that $\mathbb{P}^o(A_\ell) \in \{0,1\}$, while that last conclusion is false for certain mixing environments with elliptic, but not uniformly elliptic, environments.*

3.2 A Law of Large Numbers in $\mathbb{Z}^d$

Our next goal is to prove a law of large numbers. Unfortunately, at this point we are not able to deal with general non i.i.d. environments (see however Remark 2 following the proof of Theorem 3.2.2), and further the case of i.i.d. environments does offer some simplifications. Thus, throughout this section we make the following assumptions:

Assumption 3.2.1 *P is a uniformly elliptic, i.i.d. law on Ω.*

The main result of this section is the following:

Theorem 3.2.2 *Assume Assumption 3.2.1 and that $\mathbb{P}^o(A_\ell \cup A_{-\ell}) = 1$. Then, there exist deterministic $v_\ell, v_{-\ell}$ (possibly zero) such that*

$$\lim_{n\to\infty} \frac{X_n \cdot \ell}{n} = v_\ell \mathbf{1}_{A_\ell} + v_{-\ell}\mathbf{1}_{A_{-\ell}}, \quad \mathbb{P}^o\text{-}a.s.$$

(See (3.2.8) for an expression for v_ℓ. When $v_\ell \neq 0$ for some ℓ, we say that the walk is *ballistic*).

Proof. As in Section 3.1 we will take here $\ell = (1, 0, \cdots, 0)$. Further, we assume throughout that $\mathbb{P}^o(A_\ell) > 0$. The proof is based on introducing a renewal structure, as follows: Define $\overline{S}_0 = 0, M_0 = \ell \cdot X_0$,

$$\begin{aligned}
&\overline{S}_1 = T_{M_0+1} \le \infty,\ \overline{R}_1 = D \circ \theta_{\bar{S}_1} + \overline{S}_1 \le \infty,\\
&M_1 = \sup\{\ell \cdot X_m, \quad 0 \le m \le \overline{R}_1\} \le \infty
\end{aligned}$$

and by induction, for $k \ge 1$,

$$\begin{aligned}
&\overline{S}_{k+1} = T_{M_k+1} \le \infty,\ \overline{R}_{k+1} = D \circ \theta_{\bar{S}_{k+1}} + \overline{S}_{k+1} \le \infty,\\
&M_{k+1} = \sup\{\ell \cdot X_m, \quad 0 \le m \le \overline{R}_{k+1}\} \le \infty\,.
\end{aligned}$$

The times $\overline{S}_1, \overline{S}_2, \ldots,$ are called "fresh times", and the locations $X_{\overline{S}_1}$, $X_{\overline{S}_2}$, $\cdots$, are "fresh points": at the time $\overline{S}_k$, the path $X_\cdot$ visits for the first time after $\overline{S}_{k-1}$ and after hitting again the hyperplane $X_{\overline{S}_{k-1}} \cdot \ell - 1$, a fresh part of the environment. Note that $(\overline{S}_i, \overline{R}_i)$ are related to, but differ slightly from, (S_i, R_i) introduced in Section 3.1. Clearly,

$$0 = \overline{S}_0 \le \overline{S}_1 \le \overline{R}_1 \le \overline{S}_2 \le \cdots \le \infty$$

and the inequalities are strict if the left member is finite. Define:

$$K = \inf\{k \geq 1 : \overline{S}_k < \infty, \overline{R}_k = \infty\} \leq \infty,$$
$$\tau_1 = \overline{S}_K \leq \infty\,.$$

τ_1 is called a "regeneration time", because after τ_1, $X_\cdot \cdot \ell$ never falls behind $X_{\tau_1} \cdot \ell$.

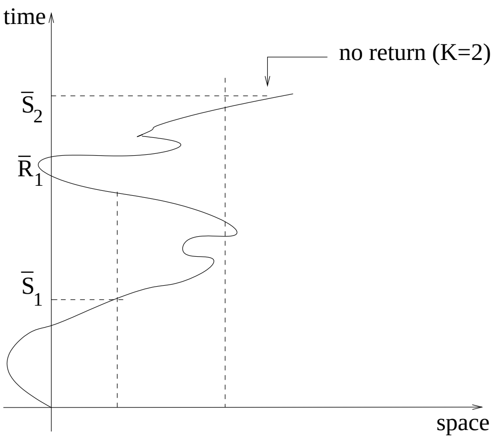


Fig. 3.2.1. Regeneration structure

By the same argument as in the proof of Theorem 3.1.2, $\mathbb{P}^o(\overline{R}_k < \infty) \leq \mathbb{P}^o(D < \infty)^k \underset{k\to\infty}{\longrightarrow} 0$ because $\mathbb{P}^o(A_\ell) > 0$ implies $\mathbb{P}^o(D < \infty) < 1$. On the other hand, on A_ℓ, $\overline{R}_k < \infty \Rightarrow S_{k+1} < \infty$, $\mathbb{P}^o$-a.s., and hence

$$\mathbb{P}^o(A_\ell \cap \{K = \infty\}) = \mathbb{P}^o(A_\ell \cap \{\tau_1 = \infty\}) = 0\,.$$

Define now the measure

$$\mathbb{Q}^o(\cdot) = \mathbb{P}^o(\cdot\,|\{\tau_1 < \infty\}) = \mathbb{P}^o(\cdot\,|A_\ell)$$

and set

$$\mathcal{G}_1 = \sigma\left(\tau_1, X_0, \cdots, X_{\tau_1}, \{\omega(y,\cdot)\}_{\ell\cdot y < \ell\cdot X_{\tau_1}}\right)\,.$$

Note that since $\{D = \infty\} \subset \{\tau_1 < \infty\}$, we have that $\{D = \infty\} \in \mathcal{G}_1$. We have the following crucial lemma, whose proof is a simple exercise in the application of the Markov property, is omitted. It is here that the i.i.d. assumption on the environment plays a crucial role:

Lemma 3.2.3 *For any measurable sets A, B,*

$$
\begin{aligned}
&\mathbb{Q}^o\Big(\{X_{\tau_1+n}-X_{\tau_1}\}_{n\geq 0}\in A, \{\omega(X_{\tau_1}+y,\cdot)\}_{y\cdot\ell\geq 0}\in B\Big)\\
=&\mathbb{P}^o\Big(\{X_n\}_{n\geq 0}\in A, \{\omega(y,\cdot)\}_{y\cdot\ell\geq 0}\in B|\{D=\infty\}\Big)\,.
\end{aligned}
$$

In fact,

$$
\begin{aligned}
&\mathbb{Q}^o\Big(\{X_{\tau_1+n}-X_{\tau_1}\}_{n\geq 0}\in A, \{\omega(X_{\tau_1}+y,\cdot)\}_{y\cdot\ell\geq 0}\in B|\mathcal{G}_1\Big)\\
=&\mathbb{P}^o\Big(\{X_n\}_{n\geq 0}\in A, \{\omega(y,\cdot)\}_{y\cdot\ell\geq 0}\in B|\{D=\infty\}\Big)\,. \qquad (3.2.4)
\end{aligned}
$$

Proof of Lemma 3.2.3 Clearly, it suffices to prove (3.2.4). Let h denote a $\mathcal{G}_1$ measurable random variable. Set $1_A := 1_{\{X_n-X_0\}_{n\geq 0}\in A}$, $1_B := 1_{\{\omega(y,\cdot)\}_{y\cdot\ell\geq 0}}$. Further, note that for each $k\in\mathbb{N}$, $x\in\mathbb{Z}^d$, there exists a random variable $h_{x,k}$, measurable with respect to $\sigma(\{\omega(y,\cdot)\}_{\ell\cdot y<x\cdot\ell}, \{X_i\}_{i\leq\overline{S}_k}, \overline{S}_k)$, such that on the event $\{\tau_1=\overline{S}_k, X_{\overline{S}_k}=x\}$, $h=h_{x,k}$ (this follows from the $\mathcal{G}_1$ measurability of h. Then, using $\bar\theta$ to denote spatial shift and θ to denote temporal shift,

$$
\begin{aligned}
&E_{\mathbb{P}^o}\left(1_A\circ\theta^{\tau_1}\cdot 1_B\circ\bar\theta^{X_{\tau_1}}\cdot h\cdot\mathbf{1}_{\tau_1<\infty}\right)\\
&\quad=\sum_{k\geq 1}\sum_{x\in\mathbb{Z}^d}E_P\left(E^o_\omega\left(\mathbf{1}_{\overline{S}_k<\infty}\mathbf{1}_{\overline{R}_k=\infty}\mathbf{1}_{X_{\overline{S}_k}=x}1_A\circ\theta^{\overline{S}_k}\cdot 1_B\circ\bar\theta^x\cdot h_{x,k}\right)\right)\\
&\quad=\sum_{k\geq 1}\sum_{x\in\mathbb{Z}^d}E_P\left(1_B\circ\bar\theta^x E^o_\omega\left(\mathbf{1}_{\overline{S}_k<\infty}\mathbf{1}_{D\circ\bar\theta_{\bar S_k}=\infty}\mathbf{1}_{X_{\overline{S}_k}=x}1_A\circ\theta^{\overline{S}_k}\cdot h_{x,k}\right)\right)\\
&\quad=\sum_{k\geq 1}\sum_{x\in\mathbb{Z}^d}E_P\left(1_B\circ\bar\theta^x E^x_\omega\left(\mathbf{1}_{D=\infty}1_A\right)E^o_\omega\left(h_{x,k}\mathbf{1}_{\overline{S}_k<\infty}\mathbf{1}_{X_{\overline{S}_k}=x}\right)\right)\\
&\quad=\sum_{k\geq 1}\sum_{x\in\mathbb{Z}^d}E_{\overline{\mathbb{P}}^o}\left(1_B\mathbf{1}_{D=\infty}1_A\right)E_{\overline{\mathbb{P}}^o}\left(h_{x,k}\mathbf{1}_{\overline{S}_k<\infty}\mathbf{1}_{X_{\overline{S}_k}=x}\right)\,,
\end{aligned}
$$

where we used the Markov property in the next to last equality and the i.i.d. structure of the environment in the last one. Substituting in the above trivial A, B, one concludes that

$$
E_{\mathbb{P}^o}\left(h\cdot\mathbf{1}_{\tau_1<\infty}\right)=\overline{\mathbb{P}}^o(\{D=\infty\})\sum_{k\geq 1}\sum_{x\in\mathbb{Z}^d}E_{\overline{\mathbb{P}}^o}\left(h_{x,k}\mathbf{1}_{\overline{S}_k<\infty}\mathbf{1}_{X_{\overline{S}_k}=x}\right)\,.
$$

Hence,

$$
E_{\mathbb{Q}^o}\left(1_A\circ\theta^{\tau_1}\cdot 1_B\circ\bar\theta^{X_{\tau_1}}\cdot h\right)=E_{\mathbb{Q}^o}(h)E_{\mathbb{P}^o}\left(1_A1_B|\{D=\infty\}\right)\,,
$$

concluding the proof of the lemma. □

Consider now τ_1 as a function of the path $(X_n)_{n\geq 0}$ and set

$$
\tau_{k+1}=\tau_k(X.)+\tau_1(X_{\tau_k+\cdot}-X_{\tau_k})\,,
$$

with $\tau_{k+1}=\infty$ on $\{\tau_k=\infty\}$ (the sequence $\{\tau_k\}$ enumerates times such that for all $k<m<n$, $X_k\cdot\ell<X_m\cdot\ell\leq X_n\cdot\ell$). By the definition and the fact that $\mathbb{P}^o(A_\ell\cap\{\tau_1=\infty\})=0$, we have that $\mathbb{P}^o(A_\ell\cap\{\tau_k=\infty\})=0$. Setting

$$\mathcal{G}_k = \sigma\Big(\tau_1, \ldots, \tau_k, \quad X_0, \cdots, X_{\tau_k}, \quad \{\omega(y,\cdot)\}_{\ell\cdot y<\ell\cdot X_{\tau_k}}\Big),$$

an obvious rerun of the proof of Lemma 3.2.3 yields that

$$\begin{aligned}&\mathbb{Q}^o\Big(\{X_{\tau_k+n} - X_{\tau_k}\}_{n\geq 0} \in A, \{\omega(X_{\tau_k}+y,\cdot)\}_{\ell\cdot y\geq 0} \in B |\ \mathcal{G}_k\Big)\\ &= \mathbb{P}^o\Big(\{X_n\}_{n\geq 0} \in A, \{\omega(y,\cdot)\}_{\ell\cdot y\geq 0} \in B | \{D=\infty\}\Big).\end{aligned}$$

We thus conclude that under $\mathbb{Q}^o$,

$$(X_{\tau_2} - X_{\tau_1}, \tau_2 - \tau_1), \cdots, (X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k)$$

are i.i.d. pairs of random variables, independent of (X_{τ_1}, τ_1), such that

$$\mathbb{Q}^o(X_{\tau_2} - X_{\tau_1} \in C_1, \tau_2 - \tau_1 \in C_2) = \mathbb{P}^o(X_{\tau_1} \in C_1, \tau_1 \in C_2 | \{D=\infty\}).$$

Next, we have the following lemma, whose proof is deferred:

Lemma 3.2.5 (Zerner)

$$\mathbb{E}^o(X_{\tau_1}\cdot \ell | \{D=\infty\}) = \frac{1}{\mathbb{P}^o(D=\infty)}.$$

We are now ready to complete the proof of Theorem 3.2.2. Assume first that $\mathbb{E}^o(\tau_1|\{D=\infty\}) < \infty$. Then, by the law of large numbers, and the renewal structure,

$$\frac{\tau_k}{k} \to_{k\to\infty} \mathbb{E}^o(\tau_1|\{D=\infty\}), \qquad \mathbb{Q}^o\text{-a.s.} \tag{3.2.6}$$

$$\frac{X_{\tau_k}\cdot \ell}{k} \to_{k\to\infty} \mathbb{E}^o(X_{\tau_1}\cdot\ell|\{D=\infty\}), \quad \mathbb{Q}^o\text{-a.s.} \tag{3.2.7}$$

(note that the finiteness of the expression in the right hand side of (3.2.7) is trivial if the right hand side of (3.2.6) is finite, and Lemma 3.2.5 is not needed in this case).

Hence,

$$\frac{X_{\tau_k}\cdot\ell}{\tau_k} \longrightarrow_{k\to\infty} \frac{\mathbb{E}^o(X_{\tau_1}\cdot\ell|\{D=\infty\})}{\mathbb{E}^o(\tau_1|\{D=\infty\})} =: v_\ell, \quad \mathbb{Q}\text{-a.s.} \tag{3.2.8}$$

Mimicking now the argument at the end of the proof of Lemma 2.1.5, we conclude that $\frac{X_n\cdot\ell}{n} \to_{n\to\infty} v_\ell, \mathbb{Q}^o$-a.s., in the case $\mathbb{E}^0(\tau_1|\{D=\infty\}) < \infty$.

On the other hand, Lemma 3.2.5 implies that (3.2.7) holds true even when $\mathbb{E}^o(\tau_1|\{D=\infty\}) = \infty$. But then, $\tau_k/k \to_{k\to\infty} \infty$, $\mathbb{Q}^o$-a.s. With $v_\ell = 0$ in this case, we conclude that

$$\frac{X_{\tau_k}\cdot\ell}{\tau_k} \longrightarrow_{k\to\infty} v_\ell = 0, \quad \mathbb{Q}^o\text{-a.s.}$$

Finally, setting k_n such that $\tau_{k_n} \le n < \tau_{k_n+1}$, we have that $k_n \to_{n\to\infty} \infty$ and $k_n/n \underset{n\to\infty}{\longrightarrow} 0$, $\mathbb{Q}^o$-a.s. because $n/k_n \ge \tau_{k_n}/k_n$. Thus,

$$\frac{X_n \cdot \ell}{n} \le \frac{X_{\tau_{k_n+1}} \cdot \ell}{k_n+1} \cdot \frac{k_n+1}{n} \underset{n\to\infty}{\longrightarrow} 0, \ \mathbb{Q}^o\text{-a.s.}$$

Since $\liminf_{n\to\infty} \frac{X_n \cdot \ell}{n} \ge 0$, $\mathbb{Q}^o$-a.s., we conclude

$$\frac{X_n \cdot \ell}{n} \underset{n\to\infty}{\longrightarrow} 0, \quad \mathbb{Q}^o\text{-a.s.}$$

□

Remarks:

1. Note that on A_ℓ, $v_\ell > 0$ if $\mathbb{E}^o(\tau_1|\{D=\infty\}) < \infty$ and $v_\ell = 0$ otherwise.
2. It is clear from the proof that in fact, if $\mathbb{E}^o(\tau_1|\{D=\infty\}) < \infty$, then the result of Theorem 3.2.2 can be strengthened to
$$\frac{X_n}{n} \longrightarrow_{n\to\infty} \frac{\mathbb{E}^o(X_{\tau_1}|\{D=\infty\})}{\mathbb{E}^o(\tau_1|\{D=\infty\})}, \ \mathbb{Q}^o\text{-.a.s.}$$
3. In the stationary, ϕ-mixing case, one can prove that the times $\{\tau_i\}$ are well defined, and form a mixing sequence. What I have not been able to show is that they are identically distributed under $\mathbb{Q}^o$ (they seem not!). Modifying slightly the definition of $(\overline{R}_k, \overline{S}_k)$ by adding an L-safeguard as in Section 3.1, the results of this section extend immediately to the case where the environment is K-dependent (i.e., $\{\omega(x,\cdot)\}_{x\cdot\ell\le 0}$ and $\{\omega(x,\cdot)\}_{x\cdot\ell > K}$ are independent). This applies, e.g., in the setup considered in [64]. The extension to a mixing setup is more complicated, and some results applicable there can be found in [13].
4. Still discussing mixing environements, some progress has been made using the approach of the environment viewed from the particle. We mention here [45] and in particular the recent preprint [63]. The latter preprint uses a-priori estimates concerning regeneration times in the ballistic case to construct an invariant measure for the environment viewed from the particle which is absolutely continuous with respect to P on certain half-spaces, and deduces a LLN using that measure.

Proof of Lemma 3.2.5 Recall that we consider $\ell = (1, 0, \dots, 0)$. Then,

$$
\begin{aligned}
&\mathbb{Q}^o\Big(\{\exists k:\ X_{\tau_k}\cdot\ell=i\}\Big)\\
&\quad=\frac{\sum_{y\in\mathbb{Z}^{d-1}}E\Big(P^o_\omega(\{\exists k:\ X_{\tau_k}=(i,y)\},A_\ell)\Big)}{\mathbb{P}^o(A_\ell)}\\
&\quad=\frac{\sum_{y\in\mathbb{Z}^{d-1}}E\Big(P^o_\omega(T_i<\infty,X_{T_i}=(i,y),D\circ\theta_{T_i}=\infty)\Big)}{\mathbb{P}^o(A_\ell)}\\
&\quad=\frac{\sum_{y\in\mathbb{Z}^{d-1}}E\Big(P^o_\omega(T_i<\infty,X_{T_i}=(i,y))P^{(i,y)}_\omega(D=\infty)\Big)}{\mathbb{P}^o(A_\ell)}\\
&\quad=\frac{\mathbb{P}^o(T_i<\infty)\mathbb{P}^o(D=\infty)}{\mathbb{P}^o(A_\ell)}\underset{i\to\infty}{\longrightarrow}\mathbb{P}^o(D=\infty)
\end{aligned}\tag{3.2.9}
$$

(since $\mathbb{P}^o(A_\ell\cup A_{-\ell})=1$ and $\lim_{i\to\infty}\mathbb{P}^o(\{T_i<\infty\}\cap A_{-\ell})=0$). On the other hand,

$$\lim_{i\to\infty}\mathbb{Q}^o\Big(\{\exists k:\ X_{\tau_k}\cdot\ell=i\}\Big)=\lim_{i\to\infty}\mathbb{Q}^o\Big(\{\exists k\geq 2:\ X_{\tau_k}\cdot\ell=i\}\Big)$$

(because $\mathbb{Q}^o(\tau_k>i)\underset{i\to\infty}{\longrightarrow}0$)

$$
\begin{aligned}
&=\lim_{i\to\infty}\sum_{n\geq 1}\mathbb{Q}^o\Big(\{\exists k\geq 2:\ X_{\tau_k}\cdot\ell=i,X_{\tau_1}\cdot\ell=n\}\Big)\\
&=\lim_{i\to\infty}\sum_{n\geq 1}\mathbb{Q}^o\Big(\{\exists k\geq 2:\ (X_{\tau_k}-X_{\tau_1})\cdot\ell=i-n,X_{\tau_1}\cdot\ell=n\}\Big)\\
&=\lim_{i\to\infty}\sum_{n\geq 1}\mathbb{Q}^o(X_{\tau_1}\cdot\ell=n)\cdot\mathbb{Q}^o\Big(\{\exists k\geq 2:(X_{\tau_k}-X_{\tau_1})\cdot\ell=i-n\}\Big)\,.
\end{aligned}
$$

But, recall that by the renewal theorem,

$$\mathbb{Q}^o\Big(\exists k\geq 2:\ (X_{\tau_k}-X_{\tau_1})\cdot\ell=i-n\Big)\underset{i\to\infty}{\longrightarrow}\frac{1}{E_{\mathbb{Q}^o}((X_{\tau_2}-X_{\tau_1})\cdot\ell)}$$

and hence, by dominated convergence,

$$\lim_{i\to\infty}\mathbb{Q}^o\Big(\{\exists k:\ X_{\tau_k}\cdot\ell=i\}\Big)=\frac{\sum_{n\geq 1}\mathbb{Q}^o(X_{\tau_1}\cdot\ell=n)}{E_{\mathbb{Q}^o}((X_{\tau_2}-X_{\tau_1})\cdot\ell)}=\frac{1}{E_{\mathbb{Q}^o}((X_{\tau_2}-X_{\tau_1})\cdot\ell)}.\tag{3.2.10}$$

Comparing (3.2.9) and (3.2.10), we conclude that

$$E_{\mathbb{Q}^o}\Big((X_{\tau_2}-X_{\tau_1})\cdot\ell\Big)=\frac{1}{\mathbb{P}^o(D=\infty)}<\infty\,. \qquad \square$$

Theorem 3.2.2 assumes that $\mathbb{P}^o(A_\ell\cup A_{-\ell})=1$, and in that situation provided a LLN if $\mathbb{P}^o(A_\ell)\in\{0,1\}$. A recent improvement to Theorem 3.2.2, due to Zerner [84], actually shows that if a 0-1 law holds true, a LLN holds, at least for i.i.d. environments. More precisely, one has the following:

Theorem 3.2.11 *There exist deterministic $v_\ell, v_{-\ell}$ (possibly zero) such that*

$$\lim_{n\to\infty} \frac{X_n \cdot \ell}{n} = v_\ell \mathbf{1}_{A_\ell} + v_{-\ell}\mathbf{1}_{A_{-\ell}}, \quad \mathbb{P}^o\text{-}a.s. \tag{3.2.12}$$

An immediate corollary, obtained by applying Theorem 3.2.11 d times with respect to the basis $\ell = e_i$, $i = 1, \dots, d$, is the following:

Corollary 3.2.13 *Assume that $\mathbb{P}^o(A_\ell) \in \{0, 1\}$ for every ℓ. Then there exists a deterministic v (possibly zero) such that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v, \quad \mathbb{P}^o\text{-}a.s.$$

Proof of Theorem 3.2.11: (sketch) In view of the 0-1 law Theorem 3.1.2 and of Theorem 3.2.2, all that remains to prove is that if $\mathbb{P}^o(A_\ell \cup A_{-\ell}) = 0$ then $X_n \cdot \ell/n \to 0$, $\mathbb{P}^o$-a.s. The complete proof for that is given in [84], and we provide next a brief description.

Consider the set of visits to the hyperplane $\mathcal{H}_m := \{z : z \cdot \ell = m\}$, defining $\tau_m^0 = T_m$ and $\tau_m^i = \min\{n > \tau_m^{i-1} : X_n \cdot \ell = m\}$. Fixing an integer L, let

$$h_{m,L} = \sup_{i \geq 0} \{\tau_m^i - \tau_m^0 : \tau_m^i < T_{m+L}\}$$

be the diameter of the set of visits to $\mathcal{H}_m$ before T_{m+L}. For any constant $c > 0$, let

$$F_{M,L}(c) = \frac{\#\{0 \leq m \leq M : h_{m,L} \leq c\}}{M+1}$$

denote the fraction of m's smaller than M such that the time between the first and last visit to $\mathcal{H}_m$ before T_{m+L} is smaller than c. The first observation, which is a deterministic (combinatorial) computation that we skip, is that for any path with $\limsup_{n\to\infty} X_n \cdot \ell/n > 0$ there exists a constant c such that

$$\inf_{L\geq 1} \limsup_{M\to\infty} F_{M,L}(c) > 0\,,$$

that is, roughly, there is a fraction of m's for which the time between first and last visits of $\mathcal{H}_m$ (before hitting $\mathcal{H}_{m+L}$) is not too large.

Assume now that $\mathbb{P}^o(\limsup X_n \cdot \ell/n > 0) > 0$. Then, by the above observation, there is some $c > 0$ such that

$$\mathbb{P}^o(\limsup_{L\to\infty} \limsup_{M\to\infty} F_{M,L}(c) > 0) > 0\,. \tag{3.2.14}$$

But on the event $\{h_{m,L} \leq c\}$, the last point visited in $\mathcal{H}_m$ before hitting $\mathcal{H}_{m+L}$ is at most at distance c from X_{T_m} and has been visited at most c times before T_{m+L}. Thus, there is a $z \in \mathcal{H}_0$ with $|z|_1 \leq c$, and an $1 \leq r \leq c$ such that the r-th visit to $X_{T_m} + z$ occurs before T_{m+L} and the walk does not backtrack from $\mathcal{H}_m$ after this r-th visit. Denoting the last event by $B^1_{m,L}(z, r)$, it follows that

$$F_{M,L}(c) \le \frac{1}{M+1} \sum_{z\in\mathcal{H}_0, |z|_1\le c} \sum_{r=1}^{c} \sum_{m=0}^{M} \mathbf{1}_{B^1_{m,L}(z,r)} \,.$$

Noting that the summation over r and z is over a finite set, and combining the last inequality with (3.2.14), it follows that for some z and r,

$$\mathbb{P}^o(\limsup_{L\to\infty} \limsup_{M\to\infty} \frac{1}{M+1} \sum_{m=0}^{M} \mathbf{1}_{B^1_{m,L}(z,r)} > 0) > 0 \,. \tag{3.2.15}$$

While the events $\{B^1_{m,L}(z,r)\}_m$ are not independent, some independence can be restored in the following way: construct independent (given the environment) copies $Y^y_\cdot$ of the RWRE, starting at y. Define the event $B_{m,L}(z,r)$ as the union of $B^1_{m,L}(z,r)$ with the event that $X_\cdot$ does not hit $X_{T_m}+z$ for the r-th time before T_{m+L}, but $Y^{X_{T_m}+z}_\cdot$ does not backtrack from $\mathcal{H}_m$ before it hits $\mathcal{H}_{m+L}$. An easy computation involving the Markov property shows that for each fixed $i=0,1,\ldots,L-1$, the events $\{B_{jL+i,L}(z,r)\}_j$ are independent, with

$$\mathbb{P}^o(B_{jL+i,L}(z,r)) = \mathbb{P}^o(D\ge T_L)\,.$$

(Here and in the sequel, we abuse notations by still using $\mathbb{P}^o$ to denote the annealed law on the enlarged probability space that supports the extra Y^y walks). Hence, since we have from (3.2.15) that

$$\mathbb{P}^o(\limsup_{L\to\infty} \limsup_{M\to\infty} \frac{1}{M+1} \sum_{i=0}^{L-1} \sum_{j=0}^{[M/L]} \mathbf{1}_{B^1_{jL+i,L}(z,r)} > 0) > 0\,, \tag{3.2.16}$$

it follows, by the standard law of large numbers, that

$$\mathbb{P}^o(D=\infty) = \limsup_{L\to\infty} \mathbb{P}^o(D\ge T_L) > 0\,.$$

But from (3.1.4), we have that $\mathbb{P}^o(A_\ell) \ge \mathbb{P}^o(D=\infty) > 0$. In particular, this shows that $\mathbb{P}^o(A_\ell)=0$ implies that $\limsup X_n\cdot \ell/n \le 0$, $\mathbb{P}^o$-a.s. Repeating this argument with $-\ell$ instead of ℓ completes the proof of the theorem. □

Bibliographical notes:

The proof here follows closely [77], except that Lemma 3.2.5 is due to private communication with Martin Zerner. The improvement Theorem 3.2.11 is based on [84].

The ballistic LLN has been proved for certain non iid environments in [13]. Alternative approaches to ballistic LLN's using the environment viewed from the particle were developped in [45] and in great generality in [63].

There are only a few LLN results in the non-ballistic case, see the bibliographical notes of Section 3.3.

3.3 CLT for walks in balanced environments

The setup in this section is the following:

Assumption 3.3.1

(B1) *P is stationary and ergodic.*
(B2) *P is balanced: for $i = 1, \cdots, d$, $P(\omega(x, x+e_i) = \omega(x, x-e_i)) = 1$.*
(B3) *P is uniformly elliptic: there exists an $\varepsilon > 0$ such that for $i = 1, \cdots, d$,*

$$P(\omega(x, x+e_i) > \varepsilon) = 1\,.$$

Unlike the situation in Section 2.1, we do not have an explicit construction of invariant measures at our disposal. The approach toward the LLN and CLT uses however (B2) in an essential way: indeed, note that in the notations of (2.1.28),

$$d(x,\omega) = \sum_{i=1}^{d} e_i \Big[\omega(x, x+e_i) - \omega(x, x-e_i)\Big] = 0\,.$$

Hence, the processes $(X_n(i))_{n\geq 0}, i = 1, \cdots, d$, are martingales, with, denoting $\mathcal{F}_n = \sigma(X_0, \cdots X_n)$,

$$E^o_\omega((X_n(i) - X_{n-1}(i))(X_n(j) - X_{n-1}(j))|\mathcal{F}_{n-1}) = 2\delta_{ij}\omega(X_{n-1}, X_{n-1}+e_i)\,.$$

Since $|\omega(\cdot,\cdot)| \leq 1$ P-a.s., it immediately follows that $X_n/n \underset{n\to\infty}{\longrightarrow} 0$, $\mathbb{P}^o$-a.s. Further, the multi-dimensional CLT (compare with Lemma 2.2.4) yields that if there exists a deterministic vector $\mathbf{a} = (a_1, \cdots, a_d)$ such that

$$\frac{1}{n}\sum_{k=1}^{n} \omega(X_{k-1}, X_{k-1}+e_i) \underset{n\to\infty}{\longrightarrow} \frac{a_i}{2} > 0, \quad \mathbb{P}^o\text{-a.s.}, \tag{3.3.2}$$

then, for any bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$, and any $y \in \mathbb{R}$,

$$\begin{aligned}
&\lim_{n\to\infty} P^o_\omega\left(f\left(\frac{X_n}{\sqrt{n}}\right) \leq y\right) \qquad\qquad (3.3.3)\\
&= \frac{1}{(2\pi)^{d/2}\prod_{i=1}^{d}\sqrt{a_i}} \int_{\mathbb{R}^d} \mathbf{1}_{\{f(\mathbf{x})\leq y\}} \exp\left(-\sum_{i=1}^{d}\frac{x_i^2}{2a_i}\right)\prod_{i=1}^{d} dx_i, \quad P\text{-a.s.}
\end{aligned}$$

Our goal in this section is to demonstrate such a CLT, and to study transience and recurrent questions for the RWRE.

Central limit theorems

Theorem 3.3.4 *Assume Assumption 3.3.1. Then, there exists a deterministic vector* $\mathbf{a}$ *such that (3.3.2) holds true. Consequently, the quenched CLT (3.3.3) holds true.*

Remark 3.3.5 *In fact, the above observations yield not only a CLT in the form of (3.3.3) but also a trajectorial CLT for the process $\{X_{[nt]}/\sqrt{n},\ t \in [0,1]\}$.*

Proof of Theorem 3.3.4 As in Section 2.1, the key to the proof of (3.3.2) is to consider the environment viewed from the particle. Define $\overline{\omega}(n) = \theta^{X_n}\omega$, and the Markov transition kernel

$$M(\omega, d\omega') = \sum_{e_i} \Big[\omega(0, e_i)\delta_{\theta^{e_i}\omega=\omega'} + \omega(0, -e_i)\delta_{\theta^{-e_i}\omega=\omega'}\Big] . \tag{3.3.6}$$

As in Lemma 2.1.18, the process $\overline{\omega}(n)$ is Markov under either P^o_ω or $\mathbb{P}^o$. Mimicking the proof of Corollary 2.1.25, if we can construct a measure Q on Ω which is absolutely continuous with respect to P and such that it is invariant under the Markov transition M, we will conclude, as in Corollary 2.1.25, that $\overline{\omega}(n)$ is stationary and ergodic and hence

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\omega(X_{k-1}, X_{k-1}+e_i) = \frac{1}{n}\sum_{k=1}^{n}\overline{\omega}(k)(0, e_i) \underset{n\to\infty}{\longrightarrow} \frac{a_i}{2} \\ := E_Q\overline{\omega}(0, e_i) \geq \varepsilon, \mathbb{P}^o\text{-a.s.},\end{aligned} \tag{3.3.7}$$

yielding (3.3.2). Our effort therefore is directed towards the construction of such a measure. Naturally, such measures will be constructed from periodic modifications of the RWRE, and require certain a-priori estimates on harmonic functions. We state these now, and defer their proof to the end of the section. The estimates we state are slightly more general than needed, but will be useful also in the study of transience and recurrence.

We let $|x|_\infty := \max_{i=1}^d |x_i|$ and define $D = D_R(x_0) = \{x \in \mathbb{Z}^d : |x - x_0|_\infty < R\}$. The generator of the RWRE, under P_ω, is the operator

$$(L_\omega f)(x) = \sum_{i=1}^{d}\omega(x, x+e_i)\Big[f(x+e_i) + f(x-e_i) - 2f(x)\Big] .$$

For any bounded $E \subset \mathbb{Z}^d$ of cardinality $|E|$, set $\partial E = \{y \in E^c : \exists x \in E, |x-y|_\infty = 1\}$, $\overline{E} = E \cup \partial E$, and $\mathrm{diam}(E) = \max\{|x-y|_\infty : x, y \in \overline{E}\}$. For any function $u : \mathbb{Z}^d \to \mathbb{R}$, we define the *normal set* at a point $x \in E$ as

$$I_u(x) = \{s \in \mathbb{R}^d : \ u(z) \leq u(x) + s\cdot(z-x), \ \forall z \in \overline{E}\} .$$

Finally, for any $q > 0$, E and u as above, define

$$\|g\|_{E,q,u} := \Big(\frac{1}{|E|}\sum_{x\in E} 1_{\{I_u(x)\neq\emptyset\}}|g(x)|^q\Big)^{1/q}, \|g\|_{E,q} := \Big(\frac{1}{|E|}\sum_{x\in E}|g(x)|^q\Big)^{1/q} .$$

Then we have the following:

Lemma 3.3.8 *There exists a constant $C = C(\varepsilon, d)$ such that*

(a)(maximum principle) For any $E \subset \mathbb{Z}^d$ bounded, any functions u and g such that

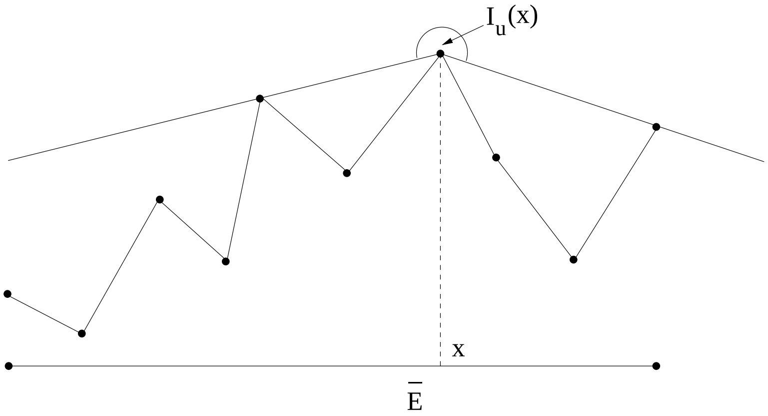


Fig. 3.3.1. The normal set at $x \in E$

$$L_\omega u(x) \geq -g(x), \;\; x \in E$$

satisfy

$$\max_{x \in E} u(x) \leq C\mathrm{diam}(E)|E|^{1/d}\|g\|_{E,d,u} + \max_{x \in \partial E} u^+(x)\,.$$

(b)(Harnack inequality) Any function $u \geq 0$ *such that*

$$L_\omega u(x) = 0, \qquad x \in D_R(x_0)\,, \tag{3.3.9}$$

satisfies

$$\frac{1}{C}u(x_0) \leq u(x) \leq Cu(x_0), \quad x \in D_{R/2}(x_0)\,.$$

We now introduce a periodic structure. Set $\Delta_N = \{-N, \cdots, N\}^d \subset \mathbb{Z}^d$ and identify elements of $T_N = \mathbb{Z}^d/(2N+1)\mathbb{Z}^d$ with a point of Δ_N, setting $\pi_N : \mathbb{Z}^d \to T_N$ and $\hat{\pi}_N : \mathbb{Z}^d \to \Delta_N$ to be the canonical projections. Set $\Omega^N = \{\omega \in \Omega : \theta^x\omega = \omega, \forall x \in (2N+1)\mathbb{Z}^d\}$. For any $\omega \in \Omega$, define $\omega^N \in \Omega^N$ by $\omega^N(x) = \omega(\hat{\pi}_N x)$. Note that ω^N is then a well defined function on T_N too.

Due to the ergodicity of P, it holds that in the sense of weak convergence,

$$P_N := \frac{1}{(2N+1)^d} \sum_{x \in \Delta_N} \delta_{\theta^x \omega^N} \underset{N \to \infty}{\longrightarrow} P, \quad P\text{-a.s.} \tag{3.3.10}$$

Let $\Omega_0 \subset \Omega$ denote those environments ω for which the convergence holds in (3.3.10) (clearly, $P(\Omega_0) = 1$).

Fixing $\omega \in \Omega_0$, let $(X_{n,N})_{n \geq 0}$ denote the RWRE on $\mathbb{Z}^d$ with law $P_{\omega^N}^{X_{0,N}}$. Then, $\overline{X}_{n,N} := \pi_N X_{0,N}$ is an irreducible Markov chain with finite state space T_N, and hence it possesses a unique invariant measure $\mu_N = \frac{1}{(2N+1)^d} \sum_{x \in T_N} \phi_N(x)\delta_x$. Setting $\overline{\omega}^N(n) := \theta^{X_{n,N}}\omega^N$, it follows that $\overline{\omega}^N(n)$ is an irreducible Markov chain with finite state space $S_N := \{\theta^x\omega^N\}_{x \in \Delta_N}$ and transition kernel M. Its unique invariant measure, supported on Ω^N, is then easily checked to be of the form

$$Q_N = \frac{1}{(2N+1)^d} \sum_{x \in \Delta_N} \phi_N(\pi_N x) \delta_{\theta^x \omega^N} \,.$$

Partitioning the state space S_N into finitely many *disjoint* states $\{\omega_\alpha^N\}_{\alpha=1}^K$, set $C_N(\alpha) = \{x \in \Delta_N : \theta^x \omega^N = \omega_\alpha^N\}$. Then,

$$f_N := \frac{dQ_N}{dP_N} = \sum_{\alpha=1}^K \mathbf{1}_{\{\omega = \omega_\alpha^N\}} \frac{1}{|C_N(\alpha)|} \sum_{x \in C_N(\alpha)} \phi_N(\pi_N x) \,.$$

We show below, as a consequence of part (a) of Lemma 3.3.8, that there exists a constant $C_2 = C_2(\varepsilon, d)$, independent of N, such that

$$\|\phi_N(\pi_N \cdot)\|_{D_{N+1}(0), d/d-1} \le C_2 \,. \tag{3.3.11}$$

Thus, using Jensen's inequality in the first inequality and (3.3.11) in the second,

$$\begin{aligned}
\int f_N^{d/d-1} dP_N &= \sum_{\alpha=1}^K \left[\frac{1}{|C_N(\alpha)|} \sum_{x \in C_N(\alpha)} \phi_N(\pi_N x) \right]^{d/d-1} \frac{|C_N(\alpha)|}{(2N+1)^d} \\
&\le \sum_{\alpha=1}^K \sum_{x \in C_N(\alpha)} \phi_N(\pi_N(x))^{d/d-1} \frac{1}{(2N+1)^d} \\
&= \frac{1}{(2N+1)^d} \sum_{x \in \Delta_N} \phi_N(\pi_N(x))^{d/d-1} \le C_2^{(d-1)/d} \,. \qquad (3.3.12)
\end{aligned}$$

Note that f_N extends to a measurable function on Ω, and the latter is, due to (3.3.12), uniformly integrable with respect to P_N. Thus, any weak limit of Q_N is absolutely continuous with respect to P, and further it is invariant with respect to the Markov kernel M.

Let $E = \{\omega : \frac{dQ}{dP} = 0\}$. By invariance, $E_Q M \mathbf{1}_E = E_Q \mathbf{1}_E = 0$, and hence $M\mathbf{1}_E \le \mathbf{1}_E$, P-a.s. But, $M\mathbf{1}_E \ge \varepsilon \sum_{i=1}^d (\mathbf{1}_E \circ \theta^{e_i} + \mathbf{1}_E \circ \theta^{-e_i})$. Hence, $\mathbf{1}_E \ge \mathbf{1}_E \circ \theta^{\pm e_i}$, P-a.s. Since P is stationary, $\mathbf{1}_E = \mathbf{1}_E \circ \theta^{\pm e_i}$, P-a.s., and hence by ergodicity (considering the invariant event $\cap_{x \in \mathbb{Z}^d} (\theta^x)^{-1} E$) $P(E) \in \{0, 1\}$. But $Q \ll P$ implies $P(E) = 0$. Hence, $Q \sim P$, as claimed (further, by (3.3.7), Q is then uniquely defined).

It thus only remains to prove (3.3.11). Fix a function g on T_N, and define the resolvent

$$\begin{aligned}
R^{\omega^N} g(x) &:= \sum_{j=0}^\infty \left(1 - \frac{1}{N^2}\right)^j E_{\omega^N}^x g(\overline{X}_{j,N}) \\
&= \sum_{j=0}^\infty \left(1 - \frac{1}{N^2}\right)^j E_{\omega^N}^x g \circ \pi_N(X_{j,N}), \quad x \in T_N
\end{aligned}$$

and the stopping times $\tau_0 = 0, \tau_1 = \tau := \min\{k \geq 1 : |X_{k,N} - X_{0,N}| \geq N\}$ and $\tau_{k+1} = \tau \circ \theta^k + \tau_k$. Since for $x \in \mathbb{Z}^d$ with $|x - X_{0,N}| < N$ it holds that $L_{\omega^N} E^x_{\omega^N} \left(\sum_{j=0}^{\tau-1} g \circ \pi_N(X_{j,N})\right) = -g(x)$, we have by Lemma 3.3.8(a) that for some constant $C = C(\varepsilon, d)$,

$$\sup_{|x-X_{0,N}|<N} \left| E^x_{\omega^N} \left(\sum_{j=0}^{\tau-1} g \circ \pi_N(X_{j,N}) \right) \right| \leq CN^2 \|g\|_{D_{N+1}(0),d} \,. \tag{3.3.13}$$

Since $(X_{n,N})_{n\geq 0}$ is a martingale, it follows from Doob's inequality that, for any $K \geq 1$,

$$\begin{aligned} P^o_{\theta^x \omega^N}[\tau \leq K] &\leq 2 \sum_{i=1}^d P^o_{\theta^x \omega^N} \left[\sup_{n \leq K} X_n(i) \geq N \right] \\ &\leq \frac{2}{N} \sum_{i=1}^d E^o_{\theta^x \omega^N} \Big((X_K(i))_+ \Big) \leq \frac{2d}{N} \sqrt{K} \,. \end{aligned}$$

Hence, using $K = N^2/8d^2$,

$$E^o_{\theta^x \omega^N} \left(\left(1 - \frac{1}{N^2}\right)^\tau \right) \leq \frac{2d}{N}\sqrt{K} + \left(1 - \frac{1}{N^2}\right)^K \leq C_3 \tag{3.3.14}$$

where $C_3 = C_3(d) < 1$ is independent of N. Thus, using the strong Markov property, (3.3.13) and (3.3.14),

$$\begin{aligned} |R^{\omega^N} g(x)| &= \sum_{m\geq 0} E^x_{\omega^N} \left(\sum_{\tau_m \leq j < \tau_{m+1}} \left(1 - \frac{1}{N^2}\right)^j g \circ \pi_N(X_{j,N}) \right) \\ &\leq \sum_{m\geq 0} E^x_{\omega^N} \left(\left(1 - \frac{1}{N^2}\right)^{\tau_m} E^{X_{\tau_m,N}}_{\omega^N} \sum_{j=0}^{\tau-1} g \circ \pi_N(X_{j,N}) \right) \\ &\leq \sum_{m\geq 0} \left(\sup_{x\in\mathbb{Z}^d} E^x_{\omega^N} \left(\left(1 - \frac{1}{N^2}\right)^\tau \right) \right)^m \cdot \sup_{x\in\mathbb{Z}^d} E^x_{\omega^N} \left(\sum_{j=0}^{\tau-1} g \circ \pi_N(X_{j,N}) \right) \\ &\leq C_4 N^2 \|g\|_{D_{N+1}(0),d} \end{aligned}$$

where $C_4 = C_4(d, \varepsilon)$. Using the invariance of ϕ_N, we now get

$$\begin{aligned} \|\phi_N(\pi_N \cdot)\|_{D_{N+1}(x_0),d/d-1} &= \|\phi_N(\pi_N \cdot)\|_{D_{N+1}(0),d/d-1} \\ &= \sup_{g:\|g\|_{D_{N+1}(0),d} \leq 1} \frac{1}{|D_{N+1}(0)|} \sum_{y\in D_{N+1}(0)} \phi_N(\pi_N y) g(y) \end{aligned}$$

$$\begin{aligned} = \frac{1}{N^2} \sup_{g:\|g\|_{D_{N+1}(0),d} \leq 1} \sum_{k\geq 0} \left(1 - \frac{1}{N^2}\right)^k \frac{1}{(2N+1)^d} \sum_{x\in\Delta_N} \phi_N(x) E^x_{\omega_N} (g \circ \pi_N(X_{k,N})) \\ \leq C_2 \end{aligned}$$

with $C_2 = C_2(d, \varepsilon)$, proving (3.3.11). $\square$

Proof of Lemma 3.3.8 (a) We may assume without loss of generality that $\max_{x\in\partial E} u(x) \le 0$, $g \ge 0$, $g \ne 0$ and that $u \ge 0$ is not identically 0. Let $\overline{u} = \max_{x\in\overline{E}} u = u(x_0)$, some $x_0 \in E$. Then, for s satisfying $|s|_\infty < \overline{u}/\mathrm{diam}(\overline{E})$, it holds that

$$u(x_0) + s\cdot(x - x_0) > 0, \quad \forall x \in \overline{E}\,.$$

Hence, with $t = \inf\{\rho \ge 0 :\, u(x_0) + s(x - x_0) + \rho > u(x), \forall x \in \overline{E}\}$, we have that $u(x) = u(x_0) + s\cdot(x - x_0) + t$, some $x \in \overline{E}$, and hence $u(x) + s\cdot(z - x) = u(x_0) + s\cdot(z - x_0) + t \ge u(z)$, $\forall z \in \overline{E}$. Hence,

$$s \in I_u(x) \subset \bigcup_{x\in E} I_u(x), \text{ for all } s \text{ with } \ |s|_\infty < \frac{\overline{u}}{\mathrm{diam}(\overline{E})}\,. \tag{3.3.15}$$

Assume $s \in I_u(x)$. Then, with $e \in \{\pm e_i\}$, and $v(y) = u(x) + s\cdot(y - x)$,

$$\begin{aligned} 0 &= \omega(x, x+e)\Big(2v(x) - v(x+e) - v(x-e)\Big) \\ &\le \omega(x, x+e)(2u(x) - u(x+e) - u(x-e))\,, \end{aligned}$$

and hence,

$$\begin{aligned} 0 &\le \omega(x, x+e)(2u(x) - u(x+e) - u(x-e)) \\ &\le \sum_{i=1}^d \omega(x, x+e_i)(2u(x) - u(x+e_i) - u(x-e_i)) = -L_\omega u(x) \le g(x)\,. \end{aligned}$$

Hence,

$$\Big(u(x) - u(x-e)\Big) - \Big(u(x+e) - u(x)\Big) \le \frac{g(x)}{\omega(x, x+e)} \le \frac{g(x)}{\varepsilon}\,.$$

Because $s \in I_u(x)$, it holds that

$$u(x+e) - u(x) \le s \cdot e \le u(x) - u(x-e)$$

and hence

$$u(x) - u(x-e) - \frac{g(x)}{\varepsilon} \le s \cdot e \le u(x) - u(x-e),\ \forall s \in I_u(x)\,. \tag{3.3.16}$$

Using (3.3.15) in the first inequality and (3.3.16) in the second, we have that

$$\left(\frac{2\overline{u}}{\mathrm{diam}(\overline{E})}\right)^d \le \left|\bigcup_{x\in E} I_u(x)\right| \le \sum_{x\in E} \left(\frac{g(x)}{\varepsilon}\right)^d 1_{\{I_u(x)\ne\emptyset\}}\,.$$

Hence,

$$\overline{u} \le C_0(d,\varepsilon)\mathrm{diam}(\overline{E})|E|^{1/d}\left(\frac{1}{|E|}\sum_{x\in E}|g(x)|^d 1_{\{I_u(x)\neq\emptyset\}}\right)^{\frac{1}{d}},$$

completing the proof of part (a).

(b) It is enough to consider $x_0 = 0$. We begin with some estimates. For parts of the proof, it is easier to work with L_2 (instead of L_∞) balls. Set $B_R = \{x \in \mathbb{Z}^d : |x|_2 < R\}$. We first deduce from part (a) that for any $p \le d$ and $\sigma < 1$, there exists a constant $C_1 = C_1(p,\sigma,d)$ such that

$$\max_{x\in B_{\sigma R}} u(x) \le C_1\left(\frac{1}{|B_R|}\sum_{x\in B_R}|u^+(x)|^p\right)^{\frac{1}{p}}. \tag{3.3.17}$$

Indeed, define $\eta(x) = \left(1-\frac{|x|^2}{R^2}\right)^{2d/p}$. A Taylor expansion reveals that for some $C_2 = C_2(p,d)$, it holds that

$$|\eta(x\pm e_i)-\eta(x)| < \frac{C_2}{R},\ |\eta(x+e_1)+\eta(x-e_1)-2\eta(x)| \le \frac{C_2}{R^2}. \tag{3.3.18}$$

Fix $\kappa_i = \kappa_i(x) \in [0,1]$, $i = 1,\dots,d$, set $\nu(x) = \eta(x)u(x), x \in B_R$, and

$$\hat{L}_\omega\nu(x) = \sum_{i=1}^d \hat\omega(x,x+e_i)(\nu(x+e_i)+\nu(x-e_i)-2\nu(x))$$

where

$$\hat\omega(x,x+e_i) = \begin{cases}\omega(x,x+e_i)\left[\frac{\kappa_i}{\eta(x-e_i)}+\frac{1-\kappa_i}{\eta(x+e_i)}\right], & |x|^2 \le R^2-4R\\ \omega(x,x+e_i), & R^2 \ge |x|^2 > R^2-4R\end{cases}.$$

Then, a tedious computation reveals that, on the set $|x|^2 \le R^2-4R$,

$$\begin{aligned}&-\hat{L}_\omega\nu(x) = -L_\omega u(x)\\ &-2\sum_i\frac{\kappa_i(\nu(x+e_i)-\nu(x))+(1-\kappa_i)(\nu(x)-\nu(x-e_i))}{\eta(x+e_i)\eta(x-e_i)}[\eta(x+e_i)-\eta(x-e_i)]\\ &+\sum_i\frac{u(x)}{\eta(x+e_i)\eta(x-e_i)}\\ &\Big[2\left(\eta(x+e_i)-\eta(x)\right)\left(\eta(x)-\eta(x-e_i)\right)-\eta(x)(\eta(x+e_i)+\eta(x-e_i)-2\eta(x)\Big]\\ &\le C_3(d,p)\Big[\sum_i\frac{|\kappa_i(\nu(x+e_i)-\nu(x))+(1-\kappa_i)(\nu(x)-\nu(x-e_i))|}{R}+\frac{u(x)}{R^2}\Big]\end{aligned}$$

where we used (3.3.9) in the first equality.

If for such x, $I_\nu(x) \neq \phi$, then by the proof in part (a), there exists a vector $q \in I_\nu(x)$ with $|q| \le \frac{\nu(x)}{R-|x|_\infty}$, and one may find a $\kappa_i \in [0,1]$ such that

$$\kappa_i(\nu(x+e_i)-\nu(x))+(1-\kappa_i)(\nu(x)-\nu(x-e_i))=q_i\,.$$

Thus, on $\{I_\nu(x)\neq\phi\}\cap\{x:|x|^2\leq R^2-4R\}$, it holds that $-\hat{L}_\omega\nu(x)\leq C_4(d,p)\frac{u(x)}{R^2}$. On the other hand, when $|x|^2\geq R^2-4R$, recalling that $u\geq 0$, it holds that

$$-\Big(\nu(x+e_i)+\nu(x-e_i)-2\nu(x)\Big)\leq 2\eta(x)u(x)\leq C_5(d,p)\frac{u(x)}{R^2}\eta(x)^{\frac{d-p}{d}}$$

and in conclusion,

$$-\hat{L}_\omega\nu(x)\leq g(x),\quad x\in B_R$$

where

$$\Big|g(x)\mathbf{1}_{I_\nu(x)\neq\phi}\Big|\leq\frac{C_6(d,p)u(x)}{R^2}\,.$$

Applying part (a) of the lemma, we get (3.3.17).

Next, let $\sigma<\tau<1$, and set

$$\mathbf{u}_\sigma=\min_{x\in B_{\sigma R}}u(x),\quad \mathbf{u}_\tau=\min_{x\in B_{\tau R}}u(x)\,.$$

We claim that (3.3.17) implies the existence of a constant $\gamma=\gamma(d,\sigma,\tau,\varepsilon)$ such that

$$\mathbf{u}_\tau\geq\gamma\mathbf{u}_\sigma\,. \tag{3.3.19}$$

Indeed, set $\overline{\eta}(x)=(R^2-|x|^2)^\beta$ with $\beta>2\vee 1/\sigma$ and $w(x)=\mathbf{u}_\sigma R^{-2\beta}\overline{\eta}(x)-u(x)$. Then, $w(x)\leq 0$ on $B_{\sigma R}\cup B_R^c$, and $L_\omega w=\mathbf{u}_\sigma R^{-2\beta}L_\omega\overline{\eta}$ on B_R. But, there is an $R_1(\beta)$ such that on $B_R\backslash B_{\sigma R}, R>R_1$,

$$L_\omega\overline{\eta}(x)\geq\begin{cases}0, & |x|<R\\ -C(\beta,d,\varepsilon)R^{2(\beta-1)}, & |x|=R\end{cases}\,,$$

implying by part (a) that on $B_R\backslash B_{\sigma R}$, $R>R_1(\beta)$,

$$w(x)\leq C(\beta,d,\varepsilon)\mathbf{u}_\sigma R^2R^{-2\beta}\left(\frac{1}{R^d}\sum_{|x|=R}R^{2(\beta-1)d}\right)^{\frac{1}{d}}\leq\frac{C(\beta,d,\varepsilon)}{R^{1/d}}\mathbf{u}_\sigma\,.$$

Thus,

$$\mathbf{u}_\tau\geq\mathbf{u}_\sigma\left[(1-\tau^2)^\beta-\frac{C(\beta,d,\varepsilon)}{R^{1/d}}\right]\,.$$

We conclude that there exists an $R_0=R_0(\sigma,\tau,d,\varepsilon)$ and $\gamma=\gamma(d,\sigma,\tau,\varepsilon)$ such that for all $R>R_0$, (3.3.19) holds. On the other hand, for $R<R_0$ (but $(1-\tau)R>1!$), (3.3.19) is trivial by finitely many applications of the equality $L_\omega u=0$. Thus, (3.3.19) is always satisfied.

A conclusion of (3.3.19) is that if $L_\omega u=0$ on B_R, $\sigma<1$, and $\Gamma\subset B_{\sigma R}\subset B_{\tau R}\subset B_R$, letting $\mathbf{u}_\Gamma=\min_{x\in\Gamma}u(x)$, we have that for some $\delta=\delta(\varepsilon,d)$,

$$|\Gamma|\geq\delta|B_{\sigma R}|\Longrightarrow\mathbf{u}_\tau\geq\gamma\mathbf{u}_\Gamma\,. \tag{3.3.20}$$

Indeed, define $\nu = \mathbf{u}_\Gamma - u$ and conclude from (3.3.17) that

$$\max_{x\in B_{\sigma R/2}} \nu(x) \le C_1 \left(\frac{1}{|B_{\sigma_R}|} \sum_{x\in B_{\sigma R}} \nu^+(x) \right) \le C_1(1-\delta) \max_{x\in B_{\sigma R}} \nu(x)$$

and hence, taking $\delta < 1$ such that $C_1(1-\delta) < 1/2$,

$$\mathbf{u}_\Gamma - \min_{x\in B_{\sigma R/2}} u(x) \le C_1(1-\delta)(\mathbf{u}_\Gamma - \mathbf{u}_\sigma) \le \frac{1}{2}\left(\mathbf{u}_\Gamma - \mathbf{u}_\sigma\right),$$

from which one concludes that $\mathbf{u}_\Gamma \le \mathbf{u}_{\sigma/2}$. (3.3.20) follows from combining this and (3.3.19).

We finally use the following covering argument. Fix a cube $Q \subset \mathbb{Z}^d$. For $t > 0$, set

$$\Gamma_t = \{x \in Q : \ u(x) > t\}.$$

Note that if $Q' = Q'(z,r)$ is any cube in $\mathbb{Z}^d$, centered at z and of side r, (3.3.20) implies that

$$|\Gamma_t \cap Q'| \ge \delta |Q'| \Rightarrow u(x) \ge \gamma t, \text{ some } \gamma = \gamma(\delta, d, \varepsilon). \tag{3.3.21}$$

Define, for any $A \subset Q$,

$$A_\delta = \bigcup_{\substack{\{r,z\} \\ z\in(\frac{1}{2}\mathbb{Z})^d}} \{Q'(z,3r) \cap Q : \ |A \cap Q'(z,r)| \ge |Q'(z,r)|\}.$$

Then, cf. [79, Lemma 3] for a proof, either $A_\delta = Q$ or $|A_\delta| \ge |\Gamma|/\delta$. Thus, if $|\Gamma_t| \ge \delta^s |Q|$, then iterating (3.3.21) and the above, $\inf_{x\in Q} u(x) \ge \gamma^s t$. Choosing s such that $\delta^s \le \frac{|\Gamma_t|}{|Q|} \le \delta^{s-1}$, we conclude that $\inf_{x\in D_R} u(x) \ge \gamma t \left(\frac{|\Gamma_t|}{|D_R|}\right)^{\log\gamma/\log\delta}$. Hence, with $p < \log\delta/\log\gamma := p'$, and $\mathbf{u} = \min_{D_R} u$, we have

$$\begin{aligned} \frac{1}{|D_R|} \sum_{x\in D_R} |u(x)|^p &= p \int_{\mathbf{u}}^\infty t^{p-1} \left(\frac{1}{|D_R|} \sum_{x\in D_r} \mathbf{1}_{u(x)\ge t} \right) dt \\ &= p \int_{\mathbf{u}}^\infty t^{p-1} \left(\frac{|\Gamma_t|}{|D_R|} \right) dt \\ &\le c(p)\ \mathbf{u}^{p'} \int_{\mathbf{u}}^\infty \frac{t^{p-1}}{t^{p'}} dt = c(p,p')\mathbf{u}^p, \end{aligned}$$

for some constants $c(p), c(p,p')$, since $p' + 1 - p > 1$. Combining this and (3.3.17) yields the lemma. $\square$

Transience and recurrence of balanced walks

The main result in this section is the following:

Theorem 3.3.22 *Assume Assumption 3.3.1. Then the RWRE* $(X_n)_{n\geq 0}$ *is transient if* $d \geq 3$ *and recurrent if* $d = 2$.

Proof. We begin with the transience statement. Fix $d \geq 3$, K large, and define $r_i = K^i$, with $B_i = \{x : |x|_\infty \leq r_i\}$. Set $\tau_0 = 1$ and

$$\tau_i = \min\{n > \tau_{i-1} : X_n \in \partial B_i\}\,.$$

We use the following uniform estimate on exit probabilities, that actually is stronger than needed: there exists some constant $C = C(\delta, \varepsilon, d) > 0$ such that, if $\Omega_0 = \{\omega : \omega(z, z + e_i) = \omega(z, z - e_i) > \varepsilon, i = 1, \ldots, d, \forall z \in \mathbb{Z}^d\}$,

$$\sup_{\omega\in\Omega_0} P^o_\omega(|X_n| < L, n = 1, \cdots, L^{2(1+\delta)}) \leq Ce^{-CL^{2\delta}}\,. \tag{3.3.23}$$

There are many ways to prove (3.3.23), including a coupling argument. We use here an optimal control trick. Let $\{B_n\}_{n\geq 0}$ denote a sequence of i.i.d. Bernoulli(1/2) random variables, independent of the environment, of law Q. Then, X_n can be constructed as follows:

$$P^o_{\omega,B}\Big(X_{n+1} = X_n + e_i | X_n = x, X_{n-1}, \cdots X_0\Big) = 2\omega(x, e_i)\mathbf{1}_{2B_{n+1}-1=\pm 1}\,.$$

(As in Section 3.1, $Q \times P^o_{\omega,B}$, when restricted to $(\mathbb{Z}^d)^{\mathbb{N}}$, equals P^o_ω.) Set $\mathcal{G}_n = \sigma(B_0, B_1, \cdots, B_n, X_0, \cdots, X_n, (\omega_z)_{z\in\mathbb{Z}^d})$. An admissible control $\boldsymbol{\alpha} = (\alpha_n)_{n\geq 0}$ is a sequence of $\mathcal{G}_n$ measurable function taking values in $\mathcal{A} := [2\varepsilon, 1-2\varepsilon(d-1)]$. Then define the $\mathbb{Z}$-valued controlled process $(Y^{\boldsymbol{\alpha}}_n)$ by $Y_0 = 0$ and

$$P\Big(Y^{\boldsymbol{\alpha}}_{n+1} = Y^{\boldsymbol{\alpha}}_n \pm 1 | \mathcal{G}_n, Y^{\boldsymbol{\alpha}}_0, \cdots, Y^{\boldsymbol{\alpha}}_n\Big) = \alpha_n \mathbf{1}_{2B_{n+1}-1=\pm 1}\,.$$

Note that, by taking $\hat{\alpha}_n = 2\omega(X_n, e_1)$, we may construct $(Y^{\hat{\boldsymbol{\alpha}}}_n)$ and X_n on the same probability space such that $Y^{\hat{\boldsymbol{\alpha}}}_n = X_n \cdot e_1$, $Q \times P^o_{\omega,B}$-a.s. Thus,

$$\begin{aligned}\sup_{\omega\in\Omega_0} & P^o_\omega\Big(|X_n|_\infty < L, n = 1, \ldots, L^{2(1+\delta)}\Big) \\ &\leq \sup_{\omega\in\Omega_0} \sup_{\boldsymbol{\alpha}} Q \times P^o_{\omega,B}\Big(|Y^\alpha_n| < L, n = 1, \cdots, L^{2(1+\delta)}\Big)\,.\end{aligned} \tag{3.3.24}$$

Let $g_{n,\omega}(x) = \sup_{\boldsymbol{\alpha}} Q \times P_{\omega,B}(|Y^{\boldsymbol{\alpha}}_i| < L, i = 1, \cdots, n | Y_0 = x)$ (it turns out eventually that $g_{n,\omega}$ does not depend on ω!) Then, due to the Markov property, $g_{n,\omega}(\cdot)$ must satisfy the dynamic programming equation

$$g_{n,\omega}(x) = \begin{cases} \max_{\alpha\in\mathcal{A}}\Big(\frac{\alpha}{2}\Big(g_{n-1,\omega}(x+1) + g_{n-1,\omega}(x-1)\Big) + (1-\alpha)g_{n-1,\omega}(x)\Big), & |x| < L \\ 0, & |x| \geq L \end{cases}$$

and $g_{0,\omega}(x) = \mathbf{1}_{|x|<L}$. Next, we note that $g_{n,\omega}(\cdot)$ satisfies

$$g_{n,\omega}(x+1) + g_{n,\omega}(x-1) - 2g_{n,\omega}(x) \le 0\,. \tag{3.3.25}$$

For $n = 0$ this is immediate, and hence

$$g_{1,\omega}(x) = g_{0,\omega}(x) + \varepsilon\Big(g_{0,\omega}(x+1) + g_{0,\omega}(x-1) - 2g_{0,\omega(x)}\Big)\,.$$

We then have that $g_{1,\omega}(x)$ satisfies (3.3.25), and the argument can be iterated. We further conclude that

$$g_{n,\omega}(x) = g_{n-1,\omega}(x)+\varepsilon\Big(g_{n-1,\omega}(x+1)+g_{n-1,\omega}(x-1)-2g_{n-1,\omega}(x)\Big)\,. \tag{3.3.26}$$

Thus, $g_{n,\omega}(x)$ is nothing but the probability that a simple random walk on $\mathbb{Z}$ with geometric $(1-2\varepsilon)$ holding times, stays confined in a strip of size L for $L^{2(1+\delta)}$ units of time (note that (3.3.26) possesses a unique solution, which does not depend on $\omega \in \Omega_0$!). The conclusion (3.3.23) follows from solving (3.3.26) and combining it with (3.3.24).

From (3.3.23), we conclude that $E^o_\omega(\tau_{i+2}) \le Cr_{i+2}^{2(1+\delta)}$, for all i large enough, all $\omega \in \Omega_0$, where $\mathbb{P}(\Omega_0) = 1$. Thus,

$$\begin{aligned}
Cr_{i+2}^{2(1+\delta)} &\ge E^o_\omega\Big(E^o_\omega(\#\text{ visits of } X_n \text{ at } B_{i-1} \text{ for } n \in (\tau_i+1,\cdots,\tau_{i+2})|X_{\tau_i})\Big)\\
&= E^o_\omega\left(\sum_{y\in B_{i-1}} E^{X_{\tau_i}}_\omega(\#\text{ visits at } y \text{ before } \tau_{i+2}\right)\\
&\ge \sum_{y\in B_{i-1}} E^o_\omega\left(E^{X_{\tau_i}-y}_{\theta^{-y}\omega}(\#\text{ visits at } 0 \text{ before } \tau_{i+1})\right)\\
&\ge C\sum_{y\in B_{i-1}} \max_{z\in E_i}\Big(E^z_{\theta^{-y}\omega}(\#\text{ of visits at } 0 \text{ before } \tau_{i+1})\Big)
\end{aligned}$$

where $E_i = \{x : \frac{r_i}{2} < |x|_\infty < \frac{3r_i}{2}\}$, and Harnack's inequality (Lemma 3.3.8) was used in the last step. Taking P-expectations, we conclude that

$$\begin{aligned}
Cr_{i+1}^{2(1+\delta)} &\ge C\sum_{y\in B_{i-1}} \mathbb{E}^o\Big(\max_{z\in E_i} E^z_{\theta^{-y}\omega}(\#\text{ of visits at } 0 \text{ before } \tau_{i+1})\Big)\\
&\ge C\sum_{y\in B_{i-1}} \mathbb{E}^o\Big(E^{X_{\tau_i}}_{\theta^{-y}\omega}(\#\text{ of visits at } 0 \text{ before } \tau_{i+1})\Big)\\
&= C\sum_{y\in B_{i-1}} \mathbb{E}^o\Big(E^{X_{\tau_i}}_{\omega}(\#\text{ of visits at } 0 \text{ before } \tau_{i+1})\Big)\\
&= C'(r_{i-1})^d\mathbb{E}^o\Big(E^{X_{\tau_i}}_{\omega}(\#\text{ of visits at } 0 \text{ before } \tau_{i+1})\Big)\,,
\end{aligned}$$

where the shift invariance of P was used in the next to last equality. Therefore,

$$\mathbb{E}^o(\#\text{ of visits at 0 between } \tau_i+1 \text{ and } \tau_{i+1}) \le C'' r_i^{2+\delta-d}\,.$$

Hence, for $d \ge 3$,

$$\mathbb{E}^o(\#\text{ of visits at 0}) \le C'' \sum_{i=1}^{\infty} r_i^{2+\delta-d} < \infty\,,$$

implying that P-a.s., $E^o_\omega(\#\text{ of visits at 0}) < \infty$, i.e. (X_n) is transient if $d \ge 3$.

Turning to $d=2$, we recall the following lemma:

Lemma 3.3.27 (Derrienic[21]) *Let (Y_i) be a stationary and ergodic lattice valued sequence, and set $S_n = \sum_{i=1}^n Y_i$. Define*

$$R_n = \{\#\textit{ of sites visited up to time } n\}\,.$$

Then,

$$\frac{R_n}{n} \underset{n\to\infty}{\longrightarrow} \mathrm{Prob}(S_i \neq 0,\ i \ge 1)\,.$$

Proof. The sequence R_n is sub-additive and hence, by Kingman's ergodic sub-additive theorem, $R_n/n \to_{n\to\infty} a$, a.s. and in L^1, for some constant a. Noting that $R_{n+1} = R_n \circ \theta + \mathbf{1}_{Y_1 \notin \{\cup_{i=2}^{n+1} S_i\}}$, it holds that $(R_{n+1} - R_n \circ \theta) \underset{n\to\infty}{\to} \mathbf{1}_A \circ \theta$, where $A = \{S_i \neq 0, i \ge 1)$. Thus, $ER_n/n \underset{n\to\infty}{\to} E\mathbf{1}_A =: a$. □

Under the measure on the environment Q introduced in this section, the increments $\{X_{n+1} - X_n\}$ are stationary and ergodic. Letting R_n denote the range of the RWRE up to time n, we have that

$$\frac{R_n}{n} \longrightarrow_{n\to\infty} Q \times P^o_\omega(\text{no return to } 0), \quad Q\text{-a.s.}$$

But, due to the CLT (Theorem 3.3.4 and Remark 3.3.5), for any $\delta > 0$,

$$\liminf_{n\to\infty} P^o_\omega\left(\frac{R_n}{n} < \delta\right) > 0, \quad Q\text{-a.s.}$$

Hence, for any $\delta > 0$,

$$P^o_\omega(\text{no return to } 0) < \delta, \quad Q\text{-a.s.}$$

and hence also P-a.s. This concludes the recurrence proof. □

Remark: It is interesting to note that the transience (for $d \ge 3$) and recurrence (for $d=2$) results are *false* for certain balanced, elliptic environments in Ω_0 (however, the P-probability of these environments is, of course, null). A simple example that exhibits the failure of recurrence for $d=2$ was suggested by N. Gantert: fix $0.25 < p < 0.5$ and $q = 0.5 - p$. With $x = (x_1, x_2) \in \mathbb{Z}^2$, define

$$\omega(x,e)=\begin{cases}\frac{1}{4}, & x_1=x_2, |e|=1\\ p, & \begin{cases} e=\pm e_2, |x_1|>|x_2|\\ \text{or}\\ e=\pm e_1, |x_1|<|x_2|\end{cases}\\ q, & \begin{cases} e=\pm e_1, |x_1|>|x_2|\\ \text{or}\\ e=\pm e_2, |x_1|<|x_2|\end{cases}\end{cases}$$

Define

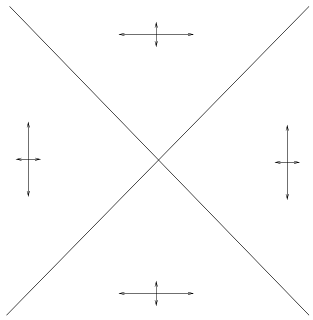


Fig. 3.3.2. Transient balanced environment, $d=2$

$$\nu(x)=\begin{cases}1, & x\neq 0\\ 4q, & x=0\,.\end{cases}$$

Then, $\nu(\cdot)$ is an excessive measure, i.e.

$$(L^*_\omega \nu)(x) := \sum_{e:|e|=1} \omega(x-e,e)\nu(x-e) \leq \nu(x), \quad x\in\mathbb{Z}^2\,.$$

If $\{X_n\}$ was recurrent, then every excessive measure needs to equal the (unique) invariant measure. But, with

$$\nu((1,0)) = 1 > (\nu L_\omega)((1,0)) = 2q+0.5\,.$$

Thus, $\nu(\cdot)$ is not invariant, contradicting the recurrence of the chain.

The intuitive idea behind the example above is that for points far from the origin, the "radial component" of the walk behaves roughly like a Bessel process of dimension $2+\delta$, some $\delta>0$, implying the transience. A similar argument, only more complicated, allows one to construct environments in $d\geq 3$ where the radial component behaves like a Bessel process of dimension $2-\delta$, some $\delta>0$. It is not hard to prove, using Lyapunov functions techniques, that there exists a $\kappa(d)<1/2d$ such that if $d\geq 3$ and the balanced environment is such that $\min_{e:|e|=1}\omega(x,e)>\kappa(d)$ then the walk is transient.

Bibliographical notes: The basic CLT under Assumption 3.3.1 is due to Lawler [48], who transfered to the discrete setting some results of Papanicolau and Varadhan. An extension to the case of non nearest neighbour walks appears in [49]. The Harnack principle (Lemma 3.3.8) was provided in [50], and in greater generality in [47], whose approach we follow, after a suggestion by Sznitman (see also [70]).

The proof of the transience part in Theorem 3.3.22 was suggested by G. Lawler in private communication. The proof of the recurrence part is due to H. Kesten, also in private communication. A recent independent proof appears in [8]. Finally, the examples mentioned at the end of the section go back to Krylov (in the context of diffusions), with this version based on discussions with Comets and Gantert.

We comment that there are very few results on LLN's and CLT's for non balanced, non ballistic walks. One exception is the result in [9], where renormalization techniques are used to prove a (quenched) CLT in symmetric (not-balanced!) environments with small disorder. Another case, in which some of the RWRE coordinates perform a simple random walk, is analysed in details in [4], using cut-times of the random walk instead of the regeneration times used in Section 3.5.

3.4 Large deviations for nestling walks

In this section, we derive an LDP for a class of nearest neighbour random walks in random environment, in $\mathbb{Z}^d$. For reasons that will become clearer below, we need to restrict attention to environments which satisfy a condition on the support of P, which we call, after M. Zerner, "nestling environments". For technical reasons, we also need to make an independence assumption (see however the remark at the end of this section).

Define $d(\omega) := \sum_{e:|e|=1} \omega(0,e)e$, and let $P_d := P \circ d^{-1}$ denote the law of $d(\omega)$ under P.

Assumption 3.4.1

(C1) *P is i.i.d.*
(C2) *P is elliptic: there exists an $\varepsilon > 0$ such that $P(\omega(z, z+e_i) \geq \varepsilon) = P(\omega(z, z-e_i) \geq \varepsilon) = 1$, $i = 1, \cdots, d$.*
(C3) *(Nestling property): $0 \in$ conv (supp(P_d)).*

We elaborate below on the nestling assumption. Clearly, balanced walks are nestling, but one may construct examples, as in $d = 1$, of nestling environments with ballistic behaviour.

For any $y \in \mathbb{R}^d$, we denote by $[y]$ the point in $\mathbb{Z}^d$ with $1 > y_i - [y]_i \geq 0$. For $z \in \mathbb{Z}^d$, we let $T_z = \inf\{n \geq 0 : X_n = z\}$. As in Section 2.3, the key to our approach to large deviation results for (X_n) is a large deviation principle for $T_{[nz]}$, $z \in \mathbb{R}^d$, stated next.

Theorem 3.4.2 *(a) Assume P is ergodic and elliptic. For any $z \in \mathbb{R}^d, |z|_1 = 1$, any $\lambda \leq 0$, the following deterministic limit exists P-a.s.*

$$a(\lambda, z) := \lim_{n\to\infty} \frac{1}{n} \log E_\omega^o(e^{\lambda T_{[nz]}} \mathbf{1}_{T_{[nz]}<\infty}) .$$

(b) Further assume Assumption 3.4.1, and define

$$I_{T,z}(s) = \sup_{\lambda<0} (\lambda s - a(\lambda, z)) .$$

Then, $T_{[nz]}/n$ satisfies, P-a.s., under P_ω^o, a (weak) LDP, with rate function $I_{T,z}(s)$. That is,

$$\begin{aligned}&\lim_{\delta\to0} \limsup_{n\to\infty} \frac{1}{n} \log P_\omega^o(T_{[nz]}/n \in (s-\delta, s+\delta)) \\ &\quad = \lim_{\delta\to0} \liminf_{n\to\infty} \frac{1}{n} \log P_\omega^o(T_{[nz]}/n \in (s-\delta, s+\delta)) = -I_{T,z}(s), \quad P-a.s.\end{aligned} \tag{3.4.3}$$

(Note that $I_{T,z}(s) = \infty$ for $s < 1$).

With Theorem 3.4.2 at hand, we may state the LDP for X_n/n. Define, for $x \in \mathbb{R}^d$,

$$I(x) = \begin{cases} |x|_1 I_{T,x/|x|_1}(1/|x|_1) , & |x|_1 \leq 1 \\ \infty , & \text{otherwise} \end{cases} .$$

Obviously, $a(\lambda, z)$ is defined for any $z \in \mathbb{R}^d \setminus \{0\}$, and is by definition homogeneous in $|z|_1$. An easy computation then reveals that $I(x) = \sup_{\lambda<0}(\lambda - a(\lambda, x))$. We have the

Theorem 3.4.4 *Assume Assumption 3.4.1. Then, P-a.s., the random variables X_n/n satisfy the LDP in $\mathbb{R}^d$ with good, convex rate function $I(\cdot)$. That is,*

$$\begin{aligned}\lim_{\delta\to0} \limsup_{n\to\infty} \frac{1}{n} \log P_\omega^o \left(\frac{X_n}{n} \in B_x(\delta) \right) &= \lim_{\delta\to\infty} \liminf_{n\to\infty} \frac{1}{n} \log P_\omega^o \left(\frac{X_n}{n} \in B_x(\delta) \right) \\ &= -I(x) , \quad P-a.s.\end{aligned}$$

Proof of Theorem 3.4.2 The idea behind the proof is relatively simple, and is related to our proof of large deviations for $d = 1$. However, there are certain complications in the proof of the lower bound, which can be overcome at present only under the nestling assumption.

a) We begin by defining, for $\lambda \leq 0$,

$$a_{n,m}(\lambda, z) := \log E_\omega^{[mz]}\left(e^{\lambda T_{[nz]}} \mathbf{1}_{T_{[nz]<\infty}}\right) .$$

We then have (since the time to reach $[nz]$ is not larger than the time to reach $[nz]$, when one is forced also to first visit $[mz]$), that

$$a_{n,0}(\lambda, z) \geq a_{m,0}(\lambda, z) + a_{n,m}(\lambda, z)\,.$$

Further, we note that due to **C2**, there exists a constant $C(\lambda, \varepsilon)$ such that $n^{-1}|a_{n,0}(\lambda, z)| \leq C(\lambda, \varepsilon)$, for all ω with $\omega(x, x+e) \geq \epsilon$, all $x \in \mathbb{Z}^d$ and e such that $|e| = 1$. Thus, by Kingman's subadditive ergodic theorem,

$$\frac{a_{n,0}(\lambda, z)}{n} \underset{n\to\infty}{\longrightarrow} a(\lambda, z)\,, \quad P\text{-a.s.} \tag{3.4.5}$$

b) By Chebycheff's inequality, (3.4.5) immediately implies the upper bound in (3.4.3). Thus, all our effort is now concentrated in proving the lower bound.

Toward this end, note that by Jensen's inequality, the deterministic function $a(\cdot, z)$ is convex, and thus differentiable a.e. We denote by $\mathcal{D}$ the set of $\lambda < 0$ such that $a(\cdot, z)$ is differentiable at λ. Recall that a point $s \in \mathbb{R}_+$ is an exposed point of $I_{T,z}(\cdot)$ if for some $\lambda < 0$ ("the exposing plane") and all $t \neq s$,

$$\lambda t - I_{T,z}(t) > \lambda s - I_{T,z}(s)\,.$$

It is straightforward to check, see e.g., [20, Lemma 2.3.9(b)] that if $y = a'(\lambda, z)$ for some $\lambda \in \mathcal{D}$, then $I_{T,z}(y) = \lambda y - a(\lambda, z)$, and further y is an exposed point of $I_{T,z}(\cdot)$, with exposing plane λ.

As we already saw, it is then standard (see, e.g., [20, Theorem 2.3.6(b)]) that the lower bound in (3.4.3) holds for any exposed point. Thus, it only remains to handle points which are not exposed. Toward this end, define (using the monotonicity to ensure the existence of the limit!)

$$s_+ := \lim_{\lambda\to 0, \lambda\in\mathcal{D}} a'(\lambda, z) \leq \infty\,.$$

Note that, for any $s \geq s_+$, $I_{T,z}(s) = -\lim_{\lambda\to 0} a(\lambda, z)$.

The approach toward the lower bound is different when $s \geq s_+$ (case a) and $s < s_+$ (case b): in case a, a strategy which will achieve a lower bound consists of spending first some time in a "trap" at the neighborhood of the origin, returning to the origin and then getting to $[nz]$ within time roughly ns_-^η, where $s_-^\eta < s_+$ is an exposed point with $|I_{T,z}(s_-^\eta) - I_{T,z}(s_+)| \leq \eta$. The nestling assumption is crucial to create the trap. In case b, the achieving strategy consists of finding an intermediate point, progressing faster than needed toward the intermediate point, and then progressing slower than expected toward $[nz]$. To control the behavior of the walk starting at intermediate points, the independence assumption comes in handy.

Turning to case a, the role of the nestling assumption is evident in the following lemma:

Lemma 3.4.6 *Assume Assumption 3.4.1. Then, there exists an $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ with the following property: for each $\delta > 0$ and each $\omega \in \Omega_0$, there exists an $R(\delta, \omega)$ and an $n_0 = n_0(\delta, \omega)$ such that, for any $n > n_0$ even,*

$$P_\omega^o(|X_m|_2 \leq R(\delta, \omega), \quad m = 1, \cdots, n-1, X_n = 0) \geq e^{-\delta n}\,.$$

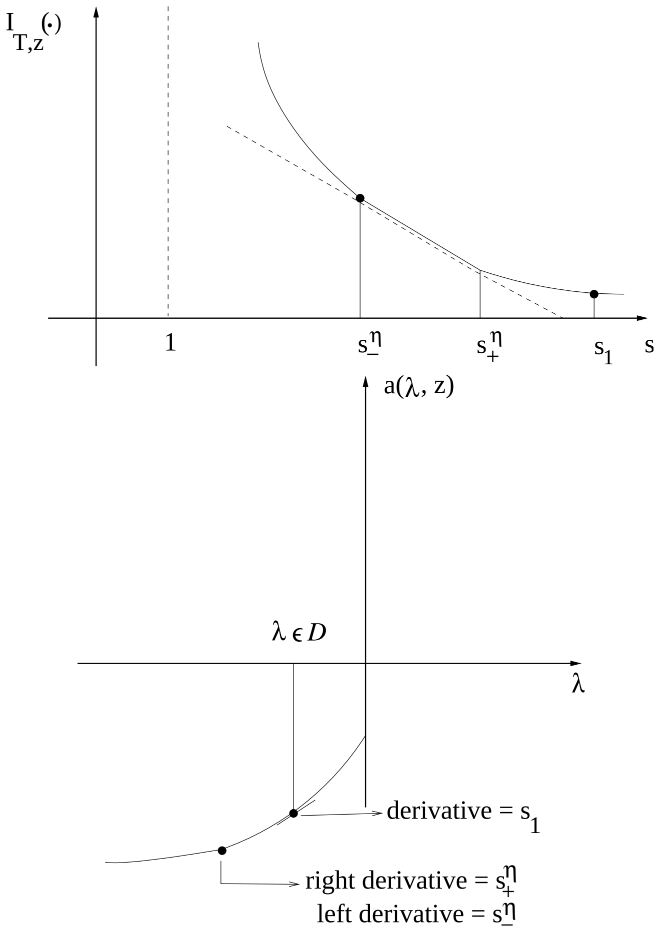


Fig. 3.4.1. exposed points and differentiability

Proof of Lemma 3.4.6 We begin by constructing a "trap". As a preliminary, with $x \in \mathbb{Z}^d$, and $\bar{\imath}$ such that $|x_{\bar{\imath}}| \geq |x_j|$, $j = 1, \cdots, d$ (and hence $|x|_2 \leq \sqrt{d}|x_{\bar{\imath}}|$) we have, defining $y_x = x -$ sign $(x_i)e_i$, that

$$|x|_2 - |y_x|_2 = \frac{|x_{\bar{\imath}}|^2 - (|x_{\bar{\imath}}| - 1)^2}{|x|_2 + |y_x|_2} \geq \frac{1}{2\sqrt{d}} .$$

Fix $\kappa = \varepsilon\delta/32\sqrt{d}$ and $F(x) = (1 - \kappa^2|x|_2^2) \vee 0$. Call *a* site $x \in \mathbb{Z}^d$ "successful" if

$$x \cdot \sum_{i=1}^{d} \Big(\omega(x, x + e_i)e_i - \omega(x, x - e_i)e_i \Big) \leq 1 .$$

Due to **(C3)**,

$$P(x \text{ is successful}) > 0\,.$$

and hence, by the independence assumption **(C1)**,

$$P(\text{all sites } x \in B_{1/\kappa}(0) \text{ are successful}) > 0\,. \tag{3.4.7}$$

Fix now $\omega \in \Omega$ such that all sites $x \in B_{1/\kappa}(0)$ are successful. We next claim that for such ω,

$$\sum_i \omega(x, x \pm e_i) F(x \pm e_i) \geq e^{-\delta/3} F(x)\,. \tag{3.4.8}$$

Indeed, for $|x|_2 \geq 1/\kappa$ this is obvious, for $1/\kappa - \varepsilon/4\sqrt{d} < |x|_2 < 1/\kappa$ this follows from the ellipticity assumption **(C2)** while for $|x|_2 < 1/\kappa - \varepsilon/4\sqrt{d}$ this follows from a Taylor expansion. Thus, $e^{\delta n/3} F(X_n)$ is, for such ω, a submartingale under P^o_ω, and we have that for all $n \geq 1$,

$$e^{-\delta n/3} = e^{-\delta n/3} E^o_\omega F(X_0) \leq e^{-\delta n/3} E^o_\omega \Big(e^{\delta n/3} F(X_n) \Big) \leq P^o_\omega \left(|X_n|_2 < \frac{1}{\kappa} \right)\,. \tag{3.4.9}$$

Fixing n_1 even large enough such that $e^{-\delta n_1/3} \varepsilon^{\sqrt{d}/\kappa} \geq e^{-2\delta n_1/3}$, we conclude that for such ω,

$$P^o_\omega \Big(|X_m|_\infty \leq n_1, m = 1, \ldots, n_1 - 1, \quad X_{n_1} = 0 \Big) \geq e^{-2\delta n_1/3}\,.$$

Due to (3.4.7) and **(C1)**, there exists (P-a.s.) an $x_0 = x_0(\omega, \delta)$ such that all sites in $B_{1/\kappa}(x_0)$ are successful. Set $m_0 = m_0(\omega, \delta) := \sum_{i=1}^d |x_0(\omega, \delta)(i)|$. Due to the ellipticity assumption **(C2)**, we have

$$P^o_\omega \Big(X_{m_0} = x_0(\omega, \delta) \Big) \geq \varepsilon^{m_0}\,, \quad P^{x_0}_\omega (X_{m_0} = 0) \geq \varepsilon^{m_0}\,.$$

Next set $R(\delta, \omega) := n_1 + 2m_0 + 1$. Define $K = \lfloor (n - 2m_0)/n_1 \rfloor$. We then have, using the Markov property, that

$$\begin{aligned}
& P^o_\omega \Big(|X_m|_2 \leq R(\delta, \omega), \quad m = 1, \cdots, n, \quad X_n = 0 \Big) \\
& \geq P^o_\omega \Big(X_{m_0} = x_0(\omega, \delta) \Big) P^{x_0}_\omega \Big(|X_m - x_0|_\infty \leq n_1, X_{n_1} = 0 \Big)^K \\
& \qquad P^{x_0}_\omega (X_{m_0} = 0) P^o_\omega (X_{n - K n_1 - 2m_0} = 0) \\
& \geq \varepsilon^{2m_0} \varepsilon^{n_1} \cdot e^{-\frac{2\delta}{3} K n_1} \geq e^{-\delta n}\,,
\end{aligned}$$

for all $n > n_0(\delta, \varepsilon, \omega)$. $\square$

Equipped with Lemma 3.4.6 we may complete the proof in case a. Indeed, all we need to prove is that for any $\delta > 0$,

$$\liminf_{n \to \infty} \frac{1}{n} \log P^o_\omega \Big(T_{[nz]}/n \in (s - \delta, s + \delta) \Big) = -I_{T,z}(s_+)\,, \quad P - a.s.$$

Fix $\eta > 0$ and an exposed point s_-^η with $|I_{T,z}(s_-^\eta) - I_{T,z}(s_+)| \le \eta$. Due to the Markov property, for all n such that $|nz|_\infty > R(\delta,\omega)$,

$$\begin{aligned}
&P_\omega^o\Big(T_{[nz]}/n \in (s-\delta, s+\delta)\Big)\\
&\ge P_\omega^o\Big(|X_m|_2 \le R(\delta,\omega),\, m = 1,\cdots,\lceil n(s-s_-^\eta)\rceil, X_{\lceil n(s-s_-^\eta)\rceil} = 0\Big)\\
&\qquad P_\omega^o\Big(T_{[nz]}/n \in (s_-^\eta - \delta', s_-^\eta + \delta')\Big)
\end{aligned}$$

where $\delta' = \delta \cdot s_-^\eta/2s$, and hence,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_\omega^o\Big(T_{[nz]}/n \in (s-\delta, s+\delta)\Big) \ge -\delta - I_{T,z}(s_-^\eta)\,.$$

Since δ is arbitrary and $I_{T,z}(s_-^\eta) \underset{\eta\to 0}{\to} I_{T,z}(s_+)$, the proof is concluded for $s \ge s_+$.

Turning to case b, recall that our plan is to consider intermediate points. This requires a slight strengthening of the convergence of $a(\lambda, z)$. We state this in the following Lemma, whose proof is deferred.

Lemma 3.4.10 *Assume Assumption 3.4.1 and set $\nu \in (0,1)$. Then, for $z \in \mathbb{R}^d$, $|z|_1 = 1$, and any $\lambda < 0$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log E_\omega^{[\nu nz]}\Big(e^{\lambda T_{[nz]}} \mathbf{1}_{T_{[nz]}<\infty}\Big) = (1-\nu)a(\lambda, z)\,, \quad P-a.s.$$

Assuming Lemma 3.4.10, we complete the proof of part b. Note the existence, for any $\eta \ge 0$, of $s_-^\eta < s < s_+^\eta$ such that s_-^η, s_+^η are exposed, and further

$$\left| I_{T,z}(s) - \left(\frac{s - s_-^\eta}{s_+^\eta - s_-^\eta}\right) I_{T,z}(s_-^\eta) - \left(\frac{s_+^\eta - s}{s_+^\eta - s_-^\eta}\right) I_{T,z}(s_+^\eta) \right| < \eta\,. \tag{3.4.11}$$

Set $\nu := (s_+^\eta - s)/(s_+^\eta - s_-^\eta)$. By the Markov property,

$$\begin{aligned}
&P_\omega^o\left(T_{[nz]}/n \in (s-\delta, s+\delta)\right)\\
&\ge P_\omega^o\left(T_{[\nu nz]}/n \in (s_-^\eta - \delta', s_-^\eta + \delta')\right) P_\omega^{[\nu nz]}\left(T_{[nz]}/n \in (s_+^\eta - \delta', s_+^\eta + \delta')\right)
\end{aligned}$$

where $\delta' = \frac{\min(\nu, 1-\nu)\delta}{2}$. Due to Lemma 3.4.10, and the fact that s_+^η, s_-^η are exposed points of $I_{T,z}(\cdot)$, one concludes that

$$\begin{aligned}
&\lim_{\delta\to\infty} \liminf_{n\to\infty} \frac{1}{n} \log P_\omega^o\left(T_{[nz]}/n \in (s-\delta, s+\delta)\right)\\
&\qquad\qquad \ge -\left[\left(\frac{s - s_-^\eta}{s_+^\eta - s_-^\eta}\right) I_{T,z}(s_-^\eta) + \left(\frac{s_+^\eta - s}{s_+^\eta - s_-^\eta}\right) I_{T,z}(s_+^\eta)\right]\,.
\end{aligned}$$

Using (3.4.11), this completes the proof of the theorem. $\square$

Proof of Theorem 3.4.4 Fix x and δ as in the statement of the theorem. Then, using the ellipticity assumption **(C2)**, for any n large enough,

$$P_\omega^o\left(\frac{X_n}{n} \in B_x(\delta)\right) \geq P_\omega^o\left(T_{[nx]} \in n\left(1-\frac{\delta}{2}, 1+\frac{\delta}{2}\right)\right)\varepsilon^{n\delta/2}$$

and the lower bound follows from Theorem 3.4.2.

Turning to the upper bound, note that $|nB_x(\delta) \cap \mathbb{Z}^d| \leq C_\delta n^d$, and that

$$P_\omega^o\left(\frac{X_n}{n} \in B_x(\delta)\right) = \sum_{y \in nB_x(\delta)\cap\mathbb{Z}^d} P_\omega^o(X_n = y)\,.$$

Further, note that $P_\omega^o(X_n = y) \leq P_\omega^o(T_{[y]} \leq n)$, and that due to the ellipticity **(C2)**,

$$\sup_{y \in nB_x(\delta)} P_\omega^o(X_n = y) \leq \varepsilon^{-n\sqrt{d}\delta}\, P_\omega^o\Big(T_{[nx]} \leq n(1+\delta)\Big)$$

and hence,

$$\begin{aligned}
&\lim_{\delta\to\infty}\limsup_{n\to\infty}\frac{1}{n}\,\log P_\omega^o\left(\frac{X_n}{n} \in B_x(\delta)\right)\\
&\quad\leq \lim_{\delta\to\infty}\limsup_{n\to\infty}\frac{1}{n}\,\log P_\omega^o\Big(T_{[nx]} \leq n(1+\delta)\Big)\\
&\quad\leq -\inf_{0\leq\eta\leq 1} I(\eta x)\,, \quad P-a.s.
\end{aligned}$$

The monotonicity of $I(\eta\cdot)$ in η, which is induced from that of $I_{T,z}(\cdot)$, completes the proof. $\square$

Proof of Lemma 3.4.10 By the ellipticity assumption **(C2)**, $\frac{1}{n}\,\log E_\omega^{[\nu n z]}$ $(e^{\lambda T_{[nz]}}\ \mathbf{1}_{T_{[nz]}<\infty})$ is uniformly bounded. Further, it possesses the same law as $\frac{1}{n}\,\log E_\omega^o(e^{\lambda T_{[n(1-\eta)z]}}\ \mathbf{1}_{T_{[n(1-\eta)z]}<\infty})$. Thus,

$$\frac{1}{n}\,\log E_\omega^{[\nu n z]}\Big(e^{\lambda T_{[nz]}}\ \mathbf{1}_{T_{[nz]}<\infty}\Big) \xrightarrow[n\to\infty]{P} (1-\nu)a(\lambda, z)\,. \tag{3.4.12}$$

Our goal is thus to prove that the convergence in (3.4.12) is in fact a.s.

Toward this end, as a first step we truncate appropriately the expectation. Set

$$N_x = \#\{\text{ visits at } x \text{ before } T_{[nz]}\}\,,$$

and $N = \sup_{x\in\mathbb{Z}^d} N_x$. We show that, for some $\delta < 1$,

$$\limsup_{n\to\infty}\frac{1}{n}\,\log\frac{E_\omega^{[\nu n z]}(e^{\lambda T_{[nz]}}\ \mathbf{1}_{T_{[nz]}<\infty})}{E_\omega^{[\nu n z]}(e^{\lambda T_{[nz]}}\ \mathbf{1}_{T_{[nz]}<\infty}\ \mathbf{1}_{N<n^\delta})} = 0\,, \quad P-a.s. \tag{3.4.13}$$

Indeed, note first that

$$E_\omega^{[\nu nz]}\Big(e^{\lambda T_{[nz]}}\ \mathbf{1}_{T_{[nz]}<\infty}\Big) \le E_\omega^{[\nu nz]}\Big(e^{\lambda T_{[nz]}}\ \mathbf{1}_{T_{[nz]}<\infty}\ \mathbf{1}_{N<n^\delta}\Big) + \sum_{x\in\mathbb{Z}^d} E_\omega^{[\nu nz]}\Big(e^{\lambda T_{[nz]}}\ \mathbf{1}_{T_[nz]<\infty}\ \mathbf{1}_{N_x>n^\delta}\Big)\,.$$

But, due to the Markov property,

$$\begin{aligned} &E_\omega^{[\nu nz]}\Big(e^{\lambda T_{[nz]}}\ \mathbf{1}_{T_[nz]<\infty}\ \mathbf{1}_{N_x>n^\delta}\Big)\\ &\le \sum_{k=[n^\delta]+1}^{\infty} E_\omega(e^{\lambda T_x}\mathbf{1}_{T_x<T_{[nz]}})^k E_\omega^{[\nu nz]}\Big(e^{\lambda T_{[nz]}}\ \mathbf{1}_{T_x<T_[nz]<\infty}\Big)\\ &\le \frac{e^{\lambda n^\delta}}{1-e^\lambda} E_\omega^{[\nu nz]}\Big(e^{\lambda T_{[nz]}}\ \mathbf{1}_{T_[nz]<\infty}\Big)\,, \end{aligned}$$

and hence,

$$E_\omega^{[\nu nz]}\Big(e^{\lambda T_{[nz]}}\ \mathbf{1}_{T_{[nz]}<\infty}\Big) \le \frac{E_\omega^{[\nu nz]}\Big(e^{\lambda T_{[nz]}}\ \mathbf{1}_{T_{[nz]}<\infty}\ \mathbf{1}_{N<n^\delta}\Big)}{1-n^d e^{\lambda n^\delta}/(1-e^\lambda)}\,,$$

yielding (3.4.13). Further, due to the ellipticity assumption **(C2)**, it holds that for some constant $K=K(\lambda)$ large enough,

$$E_\omega^{[\nu nz]}(e^{\lambda T_{[nz]}}\ \mathbf{1}_{T_{[nz]}<Kn}\ \mathbf{1}_{N<n^\delta}) \ge E_\omega^{[\nu nz]}(e^{\lambda T_{[nz]}}\ \mathbf{1}_{T_{[nz]}<\infty}\ \mathbf{1}_{N<n^\delta})/2\,.$$

Thus, it suffices to consider

$$g_\omega^\delta = \log E_\omega^{[\nu nz]}(e^{\lambda T_{[nz]}}\ \mathbf{1}_{T_{[nz]}<Kn}\ \mathbf{1}_{N<n^\delta})\,.$$

Denote by $\mathcal{P}_{k,\delta}$ the set of nearest neighbour paths (γ_n) on $\mathbb{Z}^d$ with $\gamma_0=[\nu nz]$, $\gamma_k=[nz]$, and $N(\gamma)\le n^\delta$. For $e\in\{\pm e_i\}_{i=1}^d =: \mathcal{E}$, set

$$N_{x,e}(\gamma) = \#\{\text{ steps from } x \text{ to } x+e \text{ of } \gamma \text{ before } T_{[nz]}(\gamma)\}\,.$$

Then, with $\beta(x,x+e)=\log\omega(x,x+e)$ and $D_{Kn}=\{x\in\mathbb{Z}^d : |x|_\infty\le Kn\}$,

$$g_\omega^\delta = \log\sum_{k\le Kn} e^{\lambda k}\sum_{\gamma\in\mathcal{P}_{k,\delta}}\prod_{x\in D_{Kn}}\prod_{e\in\mathcal{E}} e^{\beta(x,x+e)N_{x,e}(\gamma)}\,.$$

We use the following concentration inequality, which is a slight variant of [78, Theorem 6.6].

Lemma 3.4.14 (Talagrand) *Let $\mathcal{K}\subset\mathbb{R}^{d_1}$ be compact and convex. Let μ be a law supported on $\mathcal{K}$, and let $f:\mathcal{K}^N\to\mathbb{R}$ be convex and of Lipschitz constant L. Finally, let M_N denote the median of f with respect to $\mu^{\otimes N}$, i.e. M_N is the smallest number such that*

$$\mu^{\otimes N}(f\le M_N)\ge\frac{1}{2}\,,\ \mu(f\ge M_N)\ge\frac{1}{2}\ .$$

Then, there exists a constant $C = C(\mathcal{K})$, independent of f, μ, such that for all $t > 0$,

$$\mu^{\otimes N}(|f - M_N| \geq t) \leq C \exp(-Ct^2/L^2)\,.$$

To apply Lemma 3.4.14, note that

$$\left|\frac{\partial g_\omega^\delta}{\partial \beta(x, x+e)}\right| \leq \frac{1}{\varepsilon} \frac{\sum_{k \leq Kn} e^{\lambda k} \sum_{\gamma \in \mathcal{P}_{k,\delta}} N_{x,e}(\gamma) \prod_{x' \in D_{Kn}} \prod_{e \in \mathcal{E}} e^{\beta(x', x'+e) N_{x',e}(\gamma)}}{\sum_{k \leq Kn} e^{\lambda k} \sum_{\gamma \in \mathcal{P}_{k,\delta}} \prod_{x' \in D_{Kn}} \prod_{e \in \mathcal{E}} e^{\beta(x', x'+e) N_{x',e}(\gamma)}}\,. \tag{3.4.15}$$

Thus, using Jensen's inequality in the first inequality,

$$\sum_{x \in D_{Kn}} \left|\frac{\partial g_\omega^\delta}{\partial \beta(x, x+e)}\right|^2 \leq \frac{1}{\varepsilon} \sum_{x \in D_{Kn}} \frac{\sum_{k \leq Kn} e^{\lambda k} \sum_{\gamma \in \mathcal{P}_{k,\delta}} N_{x,e}(\gamma)^2 \prod_{x' \in D_{Kn}} \prod_{e \in \mathcal{E}} e^{\beta(x', x'+e) N_{x',e}(\gamma)}}{\sum_{k \leq Kn} e^{\lambda k} \sum_{\gamma \in \mathcal{P}_{k,\delta}} \prod_{x' \in D_{Kn}} \prod_{e \in \mathcal{E}} e^{\beta(x', x'+e) N_{x,e}(\gamma)}} \leq \frac{Kn^{1+\delta}}{\epsilon}\,.$$

It is immediate to see that on the other hand g_ω^δ is a convex function of $\{\beta(x, x+e)\}$. Hence, by Lemma 3.4.14 and the above,

$$P(|g_\omega^\delta - E g_\omega^\delta| > tn) \leq C_1 e^{-C_1 n^{1-\delta}}\,,$$

where $C_1 = C_1(\epsilon, \delta)$. The Borel-Cantelli lemma then completes the proof of Lemma 3.4.10. □

Remarks: 1. In the proof above, the independence assumption **(C1)** was used in two places. The first is the construction of traps (Lemma 3.4.6), where the independence assumption may be replaced by the requirement that P, when restricted to finite subsets, be equivalent to a product measure. More seriously, the product structure was used in the application of Talagrand's Lemma 3.4.14. It is plausible that this can be bypassed, e.g. using the techniques in [69].

2. S. R. S. Varadhan has kindly indicated to me a direct argument which gives the quenched LDP for the position, for ergodic environments, without passing through hitting times. Fix $\epsilon > 0$, and define X_n^ϵ to be the RWRE with geometric holding times of parameter $1/\epsilon$. Fix a deterministic v, with $|v|_1 < 1$, and define

$$g(m, n) = P^0_{\theta^{[mv]}\omega}(X^\epsilon_{m-n} - X^\epsilon_0 = [(n-m)v])\,.$$

Then, $g(0, n+m) \geq g(0, m) g(m, n+m) > 0$ for all $n, m \geq 1$. Consequently, by Kingman's ergodic sub-additive theorem,

$$\frac{1}{n}\log g(0,n) \to_{n\to\infty} -I^{\epsilon}(v)\,, P-a.s.,$$

for some deterministic $I^{\epsilon}(v)$. From this it follows (see e.g. [20, Theorem 4.1.11]) that X_n^{ϵ}/n satisfies the (quenched) LDP with convex, good rate function $I^{\epsilon}(\cdot)$. Finally, it is easy to check that $I^{\epsilon}(\cdot) \to_{\epsilon\to 0} I(\cdot)$ (even uniformly on compacts) and that

$$\limsup_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P_{\omega}^{o}(|X_n - X_n^{\epsilon}| > \delta n) = -\infty\,, P-a.s.,$$

from which it follows that X_n/n satisfies the quenched LDP with deterministic, convex, good rate function $I(\cdot)$.

3. Returning to the i.i.d. nestling setup, a natural question is whether one may prove an annealed large deviations principle for the position. A partial answer is given by the following. Fix a direction ℓ and recall the time $D = D(\ell)$ introduced in Section 3.2. Define $T_k^{\ell} = \min\{n : (X_n - X_0)\cdot\ell \geq k\}$. Then, for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}^{o}(e^{\lambda T_{k+m}^{\ell}}\mathbf{1}_{\{D(\ell)=\infty\}}) &\geq E_P\Big(E_{\omega}^{o}\Big(e^{\lambda T_k^{\ell}}\mathbf{1}_{\{D(\ell)>T_k^{\ell}\}}\Big)E_{\omega}^{X_{T_k^{\ell}}}\Big(e^{\lambda T_m^{\ell}}\mathbf{1}_{\{D(\ell)=\infty\}}\Big)\Big)\\ &\geq \mathbb{E}^{o}(e^{\lambda T_k^{\ell}}\mathbf{1}_{\{D(\ell)=\infty\}})\mathbb{E}^{o}(e^{\lambda T_m^{\ell}}\mathbf{1}_{\{D(\ell)=\infty\}})\,,\end{aligned}$$

and hence, by sub-additivity, the following limit exists:

$$\lim_{k\to\infty}\frac{1}{k}\log\mathbb{E}^{o}(e^{\lambda T_k^{\ell}}\mathbf{1}_{\{D(\ell)=\infty\}}) =: g(\ell,\lambda)\,.$$

One can check that if the conclusions of Lemma 3.5.11 hold then also, for $-\lambda > 0$ small enough,

$$\limsup_{k\to\infty}\frac{1}{k}\log\mathbb{E}^{o}(e^{\lambda T_k^{\ell}}) = \lim_{k\to\infty}\frac{1}{k}\log\mathbb{E}^{o}(e^{\lambda T_k^{\ell}}\mathbf{1}_{\{D'=\infty\}})\,,$$

and hence for such λ,

$$g(\ell,\lambda) = \limsup_{k\to\infty}\frac{1}{k}\log\mathbb{E}^{o}(e^{\lambda T_k^{\ell}})\,.$$

An interesting open question is to use this argument, in the nestling setup, to deduce a LDP and to relate the annealed and quenched rate functions.

Bibliographical notes: Large deviations for the position X_n of nestling RWRE in $\mathbb{Z}^d, d>1$ were first derived in Zerner's thesis [81]. Zerner uses a martingale differences argument instead of Lemma 3.4.14. With the same technique, he also derives a more general version of Lemma 3.4.10, under the name "uniform shape theorem". The large deviations for the hitting times $T_{[nz]}$ are implicit in his approach.

A recent paper of Varadhan [80] develops the quenched large deviations alluded to in remark 2 above, and a corresponding annealed LDP. He also obtains information on the zero set of the annealed and quenched rate functions, and in particular proves in a great generality that they coincide. The techniques are quite different from those presented here.

3.5 Kalikow's condition

We introduce in this section a condition on the environment, due to Kalikow, which ensures that the RWRE is "ballistic". Suppose P is elliptic, and let U be a strict subset of $\mathbb{Z}^d$, with $0 \in U$, and define on $U \cup \partial U$ an auxiliary Markov chain with transition probabilities

$$\hat{P}_U(x, x+e) = \begin{cases} \frac{\mathbb{E}^o\left[\sum_{n=0}^{\tau_{U^c}} \mathbf{1}_{\{X_n=x\}}\omega(x,x+e)\right]}{\mathbb{E}^o\left[\sum_{n=0}^{\tau_{U^c}} \mathbf{1}_{\{X_n=x\}}\right]}, & x \in U, |e| = 1 \\ 1 & x \in \partial U, e = 0 \end{cases} \tag{3.5.1}$$

where $\tau_{U^c} = \min\{n \geq 0 : X_n \in \partial U\}$ (note that the expectations in (3.5.1) are finite due the Markov property and ellipticity). The transition kernel $\hat{P}_U$ weights the transitions $x \mapsto x + e$ according to the occupation time of the vertex x before exiting U. We denote by $\hat{E}_U$ expectations with respect to the measure $\hat{P}_U$.

The following is a basic consequence of the definition of $\hat{P}_U(\cdot, \cdot)$:

Lemma 3.5.2 (Kalikow) *Assume $\hat{P}_U(\tau_{U^c} < \infty) = 1$. Then, $\hat{P}_U(X_{\tau_{U^c}} = v) = \mathbb{P}^o(X_{\tau_{U^c}} = v), v \in \partial U$. In particular, $\mathbb{P}^o(\tau_{U^c} < \infty) = 1$.*

Proof of Lemma 3.5.2: Set $g_\omega(x) = E^o_\omega\left(\sum_{n=0}^{\tau_{U^c}} \mathbf{1}_{\{X_n=x\}}\right)$. Then

$$\hat{P}_U(x, y) = \frac{E(g_\omega(x)\omega(x, y))}{E(g_\omega(x))}, \quad x \in U,\ y \in U \cup \partial U\,. \tag{3.5.3}$$

But, due to the Markov property,

$$g_\omega(x) = \mathbf{1}_{\{x=0\}} + \sum_{z \in U} \omega(z, x) g_\omega(z)\,,$$

and hence, using (3.5.3),

$$\sum_{x \in U} \Big(E(g_\omega(x))\Big) \hat{P}_U(x, y) + \mathbf{1}_{\{y=0\}} = E(g_\omega(y))\,.$$

Set $\hat{\pi}_n(y) = \hat{E}_U\left(\sum_{j=0}^{\tau_{U^c}\wedge n} \mathbf{1}_{\{X_j=y\}}\right)$. Then, $\hat{\pi}_0(y) = \mathbf{1}_{\{y=0\}}$ and

$$\hat{\pi}_{n+1}(y) = \mathbf{1}_{\{y=0\}} + \sum_{x \in U} \hat{P}_U(x, y)\hat{\pi}_n(x)\,.$$

Then, for $y \in U \cup \partial U$,

$$E\Big(g_\omega(y)\Big) - \hat{\pi}_{n+1}(y) = \sum_{x \in U} \hat{P}_U(x,y)\Big(E(g_\omega(x)) - \hat{\pi}_n(x)\Big)\,.$$

Since $E(g_\omega(y)) - \hat{\pi}_0(y) \geq 0$, it follows by the positivity of $\hat{P}_U(x,y)$ that for $y \in U \cup \partial U$,

$$\hat{E}_U\left(\sum_{n=0}^{\tau_{U^c}} \mathbf{1}_{\{X_n = y\}}\right) = \lim_{n\to\infty} \hat{\pi}_n(y) \leq E(g_\omega(y))\,.$$

Taking $y \in \partial U$ yields

$$\hat{P}_U(X_{\tau_{U^c}} = y) \leq \mathbb{P}^o(X_{\tau_{U^c}} = y), \quad y \in \partial U\,.$$

On the other hand, $\sum_{y\in\partial U} \hat{P}_U(X_{\tau_{U^c}} = y) = 1$ because $\hat{P}_U(\tau_{U^c} < \infty) = 1$ by assumption. Hence

$$\mathbb{P}^o(X_{\tau_{U^c}} = y) = \hat{P}_U(X_{\tau_{U^c}} = y), \quad \forall y \in \partial U\,. \qquad \square$$

We are now ready to introduce Kalikow's condition. Fix a hyperplane by picking a point $\ell \in \mathbb{R}^d \backslash \{0\}$, $|\ell|_1 \leq 1$. Define

$$\varepsilon_\ell := \inf_{U, x \in U} \sum_{|e|=1} (\ell \cdot e)\hat{P}_U(x, x+e)$$

where the infimum is over all connected strict subsets of $\mathbb{Z}^d$ containing 0. We say that *Kalikow's condition with respect to* ℓ holds if $\varepsilon_\ell > 0$. Note that ε_ℓ acts as a drift in the direction ℓ for the Markov chain $\hat{P}_U$.

A consequence of Lemma 3.5.2 is the following:

Theorem 3.5.4 *Assume that P satisfies Assumption 3.1.1. If Kalikow's condition with respect to ℓ holds, then $\mathbb{P}^o(A_\ell) = 1$. If further P is an i.i.d. measure then $v_\ell > 0$.*

Proof. Fix $U_L = \{z \in \mathbb{Z}^d : \ |z \cdot \ell| \leq L\}$. Let $\hat{X}_{n,L}$ denote the Markov chain with $\hat{X}_{0,L} = 0$ and transition law $\hat{P}_{U_L}(x, x+e)$. Set the local drift at x, $\hat{d}(x) = \sum_{|e|=1} e\hat{P}_{U_L}(x, x+e)$, and recall that $\hat{X}_{n,L} - \sum_{i=0}^{n-1} \hat{d}(\hat{X}_{i,L})$ is a martingale, with bounded increments. It follows that for some constant C,

$$\hat{P}_{U_L}\left(\sup_{0\leq n\leq N} |\hat{X}_{n,L} - \sum_{i=0}^{n-1} \hat{d}(\hat{X}_{i,L})| > \delta N\right) \leq Ce^{-C\delta^2 N}\,. \tag{3.5.5}$$

On the other hand, $\sum_{i=0}^{n-1} \hat{d}(\hat{X}_{i,L}) \cdot \ell \geq \varepsilon_\ell(n \wedge \tau_{U_L^c})$ while $|\hat{X}_{n,L} \cdot \ell| \leq L+1$. We thus conclude from (3.5.5) that

$$\hat{P}_{U_L}\left(\tau_{U_L^c} > \frac{L+1}{\varepsilon_\ell} + \frac{\delta}{\varepsilon_\ell}N\right) \le Ce^{-C\delta^2 N/\varepsilon_\ell^2},$$

and hence, for some C_1 independent of L, and all L large,

$$\hat{P}_{U_L}\left(|\hat{X}_{\tau_{U_L^c}} \cdot \ell - L| > 1\right) \le C_1 e^{-C_1 L}.$$

It follows from Lemma 3.5.2 that

$$\mathbb{P}^o\left(|X_{\tau_{U_L^c}} \cdot \ell - L| > 1\right) \le C_1 e^{-C_1 L}.$$

A similar argument shows that $\mathbb{P}^o(D = \infty) > 0$: indeed, take now $U_{L,+} = \{z \in \mathbb{Z}^d : 0 \le z \cdot \ell \le L\}$. Arguing as above, one finds that for some $C_2 > 0$ independent of L,

$$\hat{P}_{U_{L,+}}\left(|\hat{X}_{\tau_{U_{L,+}^c}} - L| \le 1\right) > C_2,$$

implying that

$$\mathbb{P}^o\left(|X_{\tau_{U_{L,+}^c}} - L| \le 1\right) > C_2.$$

Thus, $\mathbb{P}^o(D = \infty) > 0$, and then, by an argument as in the proof of Theorem 3.1.2, $\mathbb{P}^o(A_\ell) > 0$. By Theorem 3.1.2, it follows that $\mathbb{P}^o(A_\ell \cup A_{-\ell}) = 1$. Due to (3.5.5), it holds that $\mathbb{P}^o(\limsup_{n\to\infty} X_n \cdot \ell = \infty) = 1$. We thus conclude that $\mathbb{P}^o(A_\ell) = 1$.

To see that if P is i.i.d. then $v_\ell > 0$, recall the regeneration times $\{\tau_i\}_{i\ge 1}$ introduced in Section 3.2. By Lemma 3.2.5, it suffices to prove that $\mathbb{E}^o(\tau_1 | D = \infty) < \infty$. Let $U_{m,k,-} = \{z \in \mathbb{Z}^d : |z| < k, z \cdot \ell < m\}$, and set $T_{m,k} := T_{U_{m,k,-}}$ with $T_m = \lim_{k\to\infty} T_{m,k} = \min\{n : X_n \cdot \ell \ge m\}$, $m \ge 1$. By Kalikow's condition,

$$\mathbb{E}^o\left(\sum_{n=0}^{T_{m,k}} \mathbf{1}_{\{X_n = x\}} \sum_{e:|e|=1} \omega(x, x+e)\ell \cdot e\right) \ge \varepsilon_\ell \mathbb{E}^o\left(\sum_{n=0}^{T_{m,k}} \mathbf{1}_{\{X_n = x\}}\right),$$

and hence, summing over $x \in U_{m,k,-}$ and recalling that $X_{i+1} - X_i - d(\theta^{X_i}\omega)$ is a martingale difference sequence, one gets

$$1 + m \ge \mathbb{E}^o(X_{T_{m,k}} \cdot \ell) \ge \varepsilon_\ell \mathbb{E}^o(T_{m,k}),$$

and taking $k \to \infty$ one concludes that $m + 1 \ge \varepsilon_\ell \mathbb{E}^o(T_m)$. In particular,

$$\mathbb{E}^o(\liminf_{m\to\infty} T_m/m) \le \liminf_{m\to\infty} \mathbb{E}^o(T_m/m) \le 1/\varepsilon_\ell. \tag{3.5.6}$$

Since $\tau_i \to_{i\to\infty} \infty$, $\mathbb{P}^o$-a.s., one may find a (random) sequence k_m such that $\tau_{k_m} \le T_m < \tau_{k_m+1}$. By definition,

$$\ell \cdot X_{\tau_{k_m}} \le \ell \cdot X_{T_m} \le \ell \cdot X_{\tau_{k_m+1}},$$

and further,

$$\ell \cdot X_{\tau_k}/k \to_{k\to\infty} \mathbb{E}^o(\ell \cdot X_{\tau_1}|\{D=\infty\}) = \frac{1}{\mathbb{P}^o(D=\infty)} < \infty\,, \mathbb{P}^o - a.s.\,,$$

due to Lemma 3.2.5 and (3.2.7). Thus, $k_m/m \to_{m\to\infty} \mathbb{E}^o(X_{\tau_1}\cdot\ell|\{D=\infty\})^{-1}$, $\mathbb{P}^o$-a.s. But, since $\tau_{k_m}/k_m \to_{m\to\infty} \mathbb{E}^o(\tau_1|\{D=\infty\}) \in [1,\infty]$, $\mathbb{P}^o$-a.s., it follows that

$$\begin{aligned}\liminf_{m\to\infty}\frac{T_m}{m} \geq \liminf_{m\to\infty}\frac{\tau_{k_m}}{k_m}\frac{k_m}{m} = \lim_{m\to\infty}\frac{\tau_{k_m}}{k_m}\frac{k_m}{m} &= \frac{\mathbb{E}^o(\tau_1|\{D=\infty\})}{\mathbb{E}^o(X_{\tau_1}\cdot\ell|\{D=\infty\})}\\ &= \mathbb{E}^o(\tau_1|\{D=\infty\})\mathbb{P}^o(D=\infty)\,.\end{aligned}$$

Since (3.5.6) implies that $\liminf_{m\to\infty} T_m/m < \infty$, we conclude that $\mathbb{E}^o(\tau_1|\{D=\infty\}) < \infty$, and hence $v_\ell > 0$. $\square$

By noting that if Kalikow's condition holds for some ℓ_0 then it holds for all ℓ in a neighborhood of ℓ_0, one gets immediately the

Corollary 3.5.7 *Assume that P satisfies assumption 3.2.1. If Kalikow's condition with respect to some ℓ holds, then there exists a deterministic v such that*

$$\frac{X_n}{n} \to_{n\to\infty} v\,, \quad P - a.s.$$

The following is a sufficient condition for Kalikow's condition to hold true:

Lemma 3.5.8 *Assume P is i.i.d. and elliptic. Then Kalikow's condition with respect to ℓ holds if*

$$\inf_{f\in\mathcal{F}} \frac{E\left(\frac{\sum_{e:|e|=1}\omega(0,e)\ell\cdot e}{\sum_{e:|e|=1}\omega(0,e)f(e)}\right)}{E\left(\frac{1}{\sum_{e:|e|=1}\omega(0,e)f(e)}\right)} > 0\,, \tag{3.5.9}$$

where $\mathcal{F}$ denotes the collection of nonzero functions on $\{e : |e| = 1\}$ taking values in $[0,1]$.

Proof. Fix U a strict subset of $\mathbb{Z}^d$, $x \in U$, and let $\tau_x = \min\{n \geq 0 : X_n = x\}$. Define $g(x,y,\omega) := E^y_\omega(\mathbf{1}_{\{\tau_x<\tau_{U^c}\}})$. Note that $g(x,y,\omega)$ is independent of ω_x. Next,

$$\begin{aligned}&\sum_{|e|=1}(\ell\cdot e)\hat{P}_U(x,x+e)\\ &\quad= \frac{E\left(E^o_\omega\left(\sum_{n=0}^{\tau_{U^c}}\mathbf{1}_{\{X_n=x\}}\sum_{e:|e|=1}\omega(x,x+e)e\cdot\ell\right)\right)}{E\left(E^o_\omega\left(\sum_{n=0}^{\tau_{U^c}}\mathbf{1}_{\{X_n=x\}}\right)\right)}\\ &\quad= \frac{E\left(g(x,0,\omega)E^x_\omega\left(\sum_{n=0}^{\tau_{U^c}}\mathbf{1}_{\{X_n=x\}}\sum_{e:|e|=1}\omega(x,x+e)e\cdot\ell\right)\right)}{E\left(g(x,0,\omega)E^x_\omega\left(\sum_{n=0}^{\tau_{U^c}}\mathbf{1}_{\{X_n=x\}}\right)\right)}\,.\end{aligned}$$

Under P_ω^x, the process X_n is a Markov chain with Geometric$(\sum_{e:|e|=1}\omega(x,x+e)g(x,x+e,\omega))$ number of visits at x. The last equality and the Markov property then imply

$$\begin{aligned}
&\sum_{|e|=1}(\ell\cdot e)\hat{P}_U(x,x+e)\\
&\quad=\frac{E\left(g(x,0,\omega)\frac{\sum_{e:|e|=1}\omega(x,x+e)e\cdot\ell}{\sum_{e:|e|=1}\omega(x,x+e)g(x,x+e,\omega)}\right)}{E\left(g(x,0,\omega)\frac{1}{\sum_{e:|e|=1}\omega(x,x+e)g(x,x+e,\omega)}\right)}\\
&\quad=\frac{E\left(\frac{\sum_{e:|e|=1}\omega(x,e)\ell\cdot e}{\sum_{e:|e|=1}\omega(x,e)g(x,x+e,\omega)/g(x,0,\omega)}\right)}{E\left(\frac{1}{\sum_{e:|e|=1}\omega(x,e)g(x,x+e,\omega)/g(x,0,\omega)}\right)}\\
&\quad\geq\inf_{f\in\mathcal{F}}\frac{E\left(\frac{\sum_{e:|e|=1}\omega(x,e)\ell\cdot e}{\sum_{e:|e|=1}\omega(x,e)f(e)}\right)}{E\left(\frac{1}{\sum_{e:|e|=1}\omega(x,e)f(e)}\right)}=\inf_{f\in\mathcal{F}}\frac{E\left(\frac{\sum_{e:|e|=1}\omega(0,e)\ell\cdot e}{\sum_{e:|e|=1}\omega(0,e)f(e)}\right)}{E\left(\frac{1}{\sum_{e:|e|=1}\omega(0,e)f(e)}\right)}>0\,,
\end{aligned}$$

where the first inequality is due to the independence of $g(x,x+e,\omega)/g(x,0,\omega)$ in ω_x. □

An easy corollary is the following

Corollary 3.5.10 *Assume P satisfies Assumption 3.2.1. If either*
(a) $\operatorname{supp}(P_d)\subset\{z\in\mathbb{R}^d:\ell\cdot z\geq 0\}$ *but* $\operatorname{supp}(P_d)\not\subset\{z\in\mathbb{R}^d:\ell\cdot z=0\}$,
or
(b) $E((\sum_{e:|e|=1}\omega(0,e)e\cdot\ell)_+)>\frac{1}{\varepsilon}E((\sum_{e:|e|=1}\omega(0,e)e\cdot\ell)_-)\,,$
then Kalikow's condition with respect to ℓ holds.

In particular, non-nestling walks or walks with drift "either neutral or pointing to the right" satisfy Kalikow's condition with respect to an appropriate hyperplane ℓ. Further there exist truly nestling walks which do satisfy Kalikow's condition.

Remark: It is interesting to note that when P is elliptic and i.i.d. and $d=1$, Kalikow's condition is equivalent to $v\neq 0$, i.e. to the walk being "ballistic". For $d>1$, it is not clear yet whether there exist walks with $\mathbb{P}^o(A_\ell)>0$ but with zero speed. Such walks, if they exist, necessarily cannot satisfy Kalikow's condition.

Our next goal is to provide tail estimates on $X_{\tau_1}\cdot\ell$ and on τ_1. Our emphasis here is in providing a (relatively) simple proof and not the sharpest possible result. For the latter we refer to [72]. In this spirit we throughout assume $\ell=e_1$.

Lemma 3.5.11 *Assume P is i.i.d. and satisfies Kalikow's condition. Then, there exists a constant c such that*

$$\mathbb{E}^o(\exp c\,X_{\tau_1}\cdot\ell)<\infty\,.$$

Proof. Recall the notations of Section 3.2 and write

$$\begin{aligned}
\mathbb{E}^o(\exp(c\,X_{\tau_1}\cdot\ell)) &= \sum_{k\geq 1}\mathbb{E}^o\Big(\exp(c\,X_{\tau_1}\cdot\ell)\mathbf{1}_{\{K=k\}}\Big)\\
&= \sum_{k\geq 1}\sum_{x\in\mathbb{Z}^d} e^{c\,x\cdot\ell} E\Big(E^o_\omega\left(\mathbf{1}_{\{X_{\overline{S}_k}=x\}}\mathbf{1}_{\{\overline{S}_k<\infty\}}\right)\cdot P^x_\omega(D=\infty)\Big)\\
&= \mathbb{P}^o(D=\infty)E\left(\sum_{k\geq 1}\sum_{x\in\mathbb{Z}^d} e^{cx\cdot\ell}E^o_\omega\Big(\mathbf{1}_{\{X_{\overline{S}_k}=x\}}\mathbf{1}_{\{\overline{S}_k<\infty\}}\Big)\right)\\
&= \mathbb{P}^o(D=\infty)\sum_{k\geq 1}\mathbb{E}^o\Big(\exp(c\,X_{\overline{S}_k}\cdot\ell)\mathbf{1}_{\{\overline{S}_k<\infty\}}\Big)\,. \qquad (3.5.12)
\end{aligned}$$

But, using the Markov property,

$$\begin{aligned}
&\mathbb{E}^o\Big(\exp(c\,X_{\overline{S}_k}\cdot\ell)\;\mathbf{1}_{\{\overline{S}_k<\infty\}}\Big)\\
&\qquad\leq \mathbb{E}^o\Big(\exp(c\,X_{\overline{S}_{k-l}}\cdot\ell)\;\mathbf{1}_{\{\overline{S}_{k-1}<\infty\}}\Big)\mathbb{E}^o\Big(\exp(c\,M_0\cdot\ell)\;\mathbf{1}_{\{D<\infty\}}\Big)\,.
\end{aligned}$$

Hence,

$$\mathbb{E}^o\Big(\exp(c\,X_{\overline{S}_k}\cdot\ell)\mathbf{1}_{\{\overline{S}_k<\infty\}}\Big)\leq \mathbb{P}^o(D=\infty)\sum_{k\geq 0}\Big(\mathbb{E}^o(\exp(c\,M_0\cdot\ell)\mathbf{1}_{\{D<\infty\}}\Big)^k\,.$$

Note, using $\ell = e_1$, that

$$\begin{aligned}
&\mathbb{E}^o\Big(e^{cM_1\cdot\ell}\mathbf{1}_{\{D<\infty\}}\Big)\\
&\quad= \sum_{k=1}^{\infty} e^{ck}\mathbb{E}^o\Big(\mathbf{1}_{\{M_0\cdot\ell=k\}}\mathbf{1}_{\{D<\infty\}}\Big)\\
&\quad= \sum_{k=1}^{\infty} e^{ck}\sum_{y\in\mathbb{Z}^{d-1}}\mathbb{E}^o\Big(E^o_\omega\Big(\mathbf{1}_{\{X_{T_k}=(k,y)\}}E^{(k,y)}_\omega(T_0<T_{k+1})\Big)\,. \qquad (3.5.13)
\end{aligned}$$

Using Kalikow's condition and a computation as in Theorem 3.5.4, one has that for some $c_1>0$, $\mathbb{P}^o\left(\sum_{|y|>\frac{2k}{\varepsilon_\ell}}\mathbf{1}_{\{X_{T_k}=(k,y)\}}\right)\leq e^{-c_1k}$, while $\mathbb{P}^o(T_{-k}<T_1)\leq e^{-c_1k}$. Hence, substituting in (3.5.13), one has

$$\mathbb{E}^o\Big(e^{cM_1\cdot\ell}\mathbf{1}_{\{D<\infty\}}\Big)\leq\sum_{k=1}^{\infty}e^{ck}\left(\left(\frac{4k}{\varepsilon_\ell}\right)^{d-1}+1\right)e^{-c_1k}\,.$$

Taking $c<c_1$, the lemma follows. □

A direct consequence of Lemma 3.5.11 is that

$$\limsup_{n\to\infty}\frac{1}{n}\,\log\mathbb{P}^o(X_{\tau_1}\cdot\ell>vn)\leq-\beta(v) \qquad (3.5.14)$$

where $\beta(v)>0$ for $v>0$.

With a proof very similar to that of Lemma 3.5.11, we have in fact the

Lemma 3.5.15 *Assume P is i.i.d. and satisfies Kalikow's condition. Set $X^* = \sup_{0\le n\le \tau_1} |X_n|$. Then, there exists a constant c' such that*

$$\mathbb{E}^o(\exp c' X^*) < \infty\,.$$

We next turn to obtaining tail estimates on τ_1. Here, due to the presence of “traps”, one cannot in general expect exponential decay as in (3.5.14). We aim at proving the following result.

Theorem 3.5.16 *Assume P satisfies Assumption 3.2.1. and Kalikow's condition. Then, with $d \ge 2$, there exists an $\alpha > 1$ such that for all u large,*

$$\mathbb{P}^o(\tau_1 > u) \le e^{-(\log u)^\alpha}\,.$$

In particular, τ_1 possesses all moments.

Proof. Recall that we take $\ell = e_1$ and, for $L > 0$, set $c_L = (-L, L) \times \left(-\frac{2L}{\varepsilon_\ell}, \frac{2L}{\varepsilon_\ell}\right)^{d-1}$. Note that, with c as in Lemma 3.5.11,

$$\begin{aligned}\mathbb{P}^o(\tau_1 \ge u) &\le \mathbb{P}^o\Big(\tau_1 \ge u, X_{\tau_1}\cdot \ell \le L\Big) + \mathbb{P}^o\Big(X_{\tau_1}\cdot\ell \ge L\Big)\\ &\le e^{-cL/2} + \mathbb{P}^o(\tau_L > \tau_{c_L}) + \mathbb{P}^o(\tau_{c_L} = \tau_L \ge u)\end{aligned}$$

where

$$\begin{aligned}\tau_L &= \inf\{t:\ X_t\cdot\ell \ge L\} \text{ and}\\ \tau_{c_L} &= \inf\{t:\ X_t \notin c_L\}\,.\end{aligned}$$

Hence, by Kalikow's condition,

$$\mathbb{P}^o(\tau_1 \ge u) \le e^{-\overline{c}L/2} + \mathbb{P}^o\Big(\tau_{c_L} = \tau_L \ge u\Big) \tag{3.5.17}$$

for some constant $\overline{c} > 0$.

The heart of the proof of Theorem 3.5.16 lies in the following lemma, whose proof is deferred.

Lemma 3.5.18 *There exist a $\beta < 1$ and $\xi > 1$ such that for any $c > 0$,*

$$\limsup_{L\to\infty} \frac{1}{L^\xi} \log \mathbb{P}\Big(P^o_\omega(X_{\tau_{U_L}}\cdot \ell \ge L) \le e^{-cL^\beta}\Big) < 0\,,$$

where $U_L = \{z \in \mathbb{Z}^d : |z\cdot\ell| \le L\}$.

Accepting Lemma 3.5.18, let us complete the proof of Theorem 3.5.16. Toward this end, set $\Delta(u) = a\log u$ with a small enough such that $\varepsilon^{\Delta(u)} > \frac{1}{u^{1/6}}$, and set $L = L(u) = (\log u)^{\overline{\alpha}}$, with $N = L/\Delta = \frac{1}{a}(\log u)^{\overline{\alpha}-1}$ and $2 - \overline{\alpha} = \beta$ where β is as in Lemma 3.5.18. Observe that

$$\mathbb{P}^o(\tau_{c_L} \geq u) \leq \left(\frac{1}{2}\right)^{(\log u)^{\overline{\alpha}}} + P^o\left(\exists\, x_1 \in c_L, P_\omega^{x_1}\left(\tau_{c_L} > \frac{u}{(\log u)^{\overline{\alpha}}}\right) \geq \frac{1}{2}\right)$$
$$=: \left(\frac{1}{2}\right)^{(\log u)^{\overline{\alpha}}} + P(\mathcal{R})\,. \tag{3.5.19}$$

Note that

$$\begin{aligned}&\frac{u}{(\log u)^{\alpha}} \cdot P_\omega^{x_1}\Big(\tau_{c_L} > \frac{u}{(\log u)^{\overline{\alpha}}}\Big)\\ &\quad\leq E_\omega^{x_1}(\tau_{c_L}) = E_\omega^{x_1}\left(\sum_{i=1}^{\tau_{c_L}} 1\right) = E_\omega^{x_1}\left(\sum_{y\in c_L}\sum_{n=1}^{\tau_{c_L}} \mathbf{1}_{\{X_n=y\}}\right)\\ &\quad= \sum_{y\in c_L} \frac{E_\omega^{x_1}\Big(\mathbf{1}_{\{\tau_y<\tau_{c_L}\}}\Big)}{E_\omega^{y}\Big(\mathbf{1}_{\{\tau_y>\tau_{c_L}\}}\Big)} \leq |c_L| \frac{1}{\inf_{y\in c_L} P_\omega^{y}\Big(\mathbf{1}_{\{\tau_y\}} > \tau_{c_L}\Big)}\,,\end{aligned}$$

with $\tau_y = \inf\{t : X_t = y\}$. Hence,

$$\inf_{y\in c_L} P_\omega^y(\tau_y > \tau_{c_L}) \leq \frac{|c_L|(\log u)^{\overline{\alpha}}}{u \sup_{x_1\in c_L} P_\omega^{x_1}\left(\tau_{c_L} > \frac{u}{(\log u)^{\bar{\alpha}}}\right)}\,.$$

Hence, on $\mathcal{R}$ there exists a y with $\mathbb{P}_\omega^y(\tau_y > \tau_{c_L}) \leq \frac{1}{u^{4/5}}$, for all u large enough.

Set $A_i = \{z \in \mathbb{Z}^d : z\cdot \ell = i\Delta\}$. By ellipticity (recall $\varepsilon^{\Delta(u)} > 1/u^{1/6}$), it follows that on the event $\mathcal{R}$, there exists an $i_0 \in [-N+2, N-1]$ and an $x \in A_{i_0}$ such that

$$P_\omega^x\Big(\tau_{(i_0-1)\Delta} > \tau_{c_L}\Big) \leq \frac{1}{\sqrt{u}} \tag{3.5.20}$$

where $\tau_{(i_0-1)\Delta} = \inf\{t:\ X_t\cdot \ell = (i_0-1)\Delta\}$. Set

$$\xi_i = -\log \min_{z\in A_i\cap c_L} P_\omega^z\Big(\tau_{(i-1)\Delta} > \tau_{(i+1)\Delta}\Big)\,.$$

Then, the Markov property implies that

$$\begin{aligned}P_\omega^x\Big(\tau_{(i_0-1)\Delta} > \tau_{c_L}\Big) &\geq \exp\left(-\sum_{j=i_0}^{N-1}\xi_j\right)\varepsilon^{\Delta(u)}\\ &\geq \exp\left(-\sum_{j=i_0}^{N-1}\xi_j\right)\frac{1}{u^{1/6}}\,.\end{aligned}$$

Thus, using (3.5.20)

$$P(\mathcal{R}) \leq P\left(\sum_{i=-N+1}^{N-1}\xi_j \geq \frac{\log u}{12}\right) \leq 2N \sup_{-N+1\leq i\leq N-1} P\left(\xi_i \geq \frac{\log u}{24N}\right)\,. \tag{3.5.21}$$

But, using the shift invariance of P and the definition of $\{\xi_i\}$,

$$2N \sup_{N+1\leq i\leq N-1} P\left(\xi_i \geq \frac{\log u}{24N}\right) \leq |c_L| P\Big(P_\omega^o(\tau_{-\Delta} > \tau_\Delta) \leq e^{-(\log u)^{2-\overline{\alpha}} b}\Big)$$

for $b = \frac{a}{24}$. The theorem follows by an application of Lemma 3.5.18. □

Proof of Lemma 3.5.18

The interesting aspect in proving the Lemma is the fact that one constructs *lower* bounds on $P_\omega^o(X_{\tau_{u_L}} \cdot \ell \geq L)$ for many configurations. Toward this end, fix $1 > \beta > \overline{\beta}$, $\gamma \in \left(\frac{1}{2}, 1\right)$, $\chi = \frac{1-\overline{\beta}}{1-\gamma} < \overline{\beta} < 1$ such that $d(\overline{\beta} - \chi) > 1$ (for $\overline{\beta}$ close enough to 1, one may always find a γ close to 1 such that this condition is satisfied, if $d \geq 2$). Set next $L_0 = L^\chi$, $L_1 = L^{\overline{\beta}}$, $N_0 = L_1/L_0$ (for simplicity, assume that L_0, L_1, L_0^γ and N_0 are all integer).

Let R be a rotation of $\mathbb{Z}^d$ such that $\tilde{R}(\ell) = \tilde{R}(e_1) = \frac{v}{|v|}$, and define

$$B_1(z) = \tilde{R}\Big(z + [0, L_0]^d\Big) \cap \mathbb{Z}^d$$
$$B_2(z) = \tilde{R}\Big(z + [-L_0^\gamma, L_0 + L_0^\gamma]^d\Big) \cap \mathbb{Z}^d$$

and $\partial_+ B_2(z) = \partial B_2(z) \cap \left\{x : \, x \cdot \frac{v}{|v|} \geq L_0 + L_0^\gamma\right\}$.

We say that $z \in L_0\mathbb{Z}^d$ is *good* if $\sup_{x\in B_1(z)} P_\omega^x(X_{\tau_{B_2(z)}} \notin \partial_+ B_2(z)) \leq \frac{1}{2}$ and say that it is *bad* otherwise. The following estimate is a direct consequence of Kalikow's condition and Lemma 3.5.15.

Lemma 3.5.22 *For* $\gamma \in (1/2, 1)$,

$$\limsup_{L_0\to\infty} L_0^{1-2\gamma} \log P(0 \textit{ is bad }) < 0\,.$$

Proof of Lemma 3.5.22

Set $u = \max\{\overline{u} : \sup_{x\in B_2(0)} x\cdot\ell \geq \overline{u}\}$ and set $L_u = \sup\{n \geq 0 : X_n\cdot\ell \leq u\}$. Define $\pi(z) = z - \frac{z\cdot v}{|v|^2} v$. Setting $K_n = \sup\{k \geq 0 : \; \tau_k < n\}$, it holds that $n \leq L_u \Rightarrow K_n \leq u$. Setting $w \in \mathbb{R}^d$ with $w \cdot v = 0$, and $|w|_1 = 1$ one has

$$\begin{aligned} X_n \cdot w &= X_{\tau_{K_n}} \cdot w + (X_n - X_{\tau_{K_n}}) \cdot w \\ &\leq X_{\tau_{K_n}} \cdot w + X^* \circ \theta_{\tau_{K_n}}\,. \end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{P}^o\Big(\sup_{0\le n\le L_u} X_n\cdot w \ge u^\gamma\Big) &\le \sum_{0\le k\le u}\mathbb{P}^o\Big(X_{\tau_k}\cdot w + X^*\circ\theta_{\tau_k} > u^\gamma\Big)\\
&\le \sum_{0\le k\le u}\mathbb{P}^o\left(\Big(X_{\tau_k}-X_{\tau_1}\Big)\cdot w > \frac{u^\gamma}{3}\right) + u\,\mathbb{P}^o\left(X_{\tau_1}\cdot w > \frac{u^\gamma}{3}\right)\\
&\quad + u\,\mathbb{P}^o\left(X^* > \frac{u^\gamma}{3}|D=\infty\right)\\
&\le \sum_{0\le k\le u}\mathbb{P}^o\left(\Big(X_{\tau_k}-X_{\tau_1}\Big)\cdot w > \frac{u^\gamma}{3}\right) + \frac{2u}{\mathbb{P}^o(D=\infty)}\mathbb{P}^o\left(X^* > \frac{u^\gamma}{3}\right).
\end{aligned}$$

Note that by Lemma 3.5.15, $\mathbb{P}^o\left(X^* > \frac{u^\gamma}{3}\right) \le e^{-c_0 u^\gamma}$ while the random variables $(X_{\tau_{i+1}} - X_{\tau_i})\cdot w$ are i.i.d., of zero mean and finite exponential moments. In particular,

$$\mathbb{P}^o\left(\left|\frac{(X_{\tau_k}-X_{\tau_1})\cdot w}{k}\right| > \frac{u^\gamma}{3k}\right) \le e^{-c_0 k\frac{u^{2\gamma}}{9k^2}} \le e^{-c_0 u^{2\gamma-1}}$$

by moderate deviations (see e.g., [20, Section 3.7]).

Since $\gamma > 2\gamma - 1$, we conclude that

$$\limsup_{u\to\infty}\frac{1}{u^{2\gamma-1}}\log\mathbb{P}^o\left(\sup_{0\le n\le L_u} X_n\cdot w \ge u^\gamma\right) < 0$$

and hence

$$\limsup_{u\to\infty}\frac{1}{u^{2\gamma-1}}\log\mathbb{P}^o\left(\sup_{0\le n\le L_u} |\pi(X_n)| \ge u^\gamma\right) < 0\,. \tag{3.5.23}$$

Fix now $x\in B_1(0)$. Then, for some $c = c(d)$,

$$\begin{aligned}
\mathbb{P}^x\Big(X_{\tau_{B_2(0)}} \notin \partial_+ B_2(0)\Big) &\le \mathbb{P}^o\left(\sup_{0\le n\le L_{cu}} \pi(X_n\cdot w) \ge u^\gamma\right)\\
&\quad + \mathbb{P}^o\Big(X_{\tau_{V_u}}\cdot\ell < 0\Big)
\end{aligned}$$

where $V_u = \left\{z : \frac{-L_0^\gamma}{c} \le z\cdot\ell \le cL_0^\gamma\right\}$ and the conclusion follows from (3.5.23) and Kalikow's condition. □

Construct now the following subsets of U_L:

Set $M = \left\{z\in L_0\mathbb{Z}^d, z=(0,\overline{z}), \overline{z}\in\left\{-\frac{L_1}{L_0},\cdots,0,\frac{L_1}{L_0}\right\}^{d-1} L_0\right\}$. For $z\in M$, set $\mathrm{Row}(z) = \cup_{j=N_-(z)}^{N_+(z)} B_1(z+jL_0\ell)$ where

$$N_-(z) = \min\{j : B_1(z+jL_0\ell)\cap\{x:\ x\cdot\ell\ge 0\}\ne\emptyset\} \ge -c\frac{L}{L_0}$$
$$N_+(z) = \max\{j : B_1(z+jL_0\ell)\cap\{x:\ x\cdot\ell\ge 0\}\ne\emptyset\} \le c\frac{L}{L_0}$$

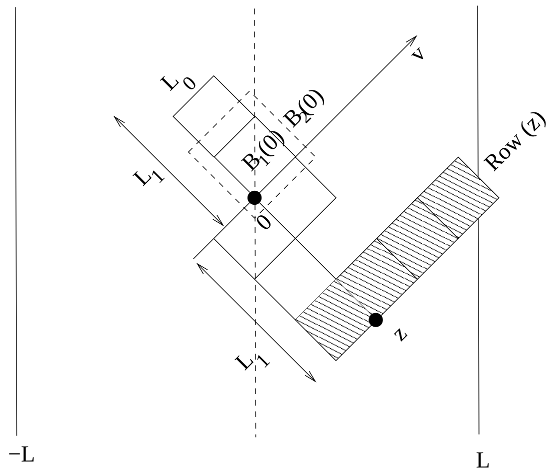


Fig. 3.5.1.

for some constant c depending on v and uniformly bounded, and set $T = \cup_{j\in\left\{-\frac{L_1}{L_0},\cdots,0,\cdots,\frac{L_1}{L_0}\right\}^{d-1}}\{jL_0\}$.

The idea behind the proof is that if one of the rows $\{R(z)\}_{z\in M}$, say $R(z_0)$, contains mostly good blocks, a good strategy for the event $(X_{\tau_{u_L}}\cdot\ell)\geq L$ is to force the walker started at x to first move to z, then move to the right successively without leaving $\cup_{z\in\mathrm{Row}(z_0)}B_2(z)$ until exiting from u_L. More precisely, let $N(z_0)$ denote the number of bad blocks in $\cup_{z\in\mathrm{Row}(z_0)}B_1(z)$. Then, for some constants c_i, using ellipticity and the definition of good boxes,

$$
\begin{aligned}
P_\omega^o(X_{\tau_{u_L}\cdot\ell}\geq L) &\geq \varepsilon^{L_1}\left(\frac{1}{2}\,\varepsilon^{2(d-1)}L_0^\gamma\right)^{cL/L_0}(\varepsilon^{L_0})^{N(z_0)}\\
&= e^{-c_4(L^{\overline{\beta}}+L^\chi N(z_0))}\,.
\end{aligned}
$$

Hence, for an arbitrary constant c_6 and all L large enough ($L>g(c_6,c)$ for some fixed function $g(\cdot)$),

$$
\begin{aligned}
P\left(P_\omega^o\Big(X_{\tau_{u_L}\cdot\ell}\geq L\Big)\leq e^{-cL^\beta}\right) &\leq P\Big(\{\nexists z_0\in M:\ N(z_0)\leq c_6L^{\overline{\beta}-\chi}\}\Big)\\
&\leq \Big[P\Big(N(0)\geq c_6L^{\overline{\beta}-\chi}\Big)\Big]^{\left(\frac{cL_1}{L_0}\right)^{(d-1)}}
\end{aligned}
$$

using the independence between even rows in Figure 3.5.1. But note that $N(0)=\sum_{i=1}^{L/2L_0}(Y_i+Z_i)$ where $\{Y_i\}$ are i.i.d., $\{Z_i\}$ are i.i.d., $\{0,1\}$ valued, $P(Y_i=1)=P(B_1(0))$ is a bad block (the division to (Y_i,Z_i) reflects the division to even and odd blocks, which creates independence). Hence,

$$P\Big(N(0) \geq c_6 L^{\overline{\beta}-x}\Big) \leq 2P\left(\sum_{i=1}^{L/2L_0} Y_i \geq \frac{c_6}{2} L^{\overline{\beta}-\chi}\right).$$

Since, by Lemma 3.5.22,

$$\log E(e^{Y_i}) \leq \log\Big(1 + e^{-c_7 L_0^{2\gamma-1}}\Big) \leq e^{-c_7 L_0^{2\gamma-1}},$$

we conclude from the independence of the Y_i's that

$$\begin{aligned} P\Big(N(0) \geq c_6 L^{\overline{\beta}-\chi}\Big) &\leq 2e^{-\frac{c_6}{2} L^{\overline{\beta}-\chi}} e^{e^{-c_7 L_0^{2\gamma-1}} \cdot \frac{L}{2L_0}} \\ &\leq e^{-c_8 L^{\overline{\beta}-\chi}}. \end{aligned}$$

Hence,

$$P\left(P_\omega^o\Big(X_{\tau_{u_L}} \cdot \ell \geq L\Big) \leq e^{-cL^\beta}\right) \leq e^{-c_9 L^{(\overline{\beta}-\chi)d}} \leq e^{-L^\xi}$$

for some $\xi > 1$, as claimed. □

Remark The restriction to $d \geq 2$ in Theorem 3.5.16 is essential: as we have seen in the case $d = 1$, one may have ballistic walks (and hence, in $d = 1$, satisfying Kalikow's condition) with moments $m_r := \mathbb{E}^o(\tau_1^r)$ of the regeneration time τ_1 being finite only for small enough $r > 1$.

We conclude this section by showing that estimates of the form of Theorem 3.5.16 lead immediately to a CLT. The statement is slightly more general than needed, and does not assume Kalikow's condition but rather some of its consequences.

Theorem 3.5.24 *Assume Assumption 3.2.1, and further assume that* $\mathbb{P}^o(A_\ell) = 1$ *and that the regeneration time* τ_1 *satisfies* $\mathbb{E}^o(\tau_1^{(2+\delta)}) < \infty$ *for some* $\delta > 0$. *Then, under the annealed measure* $\mathbb{P}^o$,

$$X_n/n \to_{n\to\infty} v := \frac{\mathbb{E}^o(X_{\tau_2} - X_{\tau_1})}{\mathbb{E}^o(\tau_2 - \tau_2)} \neq 0, \quad \mathbb{P}^o - a.s., \tag{3.5.25}$$

and $(X_n - nv)/\sqrt{n}$ *converges in law to a centered Gaussian vector.*

Proof. The LLN (3.5.25) is a consequence of Theorem 3.2.2 and its proof. To see the CLT, set

$$\xi_i = X_{\tau_{i+1}} - X_{\tau_i} - (\tau_{i+1} - \tau_i)v, \quad S_n := \sum_{i=1}^n \xi_i,$$

and $\Xi = \mathbb{E}^o(\xi_1 \xi_1^T)$. It is not hard to check that Ξ is non-degenerate, simply because $\mathbb{P}^o(|\xi_1| > K) > 0$ for each $K > 0$. Then S_n is under $\mathbb{P}^o$ a sum of i.i.d. random variables possessing finite $2+\delta$-th moments, and thus $S_{[nt]}/\sqrt{n}$ satisfies the invariance principle, with covariance matrix Ξ. Define

$$\nu_n = \min\left\{j : \sum_{i=1}^{j}(\tau_{i+1} - \tau_i) > n\right\} .$$

Note that in $\mathbb{P}^o$ probability, $n/\nu_n \to \mathbb{E}^o(\tau_2 - \tau_1) < \infty$. Hence, by time changing the invariance principle, see e.g. [2, Theorem 14.4], $S_{\nu_n}/\sqrt{\nu_n}$ converges in $\mathbb{P}^o$ probability to a centered Gaussian variable of covariance Ξ. On the other hand, for any positive η,

$$\begin{aligned}\mathbb{P}^o(|S_{\nu_n} - (X_n - nv)| > \eta\sqrt{n}) &\le \mathbb{P}^o(\exists i \le n : (\tau_{i+1} - \tau_i) > \eta\sqrt{n}/2)\\ &\quad + \mathbb{P}^o(\tau_1 > \eta\sqrt{n}/2)\\ &\le \frac{(n+1)\mathbb{P}^o(\tau_1 > \eta\sqrt{n}/2)}{\mathbb{P}^o(D')} \to_{n\to\infty} 0 ,\end{aligned}$$

where we used the moment bounds on $\mathbb{E}^o(\tau_2 - \tau_1)^{2+\delta}$ and the fact that $\mathbb{P}^o(D') > 0$ in the last limit. This yields the conclusion. Further, one observes that the limiting covariance of $X_n/\sqrt{n}$ is $\Xi/(\mathbb{E}^o(\tau_2 - \tau_1))$. $\square$

A direct conclusion of Theorem 3.5.24 is that under Kalikow's condition, $X_n/\sqrt{n}$ satisfies an annealed CLT.

Bibliographical notes: Lemma 3.5.2, Kalikow's condition, the fact that it implies $\mathbb{P}^o(A_\ell) = 1$, *and Lemma 3.5.8 appeared in [39]. The argument for* $v_\ell > 0$ *under Kalikow's condition is due to Sznitman and Zerner [77], who also observed Corollary 3.5.10. [72] proves that in the i.i.d. environment case,* $a(0, z) = 0$ *if and only if* $z = tv$, *some* $t > 0$. *The estimates in Theorem 3.5.16 are a weak form of estimates contained in [72]. Finally, [82] characterizes, under Kalikow's condition, the speed* v *as a function of Lyapunov exponents closely related to the functions* $a(\lambda, z)$.

In a recent series of papers, Sznitman has shown that many of the conclusions of this section remain valid under a weaker condition, Sznitman's (T) or (T') conditions, see [75, 74, 76]

Appendix
Markov chains and electrical networks: a quick reminder

With (V, E) as in Section 1.1, let $C_e \ge 0$ be a *conductance* associated to each edge $e \in E$. Assume that we can write

$$\omega_v(w) = \frac{C_{vw}}{\sum_{w\in N_v} C_{vw}} := \frac{C_{vw}}{C_v} .$$

To each such graph we can associate an electrical network: edges are replaced by conductors with conductance C_{vw}. The relation between the electrical network and the random walk on the graph is described in a variety of texts, see e.g. [26] for an accessible summary or [58] for a crash course. This relation is based on the uniqueness of harmonic functions on the network, and is best described as follows: fix two vertices $v, w \in V$, and apply a unit voltage between v and w. Let $V(z)$ denote the resulting voltage at vertex z. Then,

$$P_\omega^z(\{X_n\} \text{ hits } v \text{ before hitting } w) = V(z)\,.$$

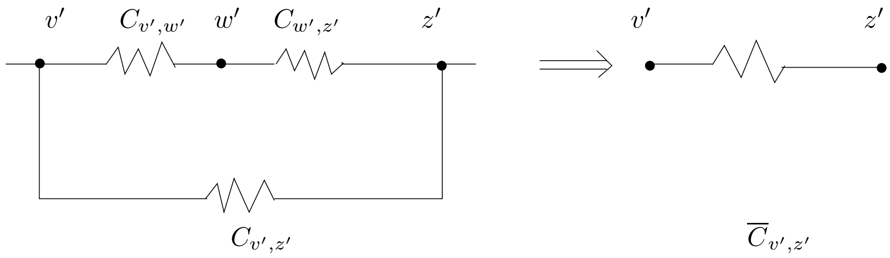
Recall that for any two vertices v, w, the *effective conductance* $C^{\text{eff}}(v \leftrightarrow w)$ is defined by applying a unit voltage between v and w and measuring the outflow of current at v. In formula, this is equivalent to

$$C^{\text{eff}}(v \leftrightarrow w) = \sum_{v' \in N_v} [1 - V(v')] C_{vv'} = \sum_{w' \in N_w} V(w') C_{ww'}\,.$$

For any integer r, the effective conductance $C_{v,r}$ between v and the horocycle of distance r from v is the effective conductance between v and the vertex r' in a modified graph where all vertices in the horocycle have been identified. We set then $C_{v,\infty} := \lim_{r\to\infty} C_{v,r}$. The effective conductance obeys the following rule:
Combination rule: Edges in parallel can be combined by summing their conductances. Futher, the effective conductance between vertices v, w is not affected if, at any vertex $w' \notin \{v, w\}$ with $N_{w'} = \{v', z'\}$, one removes the edges (v', w') and (z', w') and replaces the conductance $C_{v',z'}$ by

$$\overline{C}_{v',z'} = C_{v',z'} + \left(\frac{1}{C_{v',w'}} + \frac{1}{C_{z',w'}} \right)^{-1}\,.$$

(This formula applies even if an edge $C_{v'.w'}$ is not present, by taking $C_{v',z'} = 0$.)



Exercise A.1 *Prove formulae (2.1.3) and (2.1.4).*

Markov chains of the type discussed here possess an easy criterion for recurrence: a vertex v is recurrent if and only if the effective conductance $C_{v,\infty}$ between v and ∞ is 0. A sufficient condition for recurrence is given by means of the Nash-Williams criterion (see [58, Corollary 9.2]). Recall that an edge-cutset Π separating v from ∞ is a set of edges such that any path starting at v which includes vertices of arbitrarily large distance from v must include some edge in Π.

Lemma A.2 (Nash-Williams) *If Π_n are disjoint edge-cutsets which separate v from ∞, then*

$$C_{v,\infty} \le \left(\sum_n \left(\sum_{e \in \Pi_n} C_e \right)^{-1} \right)^{-1} .$$

As an application of the Nash-Williams criterion, we prove that a product of independent Sinai's walks is recurrent. Recall that a Sinai walk (in dimension 1) is a RWRE satisfying Assumption 2.5.1. For simplicity, we concentrate here on Sinai's walk without holding times and define a product of Sinai's walk in dimension d as the RWRE on $\mathbb{Z}^d$ constructed as follows: for each $v \in \mathbb{Z}^d$, set $N_v = \times_{i=1}^d (v_i - 1, v_i + 1)$ and let $\Omega = \times_{i=1}^d (M_1(N_v))^{\mathbb{Z}}$. For $z \in \mathbb{Z}^d$, we set $\omega_{i,z}^+ = \omega_i(z_i, z_i + 1)$, $\omega_{i,z}^- = \omega_i(z_i, z_i - 1)$ and $\rho_i(z) = \omega_{i,z}^- / \omega_{i,z}^+$. We equip Ω with a product of measures $P = \times_{i=1}^d P_i$, such that each P_i is a product measure which also satisfies Assumption 2.5.1. For a fixed $\omega \in \Omega$, define the RWRE in environment ω as the Markov chain (of law P_ω^o) such that $P_\omega^o(X_0 = 0) = 1$ and, for $v \in \{-1, 1\}^d$, $P_\omega^o(X_{n+1} = x + v | X_n = x) = \prod_{i=1}^d \omega_i(x_i, x_i + v_i)$. Define

$$C(x,v) = \prod_{i=1}^d \left(\left(\prod_{j_i = x_i}^{-1} \rho_i(j_i) \right) \left(\prod_{j_i=0}^{x_i - 1} \rho_i(j_i)^{-1} \right) \left(\rho_i(x_i)^{-1} \right)^{(v_i+1)/2} \right) ,$$

where by definition a product over an empty set of indices equals 1. Then, the resistor network with conductances $C(x,v)$ is a model for the product of Sinai's RWRE. Define

$$B_i^n(t) = -\frac{1}{\sqrt{n}} \sum_{j=0}^{\lfloor nt \rfloor} \log \rho_i(j) \cdot (\text{sign } t) .$$

Then,

$$C(x,v) \le \varepsilon^{-d} \prod_{i=1}^d e^{\sqrt{n} B_i^n(x_i/n)} .$$

Taking as cutsets Π_n the set of edges $(x, x+v)$ with $|x|_\infty = n$, $v_i \in -1, 1$ and $|x+v|_\infty = n+1$, we thus conclude that

$$\left(\sum_{e \in \Pi_n} C_e \right) \le \varepsilon^{-d} \sum_{i=1}^d \left(e^{\sqrt{n} B_i^n(1)} + e^{\sqrt{n} B_i^n(-1)} \right) \prod_{j=1, j \ne i}^d \left(\sum_{k=-n}^n e^{\sqrt{n} B_j^n(k/n)} \right) =: D_n .$$

Since P_i are product measures, we have by Kolmogorov's 0-1 law that $P(\liminf_{n\to\infty} D_n = 0) \in \{0,1\}$. On the other hand, for all n large enough, we have by the CLT that

$$P(D_n < e^{-n^{1/4}}) \geq P(B_i^n(1) \leq -1, B_i^n(-1) \\ \leq -1, \sup_{-1\leq t\leq 1} B_i^n \leq 1/2d, i = 1,\dots,d) \geq c\,,$$

for some constant $c > 0$ independent of n. Thus, by Fatou's lemma, $P(\liminf_{n\to\infty} D_n = 0) > 0$, and hence $= 1$ by the above mentioned 0-1 law. We conclude from Nash's criterion (Lemma A.2) that $C_{0,\infty} = 0$, establishing the recurrence as claimed.

Exercise A.3 *Extend the above considerations to Sinai's walk with holding times and non product measures P_i.*

Bibliographical notes: The classical reference for the link between electrical networks and Markov chain is the lovely book [26]. The application to the proof of recurrence of products of Sinai's walks was prompted by a question of N. Gantert and Z. Shi.

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