RANDOM WALKS IN RANDOM ENVIRONMENT
Notes for Minerva Foundation Lectures # 1,2,3,4

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Summary. These notes are designed to accompany the lectures at Columbia University (April 12-22, 2010). Some aspects of the lecture overlap with the published St. Flour notes [53], to which I often refer for more details and generalizations. The tension between making these notes self contained and the length of the lecture-series is resolved by seeking less generality on the one hand, and skiping routine computations on the other.

Introduction

The definition of a RWRE involves two components: first, the *environment*, which is randomly chosen but kept fixed throughout the time evolution, and second, the random walk, which, given the environment, is a time homogeneous Markov chain whose transition probabilities depend on the environment. Throughout, we will be concerned only with nearest neighbor random walks on $\mathbb{Z}^d$ (some comments on extensions will still be made). That is, let $\mathcal{M}^d$ denote the space of probability measures on $\{\pm e_i\}_{i=1}^d$, set $\Omega = (\mathcal{M}^d)^{\mathbb{Z}^d}$, and equip Ω with the natural Borel structure, denoted $\mathcal{F}$. An *environment* is simply a point $\omega \in \Omega$, with $x \mapsto \omega(x, \cdot) \in \mathcal{M}^d$ denoting the coordinate map (for $x \in \mathbb{Z}^d$).

We let P denote a probability measure on Ω, the law of the (random) environment. Without further mentioning it, we assume that P is a stationary and ergodic measure, and that it is *elliptic*, that is, we assume the existence of a constant $\epsilon > 0$ such that

$$P(\omega(0, e) \in [\varepsilon, 1 - \varepsilon]) = 1\,, \quad \text{if } |e| = 1.$$

We turn next to define the class of random walks of interest to us. For *each* $\omega \in \Omega$, we define the *random walk in the environment* ω (RWRE) as the time-homogeneous Markov chain $\{X_n\}$ taking values in $\mathbb{Z}^d$ with transition probabilities

$$P_\omega(X_{n+1} = x + e | X_n = x) = \omega(x, e) .$$

We use P_ω^x to denote the law induced on $((\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{G})$ where $\mathcal{G}$ is the σ-algebra generated by cylinder functions, and

$$P_\omega^x(X_0 = x) = 1 .$$

In the sequel, we refer to $P_\omega^x(\cdot)$ as the *quenched* law of the random walk $\{X_n\}$. Note that for each $G \in \mathcal{G}$, the map

$$\omega \mapsto P_\omega^x(G)$$

is $\mathcal{F}$-measurable. Hence, we may define the measure $\mathbb{P}^x := P \otimes P_\omega^x$ on $(\Omega \times \mathbb{Z}^d)^{\mathbb{N}}, \mathcal{F} \times \mathcal{G})$ from the relation

$$\mathbb{P}^x(F \times G) = \int_F P_\omega^x(G) P(d\omega), \quad F \in \mathcal{F}, G \in \mathcal{G} . \tag{0.1.1}$$

The marginal of $\mathbb{P}^x$ on $V^{\mathbb{N}}$, denoted also $\mathbb{P}^x$ whenever no confusion occurs, is called the *annealed law* of the random walk $\{X_n\}$; note that under P_ω^x, the RWRE $\{X_n\}$ is a Markov chain but is not invariant under $\mathbb{Z}^d$-shifts, whereas under $\mathbb{P}^x$, $\{X_n\}$ is *not* a Markov chain, however it's law is invariant under $\mathbb{Z}^d$ shifts.

When $d = 1$, we write ω_x for $\omega(x, 1)$ and set $\rho_x = (1 - \omega_x)/\omega_x$.

1 Lecture 1. One dimensional RWRE

This lecture is devoted to the study of one dimensional RWRE. We review ergodic properties and (some) scaling limits. Two important ingredients make the analysis significantly easier than in higher dimension: the existence of a reversing (environment depending) measure, and the fact that hitting times have a relatively simple structure.

1.1 Ergodic theorems and hitting times

Since the RWRE is (quenched) a Markov chain, questions concerning transience or recurrence (and whether the transience/recurrence depends on the specific realization of the environment) are natural. The reversibility of the chain will be instrumental in providing an answer.

For R, L positive integers, define the function

$$v_{R,L}(x) = P_\omega^x(\text{the walk hits } +R \text{ before } -L). \tag{1.1.1}$$

The Markov property gives

$$v_{R,L}(x) = \omega_x v(x+1) + (1-\omega_x)v(x-1), x \in (-L, R)$$

with $v_{R,L}(R) = 1, v_{R,L}(-L) = 0$ (note link with voltage across resistor networks, see [53] for more on that). With $g(x) = v_{R,L}(x+1) - v_{R,L}(x)$, get that $g(x+1) = \rho_x g(x)$ and hence, setting $\bar{\rho}_x = \prod_{-L<y<x} \rho_y$, we get

$$g(x) = v(-L+1)\bar{\rho}_x, \quad x \in (-L, R). \tag{1.1.2}$$

The condition $\sum_{y<R} g(y) = 1$ allows one to write explicitly the solution $v_{R,L}(x) = \sum_{y<x} g(y)$.

We are interested in the asymptotics of $v_{R,L}(0)$ as R, L go to infinity. Assume first $E_P \log \rho_0 < 0$. The law of large numbers then implies that $\bar{\rho}_x \to_{x\to\infty} 0$ exponentially fast, a.s., for any fixed L. One concludes that on this event, which has probability 1,

$$\lim_{L\to\infty} \lim_{R\to\infty} v_{R,L}(0) = 1\,,$$

from which it follows that if $E(\log \rho_0) < 0$ then X_n is transient and $X_n \to \infty$, $\mathbb{P}^0$-a.s. (Note that the statement is now annealed, and hence quenched!).

The same argument works to show that if $E(\log \rho_0) > 0$ then X_n is transient and $X_n \to -\infty$, $\mathbb{P}^0$-a.s.

If $E \log \rho_0 = 0$ and the environment is iid, it is not hard to use the CLT (with respect to P, the law on the environment) to show that in fact, P-a.s.,

$$\lim_{L\to\infty} \lim_{R\to\infty} v_{R,L}(0) = \lim_{R\to\infty} \lim_{L\to\infty} v_{R,L}(0) = 0,$$

from which one deduces that $\{X_n\}$ is recurrent, P-a.s (for one possible proof of this conclusion under mere ergodicity of P I refer to [53, Pg. 197–198]; that proof uses Kesten's Lemma stating that stationary sequences tending to ∞ almost surely must grow linearly). We thus have the following theorem, settling at once the question whether transience is a $0-1$ phenomenon (it is) and whether it occurs (it depends on P).

Theorem 1.1.3 (Solomon [46])

(a) $E_P(\log \rho_0) < 0 \quad \Rightarrow \lim_{n\to\infty} X_n = +\infty, \qquad \mathbb{P}^0$ *a.s.*

(b) $E_P(\log \rho_0) > 0 \quad \Rightarrow \lim_{n\to\infty} X_n = -\infty, \qquad \mathbb{P}^0$ *a.s.*

(c) $E_P(\log \rho_0) = 0 \quad \Rightarrow -\infty = \liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n = \infty, \quad \mathbb{P}^0$ *a.s.*

We turn to the law of large numbers. Note that one cannot apply directly ergodic theorems to the sequence X_n/n: The sequence $\{X_n - X_{n-1}\}$ is not even stationary! Our proof of the LLN involves hitting times, and showcases the other great simplification in dimension $d = 1$: the ability to write simple recursions. We present the proof for the i.i.d. case (which is simpler), but similar ideas apply equally well in the ergodic setup, see [53, Pg. 198–203]. An alternative proof, using the approach of the "environment viewed from the point of view of the particle", is detailed in [53, Pg. 203–209]. Time permitting, we will return to this point later.

Theorem 1.1.4 (Solomon [46]; Alili [1]) *Assume P is i.i.d. Then,*

$$\begin{aligned}
&\text{(a)}\quad E_P(\rho_0) < 1 && \Rightarrow \lim_{n\to\infty} \frac{X_n}{n} = \frac{1-E_P(\rho_0)}{1+E_P(\rho_0)}, && \mathbb{P}^o \ a.s.\\
&\text{(b)}\quad E_P(\rho_0^{-1}) < 1 && \Rightarrow \lim_{n\to\infty} \frac{X_n}{n} = -\frac{1-E_P\left(\frac{1}{\rho_0}\right)}{1+E_P\left(\frac{1}{\rho_0}\right)}, && \mathbb{P}^o \ a.s.\\
&\text{(c)}\quad \frac{1}{E_P(\rho_0)} \le 1 \le E_P(\rho_0^{-1}) && \Rightarrow \lim_{n\to\infty} \frac{X_n}{n} = 0, && \mathbb{P}^o \ a.s..
\end{aligned}$$

Since $E_P \log \rho_0 \le \log E_P \rho_0$ with a strict inequality whenever P is non-degenerate, it follows that one can find examples where $X_n \to \infty$ $\mathbb{P}^o$-a.s. but $X_n/n \to 0$, $\mathbb{P}^o$-a.s. This does not contradict Kesten's Lemma because $\{X_n - X_{n-1}\}$ is not in general a stationary sequence under $\mathbb{P}^o$.

Proof of Theorem 1.1.4 It is enough to consider the case where $E_P(\log \rho_0) \le 0$, that is $\limsup X_n = \infty$, $\mathbb{P}^0$-a.s.. We introduce certain hitting times. Let $T_0 = 0$, and

$$T_n = \min\{k : X_k = n\}.$$

Set $\tau_0 = 0$ and

$$\tau_n = T_n - T_{n-1}, \quad n \ge 1\,.$$

Lemma 1.1.5 *If $\limsup_{n\to\infty} X_n = +\infty$, $\mathbb{P}^o$-a.s., then $\{\tau_i\}_{i\ge 1}$ is a stationary and ergodic sequence.*

Proof of Lemma 1.1.5 The stationarity of $\{\tau_i\}_{i\ge1}$ follows from the stationarity of the environment. To see the ergodicity, let $\Xi = [0,1]^{\mathbb{N}}$, let U_Ξ denote the measure on Ξ making all coordinates $\{\xi_i\}$ independent and of uniform law on $[0,1]$, and note that $\{X_n\}$ may be constructed by writing

$$X_{n+1} = X_n - \mathbf{1}_{\{\omega_{X_n} < \xi_{n+1}\}} + \mathbf{1}_{\{\omega_{X_n} \ge \xi_{n+1}\}}\,.$$

Suppose $A = A(\omega,\xi) = A(\tau)$ is an event, measurable w.r.t. $\mathcal{G}_n = \sigma\{\tau_i, i \ge 1\}$, which is invariant with respect to the shift $(\theta\tau)_i = \tau_{i+1}$ (we write in the sequel $\theta A = A(\theta\tau)$). We need only show that $P \otimes U_\Xi(A) \in \{0,1\}$. Note however that $\theta^k A$, conditioned on $\sigma\{\omega_i, i \in \mathbb{Z}\}$, is independent of $\xi_1, \ldots, \xi_k$. Thus, since U_Ξ is an i.i.d. law and hence the tail sigma-field of $\{\xi_i\}$ is trivial, it follows that $A = \theta^k A$ is, under the above conditioning, independent of $\sigma\{\xi_i, i \ge 1\}$. Thus A depends only on ω. But the shift θ on the sequence $\{\tau_i\}$ induces the usual shift θ on Ω. Thus, $\theta A = A(\theta\omega) = A(\omega)$ and hence $P(A) \in \{0,1\}$. $\square$

Remarks: 1. Note that an attempt to mimic this argument in $\mathbb{Z}^d$, $d \geq 1$, with T_i denoting the hitting times of hyperplanes at distance i from the origin, fails because of the extra information contained in the hitting location.
2. It is not hard to check, see [53, Pg. 200], that if P is strongly mixing then so is the sequence $\{\tau_i\}_{i \geq 0}$.

To get a LLN for X_n, one first proves a LLN for T_n. As a first step, we have the following lemma.

Lemma 1.1.6 *If* $E_P(\rho_0) < 1$ *then*

$$\mathbb{E}^0(\tau_1) = \begin{cases} \frac{1+E_P(\rho_0)}{1-E_P(\rho_0)} =: v_P^{-1} & \text{if } E_P(\rho_0) < 1 \\ \infty & \text{if } E_P(\rho_0) \geq 1 \end{cases}$$

Proof. Decompose, with $X_0 = 0$,

$$\tau_1 = \mathbf{1}_{X_1=1} + \mathbf{1}_{X_1=-1}(1 + \tau_0'' + \tau_1'') \,. \tag{1.1.7}$$

Here, $(1 + \tau_0'')$ is the first hitting time of 0 after time 1, and $1 + \tau_0'' + \tau_1''$ is the first hitting time of 1 after time $1 + \tau_0''$.

Under P_ω^o, the law of τ_0'' conditioned on the event $\{X_1 = -1\}$ is $P_{\theta^{-1}\omega}^o(\tau_1 \in \cdot)$, while conditioned on the event $\{X_1 = -1\} \cap \{\tau_0'' < \infty\}$, τ_1'' also has law identical to that of τ_1.

Consider first the case $\mathbb{E}^0(\tau_1) < \infty$. Then, both $E_\omega^o(\tau_1) < \infty$ and $E_{\theta^{-1}\omega}(\tau_1) < \infty$, P-a.s. Taking expectations in (1.1.7), one gets then

$$\omega_0 E_\omega^o(\tau_1) = 1 + (1 - \omega_0) E_{\theta^{-1}\omega}^o(\tau_1) \,.$$

Hence,

$$E_\omega^o(\tau_1) = \frac{1}{\omega_0} + \rho_0 E_{\theta^{-1}\omega}^o(\tau_1) \,.$$

Note that $E_{\theta^{-1}\omega}^o(\tau_1)$ depends on $\{\omega_i\}_{i \leq -1}$ and in particular is independent of ρ_0. Thus, if $\mathbb{E}^0(\tau_1) < \infty$, taking expectations give the equality $(1 - E_P(\rho_0))\mathbb{E}^0(\tau_1) = 1 + E_P(\rho_0)$. In particular, since the right side is positive, $\mathbb{E}^0(\tau_1) < \infty$ implies that $E_P(\rho_0) < 1$, and the stated equality. On the other hand, replacing the equality (1.1.7) by an inequality involving the truncated variables $\tau_1^M = \tau_1 \mathbf{1}_{\{\tau_1 \leq M\}}$ and its shifts allows one to conclude that if $E_P(\rho_0) < 1$ then $\sup_M \mathbb{E}^0(\tau_1^M) \leq v_P^{-1}$, and hence, by monotone convergence, $\mathbb{E}^0(\tau_1) < \infty$. $\square$

An application of Lemmas 1.1.5 and 1.1.6 yields that in case (a)

$$\frac{T_n}{n} = \frac{\sum_{i=1}^n \tau_i}{n} \longrightarrow_{n\to\infty} E_{\mathbb{P}^o}(\tau_1) < \infty, \quad \mathbb{P}^o\text{-a.s.} \,. \tag{1.1.8}$$

On the other hand, we have the following (recall that we assumed $E_P(\log \rho_0) \leq 0$, which implies $T_n < \infty$, a.s.:

Lemma 1.1.9 *Assume* $T_n/n \to \alpha \in (0, \infty]$. *Then,*

$$\frac{X_n}{n} \underset{n\to\infty}{\longrightarrow} \frac{1}{\alpha}.$$

Proof of Lemma 1.1.9 Let k_n be the unique (random) integers such that

$$T_{k_n} \le n < T_{k_n+1}\,.$$

Note that $X_n < k_n + 1$ while $X_n \ge k_n - (n - T_{k_n})$. Hence,

$$\frac{k_n}{n} - \left(1 - \frac{T_{k_n}}{n}\right) \le \frac{X_n}{n} \le \frac{k_n+1}{n}\,.$$

But, $\lim_{n\to\infty} k_n/n = \lim_{n\to\infty} n/T_n$ (due to the existence of the second limit and the definition of k_n). Thus,

$$\frac{1}{\alpha} \ge \limsup_{n\to\infty} \frac{X_n}{n} \ge \liminf_{n\to\infty} \frac{X_n}{n} \ge \frac{1}{\alpha}\,. \qquad \square$$

Lemma 1.1.9 and (1.1.8) complete the proof of Theorem 1.1.4 in case (a). Case (b) is similar, while case (c) is a minor modification of the above argument and is left out. $\square$

1.2 Limit laws - transient walks

The heart of the proof of the LLN was Lemma 1.1.6. It is instructive (and useful) to reinterpret it in terms of a branching process in a random environment. Namely, assume $E_P(\log \rho_0) < 0$ and, for $x < 0$, consider the excursions *left* from x, denoted N_x, with the convention that $N_x = 0$ if the path $\{X_n\}$ does not hit x. Since $X_n \to \infty$ a.s., we have that $N_x < \infty$ a.s. Further, conditioned on the environment, the excursions form a branching structure: for $i = 1, \ldots, N_x$, the number N^i_{x-1} of excursions from x to $x-1$ before hitting $x+1$ during the ith excursion reaching x from $x+1$ is geometric with mean ρ_x, and the collection $\{N^i_{x-1}\}_{i=1}^{N_x}$ consists of iid variables, with $N_{x-1} = \sum_{i=1}^{N_x} N^i_{x-1}$ (for consistency, take $N_1 = 1$). This structure is identical to the structure of a branching process in a random environment with a geometric offspring law whose mean depends on the generation. Further, since $T_1 = 1 + 2\sum_{x\le 0} N_x < \infty$ a.s., this branching process almost surely dies out, and further

$$\begin{aligned}\mathbb{E}^0(T_1) &= 1 + 2\sum_{x\le 0} \mathbb{E}^0(N_x) = 1 + 2\sum_{x\le 0} \mathbb{E}^0(N_{x-1}\rho_x)\\ &= 1 + 2\sum_{x\le 0} E(\rho_0)\mathbb{E}^0(N_{x-1}) = 1 + 2\sum_{k=1}^{\infty}[E(\rho_0)]^k = v_P^{-1}\,. \quad (1.2.1)\end{aligned}$$

This representation (or the previous recursions) allow one to compute higher moments of T_1. In particular, one gets the following result.

Lemma 1.2.2 *Assume P i.i.d. and $E_P(\log \rho_0) < 0$. Define $s = \sup\{r > 0 : E_P \rho_0^r < 1\}$. Then, $\mathbb{E}^0(T_1^t) < \infty$ if and only if $t < s$.*

Of course, the random variables τ_i, although distributed like T_1, are not i.i.d.; however, they are mixing, and it is not too hard to evaluate their mixing properties. In particular, since for $m > 0$ τ_m depends on the environment to the left of 0 only through the event that the process $\{X_n\}_{n\geq 0}$ visits 0 after T_m, we have for $r > 2$ and $r' = r/(r-2)$

$$\begin{aligned}
|\mathbb{E}^0(\tau_m\tau_1) - \mathbb{E}^0(\tau_1)^2| &\leq 2\mathbb{E}^0(\tau_m\tau_1\mathbf{1}_{\{\{X_n\}_{n>T_m}\cap\{0\}\neq\emptyset\}}) \\
&\leq 2[\mathbb{E}^0(\tau_0^r)]^{2/r}\mathbb{P}^0(\{X_n\}_{n>T_m}\cap\{0\}\neq\emptyset)^{r'} \\
&= 2[\mathbb{E}^0(\tau_0^r)]^{2/r}\mathbb{P}^0(\{X_n\}_{n>0}\cap\{-m\}\neq\emptyset)^{r'} \\
&\leq CE_P[v_{\infty,m})]^{r'},
\end{aligned}$$

where we used Hölder's inequality and the functions $v_{R,L}(x)$, see (1.1.1). It is not hard to check that, for P i.i.d., the latter decays exponentially in m, and therefore, defining $\bar{\tau}_i = \tau_i - \mathbb{E}^0(\tau_i) = \tau_i - v_P^{-1}$, if $s > 2$ then by choosing $r \in (2, s)$ we get that

$$s > 2 \Longrightarrow \quad \sigma^2 := \mathbb{E}^0(\bar{\tau}_1^2) + 2\sum_{m=2}^{\infty}\mathbb{E}^0(\bar{\tau}_1\bar{\tau}_m) < \infty. \tag{1.2.3}$$

By standard results on the invariance principle for mixing sequences, this implies the following theorem.

Theorem 1.2.4 (Annealed CLT: Kesten, Kozlov, Spitzer [31]) . *Assume that P is i.i.d., that $E_P \log\rho_0 < 0$ and that $s > 2$. Then,*
(a) *the process $\mathbb{T}_t := (T_{[nt]} - ntv_P^{-1})/\sqrt{n}$ converges weakly (under the annealed measure $\mathbb{P}^0$) to standard Brownian motion with variance σ^2,*
(b) *the process $Z_t := (X_{[nt]} - ntv_P)/\sqrt{n}$ converges weakly (under the annealed measure $\mathbb{P}^0$) to standard Brownian motion with variance σ_a^2.*

(Going from point (a) to point (b) is similar to the argument used in the proof of the LLN, see [31] for details.)

It is worthwhile at this point to expand slightly on the role of the parameter s, which is closely related to *traps* in the environment. Recall from (1.1.2) the role played by large values of partial sums $S_{x,T} := \sum_{i=x}^{x+T}\log\rho_i$. The relevant scale for T is $T = \alpha\log n$. Define

$$S_n^* = \max_{x\in\{0,\dots,n\},T>0}\frac{S_{x,T}}{\log n}.$$

Let $I(\cdot)$ denote the rate function of the large deviations associated by Cramer's theorem to the sum of i.i.d. random variables $\{\log\rho_x\}$, that is

$$I(x) = \max_\lambda(\lambda x - \log E_P(e^{\lambda\log\rho_0})) = \max_\lambda(\lambda x - E_P\rho^\lambda)\,.$$

One then has

$$\frac{1}{\log n} P(\sum_{i=0}^{\alpha \log n} \log \rho_i > c \log n) \to -\alpha I(c/\alpha)\,.$$

Optimizing over α (using some calculus and the definitions of I and of s) reveals that $\min_\alpha \alpha I(c/\alpha) = sc$. Since there are about $n/\log n$ disjoint i.i.d. blocks, for large n one sees that S_n^* behaves like the maximum of c for which $sc = 1$, that is one has $S_n^* \to 1/s$ (first in probability and then, with some additional effort, also almost-surely). In particular:

• if $s < 1$ then inside $[0, \ldots, n]$ there is a stretch (which we refer to as *trap*) for which $S_{x,T} \sim s^{-1} \log n$, and it will take the walker order of $n^{1/s}$ steps to cross it, hence the velocity vanishes,

• if $s \in (1, 2)$ the time spent in the largest trap is of order $n^{1/s}$, which is $>> n^{1/2}$ but $<< n$, and therefore the velocity does not vanish but the CLT (in scale $n^{1/2}$) fails,

• if $s > 1/2$ traps are not strong enough to delay the walk (in scale $n^{1/2}$) and hence the crossing time of $[0, \ldots, n]$ is essentially dominated by a sum of strongly mixing random variables (crossing times of blocks), implying the CLT.

These remarks hint that when $s \in (0, 2)$, one should obtain stable limit laws, instead of CLT's. This is indeed the case, at least in the annealed setup; the proof (which is delicate since stable laws for sums of mixing random variables require more than just control on moments, and in particular require some "regularly varying tails) can be found in [31] (in the i.i.d. setup) and [36] (in the Markovian environment setup).

Before dropping this topic, a natural point to consider is the existence of *quenched* limit laws. That is, is it true that, for P-a.e. ω and say $s > 2$, $P_\omega(X_n - nv_P < x\sigma_a\sqrt{n})$ converges to $\Phi(x)$ where Φ denotes the Gaussian cumulative distribution? Maybe surprisingly, the answer is *no*. The reason is that $E_\omega X_n$ itself fluctuates, and so the correct centering should be the value $E_\omega X_n$ and not $v_P n$. With this modification, the quenched invariance principle holds, with a smaller variance σ_q^2 (as it turns out, none of the traps dominates the others, and a version of the Lindberg–Feller CLT can be applied). That is, one has the following.

Theorem 1.2.5 (Peterson [37], Goldsheid [22]) *Assume that P is i.i.d., that $E_P \log \rho_0 < 0$ and that $s > 2$. Then, for P-a.e. ω, the process $Z_t^q := (X_{[nt]} - E_\omega(X_{[nt]}))/\sqrt{n}$ converges weakly to standard Brownian motion with variance $\sigma_q^2 < \sigma_a^2$.*

The situation dramatically changes when one considers $s < 2$: in that case, no centering can restore the stable limits under the quenched law. For details, refer to [39, 38].

1.3 Limits for recurrent walks

In case $E_P \log \rho_0 = 0$, we have seen that the RWRE is recurrent. By the CLT (under P) for the partial sums of i.i.d. random variables $\{\rho_x\}_{x\in\mathbb{Z}}$, traps of height $\log n$ occur within distance of order $(\log n)^2$ from the origin. It is therefore no surprise that the scaling of the RWRE differs dramatically from that of standard random walks.

Theorem 1.3.1 (Sinai [45]) *Assume P i.i.d., and that $E_P \log \rho_0 = 0$. Then $X_n/(\log n)^2$ converges in distribution, under the annealed law $\mathbb{P}$, to a random variable $\bar{X}$.*

The law of $\bar{X}$ has been identified by Kesten [30]. Various properties of the walk in the recurrent regime, including localization and aging, can be found in [24, 27, 18, 14], see [53] for more details.

1.4 Further results

Much more is known concerning the one dimensional case. Slowdown estimates, both quenched and annealed, are detailed in [15, 21, 20, 11, 40, 41], see the review in [53]. Large deviations (quenched and annealed) are described in [25, 11, 13] (we will return to this topic in the multi-dimensional setup).

A related model is that of RWRE on $\mathbb{Z}$ with non nearest-neighbor jumps, and the closely related model of RWRE on a strip. The seminal work here is due to Key [32], who described the relation between ergodic properties of the walk and certain Lyapunov exponents for products of random matrices, see [35] for a detailed exposition. More recent developments can be found in [8, 9, 10, 4, 23].

2 Lecture 2. $d > 1$: $0-1$ laws and law of large number

We turn our attention to the study of RWRE in $\mathbb{Z}^d$. Even if considerable progress was achieved in recent years, the situation here is, unfortunately, much less satisfying than for $d = 1$ (and both parts of this sentence, copied from notes I wrote almost 10 years ago, are still true!).

We will assume throughout that P is a uniformly elliptic product measure (some hints concerning extensions are briefly discussed at the end).

Fix $\ell \in \mathbb{R}^d \setminus \{0\}$, and consider the events

$$A_{\pm\ell} = \{\lim_{n\to\infty} X_n \cdot \ell = \pm\infty\}.$$

We have the

Theorem 2.1.1 (Kalikow[28])

$$\mathbb{P}^o(A_\ell \cup A_{-\ell}) \in \{0, 1\}.$$

Proof. We will prove the theorem for $\ell = (1, 0 \ldots 0)$, the general case being similar but requiring more cumbersome notations.

For any $u < v$, the walk cannot visit infinitely often the strip $u \le z \cdot \ell \le v$ without crossing the line $z \cdot \ell = v$. More precisely, with

$$T_v = \inf\{n \ge 0 : \quad X_n \cdot \ell \ge v\}, \tag{2.1.2}$$

we have

$$\mathbb{P}^o\Big(\#\{n > 0 : \; X_n \cdot \ell \ge u\} = \infty, T_v = \infty\Big) = 0 \,. \tag{2.1.3}$$

Indeed, note that for any z with $u \le z \cdot \ell \le v$, and any ω,

$$P^z_\omega(X_{v-u} \cdot \ell \ge v) \ge \varepsilon^{v-u} \,,$$

yielding (2.1.3) by the strong Markov property.

Assume next that $\mathbb{P}^o(A_\ell) > 0$. Set $D = \inf\{n \ge 0 : \; X_n \cdot \ell < X_0 \cdot \ell\}$. $\mathbb{P}^o(D = \infty) > 0$ because if $\mathbb{P}^o(D = \infty) = 0$ then $\mathbb{P}^z(D < \infty) = 1 \; \forall z \in \mathbb{Z}^d$, and thus P-a.s., for all $z \in \mathbb{Z}^d$, $P^z_\omega(D < \infty) = 1$. This implies by the Markov property that

$$\liminf_{n \to \infty} X_n \cdot \ell \le 0, \quad \mathbb{P}^o\text{-a.s.}\,,$$

contradicting $\mathbb{P}^o(A_\ell) > 0$.

Define $\mathcal{O}_\ell$ to be the event that $X_n \cdot \ell$ changes its sign infinitely often. We next show that whenever $\mathbb{P}^o(A_\ell) > 0$, then $\mathbb{P}^o(\mathcal{O}_\ell) = 0$. Set $M = \sup_n X_n \cdot \ell$, fix $v > 0$ and note by (2.1.3) that

$$\mathbb{P}^o(\mathcal{O}_\ell \cap \{M < v\}) = 0 \,. \tag{2.1.4}$$

We next prove that if $\mathbb{P}^o(A_\ell) > 0$ then $\mathbb{P}^o(\mathcal{O}_\ell \cap \{M = \infty\}) = 0$. Toward this end, set $\mathcal{G}_n = \sigma(X_i), \, i \le n)$ and setting $S_0 = 0$, define recursively $\mathcal{G}_n$ stopping times as follows:

$$\begin{aligned} R_k &= \inf\Big\{n \ge S_k : \; X_n \cdot \ell < 0\}\Big\} \,, \\ S_{k+1} &= \inf\Big\{n \ge R_k : X_n \cdot \ell > \max\{X_m \cdot \ell : \; m < n\}\Big\} \,. \end{aligned}$$

On $\mathcal{O}_\ell \cap \{M = \infty\}$, all these stopping times are finite. Now, at each time S_k the walk enters a half space it never visited before. In particular, events in the σ-algebra $\mathcal{G}_{S_k - 1}$ are independent of $\sigma(\omega_z : \; z \cdot \ell \ge X_{S_k} \cdot \ell)$. Note that, using $\mathbb{P}^o(A_\ell) > 0$ in the second inequality,

$$\mathbb{P}^o(R_0 < \infty) = \mathbb{P}^o(D < \infty) < 1 \,,$$

whereas, using θ to denote both time and space shifts as needed from the context,

$$\begin{aligned}\mathbb{P}^o(R_1<\infty) &\le \mathbb{P}^o(R_0<\infty, R_0\circ\theta_{X_{S_1}}<\infty)\\ &= \sum_{z\in\mathbb{Z}^d}\mathbb{P}^o(R_0<\infty, R_0\circ\theta_z<\infty, X_{S_1}=z)\\ &= \sum_{z\in\mathbb{Z}^d} E_P\Big(P^o_\omega(R_0<\infty, X_{S_1}=z)\cdot P^o_{\theta^z\omega}(R_0<\infty)\Big)\,.\end{aligned}$$

Note that $P^o_{\theta^z\omega}(R_0<\infty)$ is measurable on $\sigma(\omega_x\,:\, x\cdot\ell\ge z\cdot\ell)$ whereas $P^o_\omega(R_0<\infty, X_{S_1}=z)$ is measurable on $\sigma(\omega_x: x\cdot\ell\le z\cdot\ell-1)$. Hence, by the product structure of P,

$$\begin{aligned}\mathbb{P}^o(R_1<\infty) \le \sum_{z\in\mathbb{Z}^d}&\Big[E_P\Big(P^o_\omega(R_0<\infty, X_{S_1}=z)\Big)\\ &\qquad\cdot E_P\Big(P^o_\omega(R_0<\infty)\Big)\Big]\\ &\le (\mathbb{P}^o(R_0<\infty))^2\le(\mathbb{P}^o(D<\infty))^2\,.\end{aligned}$$

Repeating this procedure, we conclude that $P^o(\mathcal{O}_\ell\cap\{M=\infty\})\le\mathbb{P}^o(R_k<\infty)\le(\mathbb{P}^o(D<\infty))^{k+1}$. Since k is arbitrary, we conclude that $\mathbb{P}^o(\mathcal{O}_\ell\cap\{M=\infty\})=0$, yielding with the above that $\mathbb{P}^o(\mathcal{O}_\ell)=0$ as soon as $\mathbb{P}^o(A_\ell)>0$. In a similar manner one proves that $\mathbb{P}^o(A_{-\ell})>0$ also implies $\mathbb{P}^o(\mathcal{O}_\ell)=0$.

Assume now $1>\mathbb{P}^o(A_\ell\cup A_{-\ell})$. Then one can find a v such that $\mathbb{P}^o(X_n\cdot\ell\in[-v,v]$ infinitely often$)>0$. Therefore, $\mathbb{P}^o(\mathcal{O}_\ell)>0$, implying by the above $\mathbb{P}^o(A_\ell)=\mathbb{P}^o(A_{-\ell})=0$. $\square$

In the particular case of $d=2$, [56] proved that in fact, $\mathbb{P}^o(A_\ell)\in\{0,1\}$, and provided counter examples to this claim for certain ergodic (but not uniformly elliptic!) environments (we explore these in Section 2.2 below). Whether or not this $0-1$ law holds for uniformly elliptic product measures with $d>2$ is one of the major open problems in the study of multi-dimensional RWRE. As we are about to see, the resolution of this question is the only obstacle toward proving a general law of large numbers, to which we turn next.

Theorem 2.1.5 (Sznitman-Zerner[52], Zerner[54]) *There exist deterministic $v_\ell, v_{-\ell}$ (possibly zero) such that*

$$\lim_{n\to\infty}\frac{X_n\cdot\ell}{n}=v_\ell\mathbf{1}_{A_\ell}+v_{-\ell}\mathbf{1}_{A_{-\ell}},\quad \mathbb{P}^o\text{-}a.s. \qquad (2.1.6)$$

(See (2.1.13) for some information on v_ℓ. When $v_\ell\ne0$ we say that the walk is *ballistic* in the direction ℓ).

An immediate corollary, obtained by applying Theorem 2.1.5 d times with respect to the basis $\ell=e_i$, $i=1,\ldots,d$, is the following:

Corollary 2.1.7 *Assume that $\mathbb{P}^o(A_\ell)\in\{0,1\}$ for every ℓ. Then there exists a deterministic v (possibly zero) such that*

$$\lim_{n\to\infty}\frac{X_n}{n}=v,\quad \mathbb{P}^o\text{-}a.s.$$

Proof of Theorem 2.1.5: (sketch) As before, to prove (2.1.6) we will only consider $\ell = (1, 0, \cdots, 0)$. First, assume that $\mathbb{P}^o(A_\ell) > 0$ (and hence, by the $0-1$ law, $\mathbb{P}^o(A_\ell \cup A_{-\ell}) = 1$). The proof is based on introducing a renewal structure, as follows: Define $\overline{S}_0 = 0, M_0 = \ell \cdot X_0$,

$$\overline{S}_1 = T_{M_0+1} \le \infty,\ \overline{R}_1 = D \circ \theta_{\bar{S}_1} + \overline{S}_1 \le \infty,$$
$$M_1 = \sup\{\ell \cdot X_m, \quad 0 \le m \le \overline{R}_1\} \le \infty$$

and by induction, for $k \ge 1$,

$$\overline{S}_{k+1} = T_{M_k+1} \le \infty,\ \overline{R}_{k+1} = D \circ \theta_{\bar{S}_{k+1}} + \overline{S}_{k+1} \le \infty,$$
$$M_{k+1} = \sup\{\ell \cdot X_m, \quad 0 \le m \le \overline{R}_{k+1}\} \le \infty\,.$$

The times $\overline{S}_1, \overline{S}_2, \ldots$, are called "fresh times", and the locations $X_{\overline{S}_1}$, $X_{\overline{S}_2}$, $\cdots$, are "fresh points": at the time $\overline{S}_k$, the path $X.$ visits for the first time after $\overline{S}_{k-1}$ and after hitting again the hyperplane $X_{\overline{S}_{k-1}} \cdot \ell - 1$, a fresh part of the environment. Note that $(\overline{S}_i, \overline{R}_i)$ are related to, but differ slightly from, (S_i, R_i) introduced earlier. Clearly,

$$0 = \overline{S}_0 \le \overline{S}_1 \le \overline{R}_1 \le \overline{S}_2 \le \cdots \le \infty$$

and the inequalities are strict if the left member is finite. Define:

$$K = \inf\{k \ge 1 : \overline{S}_k < \infty, \overline{R}_k = \infty\} \le \infty,$$
$$\tau_1 = \overline{S}_K \le \infty\,.$$

τ_1 is called a "regeneration time", because after τ_1, $X. \cdot \ell$ never falls behind $X_{\tau_1} \cdot \ell$.

By the same argument as in the proof of Theorem 2.1.1, $\mathbb{P}^o(\overline{R}_k < \infty) \le \mathbb{P}^o(D < \infty)^k \underset{k\to\infty}{\longrightarrow} 0$ because $\mathbb{P}^o(A_\ell) > 0$ implies $\mathbb{P}^o(D < \infty) < 1$. On the other hand, on A_ℓ, $\overline{R}_k < \infty \Rightarrow S_{k+1} < \infty$, $\mathbb{P}^o$-a.s., and hence

$$\mathbb{P}^o(A_\ell \cap \{K = \infty\}) = \mathbb{P}^o(A_\ell \cap \{\tau_1 = \infty\}) = 0\,.$$

Define now the measure

$$\mathbb{Q}^o(\cdot) = \mathbb{P}^o(\cdot\,|\{\tau_1 < \infty\}) = \mathbb{P}^o(\cdot\,|A_\ell)$$

and set

$$\mathcal{G}_1 = \sigma\left(\tau_1, X_0, \cdots, X_{\tau_1}, \{\omega(y, \cdot)\}_{\ell\cdot y < \ell\cdot X_{\tau_1}}\right)\,.$$

Note that since $\{D = \infty\} \subset \{\tau_1 < \infty\}$, we have that $\{D = \infty\} \in \mathcal{G}_1$. We have the following crucial lemma, whose proof is a simple exercise in the application of the Markov property, and is omitted. It is here that the i.i.d. assumption on the environment plays a crucial role:

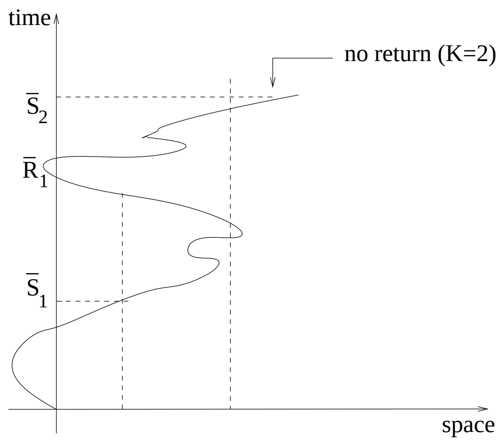

Fig. 2.1.1. Regeneration structure

Lemma 2.1.8 *For any measurable sets A, B,*

$$\mathbb{Q}^o\Big(\{X_{\tau_1+n}-X_{\tau_1}\}_{n\geq 0}\in A, \{\omega(X_{\tau_1}+y,\cdot)\}_{y\cdot\ell\geq 0}\in B\Big)$$
$$=\mathbb{P}^o\Big(\{X_n\}_{n\geq 0}\in A, \{\omega(y,\cdot)\}_{y\cdot\ell\geq 0}\in B|\{D=\infty\}\Big)\,.$$

In fact,

$$\mathbb{Q}^o\Big(\{X_{\tau_1+n}-X_{\tau_1}\}_{n\geq 0}\in A, \{\omega(X_{\tau_1}+y,\cdot)\}_{y\cdot\ell\geq 0}\in B|\mathcal{G}_1\Big)$$
$$=\mathbb{P}^o\Big(\{X_n\}_{n\geq 0}\in A, \{\omega(y,\cdot)\}_{y\cdot\ell\geq 0}\in B|\{D=\infty\}\Big)\,. \tag{2.1.9}$$

Consider now τ_1 as a function of the path $(X_n)_{n\geq 0}$ and set

$$\tau_{k+1}=\tau_k(X.)+\tau_1(X_{\tau_k+\cdot}-X_{\tau_k})\,,$$

with $\tau_{k+1}=\infty$ on $\{\tau_k=\infty\}$ (the sequence $\{\tau_k\}$ enumerates times such that for all $k<m<n$, $X_k\cdot\ell<X_m\cdot\ell\leq X_n\cdot\ell$). By the definition and the fact that $\mathbb{P}^o(A_\ell\cap\{\tau_1=\infty\})=0$, we have that $\mathbb{P}^o(A_\ell\cap\{\tau_k=\infty\})=0$. Setting

$$\mathcal{G}_k=\sigma\Big(\tau_1,\ldots,\tau_k,\quad X_0,\cdots,\ X_{\tau_k},\quad \{\omega(y,\cdot)\}_{\ell\cdot y<\ell\cdot X_{\tau_k}}\Big)\,,$$

an obvious restatement of Lemma 2.1.8 yields that

$$\mathbb{Q}^o\Big(\{X_{\tau_k+n}-X_{\tau_k}\}_{n\geq 0}\in A, \{\omega(X_{\tau_k+y},\cdot)\}_{\ell\cdot y\geq 0}\in B|\ \mathcal{G}_k\Big)$$
$$=\mathbb{P}^o\Big(\{X_n\}_{n\geq 0}\in A, \{\omega(y,\cdot)\}_{\ell\cdot y\geq 0}\in B|\{D=\infty\}\Big)\,.$$

We thus conclude that under $\mathbb{Q}^o$,

$$(X_{\tau_2} - X_{\tau_1}, \tau_2 - \tau_1), \cdots, (X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k)$$

are i.i.d. pairs of random variables, independent of (X_{τ_1}, τ_1), such that

$$\mathbb{Q}^o(X_{\tau_2} - X_{\tau_1} \in C_1, \tau_2 - \tau_1 \in C_2) = \mathbb{P}^o(X_{\tau_1} \in C_1, \tau_1 \in C_2 | \{D = \infty\}).$$

Next, we have the following lemma, whose proof is deferred:

Lemma 2.1.10 (Zerner)

$$\mathbb{E}^o(X_{\tau_1} \cdot \ell | \{D = \infty\}) = \frac{1}{\mathbb{P}^o(D = \infty)}.$$

We are now ready to complete the proof of (2.1.6). Assume first that $\mathbb{E}^o(\tau_1 | \{D = \infty\}) < \infty$. Then, by the law of large numbers, and the renewal structure,

$$\frac{\tau_k}{k} \to_{k\to\infty} \mathbb{E}^o(\tau_1 | \{D = \infty\}), \qquad \mathbb{Q}^o\text{-a.s.} \tag{2.1.11}$$

$$\frac{X_{\tau_k} \cdot \ell}{k} \to_{k\to\infty} \mathbb{E}^o(X_{\tau_1} \cdot \ell | \{D = \infty\}), \quad \mathbb{Q}^o\text{-a.s.} \tag{2.1.12}$$

(note that the finiteness of the expression in the right hand side of (2.1.12) is trivial if the right hand side of (2.1.11) is finite, and Lemma 2.1.10 is not needed in this case).

Hence,

$$\frac{X_{\tau_k} \cdot \ell}{k} \longrightarrow_{k\to\infty} \frac{\mathbb{E}^o(X_{\tau_1} \cdot \ell | \{D = \infty\})}{\mathbb{E}^o(\tau_1 | \{D = \infty\})} =: v_\ell, \quad \mathbb{Q}\text{-a.s.} \tag{2.1.13}$$

Mimicking now the argument at the end of the proof of Lemma 2.1.10, we conclude that $\frac{X_n \cdot \ell}{n} \to_{n\to\infty} v_\ell, \mathbb{Q}^o$-a.s., in the case $\mathbb{E}^0(\tau_1 | \{D = \infty\}) < \infty$.

On the other hand, Lemma 2.1.10 implies that (2.1.12) holds even when $\mathbb{E}^o(\tau_1 | \{D = \infty\}) = \infty$. But then, $\tau_k / k \to_{k\to\infty} \infty$, $\mathbb{Q}^o$-a.s. With $v_\ell = 0$ in this case, we conclude that

$$\frac{X_{\tau_k} \cdot \ell}{\tau_k} \longrightarrow_{k\to\infty} v_\ell = 0, \quad \mathbb{Q}^o\text{-a.s.}$$

Finally, setting k_n such that $\tau_{k_n} \le n < \tau_{k_n+1}$, we have that $k_n \to_{n\to\infty} \infty$ and $k_n/n \underset{n\to\infty}{\longrightarrow} 0$, $\mathbb{Q}^o$-a.s. because $n/k_n \ge \tau_{k_n}/k_n$. Thus,

$$\frac{X_n \cdot \ell}{n} \le \frac{X_{\tau_{k_n+1}} \cdot \ell}{k_n + 1} \cdot \frac{k_n + 1}{n} \underset{n\to\infty}{\longrightarrow} 0, \ \mathbb{Q}^o\text{-a.s.}$$

Since $\liminf_{n\to\infty} \frac{X_n \cdot \ell}{n} \ge 0$, $\mathbb{Q}^o$-a.s., we conclude

$$\frac{X_n \cdot \ell}{n} \underset{n\to\infty}{\longrightarrow} 0, \quad \mathbb{Q}^o\text{-a.s.} \tag{2.1.14}$$

A similar proof works on the event $A_{-\ell}$, and thus all that remains to prove is that if $\mathbb{P}^o(A_\ell \cup A_{-\ell}) = 0$ then $X_n \cdot \ell/n \to 0$, $\mathbb{P}^o$-a.s. The complete proof for that is given in [54], and we provide next a brief description.

Consider the set of visits to the hyperplane $\mathcal{H}_m := \{z : z \cdot \ell = m\}$, defining $\tau_m^0 = T_m$ and $\tau_m^i = \min\{n > \tau_m^{i-1} : X_n \cdot \ell = m\}$. Fixing an integer L, let

$$h_{m,L} = \sup_{i \geq 0}\{\tau_m^i - \tau_m^0 : \tau_m^i < T_{m+L}\}$$

be the diameter of the set of visit times to $\mathcal{H}_m$ before T_{m+L}. For any constant $c > 0$, let

$$F_{M,L}(c) = \frac{\#\{0 \leq m \leq M : h_{m,L} \leq c\}}{M+1}$$

denote the fraction of m's smaller than M such that the time between the first and last visit to $\mathcal{H}_m$ before T_{m+L} is smaller than c. The first observation, which is a deterministic (combinatorial) computation that we skip, is that for any path with $\limsup_{n\to\infty} X_n \cdot \ell/n > 0$ there exists a constant c such that

$$\inf_{L \geq 1} \limsup_{M \to \infty} F_{M,L}(c) > 0\,,$$

that is, roughly, there is a fraction of m's for which the time between first and last visits of $\mathcal{H}_m$ (before hitting $\mathcal{H}_{m+L}$) is not too large (intuitively, this should be clear: if the walk moves ballistically, at least on subsequences, then it cannot “afford” to be delayed for too long by visits to a given hyperplane).

Assume now that $\mathbb{P}^o(\limsup X_n \cdot \ell/n > 0) > 0$. Then, by the above observation, there is some $\bar{c} > 0$ such that

$$\mathbb{P}^o(\limsup_{L\to\infty} \limsup_{M\to\infty} F_{M,L}(\bar{c}) > 0) > 0\,. \tag{2.1.15}$$

But on the event $\{h_{m,L} \leq \bar{c}\}$, the last point visited in $\mathcal{H}_m$ before hitting $\mathcal{H}_{m+L}$ is at most at distance $\bar{c}$ from X_{T_m} and has been visited at most $\bar{c}$ times before T_{m+L}. Thus, there is a $z \in \mathcal{H}_0$ with $|z|_1 \leq \bar{c}$, and an $1 \leq r \leq \bar{c}$ such that the rth visit to $X_{T_m} + z$ occurs before T_{m+L} and the walk does not backtrack from $\mathcal{H}_m$ after this rth visit before T_{m+L}. Denoting the last event by $B^1_{m,L}(z,r)$, it follows that

$$F_{M,L}(\bar{c}) \leq \frac{1}{M+1} \sum_{z \in \mathcal{H}_0, |z|_1 \leq \bar{c}} \sum_{r=1}^{\bar{c}} \sum_{m=0}^{M} \mathbf{1}_{B^1_{m,L}(z,r)}\,.$$

Noting that the summation over r and z is over a finite set, and combining the last inequality with (2.1.15), it follows that for some z and r,

$$\mathbb{P}^o(\limsup_{L\to\infty} \limsup_{M\to\infty} \frac{1}{M+1} \sum_{m=0}^{M} \mathbf{1}_{B^1_{m,L}(z,r)} > 0) > 0\,. \tag{2.1.16}$$

In what follows, we drop z, r from the notation and write $B^1_{m,L}$ for $B^1_{m,L}(z,r)$. While the events $\{B^1_{m,L}\}_m$ are not independent (that the rth visit to $X_{T_m} + z$ occurs before T_{m+L} gives some information on the environment to the *left* of m, and hence depends on the path of the walk before T_m), some independence can be restored in the following way: construct independent (given the environment) copies $Y^y_\cdot$ of the RWRE, starting at y. Define the event $B^2_{m,L}$ as the event that $X_\cdot$ does not hit $X_{T_m} + z$ for the rth time before T_{m+L}, but $Y^{X_{T_m}+z}_\cdot$ does not backtrack from $\mathcal{H}_m$ before it hits $\mathcal{H}_{m+L}$, and set $B_{m,L} = B^1_{m,L} \cup B^2_{m,L}$. Since $B_{m,L}$ essentially involves only the occurrence of no-backtrack from $z_m = X_{T_m} + z$ before T_{m+L} for a piece of RWRE that starts at z_m, an easy computation involving the Markov property shows that for each fixed $i = 0, 1, \ldots, L-1$, the events $\{B_{jL+i,L}\}_j$ are independent, with

$$\mathbb{P}^o(B_{jL+i,L}) = \mathbb{P}^o(D \geq T_L)\,.$$

(Here and in the sequel, we abuse notations by still using $\mathbb{P}^o$ to denote the annealed law on the enlarged probability space that supports the extra $\{Y^y\}_y$ walks.) Hence, since we have from (2.1.16) that

$$\begin{aligned}
&\mathbb{P}^o(\limsup_{L\to\infty} \limsup_{M\to\infty} \frac{1}{M+1} \sum_{i=0}^{L-1} \sum_{j=0}^{[M/L]} \mathbf{1}_{B_{jL+i,L}} > 0) \\
&\geq \mathbb{P}^o(\limsup_{L\to\infty} \limsup_{M\to\infty} \frac{1}{M+1} \sum_{i=0}^{L-1} \sum_{j=0}^{[M/L]} \mathbf{1}_{B^1_{jL+i,L}(z,r)} > 0) > 0\,,
\end{aligned}$$

it follows, by the standard law of large numbers, that

$$\mathbb{P}^o(D = \infty) = \limsup_{L\to\infty} \mathbb{P}^o(D \geq T_L) > 0\,.$$

But from (2.1.3), we have that $\mathbb{P}^o(A_\ell) \geq \mathbb{P}^o(D = \infty) > 0$. In particular, this shows that $\mathbb{P}^o(A_\ell) = 0$ implies that $\limsup X_n \cdot \ell/n \leq 0$, $\mathbb{P}^o$-a.s. Repeating this argument with $-\ell$ instead of ℓ completes the proof of the theorem. $\square$

Proof of Lemma 2.1.10 Recall that we consider $\ell = (1, 0, \ldots, 0)$. Then,

$$
\begin{aligned}
&\mathbb{Q}^o\Big(\{\exists k:\ X_{\tau_k}\cdot\ell=i\}\Big)\\
&\quad=\frac{\sum_{y\in\mathbb{Z}^{d-1}}E\Big(P^o_\omega(\{\exists k:\ X_{\tau_k}=(i,y)\},A_\ell)\Big)}{\mathbb{P}^o(A_\ell)}\\
&\quad=\frac{\sum_{y\in\mathbb{Z}^{d-1}}E\Big(P^o_\omega(T_i<\infty,X_{T_i}=(i,y),D\circ\theta_{T_i}=\infty)\Big)}{\mathbb{P}^o(A_\ell)}\\
&\quad=\frac{\sum_{y\in\mathbb{Z}^{d-1}}E\Big(P^o_\omega(T_i<\infty,X_{T_i}=(i,y))P^{(i,y)}_\omega(D=\infty)\Big)}{\mathbb{P}^o(A_\ell)}\\
&\quad=\frac{\mathbb{P}^o(T_i<\infty)\mathbb{P}^o(D=\infty)}{\mathbb{P}^o(A_\ell)}\underset{i\to\infty}{\longrightarrow}\mathbb{P}^o(D=\infty)
\end{aligned}
\tag{2.1.17}
$$

(since $\mathbb{P}^o(A_\ell\cup A_{-\ell})=1$ and $\lim_{i\to\infty}\mathbb{P}^o(\{T_i<\infty\}\cap A_{-\ell})=0$). On the other hand,

$$
\lim_{i\to\infty}\mathbb{Q}^o\Big(\{\exists k:\ X_{\tau_k}\cdot\ell=i\}\Big)=\lim_{i\to\infty}\mathbb{Q}^o\Big(\{\exists k\ge 2:\ X_{\tau_k}\cdot\ell=i\}\Big)
$$

(because $\mathbb{Q}^o(\tau_k>i)\underset{i\to\infty}{\longrightarrow}0$)

$$
\begin{aligned}
&=\lim_{i\to\infty}\sum_{n\ge1}\mathbb{Q}^o\Big(\{\exists k\ge 2:\ X_{\tau_k}\cdot\ell=i,X_{\tau_1}\cdot\ell=n\}\Big)\\
&=\lim_{i\to\infty}\sum_{n\ge1}\mathbb{Q}^o\Big(\{\exists k\ge 2:\ (X_{\tau_k}-X_{\tau_1})\cdot\ell=i-n,X_{\tau_1}\cdot\ell=n\}\Big)\\
&=\lim_{i\to\infty}\sum_{n\ge1}\mathbb{Q}^o(X_{\tau_1}\cdot\ell=n)\cdot\mathbb{Q}^o\Big(\{\exists k\ge 2:(X_{\tau_k}-X_{\tau_1})\cdot\ell=i-n\}\Big).
\end{aligned}
$$

But, recall that by the renewal theorem,

$$
\mathbb{Q}^o\Big(\exists k\ge 2:\ (X_{\tau_k}-X_{\tau_1})\cdot\ell=i-n\Big)\underset{i\to\infty}{\longrightarrow}\frac{1}{E_{\mathbb{Q}^o}((X_{\tau_2}-X_{\tau_1})\cdot\ell)}
$$

and hence, by dominated convergence,

$$
\lim_{i\to\infty}\mathbb{Q}^o\Big(\{\exists k:\ X_{\tau_k}\cdot\ell=i\}\Big)=\frac{\sum_{n\ge1}\mathbb{Q}^o(X_{\tau_1}\cdot\ell=n)}{E_{\mathbb{Q}^o}((X_{\tau_2}-X_{\tau_1})\cdot\ell)}=\frac{1}{E_{\mathbb{Q}^o}((X_{\tau_2}-X_{\tau_1})\cdot\ell)}.
\tag{2.1.18}
$$

Comparing (2.1.17) and (2.1.18), we conclude that

$$
E_{\mathbb{Q}^o}\Big((X_{\tau_2}-X_{\tau_1})\cdot\ell\Big)=\frac{1}{\mathbb{P}^o(D=\infty)}<\infty\,. \qquad\square
$$

Remarks:

1. Note that on A_ℓ, $v_\ell>0$ if $\mathbb{E}^o(\tau_1|\{D=\infty\})<\infty$ and $v_\ell=0$ otherwise.

2. It is not hard to conclude from Theorems 2.1.1 and 2.1.5 (applied to different directions ℓ) that there can be at most two (antipodal) asymptotic velocities, that is there exist disjoint events A_+, A_- with $\mathbb{P}^0(A_- \cup A_+) = 1$, a vector ℓ and constants $v_-, v_+ \in [0,1)$ such that
$$X_n/n \to \mathbf{1}_{A_+} v_+ \ell - \mathbf{1}_{A_-} v_- \ell, \quad \mathbb{P}^0 - a.s. \tag{2.1.19}$$
In particular, if P is invariant under lattice isometries then $X_n/n \to 0$, $\mathbb{P}^0$-a.s..
3. Modifying slightly the definition of $(\overline{R}_k, \overline{S}_k)$ (by adding an L-safeguards and an auxiliary randomization using the uniform ellipticity), the results of this section extend immediately to the case where the environment is K-dependent (in fact, what is needed is that $\{\omega(x,\cdot)\}_{x\cdot\ell\le 0}$ and $\{\omega(x,\cdot)\}_{x\cdot\ell>K}$ are independent). The extension to a mixing setup is more complicated, and one needs to consider a modification of these times in which the event $D=\infty$ is replaced by the event that the walk does not exit a cone based at the origin, and assume that the environment is "mixing on cones". For more details, see [12].
4. Still discussing mixing environments, some progress has been made using the approach of the environment viewed from the particle. We mention here in particular the work of Rassoul-Agha [42], who uses a-priori estimates concerning regeneration times in the ballistic case to construct an invariant measure for the environment viewed from the particle which is absolutely continuous with respect to P on certain half-spaces, and deduces a LLN using that measure.
5. There are results also concerning the existence of asymptotic *directions* for the RWRE. For example, Simenhaus [44] proves that the existence of an open set of directions ℓ with $\mathbb{P}^0(A_\ell) = 1$ is equivalent to the a.s. existence of the limit $X_n/|X_n|$. See also [17].

2.2 The Zerner-Merkl $0-1$ law for $d=2$, and counterexamples

Since Corollary 2.1.7 shows that the only possible obstacle to the LLN is the question whether it is possible that $\mathbb{P}(A_\ell) \in (0,1)$, it is natural to inquire whether one could rule out the later in some generality.

For i.i.d. environments, the answer is affirmative in dimension $d=2$.

Theorem 2.2.20 (Zerner, Merkl [56]) *Assume P is i.i.d., elliptic, and $d=2$. Then $\mathbb{P}^0(A_\ell) \in \{0,1\}$.*

Proof. We follow the simple proof in [55], and to simplify notation, take $\ell = e_1$ and write $A_+ = A_\ell$ and $A_- = A_{-\ell}$. We also assume uniform ellipticity, although this is not really needed. By Theorem 2.1.1, we may and will assume $\mathbb{P}^0(A_+ \cup A_-) = 1$. With $T_- = \min\{n>0 : X_n \cdot e_1 < 0\}$ and $T_+ = \min\{n > 0 : X_n \cdot e_1 > 0\}$, we note that $\{T_- < \infty\} = \{D < \infty\}$ and therefore, as we saw in the course of the proof of Theorem 2.1.1, if $\mathbb{P}^0(A_+) > 0$ then

$\mathbb{P}^0(T_- = \infty) > 0$. Similarly, $\mathbb{P}^0(A_-) > 0$ then $\mathbb{P}^0(T_+ = \infty) > 0$. Thus the theorem will follow if we prove that

$$0 = \mathbb{P}^0(T_- = \infty)\mathbb{P}^0(T_+ = \infty) := P_1. \tag{2.2.21}$$

Now, for any $z_L = (2L, y_L)$,

$$P_1 = \lim_{L\to\infty} \mathbb{P}^0(T_L < T_-)\mathbb{P}^{z_L}(T_L < T_{2L+1}) := \lim_{L\to\infty} P_2^L$$

Note that the two probabilities in the right side of the last display depend on disjoint environments. In particular, considering two independent random walks X_n^1, X_n^2 in the *same* environment ω, started from 0 and z_L, respectively, the probability P_2^L represents the probability that $X_\cdot^1$ hits level L before falling back below 0, and $X_\cdot^2$ hits level L before crossing over to a level larger than $2L$. Thus, the event in P_2^L is contained in the union of the following three events: either $X_\cdot^1$ hits level L and then falls back to level 0, or $X_\cdot^2$ hits level L and then returns to level $2L$, or $X_\cdot^1$ crosses over to level $2L$ before returning to level 0 and $X_\cdot^2$ crosses over to level 0 before hitting level $2L+1$. In formulas,

$$P_2^L \le \mathbb{P}^0(T_L < T_- < \infty) + \mathbb{P}^{z_L}(T_L < T_{2L+1} < \infty) + P_3^L\,,$$

where

$$P_3^L = \mathbb{P}^{0,z_L}(T_{2L}^1 < T_-^1, T_-^2 < T_{2L+1}^2)\,,$$

and the indices over stopping time describe which of the walks is involved in their definition. Note that

$$\mathbb{P}^0(T_L < T_- < \infty) \le \mathbb{P}^0(\exists n \ge 2L : |X_n \cdot e_1| < 1) \to_{L\to\infty} 0$$

because $\mathbb{P}^0(A_+ \cup A_-) = 1$. Arguing similarly for $\mathbb{P}^{z_L}(T_L < T_{2L+1} < \infty)$, we conclude that

$$\lim_{L\to\infty} P_2^L = \lim_{L\to\infty} P_3^L\,.$$

So far the choice of y_L in the definition of z_L was arbitrary. Now we specify it as being the median of the distribution $\mathbb{P}^0(X_{T_{2L}} \cdot e_2 \in \cdot | T_{2L} < T_-)$. In particular, $\mathbb{P}^0(X_{T_{2L}} \cdot e_2 \overset{>}{<} y_L | T_{2L} < T_-) < 1/2$.

It is now time to use the planarity: under the event in P_3^L, the paths of $X_\cdot^1$ and $X_\cdot^2$ either intersect before they cross the levels 0 to $2L$, or they do not intersect. Write $P_{3,i}^L$ and $P_{3,n}^L$ for the respective probabilities. If they do not intersect, one must be consistently above the other, and they evolve in disjoint, and hence independent, environments. One then gets that

$$P_{3,n}^L \le \frac{1}{2}\mathbb{P}^0(T_{2L} < T_-)\mathbb{P}^{z_L}(T_0 < T_{2L+1}) \to_{L\to\infty} \frac{1}{2}P_1\,.$$

On the other hand, if the walks intersect, they must do that either in between levels 0 and L or between levels L and $2L$. Consider the first possibility, and set $r(x,\omega) = P_\omega^x(A_+)$. Then, either at the intersection point $r(x,\omega) \le \epsilon$, in which case the probability of $X^1_\cdot$ to completes its crossing to level $2L$ is asymptotically bounded by ϵ, or $r(x,\omega) > \epsilon$, in which case it follows that $r(X_n^2,\omega) > \epsilon$ for some $n > L$. But, by the Martingale convergence theorem, $r(X_n^2,\omega) \to_{n\to\infty} \mathbf{1}_{A_+}$, implying that A_+ occurs for X_n^2. Taking $L \to \infty$ one gets that $\lim_{L\to\infty} P_{3,i}^L \le \epsilon$, and thus vanishes since ϵ was arbitrary. In conclusion, we obtained $P_1 \le P_1/2$, and hence $P_1 = 0$, as claimed. $\square$

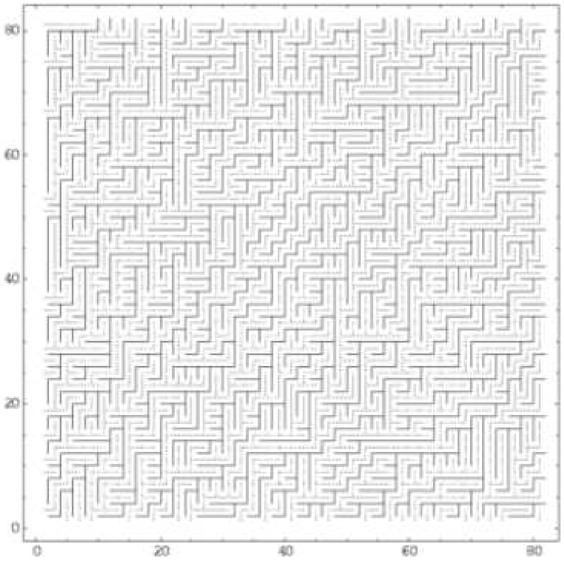

Fig. 2.2.2. 2d counter example

Equipped with Theorem 2.2.20, one may wonder whether the $0-1$ is merely an ergodic phenomenon. The answer is a resounding *no*, as Zerner and Merkl themselves show in [56]. The following is their counter-example.

Consider the sub-lattice $2\mathbb{Z}^2$ consisting of even points in $\mathbb{Z}^2$. Connect independently with probability $1/2$ each vertex to its odd eastern neighbor, connecting that neighbor to its own eastern (even) neighbor. Otherwise connect it to the northern neighbor(s). One can prove that the resulting graph is a spanning tree in $2\mathbb{Z}^2$ with a single end (this does require a proof, but is not too hard - using properties of random walks in one dimension), and further a "dual" spanning tree is determined by this procedure. This is best explained by a picture, so here it is. To make this picture ergodic, randomize the origin between the odd and even parities (4 choices).

Now, because one got a one ended tree, there is a notion of parent to each vertex, and the sub-tree of descendants of each vertex is finite almost surely. Call l_x the depth (i.e., longest line of descendants) of the sub-tree attached to x, counting x itself (i.e., $l_x \geq 1$). If x_p denotes the parent of x, then $l_{x_p} \geq l_x + 1$. If we now define $\omega(x, x_p - x) = 1 - 1/2l_x^2$ and $\omega(x, y - x) = 1/6l_x^2$ for $y \neq x_p$, the Borel-Cantelli Lemma implies that with positive probability, at each time the walk jumps to its parent; similarly, the walk might with positive probability jump in the first step toward a neighbor on the "dual" tree and then move exclusively toward its parents there. Taking $\ell = e_1$ one obtains easily that $\mathbb{P}^0(A_\ell)\mathbb{P}^0(A_{-\ell}) > 0$, contradicting the $0 - 1$ law.

Although this example is not mixing and not uniformly elliptic, with considerable effort one may modify it and provide a counter-example to the $0 - 1$ law which is uniformly elliptic and polynomially mixing, for $d \geq 3$.

2.3 Possible limiting velocities

We have seen that even if the 0-1 law fails, the only possible velocities are antipodal, see (2.1.19). When the dimension is large enough, more can be said.

Theorem 2.3.22 (Berger [2]) *Assume P i.i.d. and uniformly elliptic, and $d \geq 5$. Then $v_+ \cdot v_- = 0$ in* (2.1.19).

Of course, the only interest in Theorem 2.3.22 is as long as the 0-1 law is not proved, or if it is disproved.

Proof. Since I do not expect to cover this during the lecture, I only provide a rough sketch. Suppose $\mathbb{P}^0(A_\ell \cup A_{-\ell}) = 1$, and write again $A_+ = A_\ell$ and $A_- = A_{-\ell}$. We use the regeneration structure to construct a *backward path* X^1, which is a path of the RWRE "coming from $-\infty$ and crossing to $+\infty$, passing through the origin". It is constructed by concatenating environments and pieces of path, between regenerations, under the measure $\mathbb{P}^0(\cdot|A_+)$, creating also a law on the environment that we denote Q. (A figure here would help!). Note that off the backward path, the environment is i.i.d. and sampled according to P (that is, the marginal of Q off the backward path coincides with P). Recall the (quenched) martingale $r(X_n, \omega)$ from the proof of Theorem 2.2.20. Because $r(X_n, \omega) \to \mathbf{1}_{A_+}$, it follows from the construction that $r(0, \omega) = 1$ and hence $r(x, \omega) = 1$ for all $x \in \mathbb{Z}^d$. On the other hand, suppose that one proves that a path starting from some $z_0 \in \mathcal{H}_0$ with z_0 not on the backward path, evolving in an i.i.d. environment which is P-distributed and going to $-\infty$ with non-zero speed, has a positive probability not to intersect the backward path. Then this will imply that $E_Q(r(z_0,\omega)) < 1$, a contradiction.

So one only needs to prove that a path X^2 started at z_0 for which A_- holds with non-zero velocity has a positive probability not to intersect the backward path X^1. This is best understood by following the following two steps. First, such a path has regenerations, and the distance between regeneration has finite expectation. If one looked only at regeneration times, both of X^2 and X^1, we have two random walks in dimension $d \geq 5$, and a simple Green function estimate for such walks with i.i.d. increments shows that the expected number of intersections can be made less than 1 by taking $|z_0|$ large. Second, filling the gaps between regeneration times is possible because of the assumption of positive velocity (which implies a finite expectation of the inter-regeneration time and hence bounds the expectation of the number of sites visited between regenerations).

More precisely, let $\mathcal{T}^i$, $i = 1, 2$ denote the paths of X^i. The expected number of intersections (in independent environments) equals

$$\begin{aligned}
&\sum_{z\in\mathbb{Z}^d} Q(z \in \mathcal{T}^1)\mathbb{P}^0(z \in \mathcal{T}^2|A_-) \\
&\leq \left(\sum_{z\in\mathbb{Z}^d} Q(z \in \mathcal{T}^1)^2\right)^{1/2} \cdot \left(\sum_{z\in\mathbb{Z}^d} \mathbb{P}^0(z \in \mathcal{T}^2|A_-)^2\right)^{1/2} := A_1 \cdot A_2 \,.
\end{aligned}$$

We explain the argument involving A_1, the one concerning A_2 is similar. Note that

$$Q(z \in \mathcal{T}^1) = \sum_n Q(z \in \mathcal{T}^1_n) := \sum_n Q(z, n)\,,$$

where $\mathcal{T}^n_1$ denotes the part of $\mathcal{T}^1$ corresponding to the n-th regeneration. The finite expectation of regeneration times implies that $\sum_{z\in\mathbb{Z}^d} Q(z,n) < C$, while the standard estimates for sums of i.i.d. random variables alluded to above imply that $Q(z,n) \leq Cn^{-d/2}$ (here and in what follows, C is a constant that may change from line to line). Hence, $\sum_{z\in\mathbb{Z}^d} Q(z,n)^2 \leq Cn^{-d/2}$. We get that

$$A_1 = \left(\sum_{z\in\mathbb{Z}^d}\sum_n Q(z,n)^2\right)^{1/2} \leq \sum_n \left(\sum_{z\in\mathbb{Z}^d} Q(z,n)^2\right)^{1/2} \leq C\sum_n n^{-d/4} < C,$$

where $d \geq 5$ was used (for the first time in the proof). This completes the argument. $\square$

3 Lecture 3. More on regenerations

Recall the regeneration times $\{tau_i\}_{i\geq 1}$ introduced in Section 2, with respect to a direction ℓ. Their main advantage is that they create, under the annealed law $\mathbb{P}^0(\cdot|A_\ell)$, a sequence of i.i.d. increments $(\tau_{i+1}-\tau_i, X_{\tau_{i+1}}-X_{\tau_i})$, and further, $\mathbb{E}^0(|X_{\tau_2}-X_{\tau_1}|\,|A_\ell)<\infty$.

3.1 Central Limit Theorems in the ballistic regime

The following is a rather elementary consequence of the i.i.d. structure of regeneration times. Recall that on the event A_ℓ, one has $X_n/n\to v_\ell$, a.s..

Theorem 3.1.1 *Assume P i.i.d. and uniformly elliptic. Further assume that for some direction ℓ, $\mathbb{P}^0(A_\ell>0)$*

$$\mathbb{E}^0((\tau_2-\tau_1)^r|A_\ell)<\infty \tag{3.1.2}$$

for some $r>2$. Then, under the law $\mathbb{P}^0(\cdot|A_\ell)$, an invariance principle holds for the processes

$$Z_n(t)=\frac{1}{\sqrt{n}}\left(X_{[nt]}-ntv_\ell\right),$$

that is the sequence of processes $\{Z_n(t)\}_{t\geq 0}$ converges in distribution to a Brownian motion with a non-degenerate covariance matrix $\mathcal{R}$.

A natural question is whether the above extends to a quenched invariant principle. As discussed in Section 1, see Theorem 1.2.5, a change of centering is needed in the quenched setup. As we are about to see, this is not the case in dimension $d\geq 2$, provided appropriate moment conditions on the regeneration times are assumed.

Theorem 3.1.3 (Berger–Zeitouni [3], Rassoul-Agha–Seppalainen [43]) *Let $d\geq 2$ and let P be i.i.d. and uniformly elliptic. Assume that the random walk $\{X(n)\}_{n\geq 0}$ satisfies a law of large numbers with nonzero limiting velocity v. With ℓ so that $\langle \ell, v\rangle>0$, define τ_i to be the ith regeneration time in direction ℓ. Assume that there exists an $\epsilon>0$ such that $\mathbb{E}^0(\tau_1)^\epsilon<\infty$ and, with some $r\geq r_0$,*

$$\mathbb{E}^0[(\tau_2-\tau_1)^r]<\infty\,. \tag{3.1.4}$$

where

$$r_0=r_0(d)=\begin{cases}2, & d\geq 5\\ 8, & d=4\\ 40, & d=2,3\,.\end{cases}$$

Then, the process $Z_n(t)$ satisfies a quenched invariance principle with variance $\mathcal{R}$.

Proof. (sketch) I will discuss only the case $d \geq 5$, and will assume that $\langle v, e_1 \rangle > 0$ so that we can take $\ell = e_1$. The proof proceeds by considering two independent RWRE's in the same environment. Thus, let $\beta_n^1(t), \beta_n^2(t)$ denote two independent copies (given the environment) of linear interpolations of $Z_n(t)$. A key ingredient in the proof is the following observation.

Lemma 3.1.5 (Bolthausen–Sznitman [6]) *Assume that $Z_n(t)$ satisfies the annealed invariance principle. Assume that for all $T > 0$ and $b \in (1,2]$ and for all bounded Lipschitz functions F on $C([0,T], \mathbb{R}^d)$,*

$$\sum_m (E_P[E_\omega(F(\beta_1([b^m]))E_\omega(F(\beta_2([b^m]))] - \mathrm{e}[F(\beta_1([b^m]))]\mathrm{e}[F(\beta_2([b^m])]) < \infty \,. \tag{3.1.6}$$

Then for P a.e. ω, $Z_n(t)$ satisfies the quenched invariance principle.

Note that the assumption of Lemma 3.1.5 is simply that the variance of the quenched mean $E_\omega(F(\beta([b^m]))$ decays to 0 in a summable way. In particular, $E_\omega(F(\beta([b^m]))$ converges almost surely to $\mathbb{E}^0(F(\beta([b^m]))$, which by the assumed invariance principle converges to the corresponding expectation with respect to the Brownian motion law of covariance $\mathcal{R}$. An interpolation argument completes the proof of the lemma.

We next explain quickly how to check the conditions of Lemma 3.1.5 so as to complete the proof of Theorem 3.1.3. As a first easy step, let $H_K = \min\{x \in \mathbb{R}^d : x_1 \geq K\}$, and

$$W_K := \{\{X_1(\cdot)\} \cap \{X_2(\cdot)\} \cap H_K \neq \emptyset\} \,.$$

One reduces the verification of (3.1.6) to the verification that $\sum_m \mathbb{P}^0(W_{b^m}) < \infty$. This in turn will follow from the the following estimate:

$$E_P[P_\omega(W_K)] < CK^{-\kappa_d} \tag{3.1.7}$$

where $\kappa_d = \kappa_d(\epsilon, r) > 0$. The latter estimate follows from Green function estimates for simple random walks, in an analogous way to the proof of Theorem 2.3.22. □

3.2 Moment bounds on regeneration times: Kalikow's condition

We introduce in this section a condition on the environment, due to Kalikow, which ensures that the RWRE is ballistic. As before, we let P be a uniformly elliptic product measure. Let U be a strict subset of $\mathbb{Z}^d$, with $0 \in U$, and define on $U \cup \partial U$ an auxiliary Markov chain with transition probabilities

$$\hat{P}_U(x, x+e) = \begin{cases} \frac{\mathbb{E}^o\left[\sum_{n=0}^{\tau_{U^c}} \mathbf{1}_{\{X_n = x\}} \omega(x, x+e)\right]}{\mathbb{E}^o\left[\sum_{n=0}^{\tau_{U^c}} \mathbf{1}_{\{X_n = x\}}\right]}, & x \in U, |e| = 1 \\ 1 & x \in \partial U, e = 0 \end{cases} \tag{3.2.8}$$

where $\tau_{U^c} = \min\{n \geq 0 : X_n \in \partial U\}$ (note that the expectations in (3.2.8) are finite due the Markov property and ellipticity). The transition kernel $\hat{P}_U$ weights the transitions $x \mapsto x + e$ according to the occupation time of the vertex x before exiting U. We denote by $\hat{E}_U$ expectations with respect to the measure $\hat{P}_U$.

The following is a basic consequence of the definition of $\hat{P}_U(\cdot, \cdot)$, and hints at its usefuleness:

Lemma 3.2.9 (Kalikow) *Assume $\hat{P}_U(\tau_{U^c} < \infty) = 1$. Then, $\hat{P}_U(X_{\tau_{U^c}} = v) = \mathbb{P}^o(X_{\tau_{U^c}} = v), v \in \partial U$. In particular, $\mathbb{P}^o(\tau_{U^c} < \infty) = 1$.*

Proof of Lemma 3.2.9: Set $g_\omega(x) = E^o_\omega\left(\sum_{n=0}^{\tau_{U^c}} \mathbf{1}_{\{X_n = x\}}\right)$. Then

$$\hat{P}_U(x, y) = \frac{E(g_\omega(x)\omega(x, y))}{E(g_\omega(x))}, \quad x \in U,\ y \in U \cup \partial U\,. \tag{3.2.10}$$

But, due to the Markov property,

$$g_\omega(y) = \mathbf{1}_{\{y=0\}} + \sum_{x \in U} \omega(x, y) g_\omega(x)\,,$$

and hence, using (3.2.10),

$$\sum_{x \in U} \Big(E(g_\omega(x))\Big) \hat{P}_U(x, y) + \mathbf{1}_{\{y=0\}} = E(g_\omega(y))\,.$$

Set $\hat{\pi}_n(y) = \hat{E}_U\left(\sum_{j=0}^{\tau_{U^c} \wedge n} \mathbf{1}_{\{X_j = y\}}\right)$. Then, $\hat{\pi}_0(y) = \mathbf{1}_{\{y=0\}}$ and

$$\hat{\pi}_{n+1}(y) = \mathbf{1}_{\{y=0\}} + \sum_{x \in U} \hat{P}_U(x, y) \hat{\pi}_n(x)\,.$$

Then, for $y \in U \cup \partial U$,

$$E\Big(g_\omega(y)\Big) - \hat{\pi}_{n+1}(y) = \sum_{x \in U} \hat{P}_U(x, y) \Big(E(g_\omega(x)) - \hat{\pi}_n(x)\Big)\,.$$

Since $E(g_\omega(y)) - \hat{\pi}_0(y) \geq 0$, it follows by the positivity of $\hat{P}_U(x, y)$ that for $y \in U \cup \partial U$,

$$\hat{E}_U\left(\sum_{n=0}^{\tau_{U^c}} \mathbf{1}_{\{X_n = y\}}\right) = \lim_{n \to \infty} \hat{\pi}_n(y) \leq E(g_\omega(y))\,.$$

Taking $y \in \partial U$ yields

$$\hat{P}_U(X_{\tau_{U^c}} = y) \leq \mathbb{P}^o(X_{\tau_{U^c}} = y), \quad y \in \partial U\,.$$

On the other hand, $\sum_{y \in \partial U} \hat{P}_U(X_{\tau_{U^c}} = y) = 1$ because $\hat{P}_U(\tau_{U^c} < \infty) = 1$ by assumption. Hence

$$\mathbb{P}^o(X_{\tau_{U^c}} = y) = \hat{P}_U(X_{\tau_{U^c}} = y), \quad \forall y \in \partial U\,. \qquad \square$$

We are now ready to introduce Kalikow's condition. Fix a hyperplane by picking a point $\ell \in \mathbb{R}^d \backslash \{0\}$, $|\ell|_1 \leq 1$. Define

$$\varepsilon_\ell := \inf_{U, x \in U} \sum_{|e|=1} (\ell \cdot e) \hat{P}_U(x, x+e)$$

where the infimum is over all connected strict subsets of $\mathbb{Z}^d$ containing 0. We say that *Kalikow's condition with respect to* ℓ holds if $\varepsilon_\ell > 0$. Note that ε_ℓ acts as a drift in the direction ℓ for the Markov chain $\hat{P}_U$.

A consequence of Lemma 3.2.9 is the following:

Theorem 3.2.11 (Kalikow[28], Sznitman and Zerner[52]) *If Kalikow's condition with respect to ℓ holds, then $\mathbb{P}^o(A_\ell) = 1$, and $v_\ell > 0$.*

Proof. Fix $U_L = \{z \in \mathbb{Z}^d : \ |z \cdot \ell| \leq L\}$. Let $\hat{X}_{n,L}$ denote the Markov chain with $\hat{X}_{0,L} = 0$ and transition law $\hat{P}_{U_L}(x, x+e)$. Set the local drift at x, $\hat{d}(x) = \sum_{|e|=1} e \hat{P}_{U_L}(x, x+e)$, and recall that $\hat{X}_{n,L} - \sum_{i=0}^{n-1} \hat{d}(\hat{X}_{i,L})$ is a martingale, with bounded increments. It follows that for some constant C,

$$\hat{P}_{U_L}\left(\sup_{0 \leq n \leq N} |\hat{X}_{n,L} - \sum_{i=0}^{n-1} \hat{d}(\hat{X}_{i,L})| > \delta N \right) \leq C e^{-C\delta^2 N}\,. \tag{3.2.12}$$

On the other hand, $\sum_{i=0}^{n-1} \hat{d}(\hat{X}_{i,L}) \cdot \ell \geq \varepsilon_\ell (n \wedge \tau_{U_L^c})$ while $|\hat{X}_{n,L} \cdot \ell| \leq L+1$. We thus conclude from (3.2.12) that

$$\hat{P}_{U_L}\left(\tau_{U_L^c} > \frac{L+1}{\varepsilon_\ell} + \frac{\delta}{\varepsilon_\ell} N \right) \leq C e^{-C\delta^2 N / \varepsilon_\ell^2}\,,$$

and hence, for some C_1 independent of L, and all L large,

$$\hat{P}_{U_L}\Big(|\hat{X}_{\tau_{U_L^c}} \cdot \ell - L| > 1 \Big) \leq C_1 e^{-C_1 L}\,.$$

It follows from Lemma 3.2.9 that

$$\mathbb{P}^o\Big(|X_{\tau_{U_L^c}} \cdot \ell - L| > 1 \Big) \leq C_1 e^{-C_1 L}\,.$$

A similar argument shows that $\mathbb{P}^o(D = \infty) > 0$: indeed, take now $U_{L,+} = \{z \in \mathbb{Z}^d : \ 0 \leq z \cdot \ell \leq L\}$. Arguing as above, one finds that for some $C_2 > 0$ independent of L,

$$\hat{P}_{U_{L,+}}\Big(|\hat{X}_{\tau_{U_{L,+}^c}} - L| \leq 1 \Big) > C_2\,,$$

implying that

$$\mathbb{P}^o\Big(|X_{\tau_{U_{L,+}^c}} - L| \leq 1 \Big) > C_2\,.$$

Thus, $\mathbb{P}^o(D=\infty)>0$, and then, by an argument as in the proof of Theorem 2.1.1, $\mathbb{P}^o(A_\ell)>0$. By Theorem 2.1.1, it follows that $\mathbb{P}^o(A_\ell\cup A_{-\ell})=1$. Due to (3.2.12), it holds that $\mathbb{P}^o(\limsup_{n\to\infty} X_n\cdot\ell=\infty)=1$. We thus conclude that $\mathbb{P}^o(A_\ell)=1$.

To see that if P is i.i.d. then $v_\ell>0$, recall the regeneration times $\{\tau_i\}_{i\geq 1}$. By Lemma 2.1.10, it suffices to prove that $\mathbb{E}^o(\tau_1|D=\infty)<\infty$. Let $U_{m,k,-}=\{z\in\mathbb{Z}^d:|z|<k, z\cdot\ell<m\}$, and set $T_{m,k}:=T_{U_{m,k,-}}$ with $T_m=\lim_{k\to\infty}T_{m,k}=\min\{n:X_n\cdot\ell\geq m\}$, $m\geq 1$. By Kalikow's condition,

$$\mathbb{E}^o\left(\sum_{n=0}^{T_{m,k}}\mathbf{1}_{\{X_n=x\}}\sum_{e:|e|=1}\omega(x,x+e)\ell\cdot e\right)\geq\varepsilon_\ell\mathbb{E}^o\left(\sum_{n=0}^{T_{m,k}}\mathbf{1}_{\{X_n=x\}}\right),$$

and hence, summing over $x\in U_{m,k,-}$ and recalling that $X_{i+1}-X_i-d(\theta^{X_i}\omega)$ is a martingale difference sequence, one gets

$$1+m\geq\mathbb{E}^o(X_{T_{m,k}}\cdot\ell)\geq\varepsilon_\ell\mathbb{E}^o(T_{m,k}),$$

and taking $k\to\infty$ one concludes that $m+1\geq\varepsilon_\ell\mathbb{E}^o(T_m)$. In particular,

$$\mathbb{E}^o(\liminf_{m\to\infty}T_m/m)\leq\liminf_{m\to\infty}\mathbb{E}^o(T_m/m)\leq 1/\varepsilon_\ell\,. \qquad (3.2.13)$$

Since $\tau_i\to_{i\to\infty}\infty$, $\mathbb{P}^o$-a.s., one may find a (random) sequence k_m such that $\tau_{k_m}\leq T_m<\tau_{k_m+1}$. By definition,

$$\ell\cdot X_{\tau_{k_m}}\leq\ell\cdot X_{T_m}\leq\ell\cdot X_{\tau_{k_m+1}},$$

and further,

$$\ell\cdot X_{\tau_k}/k\to_{k\to\infty}\mathbb{E}^o(\ell\cdot X_{\tau_1}|\{D=\infty\})=\frac{1}{\mathbb{P}^o(D=\infty)}<\infty\,,\mathbb{P}^o-a.s.\,,$$

due to Lemma 2.1.10 and (2.1.12). Thus, $k_m/m\to_{m\to\infty}\mathbb{E}^o(X_{\tau_1}\cdot\ell|\{D=\infty\})^{-1}$, $\mathbb{P}^o$-a.s. But, since $\tau_{k_m}/k_m\to_{m\to\infty}\mathbb{E}^o(\tau_1|\{D=\infty\})\in[1,\infty]$, $\mathbb{P}^o$-a.s., it follows that

$$\begin{aligned}\liminf_{m\to\infty}\frac{T_m}{m}\geq\liminf_{m\to\infty}\frac{\tau_{k_m}}{k_m}\frac{k_m}{m}=\lim_{m\to\infty}\frac{\tau_{k_m}}{k_m}\frac{k_m}{m}&=\frac{\mathbb{E}^o(\tau_1|\{D=\infty\})}{\mathbb{E}^o(X_{\tau_1}\cdot\ell|\{D=\infty\})}\\&=\mathbb{E}^o(\tau_1|\{D=\infty\})\mathbb{P}^o(D=\infty)\,.\end{aligned}$$

Since (3.2.13) implies that $\liminf_{m\to\infty}T_m/m<\infty$, we conclude that $\mathbb{E}^o(\tau_1|\{D=\infty\})<\infty$, and hence $v_\ell>0$. $\square$

By noting that if Kalikow's condition holds for some ℓ_0 then it holds for all ℓ in a neighborhood of ℓ_0, one gets immediately the

Corollary 3.2.14 *If Kalikow's condition with respect to some ℓ holds, then there exists a non-zero deterministic v such that*

$$\frac{X_n}{n}\to_{n\to\infty}v\,,\quad P-a.s.$$

The following is a sufficient condition for Kalikow's condition to hold true:

Lemma 3.2.15 *Kalikow's condition with respect to ℓ holds if*

$$\inf_{f\in\mathcal{F}} \frac{E\left(\frac{\sum_{e:|e|=1}\omega(0,e)\ell\cdot e}{\sum_{e:|e|=1}\omega(0,e)f(e)}\right)}{E\left(\frac{1}{\sum_{e:|e|=1}\omega(0,e)f(e)}\right)} > 0\,, \tag{3.2.16}$$

where $\mathcal{F}$ denotes the collection of nonzero functions on $\{e : |e| = 1\}$ taking values in $[0,1]$.

Proof. Fix U a strict subset of $\mathbb{Z}^d$, $x \in U$, and let $\tau_x = \min\{n \geq 0 : X_n = x\}$. Define $g(x,y,\omega) := E^y_\omega(\mathbf{1}_{\{\tau_x<\tau_{U^c}\}})$. Note that $g(x,y,\omega)$ is independent of ω_x. Next,

$$\begin{aligned}
&\sum_{|e|=1}(\ell\cdot e)\hat{P}_U(x,x+e)\\
&\quad= \frac{E\left(E^o_\omega\left(\sum_{n=0}^{\tau_{U^c}}\mathbf{1}_{\{X_n=x\}}\sum_{e:|e|=1}\omega(x,x+e)e\cdot\ell\right)\right)}{E\left(E^o_\omega\left(\sum_{n=0}^{\tau_{U^c}}\mathbf{1}_{\{X_n=x\}}\right)\right)}\\
&\quad= \frac{E\left(g(x,0,\omega)E^x_\omega\left(\sum_{n=0}^{\tau_{U^c}}\mathbf{1}_{\{X_n=x\}}\sum_{e:|e|=1}\omega(x,x+e)e\cdot\ell\right)\right)}{E\left(g(x,0,\omega)E^x_\omega\left(\sum_{n=0}^{\tau_{U^c}}\mathbf{1}_{\{X_n=x\}}\right)\right)}\,.
\end{aligned}$$

Under P^x_ω, the process X_n is a Markov chain with Geometric($\sum_{e:|e|=1}\omega(x,x+e)g(x,x+e,\omega)$) number of visits at x. The last equality and the Markov property then imply

$$\begin{aligned}
&\sum_{|e|=1}(\ell\cdot e)\hat{P}_U(x,x+e)\\
&\quad= \frac{E\left(g(x,0,\omega)\frac{\sum_{e:|e|=1}\omega(x,x+e)e\cdot\ell}{\sum_{e:|e|=1}\omega(x,x+e)g(x,x+e,\omega)}\right)}{E\left(g(x,0,\omega)\frac{1}{\sum_{e:|e|=1}\omega(x,x+e)g(x,x+e,\omega)}\right)}\\
&\quad= \frac{E\left(\frac{\sum_{e:|e|=1}\omega(x,e)\ell\cdot e}{\sum_{e:|e|=1}\omega(x,e)g(x,x+e,\omega)/g(x,0,\omega)}\right)}{E\left(\frac{1}{\sum_{e:|e|=1}\omega(x,e)g(x,x+e,\omega)/g(x,0,\omega)}\right)}\\
&\quad\geq \inf_{f\in\mathcal{F}}\frac{E\left(\frac{\sum_{e:|e|=1}\omega(x,e)\ell\cdot e}{\sum_{e:|e|=1}\omega(x,e)f(e)}\right)}{E\left(\frac{1}{\sum_{e:|e|=1}\omega(x,e)f(e)}\right)} = \inf_{f\in\mathcal{F}}\frac{E\left(\frac{\sum_{e:|e|=1}\omega(0,e)\ell\cdot e}{\sum_{e:|e|=1}\omega(0,e)f(e)}\right)}{E\left(\frac{1}{\sum_{e:|e|=1}\omega(0,e)f(e)}\right)} > 0\,,
\end{aligned}$$

where the first inequality is due to the independence of $g(x,x+e,\omega)/g(x,0,\omega)$ in ω_x. $\square$

An easy corollary is the following

Corollary 3.2.17 *If either*
(a) $\mathrm{supp}(P_d) \subset \{z \in \mathbb{R}^d : \ell \cdot z \geq 0\}$ *but* $\mathrm{supp}(P_d) \not\subset \{z \in \mathbb{R}^d : \ell \cdot z = 0\}$,
or
(b) $E((\sum_{e:|e|=1} \omega(0,e)e \cdot \ell)_+) > \frac{1}{\varepsilon} E((\sum_{e:|e|=1} \omega(0,e)e \cdot \ell)_-)$,
then Kalikow's condition with respect to ℓ *holds.*

In particular, non-nestling walks or walks with drift "either neutral or pointing to the right" satisfy Kalikow's condition with respect to an appropriate hyperplane ℓ. Further there exist truly nestling walks which do satisfy Kalikow's condition.

Remark: It is interesting to note that when P is elliptic and i.i.d. and $d = 1$, Kalikow's condition is equivalent to $v \neq 0$, i.e. to the walk being "ballistic". For $d > 1$, it is not clear yet whether there exist walks with $\mathbb{P}^o(A_\ell) > 0$ but with zero speed. Such walks, if they exist, necessarily cannot satisfy Kalikow's condition.

Our next goal is to provide tail estimates on $X_{\tau_1} \cdot \ell$ and on τ_1. Our emphasis here is in providing a (relatively) simple proof and not the sharpest possible result. For the latter we refer to [48]. In this spirit we continue to assume that $\ell = e_1$.

Lemma 3.2.18 *Assume* P *satisfies Kalikow's condition. Then, there exists a constant* c *such that*

$$\mathbb{E}^o(\exp c\, X_{\tau_1} \cdot \ell) < \infty\,.$$

Proof. Write

$$\begin{aligned}
\mathbb{E}^o(\exp(c\, X_{\tau_1} \cdot \ell)) &= \sum_{k \geq 1} \mathbb{E}^o\Big(\exp(c\, X_{\tau_1} \cdot \ell) \mathbf{1}_{\{K=k\}}\Big) \\
&= \sum_{k\geq 1} \sum_{x \in \mathbb{Z}^d} e^{c\,x\cdot\ell} E\Big(E^o_\omega\left(\mathbf{1}_{\{X_{\overline{S}_k} = x\}} \mathbf{1}_{\{\overline{S}_k < \infty\}}\right) \cdot P^x_\omega(D = \infty)\Big) \\
&= \mathbb{P}^o(D=\infty) E\left(\sum_{k\geq 1} \sum_{x\in\mathbb{Z}^d} e^{cx\cdot\ell} E^o_\omega\Big(\mathbf{1}_{\{X_{\overline{S}_k}=x\}} \mathbf{1}_{\{\overline{S}_k<\infty\}}\Big)\right) \\
&= \mathbb{P}^o(D=\infty) \sum_{k\geq 1} \mathbb{E}^o\Big(\exp(c\, X_{\overline{S}_k} \cdot \ell) \mathbf{1}_{\{\overline{S}_k < \infty\}}\Big)\,. \qquad (3.2.19)
\end{aligned}$$

But, using the Markov property,

$$\begin{aligned}
&\mathbb{E}^o\Big(\exp(c\, X_{\overline{S}_k} \cdot \ell)\ \mathbf{1}_{\{\overline{S}_k<\infty\}}\Big) \\
&\qquad \leq \mathbb{E}^o\Big(\exp(c\, X_{\overline{S}_{k-l}} \cdot \ell)\ \mathbf{1}_{\{\overline{S}_{k-1}<\infty\}}\Big) \mathbb{E}^o\Big(\exp(c\, M_0 \cdot \ell)\ \mathbf{1}_{\{D<\infty\}}\Big)\,.
\end{aligned}$$

Hence,

$$\mathbb{E}^o\Big(\exp(c\, X_{\overline{S}_k} \cdot \ell) \mathbf{1}_{\{\overline{S}_k<\infty\}}\Big) \leq \mathbb{P}^o(D=\infty) \sum_{k\geq 0} \Big(\mathbb{E}^o(\exp(c\, M_0 \cdot \ell) \mathbf{1}_{\{D<\infty\}}\Big)^k\,.$$

Note, using $\ell = e_1$, that

$$
\begin{aligned}
&\mathbb{E}^o\Big(e^{cM_0\cdot\ell}\mathbf{1}_{\{D<\infty\}}\Big)\\
&\quad=\sum_{k=1}^{\infty} e^{ck}\mathbb{E}^o\Big(\mathbf{1}_{\{M_0\cdot\ell=k\}}\mathbf{1}_{\{D<\infty\}}\Big)\\
&\quad=\sum_{k=1}^{\infty} e^{ck}\sum_{y\in\mathbb{Z}^{d-1}}\mathbb{E}^o\Big(E^o_\omega\Big(\mathbf{1}_{\{X_{T_k}=(k,y)\}}E^{(k,y)}_\omega(T_0<T_{k+1})\Big)\,. \qquad (3.2.20)
\end{aligned}
$$

Using Kalikow's condition and a computation as in Theorem 3.2.11, one has that for some $c_1 > 0$, $\mathbb{P}^o\left(\sum_{|y|>\frac{2k}{\varepsilon_\ell}}\mathbf{1}_{\{X_{T_k}=(k,y)\}}\right) \le e^{-c_1k}$, while $\mathbb{P}^o(T_{-k} < T_1) \le e^{-c_1k}$. Hence, substituting in (3.2.20), one has

$$
\mathbb{E}^o\Big(e^{cM_0\cdot\ell}\mathbf{1}_{\{D<\infty\}}\Big) \le \sum_{k=1}^{\infty} e^{ck}\left(\left(\frac{4k}{\varepsilon_\ell}\right)^{d-1}+1\right)e^{-c_1k}\,.
$$

Taking $c < c_1$, the lemma follows. □

A direct consequence of Lemma 3.2.18 is that

$$
\limsup_{n\to\infty}\frac{1}{n}\,\log\mathbb{P}^o(X_{\tau_1}\cdot\ell > vn) \le -\beta(v) \qquad (3.2.21)
$$

where $\beta(v) > 0$ for $v > 0$.

With a proof very similar to that of Lemma 3.2.18, we have in fact the

Lemma 3.2.22 *Assume P satisfies Kalikow's condition. Set $X^* = \sup_{0\le n\le\tau_1}|X_n|$. Then, there exists a constant c' such that*

$$
\mathbb{E}^o(\exp c'X^*) < \infty\,.
$$

We next turn to obtaining tail estimates on τ_1. Here, due to the presence of "traps", one cannot in general expect exponential decay as in (3.2.21). We aim at proving the following result.

Theorem 3.2.23 (Sznitman[48]) *Assume P satisfies Kalikow's condition. Then, with $d \ge 2$, there exists an $\alpha > 1$ such that for all u large,*

$$
\mathbb{P}^o(\tau_1 > u) \le e^{-(\log u)^\alpha}\,. \qquad (3.2.24)
$$

In particular, τ_1 possesses all moments.

Proof. Recall that we take $\ell = e_1$ and, for $L > 0$, set $c_L = (-L, L) \times \left(-\frac{2L}{\varepsilon_\ell}, \frac{2L}{\varepsilon_\ell}\right)^{d-1}$. Note that, with c as in Lemma 3.2.18,

$$
\begin{aligned}
\mathbb{P}^o(\tau_1 \ge u) &\le \mathbb{P}^o\Big(\tau_1 \ge u, X_{\tau_1}\cdot\ell \le L\Big) + \mathbb{P}^o\Big(X_{\tau_1}\cdot\ell \ge L\Big)\\
&\le e^{-cL/2} + \mathbb{P}^o(\tau_L > \tau_{c_L}) + \mathbb{P}^o(\tau_{c_L} = \tau_L \ge u)
\end{aligned}
$$

where

$$\tau_L = \inf\{t : \ X_t \cdot \ell \geq L\} \text{ and}$$
$$\tau_{c_L} = \inf\{t : \ X_t \notin c_L\} \,.$$

Hence, by Kalikow's condition,

$$\mathbb{P}^o(\tau_1 \geq u) \leq e^{-\overline{c}L/2} + \mathbb{P}^o\Big(\tau_{c_L} = \tau_L \geq u\Big) \tag{3.2.25}$$

for some constant $\overline{c} > 0$.

The heart of the proof of Theorem 3.2.23 lies in the following lemma, whose proof is deferred.

Lemma 3.2.26 *There exist a $\beta < 1$ and $\xi > 1$ such that for any $c > 0$,*

$$\limsup_{L\to\infty} \frac{1}{L^\xi} \log \mathbb{P}\Big(P^o_\omega(X_{\tau_{U_L}} \cdot \ell \geq L) \leq e^{-cL^\beta}\Big) < 0\,,$$

where $U_L = \{z \in \mathbb{Z}^d : |z\cdot\ell| \leq L\}$.

Accepting Lemma 3.2.26, let us complete the proof of Theorem 3.2.23. Toward this end, set $\Delta(u) = a\log u$ with a small enough such that $\varepsilon^{\Delta(u)} > \frac{1}{u^{1/6}}$, and set $L = L(u) = (\log u)^{\overline{\alpha}}$, with $N = L/\Delta = \frac{1}{a}(\log u)^{\overline{\alpha}-1}$ and $2 - \overline{\alpha} = \beta$ where β is as in Lemma 3.2.26. Observe that

$$\begin{aligned}\mathbb{P}^o(\tau_{c_L} \geq u) &\leq \left(\frac{1}{2}\right)^{(\log u)^{\overline{\alpha}}} + P^o\left(\exists\, x_1 \in c_L, P^{x_1}_\omega\left(\tau_{c_L} > \frac{u}{(\log u)^{\overline{\alpha}}}\right) \geq \frac{1}{2}\right)\\ &=: \left(\frac{1}{2}\right)^{(\log u)^{\overline{\alpha}}} + P(\mathcal{R})\,.\end{aligned} \tag{3.2.27}$$

Note that

$$\begin{aligned}&\frac{u}{(\log u)^\alpha}\cdot P^{x_1}_\omega\Big(\tau_{c_L} > \frac{u}{(\log u)^{\overline{\alpha}}}\Big)\\ &\quad\leq E^{x_1}_\omega(\tau_{c_L}) = E^{x_1}_\omega\left(\sum_{i=1}^{\tau_{c_L}} 1\right) = E^x_\omega\left(\sum_{y\in c_L}\sum_{i=1}^{\tau_{c_L}} \mathbf{1}_{\{X_n=y\}}\right)\\ &\quad= \sum_{y\in c_L} \frac{E^x_\omega\Big(\mathbf{1}_{\{\tau_y<\tau_{c_L}\}}\Big)}{E^y_\omega\Big(\mathbf{1}_{\{\tau_y>\tau_{c_L}\}}\Big)} \leq |c_L|\frac{1}{\inf\limits_{y\in c_L} P^y_\omega\Big(\mathbf{1}_{\{\tau_y>\tau_{c_L}\}}\Big)}\,,\end{aligned}$$

with $\tau_y = \inf\{t : X_t = y\}$. Hence,

$$\inf_{y\in c_L} P^y_\omega(\tau_y > \tau_{c_L}) \leq \frac{|c_L|(\log u)^{\overline{\alpha}}}{u \inf\limits_{x_1\in c_L} P^{x_1}_\omega\left(\tau_{c_L} > \frac{u}{(\log u)^{\bar{\alpha}}}\right)}\,.$$

Hence, on $\mathcal{R}$ there exists a y with $\mathbb{P}_\omega^y(\tau_y > \tau_{c_L}) \le \frac{1}{u^{4/5}}$, for all u large enough.

Set $A_i = \{z \in \mathbb{Z}^d : z \cdot \ell = i\Delta\}$. By ellipticity (recall $\varepsilon^{\Delta(u)} > 1/u^{1/6}$), it follows that on the event $\mathcal{R}$, there exists an $i_0 \in [-N+2, N-1]$ and an $x_0 \in A_{i_0}$ such that

$$P_\omega^{x_0}\Big(\tau_{(i_0-1)\Delta} > \tau_{c_L}\Big) \le \frac{1}{\sqrt{u}} \tag{3.2.28}$$

where $\tau_{(i_0-1)\Delta} = \inf\{t : X_t \cdot \ell = (i_0-1)\Delta\}$.

Set

$$X_i = -\log \min_{z \in A_i \cap c_L} P_\omega^z\Big(\tau_{(i-1)\Delta} > \tau_{(i+1)\Delta}\Big) \ge 0\,.$$

Then, the Markov property implies that for any x with $x \cdot \ell > (i-1)\Delta$,

$$\begin{aligned} P_\omega^x\Big(\tau_{(i-1)\Delta} > \tau_{c_L}\Big) &\ge \exp\left(-\sum_{j=i}^{N-1} X_j\right) \varepsilon^{\Delta(u)} \\ &\ge \exp\left(-\sum_{j=i}^{N-1} X_j\right) \frac{1}{u^{1/6}}\,. \end{aligned}$$

Thus, using (3.2.28)

$$P(\mathcal{R}) \le P\left(\sum_{i=-N+1}^{N-1} X_j \ge \frac{\log u}{12}\right) \le 2N \sup_{-N+1 \le i \le N-1} P\left(X_i \ge \frac{\log u}{24N}\right)\,. \tag{3.2.29}$$

But, using the shift invariance of P and the definition of $\{X_i\}$,

$$2N \sup_{N+1 \le i \le N-1} P\left(X_i \ge \frac{\log u}{24N}\right) \le |c_L| P\Big(P_\omega^o(\tau_{-\Delta} > \tau_\Delta) \le e^{-(\log u)^{2-\overline{\alpha}} b}\Big)$$

for $b = \frac{a}{24}$. The theorem follows by an application of Lemma 3.2.26. $\square$

Proof of Lemma 3.2.26 The interesting aspect in proving the lemma is the fact that one constructs *lower* bounds on $P_\omega^o(X_{\tau_{u_L}} \cdot \ell \ge L)$ for many configurations. Toward this end, fix $1 > \beta > \overline{\beta}$, $\gamma \in \left(\frac{1}{2}, 1\right)$, $\chi = \frac{1-\overline{\beta}}{1-\gamma} < \overline{\beta} < 1$ such that $d(\overline{\beta} - \chi) > 1$ (for $\overline{\beta}$ close enough to 1, one may always find a γ close to 1 such that this condition is satisfied, if $d \ge 2$). Set next $L_0 = L^\chi$, $L_1 = L^{\overline{\beta}}$, $N_0 = L_1/L_0$ (for simplicity, assume that L_0, L_1, L_0^γ and N_0 are all integer).

Let R be a rotation of $\mathbb{Z}^d$ such that $\tilde{R}(\ell) = \tilde{R}(e_1) = \frac{v}{|v|}$, and define

$$\begin{aligned} B_1(z) &= \tilde{R}\Big(z + [0, L_0]^d\Big) \cap \mathbb{Z}^d \\ B_2(z) &= \tilde{R}\Big(z + [-L_0^\gamma, L_0 + L_0^\gamma]^d\Big) \cap \mathbb{Z}^d \end{aligned}$$

and $\partial_+ B_2(z) = \partial B_2(z) \cap \left\{x : \ x \cdot \frac{v}{|v|} \geq L_0 + L_0^\gamma\right\}$.

We say that $z \in L_0\mathbb{Z}^d$ is *good* if $\sup_{x \in B_1(z)} P_\omega^x(X_{\tau_{B_2(z)}} \notin \partial_+ B_2(z)) \leq \frac{1}{2}$ and say that it is *bad* otherwise. The following estimate is a direct consequence of Kalikow's condition and Lemma 3.2.22 (and would be immediate if $v/|v| = \ell$).

Lemma 3.2.30 *For $\gamma \in (1/2, 1)$,*

$$\limsup_{L_0 \to \infty} L_0^{1-2\gamma} \log P(0 \text{ is bad }) < 0 \,.$$

Proof of Lemma 3.2.30

Set $u = \max\{\overline{u} : \sup_{x \in B_2(0)} x \cdot \ell \geq \overline{u}\}$ and set $L_u = \sup\{n \geq 0 : X_n \cdot \ell \leq u\}$. Define $\pi(z) = z - \frac{z \cdot v}{|v|^2} v$. Setting $K_n = \sup\{k \geq 0 : \ \tau_k < n\}$, it holds that $n \leq L_u \Rightarrow K_n \leq u$. Setting $w \in \mathbb{R}^d$ with $w \cdot v = 0$, and $|w|_1 = 1$ one has

$$\begin{aligned} X_n \cdot w &= X_{\tau_{K_n}} \cdot w + (X_n - X_{\tau_{K_n}}) \cdot w \\ &\leq X_{\tau_{K_n}} \cdot w + X^* \circ \theta_{\tau_{K_n}} \,. \end{aligned}$$

Hence,

$$\begin{aligned} &\mathbb{P}^o\Big(\sup_{0 \leq n \leq L_u} X_n \cdot w \geq u^\gamma\Big) \leq \sum_{0 \leq k \leq u} \mathbb{P}^o\Big(X_{\tau_k} \cdot w + X^* \circ \theta_{\tau_k} > u^\gamma\Big) \\ &\leq \sum_{0 \leq k \leq u} \mathbb{P}^o\left(\left(X_{\tau_k} - X_{\tau_1}\right) \cdot w > \frac{u^\gamma}{3}\right) + u\, \mathbb{P}^o\left(X_{\tau_1} \cdot w > \frac{u^\gamma}{3}\right) \\ &\quad + u\, \mathbb{P}^o\left(X^* > \frac{u^\gamma}{3} | D = \infty\right) \\ &\leq \sum_{0 \leq k \leq u} \mathbb{P}^o\left(\left(X_{\tau_k} - X_{\tau_1}\right) \cdot w > \frac{u^\gamma}{3}\right) + \frac{2u}{\mathbb{P}^o(D = \infty)} \mathbb{P}^o\left(X^* > \frac{u^\gamma}{3}\right) \,. \end{aligned}$$

Note that by Lemma 3.2.22, $\mathbb{P}^o\left(X^* > \frac{u^\gamma}{3}\right) \leq e^{-c_0 u^\gamma}$ while the random variables $(X_{\tau_{i+1}} - X_{\tau_i}) \cdot w$ are i.i.d., of zero mean and finite exponential moments. In particular,

$$\mathbb{P}^o\left(\left|\frac{(X_{\tau_k} - X_{\tau_1}) \cdot w}{k}\right| > \frac{u^\gamma}{3k}\right) \leq e^{-c_0 k \frac{u^{2\gamma}}{9k^2}} \leq e^{-c_0 u^{2\gamma - 1}}$$

by moderate deviations (see e.g., [16, Section 3.7]).

Since $\gamma > 2\gamma - 1$, we conclude that

$$\limsup_{u \to \infty} \frac{1}{u^{2\gamma - 1}} \log \mathbb{P}^o\left(\sup_{0 \leq n \leq L_u} X_n \cdot w \geq u^\gamma\right) < 0$$

and hence

$$\limsup_{u\to\infty} \frac{1}{u^{2\gamma-1}} \log \mathbb{P}^o \left(\sup_{0\le n\le L_u} |\pi(X_n)| \ge u^\gamma \right) < 0 \,. \tag{3.2.31}$$

Fix now $x \in B_1(0)$. Then, for some $c = c(d)$,

$$\mathbb{P}^x \Big(X_{\tau_{B_2(0)}} \notin \partial_+ B_2(0) \Big) \le \mathbb{P}^o \left(\sup_{0\le n\le L_{cu}} \pi(X_n \cdot w) \ge u^\gamma \right) + \mathbb{P}^o \Big(X_{\tau_{V_u}} \cdot \ell < 0 \Big)$$

where $V_u = \left\{ z : \frac{-L_0^\gamma}{c} \le z \cdot \ell \le c L_0^\gamma \right\}$ and the conclusion follows from (3.2.31) and Kalikow's condition. □

Construct now the following subsets of U_L:

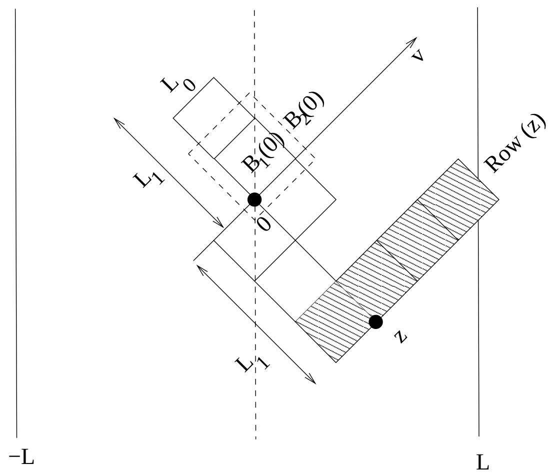

Fig. 3.2.1.

Set $M = \left\{ z \in L_0\mathbb{Z}^d, z = (0, \overline{z}), \overline{z} \in \left\{ -\frac{L_1}{L_0}, \cdots, 0, \frac{L_1}{L_0} \right\}^{d-1} L_0 \right\}$. For $z \in M$, set $\mathrm{Row}(z) = \cup_{j=N_-(z)}^{N_+(z)} B_1(z + jL_0\ell)$ where

$$N_-(z) = \min\{ j : B_1(z + jL_0\ell) \cap \{x :\ x \cdot \ell \ge 0\} \ne \emptyset \} \ge -c\frac{L}{L_0}$$
$$N_+(z) = \max\{ j : B_1(z + jL_0\ell) \cap \{x :\ x \cdot \ell \ge 0\} \ne \emptyset \} \le c\frac{L}{L_0}$$

for some constant c depending on v and uniformly bounded, and set $T = \cup_{j\in\left\{-\frac{L_1}{L_0},\cdots,0,\cdots,\frac{L_1}{L_0}\right\}^{d-1}} \{jL_0\}$.

The idea behind the proof is that if one of the rows $\{R(z)\}_{z\in M}$, say $R(z_0)$, contains mostly good blocks, a good strategy for the event $(X_{\tau_{u_L}} \cdot \ell) \geq L$ is to force the walker started at 0 to first move to z, then move to the right successively without leaving $\cup_{z\in \mathrm{Row}(z_0)} B_2(z)$ until exiting from u_L. More precisely, let $N(z_0)$ denote the number of bad blocks in $\cup_{z\in \mathrm{Row}(z_0)} B_1(z)$. Then, for some constants c_i, using ellipticity and the definition of good boxes,

$$\begin{aligned} P^o_\omega(X_{\tau_{u_L}\cdot\ell} \geq L) &\geq \varepsilon^{L_1}\left(\frac{1}{2}\,\varepsilon^{2(d-1)L_0^\gamma}\right)^{cL/L_0}(\varepsilon^{L_0})^{N(z_0)} \\ &= e^{-c_4(L^{\overline{\beta}}+L^{\chi}N(z_0))}\,. \end{aligned}$$

Hence, for an arbitrary constant c_6 and all L large enough ($L > g(c_6, c)$ for some fixed function $g(\cdot)$),

$$\begin{aligned} P\left(P^o_\omega\Big(X_{\tau_{u_L}\cdot\ell} \geq L\Big) \leq e^{-cL^\beta}\right) &\leq P\Big(\{\nexists z_0 \in M:\ N(z_0) \leq c_6 L^{\overline{\beta}-\chi}\}\Big) \\ &\leq \Big[P\Big(N(0) \geq c_6 L^{\overline{\beta}-\chi}\Big)\Big]^{\left(\frac{cL_1}{L_0}\right)^{(d-1)}} \end{aligned}$$

using the independence between even rows in Figure 3.2.1. But note that $N(0) = \sum_{i=1}^{L/2L_0}(Y_i + Z_i)$ where $\{Y_i\}$ are i.i.d., $\{Z_i\}$ are i.i.d., $\{0,1\}$ valued, $P(Y_i = 1) = P(B_1(0))$ is a bad block (the division to (Y_i, Z_i) reflects the division to even and odd blocks, which creates independence). Hence,

$$P\Big(N(0) \geq c_6 L^{\overline{\beta}-x}\Big) \leq 2P\left(\sum_{i=1}^{L/2L_0} Y_i \geq \frac{c_6}{2} L^{\overline{\beta}-\chi}\right).$$

Since, by Lemma 3.2.30,

$$\log E(e^{Y_i}) \leq \log\Big(1 + e^{-c_7 L_0^{2\gamma-1}}\Big) \leq e^{-c_7 L_0^{2\gamma-1}}\,,$$

we conclude from the independence of the Y_i's that

$$\begin{aligned} P\Big(N(0) \geq c_6 L^{\overline{\beta}-\chi}\Big) &\leq 2e^{-\frac{c_6}{2}L^{\overline{\beta}-\chi}} e^{e^{-c_7 L_0^{2\gamma-1}}\cdot\frac{L}{2L_0}} \\ &\leq e^{-c_8 L^{\overline{\beta}-\chi}}\,. \end{aligned}$$

Hence,

$$P\left(P^o_\omega\Big(X_{\tau_{u_L}\cdot\ell} \geq L\Big) \leq e^{-cL^\beta}\right) \leq e^{-c_9 L^{(\overline{\beta}-\chi)d}} \leq e^{-L^\xi}$$

for some $\xi > 1$, as claimed. □

Remark The restriction to $d \geq 2$ in Theorem 3.2.23 is essential: as we have seen in the case $d = 1$, one may have ballistic walks (and hence, in $d = 1$, satisfying Kalikow's condition) with moments $m_r := \mathbb{E}^o(\tau_1^r)$ of the regeneration time τ_1 being finite only for small enough $r > 1$.

3.3 Sznitman's conditions T,T'

Unlike $d = 1$, all attempts to build explicit traps that slow down the particle to a sub-diffusive scale quickly fail. One thus suspects that a good control of trapping properties will lead to an analysis of the RWRE.

With this motivation in mind, we follow Sznitman in introducing some condition on the environment that will eventually lead to a good understanding of the ballistic regime. Fix a direction $\ell \in S^{d-1}$, and for $b > 0$, define the region $U_{\ell,b,L} = \{x \in \mathbb{Z}^d : x \cdot \ell \in (-bL, L)\}$. Let $T_{\ell,b,L} = \min\{n > 0 : X_n \notin U_{\ell,b,L}\}$.

Definition 1. *Let $\gamma \in (0,1)$ be given. Then, P satisfies condition T_γ relative to ℓ if for all ℓ' in some neighborhood of ℓ, and all $b > 0$,*

$$\limsup_{L\to\infty} \frac{1}{L^\gamma} \log \mathbb{P}\left(X_{T_{\ell,b,L}} \cdot \ell < 0\right) < 0\,. \tag{3.3.32}$$

P satisfies condition T' relative to ℓ if it satisfies condition T_γ relative to ℓ for all $\gamma \in (0,1)$. It satisfies condition T relative to ℓ if it satisfies condition T_1 relative to ℓ.

In words, condition T relative to ℓ holds if the exit from a slab that is contained between two hyperplanes perpendicular to ℓ, located respectively at distance $+L$ in the ℓ direction and $-bL$ in the opposite direction, occurs through the "backward" direction with probability that is exponentially small in L. Condition T' relaxes the exponential decay to "almost" exponential decay (there is an alternative description of condition T' in terms of regeneration distances, see Proposition 3.3.33 below). The power of condition T' is the following.

Theorem 1 (Sznitman [49]). *Assume P is i.i.d. and uniformly elliptic, and that condition T' relative to some direction ℓ holds. Then, regeneration times with respect to direction ℓ possess all moments. In particular, the process (X_n) is ballistic, i.e. $X_n/n \to v \neq 0$ for some deterministic v with $v \cdot \ell > 0$, and there is a deterministic $\sigma^2 > 0$ such that, under the annealed measure $\mathbb{P}$, $(X_n - nv)/\sigma\sqrt{n}$ converges in distribution to a standard Gaussian random variable.*

(The convergence in distribution in Theorem 1 actually extends to an invariance principle. As we saw in Theorem 3.1.3, this also implies a quenched CLT.)

A simple martingale argument implies that condition T (and hence T') holds for a certain direction ℓ when the environment is non-nestling; they also hold under Kalikow's condition; However as shown in [51], there are examples where Sznitman's T' condition holds but Kalikow's example does not. (In [50], a criterion is given for checking whether T' holds, even if this criterion can hardly be called easy to compute.) The usefulness of condition T' is in the following result from [49].

Proposition 3.3.33 *Let $d \geq 2$, $\ell \in S^{d-1}$, and $\gamma \in (0,1]$. The following are equivalent:*
(a) Condition T_γ holds.
(b) $\mathbb{P}(A_\ell^+) = 1$ and, with $X^* := \sup_{0 \leq n \leq \tau_1} |X_n|$, there exists a $c > 0$ such that

$$\mathbb{E}\left(\exp(c(X^*)^\gamma)\right) < \infty\,.$$

The proof is detailed in [49], see also the exposition in [47]. From part (b) of Proposition 3.3.33, tail estimates on τ_1 of the form (3.2.24) follow, by a route similar to that described above when discussing Kalikow's condition. We omit further details.

Remark 3.3.34 *Regeneration times do not have to involve "non-bactracking times" with respect to a hyperplane. For walks where transitions in some dimensions are not random (some in this context means "more than 4"), regeneration can be provided using "cut times" for the components of the walk that are non-random. One then recovers conditional independence (though not identically distributed), and both LLN's and CLT's. Details are provided in [7].*

4 Lecture 4. Homogenization for balanced environments

So far we have not discussed homogenization techniques. In this lecture we pick up this topic by looking at a particular case where the method can be made to work. It has been challenging, so far, to extend this to more general situations.

4.1 Setup and environment viewed from the point of view of the particle

We begin by introducing the setup. We say that an environment is *balanced* if $P(\omega(0,e) = \omega(0,-e)) = 1$. For balanced environments, the processes $(X_n(i))_{n\geq 0}, i = 1, \cdots, d$, are martingales, with, denoting $\mathcal{F}_n = \sigma(X_0, \cdots X_n)$,

$$E_\omega^o((X_n(i) - X_{n-1}(i))(X_n(j) - X_{n-1}(j))|\mathcal{F}_{n-1}) = 2\delta_{ij}\omega(X_{n-1}, X_{n-1} + e_i)\,.$$

Since $|\omega(\cdot,\cdot)| \leq 1$ P-a.s., it immediately follows that $X_n/n \underset{n\to\infty}{\longrightarrow} 0$, $\mathbb{P}^o$-a.s. Further, the multi-dimensional CLT yields that if there exists a deterministic vector $\mathbf{a} = (a_1, \cdots, a_d)$ such that

$$\frac{1}{n}\sum_{k=1}^{n} \omega(X_{k-1}, X_{k-1} + e_i) \underset{n\to\infty}{\longrightarrow} \frac{a_i}{2} > 0, \quad \mathbb{P}^o\text{-a.s.}, \tag{4.1.1}$$

then, for any bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$, and any $y \in \mathbb{R}$,

$$\lim_{n\to\infty} P^o_\omega \left(f\left(\frac{X_n}{\sqrt{n}}\right) \le y \right) \tag{4.1.2}$$
$$= \frac{1}{(2\pi)^{d/2}\prod_{i=1}^d \sqrt{a_i}} \int_{\mathbb{R}^d} \mathbf{1}_{\{f(\mathbf{x})\le y\}} \exp\left(-\sum_{i=1}^d \frac{x_i^2}{2a_i}\right) \prod_{i=1}^d dx_i, \quad P\text{-a.s.}$$

Thus, a QCLT (or even invariance principle) follows if an appropriate ergodic theorem can be proved.

Remark 4.1.3 *The importance of the balanced environment is precisely due to the condition $E^0_\omega(X_n) = 0$. Exactly as for $d = 1$, when this is not the case then the fluctuations of the random centering may (in principle) be large and dominate limit laws. Because of that and the lack of reveresibility in the RWRE model for $d \ge 2$, the technique has so far not been useful in handling general RWRE. For $d = 1$, it can be succesfully applied, see [53].*

Theorem 4.1.4 (Lawler [34]) *Assume P ergodic, uniformly elliptic and balanced. Then, there exists a deterministic vector $\mathbf{a}$ such that (4.1.1) holds. Consequently, the quenched invariance principle holds.*

The proof of Theorem 4.1.4 that we present follows the original outline of [34], with some modifications introduced in [53] and especially [26]; In particular, we will only assume that P is ergodic, not necessarily i.i.d., and we will make explicit the dependence of estimates on the ellipticity constant (in fact, we prove Theorem 4.1.4 under the assumption 4.1.19). The proof involves some analytic machinery, but we first introduce an important ingredient. Define $\overline{\omega}(n) = \theta^{X_n}\omega$, and the transition kernel

$$M(\omega, d\omega') = \sum_{e_i} \Big[\omega(0,e_i)\delta_{\theta^{e_i}\omega=\omega'} + \omega(0,-e_i)\delta_{\theta^{-e_i}\omega=\omega'} \Big]. \tag{4.1.5}$$

It is immediate to check that $\overline{\omega}(n)$ is a Markov process, called for obvious reasons *the environment viewed from the point of view of the particle.* We have the following lemma.

Lemma 4.1.6 *Suppose a measure Q which is invariant under M and satisfies $Q \sim P$ exists. Then, under the law induced by $Q \otimes P^o_\omega$, the sequence $\{\overline{\omega}(n)\}$ is stationary and ergodic.*

Proof. The stationarity follows from the stationarity of $\overline{Q}$. Let $\overline{\theta}$ denote the shift on $\overline{\Omega} = \Omega^{\mathbb{N}}$, that is, for $\overline{\omega} \in \overline{\Omega}, \overline{\theta}\overline{\omega}(n) = \overline{\omega}(n+1)$. Denote by $\overline{P}_\omega$ the law of the sequence $\{\overline{\omega}(n)\}$ with $\overline{\omega}(0) = \omega$, that is, for any measurable sets $B_i \subset \Omega$,

$$\overline{P}_\omega\Big(\overline{\omega}(i) \in B_i, i = 1,\ldots,\ell\Big)$$
$$= \int_{B_1} \cdots \int_{B_\ell} M(\omega, d\omega^1)M(\omega^1, d\omega^2)\cdots M(\omega^{\ell-1}, d\omega^\ell)$$

and set $\overline{\mathcal{Q}} = Q \otimes \overline{P}_\omega$ (as usual, we also use $\overline{\mathcal{Q}}$ to denote the corresponding marginal induced on $\overline{\Omega}$).

We need to show that for any invariant A, that is $A \in \overline{\Omega}$ such that $\overline{\theta}^{-1} A = A$, $\overline{\mathcal{Q}}(A) \in \{0,1\}$. Set $\varphi(\omega) = \overline{P}_\omega(A)$, we claim that $\{\varphi(\overline{\omega}(n))\}$ is a martingale with respect to the filtration $\mathcal{G}_n = \sigma(\overline{\omega}(0), \ldots, \overline{\omega}(n))$: indeed,

$$\varphi(\overline{\omega}(n)) = \overline{P}_{\overline{\omega}(n)}(A) = E_{\overline{\mathcal{Q}}}\Big(\mathbf{1}_{\overline{\theta}^{-n}A} | \mathcal{G}_n\Big) = E_{\overline{\mathcal{Q}}}\Big(\mathbf{1}_A | \mathcal{G}_n\Big)\,,$$

where the second equality is due to the Markov property and the third due to the invariance of A. Hence, by the martingale convergence theorem,

$$\varphi(\overline{\omega}(n)) \underset{n\to\infty}{\longrightarrow} \mathbf{1}_A, \quad \overline{\mathcal{Q}}\text{-a.s.} \tag{4.1.7}$$

Further, $Q(\varphi(\omega) \notin \{0,1\}) = 0$ because otherwise there exists an interval $[a,b]$ with $\{0\}, \{1\} \notin [a,b]$ and $Q(\varphi(\omega) \in [a,b]) > 0$, while

$$\frac{1}{n} \sum_0^{n-1} \mathbf{1}_{\{\varphi(\overline{\omega}(n)) \in [a,b]\}} \to E_{\overline{\mathcal{Q}}}\Big(\mathbf{1}_{\{\varphi(\overline{\omega}(0)) \in [a,b]\}} | \mathcal{I}\Big)\,, \tag{4.1.8}$$

where $\mathcal{I}$ is the invariant σ-field.

Taking expectations in (4.1.8) and using (4.1.7), one concludes that

$$0 = \overline{\mathcal{Q}}\Big(\varphi(\overline{\omega}(0)) \in [a,b]\Big) = Q\Big(\varphi(\omega) \in [a,b]\Big),$$

a contradiction. Thus for some measurable $B \subset \Omega$, $\varphi(\omega) = \mathbf{1}_B\,, Q-$ a.s.. Further, the Markov property and invariance of A yield that $M\mathbf{1}_B = \mathbf{1}_B$, Q-a.s. and hence P-a.s. But then,

$$\mathbf{1}_B = M\mathbf{1}_B \geq \omega(0,e)\mathbf{1}_{\theta^e B}, P\text{-a.s.}\,.$$

Since $P(\omega(0,e) = 0) = 0$, it follows that $\mathbf{1}_B \geq \mathbf{1}_{\theta^e B}$, P-a.s., for any e with $|e| = 1$, and then $E_P(\mathbf{1}_B) = E_P(\mathbf{1}_{\theta^e B})$ implies that $\mathbf{1}_B = \mathbf{1}_{\theta^e B}$, P-a.s. But then, by ergodicity of P, $P(B) \in \{0,1\}$, and hence $Q(B) \in \{0,1\}$. Since $\overline{\mathcal{Q}}(A) = E_Q\varphi(\omega) = Q(B)$, the conclusion follows. □

It thus follows that the assumptions of Lemma 4.1.6 imply that

$$\frac{1}{n} \sum_{i=1}^n \omega(X_{n-1}, X_{n-1} + e_i) = \frac{1}{n} \sum_{i=1}^n \overline{\omega}(n)(0, e_i) \underset{n\to\infty}{\longrightarrow} \frac{a_i}{2}$$
$$:= E_Q \overline{\omega}(0, e_i) \mathbb{P}^o\text{-a.s.}, \tag{4.1.9}$$

yielding (4.1.1) (and ellipticity implies the non-degeneracy of the limiting covariance). Our effort therefore is directed towards the construction of such a measure.

Remark 4.1.10 *For transient RWRE in dimension $d = 1$, such measures can be constructed explicitely, and this provides an alternative to the proofs of the LLN and ACLT that we provided. We refer again to [53] for details.*

Let $\Delta_N(x_0) = \{x \in \mathbb{Z}^d : |x - x_0|_\infty \le N\}$ be the cube centered at x_0 of length $2N$. Let $\Delta_N = \Delta_N(0)$. For any $x \in \mathbb{Z}^d$, set

$$\bar{x} := x + (2N+1)\mathbb{Z}^d \in \mathbb{Z}^d/(2N+1)\mathbb{Z}^d.$$

For any fixed $\omega \in \Omega$, we define ω^N by setting $\omega^N(x) = \omega(x)$ for $x \in \Delta_N$ and $\omega^N(y) = \omega^N(x)$ for $y \in \mathbb{Z}^d$ whenever $\bar{y} = \bar{x}$. Let $\Omega^N = \{\omega^N : \omega \in \Omega\}$. Let $\{X_{n,N}\}$ denote the random walk on $\mathbb{Z}^d$ in the environment ω^N. Then $\{\bar{X}_{n,N}\}$ is an irreducible finite-state Markov chain with a unique invariant probability measure of the form

$$\mu_N = \frac{1}{(2N+1)^d} \sum_{x \in \Delta_N} \Phi_N(x)\delta_{\bar{x}}.$$

Also, $\{\bar{\omega}^N(n)\}$ is an irreducible finite-state Markov chain on Ω^N with an invariant probability measure (with respect to the kernel M)

$$Q_N = Q_{N,\omega} = \frac{1}{(2N+1)^d} \sum_{x \in \Delta_N} \Phi_N(x)\delta_{\theta^x\omega^N}.$$

We will show that Q_N converges weakly to some measure Q with good properties. To do this, we first introduce a sequence of measures

$$P_N = P_{N,\omega} = \frac{1}{(2N+1)^d} \sum_{x \in \Delta_N} \delta_{\theta^x\omega^N}$$

which by the multidimensional ergodic theorem converges weakly to P, P- $a.s.$

Let $\{\omega_\gamma^N\}_{\gamma=1}^k$ denote the set of distinct states in $\{\theta^x\omega^N\}_{x\in\Delta_N}$ and $C_N(\gamma) := \{x \in \Delta_N : \theta^x\omega^N = \omega_\gamma^N\}$. Set, for any finite subset $E \subset \mathbb{Z}^d$,

$$\|f\|_{E,j} := (|E|^{-1} \sum_{x\in E} |f(x)|^j)^{\frac{1}{j}}.$$

Since $\mathrm{d}Q_N/\mathrm{d}P_N = \sum_{\gamma=1}^k \delta_{\omega_\gamma^N} |C_N(\gamma)|^{-1} \sum_{x\in C_N(\gamma)} \Phi_N(x) := f_N$, we have that for any measurable function g on Ω,

$$\begin{aligned} |Q_N g| &\le (\int f_N^\alpha \,\mathrm{d}P_N)^{\frac{1}{\alpha}} (\int |g|^{\alpha'} \,\mathrm{d}P_N)^{\frac{1}{\alpha'}} \\ &\le \big(\frac{1}{|\Delta_N|} \sum_{\gamma=1}^k \sum_{x\in C_N(\gamma)} \Phi_N(x)^\alpha\big)^{\frac{1}{\alpha}} (\int |g|^{\alpha'} \,\mathrm{d}P_N)^{\frac{1}{\alpha'}} \\ &= \|\Phi_N\|_{\Delta_N,\alpha} (P_N|g|^{\alpha'})^{\frac{1}{\alpha'}}, \end{aligned} \tag{4.1.11}$$

where α' is the Hölder conjugate of α, $1/\alpha + 1/\alpha' = 1$, and we used Hölder's inequality in the first and the second inequalities. Since Ω is compact with respect to the product topology, along some subsequence $N_k \to \infty$, $\{Q_{N_k}\}$ converges weakly to a limit, denoted Q. Assume for the moment that

$$\overline{\lim_{N\to\infty}} \|\Phi_N\|_{\Delta_N,\alpha} \le C, \quad P\text{- } a.s.. \tag{4.1.12}$$

We show that then, for a.e. $\omega \in \Omega$,

$$Q \sim P. \tag{4.1.13}$$

Indeed, let $A \subset \Omega$ be measurable. Let ρ denote a metric on the Polish space Ω. For any closed subset $F \subset A$, $\delta > 0$, introduce the function $f(\omega) = [1 - \rho(\omega, F)/\delta]^+$ which is supported on $F_\delta = \{\omega \in \Omega : \rho(\omega, F) < \delta\}$. Then by (4.1.11), (4.1.12),

$$QF \le \overline{\lim_{N\to\infty}} Q_N f \le C(Pf^{\alpha'})^{\frac{1}{\alpha'}} \le C(PF_\delta)^{\frac{1}{\alpha'}}.$$

Letting $\delta \downarrow 0$, we get $QF \le C(PF)^{\frac{1}{\alpha'}}$. Taking supremums over all closed subset $F \subset A$, one concludes that $QA \le C \cdot (PA)^{\frac{1}{\alpha'}}$, which proves that $Q \ll P$.

To see the reverse implication, set $E = \{\omega : \frac{dQ}{dP} = 0\}$. By invariance, $E_Q M\mathbf{1}_E = E_Q\mathbf{1}_E = 0$, and hence $M\mathbf{1}_E \le \mathbf{1}_E$, P-a.s. But, $M\mathbf{1}_E \ge \varepsilon \sum_{i=1}^d (\mathbf{1}_E \circ \theta^{e_i} + \mathbf{1}_E \circ \theta^{-e_i})$. Hence, $\mathbf{1}_E \ge \mathbf{1}_E \circ \theta^{\pm e_i}$, P-a.s. Since P is stationary, $\mathbf{1}_E = \mathbf{1}_E \circ \theta^{\pm e_i}$, P-a.s., and hence by ergodicity (considering the invariant event $\cap_{x\in\mathbb{Z}^d}(\theta^x)^{-1}E$) $P(E) \in \{0,1\}$. But $Q \ll P$ implies $P(E) = 0$. Hence, $Q \sim P$, as claimed.

By Lemma 4.1.6 and the ergodic theorem, we therefore have that Q is uniquely determined by $Qg = \lim_{n\to\infty} E \sum_{j=0}^{n-1} g(\bar\omega_j)/n$ for every bounded measurable g, and hence Q is *the* weak limit of Q_N. Therefore, to prove Theorem 4.1.4, it suffices to prove (4.1.12). Toward this end, we introduce the analytic machinery alluded to above.

Fix an $\omega \in \Omega$. For any bounded set $E \subset \mathbb{Z}^d$, let $\partial E = \{y \in E^c : \exists x \in E, |x - y|_\infty = 1\}$, $\bar E = E \bigcup \partial E$ and $\mathrm{diam}(E) = \max\{|x-y|_\infty : x, y \in E\}$. For any function f defined on $\bar E$, let L_ω denote the operator

$$(L_\omega f)(x) = \sum_{i=1}^d \omega(x, e_i)[f(x+e_i) + f(x-e_i) - 2f(x)], \quad x \in E.$$

The following discrete maximum principle is an adaption the Alexandrov-Bakelman-Pucci PDE estimates to the discrete setup, and appear in Theorem 2.1 of [33]. We provide a proof in Section 4.2.

Theorem 4.1.14 (Maximum Principle) *Let $E \subset \mathbb{Z}^d$ be bounded, and let u be a function on $\bar E$. For all $x \in E$, assume $\varepsilon(x) := \min_i \omega(x, e_i) > 0$ and define*

$$I_u(x) := \{s \in \mathbb{R}^d : u(x) - s\cdot x \ge u(z) - s\cdot z, \forall z \in \bar E\}.$$

If $L_\omega u(x) \geq -g(x)$ for all $x \in E$ such that $I_u(x) \neq \emptyset$, then

$$\max_E u \leq C \operatorname{diam} \bar{E} \Big(\sum_{\substack{x\in E \\ I_u(x)\neq\emptyset}} |\frac{g}{\varepsilon}|^d \Big)^{\frac{1}{d}} + \max_{\partial E} u. \tag{4.1.15}$$

In particular,

$$\max_E u \leq C \operatorname{diam} \bar{E} \cdot |E|^{\frac{1}{d}} \| \frac{g}{\varepsilon} \|_{E,d} + \max_{\partial E} u.$$

Define the stopping times $\tau_0 = 0$, $\tau_1 = \tau := \min\{j \geq 1 : |X_{j,N} - X_{0,N}|_\infty > N\}$ and $\tau_{j+1} = \min\{n > \tau_j : |X_{n,N} - X_{\tau_j,N}|_\infty > N\}$.

Lemma 4.1.16 *Let ω^N, $\{X_{n,N}\}$ be as in Section 1 and τ as defined above, then there exists a constant c such that, for all N large,*

$$E^o_{\theta^x\omega^N}(1 - \frac{c}{N^2})^\tau \leq C < 1.$$

Proof. Since P is balanced, $X_{n,N}$ is a martingale and it follows from Doob's inequality that for any $K \geq 1$,

$$\begin{aligned} P^o_{\theta^x\omega^N}\{\tau \leq K\} &\leq 2\sum_{i=1}^d P^o_{\theta^x\omega^N}\{\sup_{n\leq K} X_{n,N}(i) \geq N+1\} \\ &\leq \frac{2}{N+1}\sum_{i=1}^d E^o_{\theta^x\omega^N} X_{K,N}(i)^+ \leq \frac{2d}{N+1}\sqrt{K}, \end{aligned}$$

where $X_{n,N}(i)$ is the i-th coordinate of $X_{n,N}$. Hence

$$E^o_{\theta^x\omega^N}(1 - \frac{c}{N^2})^\tau \leq (1 - \frac{c}{N^2})^K + \frac{2d}{N+1}\sqrt{K}.$$

Taking $c = 16d^2$ and $K = N^2/16d^2$, we get $E^o_{\theta^x\omega^N}(1 - \frac{c}{N^2})^\tau \leq e^{-1} + 2^{-1}$. □

Theorem 4.1.17

$$\|\Phi_N \varepsilon\|_{\Delta_N,\beta} \leq C, \tag{4.1.18}$$

where $\beta = d' = d/(d-1)$.

Proof. Let c be the same constant as in Lemma 4.1.16. For any function $h \geq 0$ on Δ_N,

$$
\begin{aligned}
&\|\Phi_N \cdot h\|_{\Delta_N,1} \\
&= \frac{c}{N^2} \sum_{x\in\Delta_N} \frac{\Phi_N(x)}{|\Delta_N|} \sum_{m\geq 0} E^x_{\omega^N} \sum_{\tau_m \leq\ j<\tau_{m+1}} (1-\frac{c}{N^2})^j h(\bar{X}_{j,N}) \\
&\leq \frac{c}{N^2} \sum_{x\in\Delta_N} \frac{\Phi_N(x)}{|\Delta_N|} \sum_{m\geq 0} E^x_{\omega^N} (1-\frac{c}{N^2})^{\tau_m} E^{\bar{X}_{\tau_m,N}}_{\omega^N} \sum_{j=0}^{\tau-1} h(\bar{X}_{j,N}) \\
&\leq \frac{c}{N^2} \sum_{x\in\Delta_N} \frac{\Phi_N(x)}{|\Delta_N|} \sum_{m\geq 0} \big[\sup_{y\in\Delta_N} E^y_{\omega^N} (1-\frac{c}{N^2})^{\tau}\big]^m \cdot \sup_{y\in\Delta_N} E^y_{\omega^N} \sum_{j=0}^{\tau-1} h(\bar{X}_{j,N}).
\end{aligned}
$$

By the maximum principle (Theorem 4.1.14),

$$
\sup_{y\in\Delta_N} E^y_{\omega^N} \sum_{j=0}^{\tau-1} h(\bar{X}_{j,N}) \leq CN^2 \|\frac{h}{\varepsilon}\|_{\Delta_N,d}.
$$

This together with Lemma 4.1.16 and $\sum_{x\in\Delta_N} \Phi_N(x)/|\Delta_N| = 1$ yield

$$
\|\Phi_N \cdot h\|_{\Delta_N,1} \leq C\|\frac{h}{\varepsilon}\|_{\Delta_N,d}.
$$

Hence by the duality of norms,

$$
\|\Phi_N \varepsilon\|_{\Delta_N,\beta} = \sup_{\|\frac{h}{\varepsilon}\|_{\Delta_N,d}=1} \|\Phi_N h\|_{\Delta_N,1} \leq C. \qquad \square
$$

Proof of (4.1.12) Instead of uniform ellipticity, assume that

$$
\mathrm{E}\varepsilon(o)^{-p} < \infty \text{ for some } p > d. \tag{4.1.19}
$$

Take $\alpha = (1-1/d+1/p)^{-1}$. We use Hölder's inequality and Theorem 4.1.17 to get

$$
\|\Phi_N\|_{\Delta_N,\alpha} \leq \|\Phi_N \varepsilon\|_{\Delta_N,\beta} \|\varepsilon^{-1}\|_{\Delta_N,p} \leq C\|\varepsilon^{-1}\|_{\Delta_N,p}.
$$

By the multidimensional ergodic theorem,

$$
\lim_{N\to\infty} \|\varepsilon^{-1}\|_{\Delta_N,p} = (E\varepsilon(o)^{-p})^{\frac{1}{p}} < \infty, \quad P\text{- a.s.}. \quad \square
$$

Remark 4.1.20 *In the i.i.d. case, one can combine the above reasoning with percolation estimates in order to prove Theorem 4.1.4 under mere ellipticity, that is no moment assumptions. The details can be found in [26].*

Remark 4.1.21 *One can show that under the assumptions of Theorem 4.1.4, the RWRE is recurrent when $d = 2$ and transient when $d \geq 3$. Details can be found in [53, Section 3.3] and [26].*

4.2 Proof of the maximum principle

Without loss of generality, assume $g \geq 0$ and

$$\max_{\bar{E}} u = u(x_0) > \max_{\partial E} u$$

for some $x_0 \in E$.

For $s \in \mathbb{R}^d$ such that $|s|_\infty \leq [u(x_0) - \max_{\partial E} u]/(d \operatorname{diam} \bar{E})$ we have $u(x_0) - u(x) \geq s \cdot (x_0 - x)$ for all $x \in \partial E$. Defining $t = \inf\{\rho \geq 0 : u(x_0) + s \cdot (x - x_0) + \rho \geq u(x)\ \forall x \in \bar{E}\}$, we get that $u(x) = u(x_0) + s \cdot (x - x_0) + t$, some $x \in E$, and then $u(x) + s \cdot (z - x) = u(x_0) + s \cdot (x - x_0) + t \geq u(z)$ for all $z \in \bar{E}$. Hence, $s \in \bigcup_{x \in E} I_u(x)$. Therefore,

$$[-\frac{u(x_0) - \max_{\partial E} u}{d \operatorname{diam} \bar{E}}, \frac{u(x_0) - \max_{\partial E} u}{d \operatorname{diam} \bar{E}}]^d \subset \bigcup_{x \in E} I_u(x). \tag{4.2.1}$$

Further, for $s \in I_u(x)$ we have $u(x + e_i) - u(x) \leq s \cdot e_i \leq u(x) - u(x - e_i)$, so

$$\bigcup_{x \in E} I_u(x) \subset \bigcup_{x \in E, I_u(x) \neq \emptyset} \prod_{i=1}^{d} [u(x + e_i) - u(x), u(x) - u(x - e_i)] \tag{4.2.2}$$

By (4.2.1), (4.2.2) we have

$$\begin{aligned}
&\Big(\frac{2u(x_0) - 2\max_{\partial E} u}{d \operatorname{diam} \bar{E}}\Big)^d \\
&\leq \sum_{x \in E, I_u(x) \neq \emptyset} \prod_{i=1}^{d} [2u(x) - u(x - e_i) - u(x + e_i)] \\
&\leq \sum_{x \in E, I_u(x) \neq \emptyset} \Big(\frac{1}{d} \sum_{i=1}^{d} \frac{\omega(x, e_i)}{\varepsilon(x)} [2u(x) - u(x - e_i) - u(x + e_i)]\Big)^d \\
&\leq \sum_{\substack{x \in E \\ I_u(x) \neq \emptyset}} |\frac{g(x)}{d\varepsilon(x)}|^d.
\end{aligned}$$

This completes the proof. □

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