RANDOM WALKS IN RANDOM ENVIRONMENT - a guided tour

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Preface

The goal of these notes is to present a guided tour of the state of knowledge concerning random walks in random environments, an active and current area of research. They borrow freely from a course I gave in St. Flour in 2001, whose (much more detailed) notes will eventually appear in print [42]. I refer to the latter notes for many detailed calculations, and put the emphasis here more on recent results and directions for further development. I have not attempted to provide here a detailed bibliography, and refer instead simply to [42].

1 Introduction

The definition of a RWRE involves two components: first, the *environment*, which is randomly chosen but kept fixed throughout the time evolution, and second, the random walk, which, given the environment, is a time homogeneous Markov chain whose transition probabilities depend on the environment. Throughout, we will be concerned only with nearest neighbour random walks on $\mathbb{Z}^d$. That is, let $\mathcal{M}^d$ denote the space of probability measures on $\{\pm e_i\}_{i=1}^d$, set $\Omega = (\mathcal{M}^d)^{\mathbb{Z}^d}$, and equip Ω with the natural Borel structure, denoted $\mathcal{F}$. An *environment* is simply a point $\omega \in \Omega$, with $x \mapsto \omega(x, \cdot) \in \mathcal{M}^d$ denoting the coordinate map (for $x \in \mathbb{Z}^d$).

We let P denote a probability measure on Ω, the law of the (random) environment. Without further mentioning it, we assume that P is a stationary and ergodic measure, and that it is *elliptic*, that is, we assume the existence of a constant $\epsilon > 0$ such that

$$P(\omega(0,e) \in [\varepsilon, 1-\varepsilon]) = 1\,, \quad \text{if } |e| = 1.$$

We turn next to define the class of random walks of interest to us. For *each* $\omega \in \Omega$, we define the *random walk in the environment* ω (RWRE) as the time-homogeneous Markov chain $\{X_n\}$ taking values in $\mathbb{Z}^d$ with transition probabilities

$$P_\omega(X_{n+1} = x + e | X_n = x) = \omega(x, x + e) .$$

We use P_ω^x to denote the law induced on $((\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{G})$ where $\mathcal{G}$ is the σ-algebra generated by cylinder functions, and

$$P_\omega^x(X_0 = x) = 1 .$$

In the sequel, we refer to $P_\omega^x(\cdot)$ as the *quenched* law of the random walk $\{X_n\}$. Note that for each $G \in \mathcal{G}$, the map

$$\omega \mapsto P_\omega^x(G)$$

is $\mathcal{F}$-measurable. Hence, we may define the measure $\mathbb{P}^x := P \otimes P_\omega^x$ on $(\Omega \times \mathbb{Z}^d)^{\mathbb{N}}, \mathcal{F} \times \mathcal{G})$ from the relation

$$\mathbb{P}^x(F \times G) = \int_F P_\omega^x(G) P(d\omega), \quad F \in \mathcal{F}, G \in \mathcal{G} . \tag{1.0.1}$$

The marginal of $\mathbb{P}^x$ on $V^{\mathbb{N}}$, denoted also $\mathbb{P}^x$ whenever no confusion occurs, is called the *annealed law* of the random walk $\{X_n\}$; note that under P_ω^x, the RWRE $\{X_n\}$ is a Markov chain but is not invariant under $\mathbb{Z}^d$-shifts, whereas under $\mathbb{P}^x$, $\{X_n\}$ is *not* a Markov chain, however it's law is invariant under $\mathbb{Z}^d$ shifts.

When $d = 1$, we write ω_x for $\omega(x, x+1)$ and set $\rho_x = (1 - \omega_x)/\omega_x$.

2 RWRE – d=1

This chapter is devoted to the study of one dimensional RWRE. We review ergodic properties, subdiffusive behaviour, and asymptotics of large deviations. Two important ingredients make the analysis significantly easier than in higher dimension: the existence of a reversing (environment depending) measure, and the fact that hitting times have a relatively simple structure.

2.1 Ergodic theorems and hitting times

Theorem 2.1.1 (Solomon[32])

$$\begin{array}{llll}
\text{(a)} & E_P(\log \rho_0) < 0 & \Rightarrow \lim_{n\to\infty} X_n = +\infty, & \mathbb{P}^o \; a.s. \\
\text{(b)} & E_P(\log \rho_0) > 0 & \Rightarrow \lim_{n\to\infty} X_n = -\infty, & \mathbb{P}^o \; a.s. \\
\text{(c)} & E_P(\log \rho_0) = 0 & \Rightarrow -\infty = \liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n = \infty, & \mathbb{P}^o \; a.s.
\end{array}$$

(Note that (a), (b) above imply that X_n is $\mathbb{P}^o$-a.s. transient, whereas (c) implies it is recurrent).

Proof. (sketch) The proof is based on the construction of a reversing measure: Fix an environment ω. For $z \in [-m_-, m_+]$, define

$$\mathcal{V}_{m_-,m_+,\omega}(z) := P^z_\omega(\{X_n\} \text{ hits } -m_- \text{ before hitting } m_+)\,.$$

The Markov property implies that $\mathcal{V}_{m_-,m_+,\omega}(\cdot)$ is harmonic, that is it satisfies

$$\begin{cases} \mathcal{V}_{m_-,m_+,\omega}(z) = (1-\omega_z)\mathcal{V}_{m_-,m_+,\omega}(z-1) + \omega_z \mathcal{V}_{m_-,m_+,\omega}(z+1), & z \in (-m_-, m_+)\,, \\ \mathcal{V}_{m_-,m_+,\omega}(-m_-) = 1, \quad \mathcal{V}_{m_-,m_+,\omega}(m_+) = 0\,. \end{cases} \tag{2.1.2}$$

Solving (2.1.2), we find

$$\mathcal{V}_{m_-,m_+,\omega}(z) = \frac{\sum_{i=z+1}^{m_+} \prod_{j=z+1}^{i-1} \rho_j}{\sum_{i=z+1}^{m_+} \prod_{j=z+1}^{i-1} \rho_j + \sum_{i=-m_-+1}^{z} \left(\prod_{j=i}^{z} \rho_j^{-1} \right)}\,. \tag{2.1.3}$$

(Note that the solution to (2.1.2) is unique due to the maximum principle, hence it is enough to verify that the function in (2.1.3) satisfies (2.1.2).)

Define $S(\omega) = \sum_{n=1}^{\infty} \rho_1 \cdots \rho_n$, $F(\omega) = \sum_{n=0}^{\infty} \rho_0^{-1} \ldots \rho_{-n}^{-1}$. Further, define the events

$$\mathcal{S}_+ = \{S(\omega) < \infty\}, \quad \mathcal{F}_+ = \{F(\omega) < \infty\}\,.$$

Then:

– on $T_+ := \{\mathcal{S}_+ \cap \mathcal{F}_+^c\}$, it holds that

$$\lim_{m\to\infty} \Big[1 - \mathcal{V}_{1,m,\omega}(0)\Big] > 0, \lim_{k\to\infty} \lim_{m\to\infty} \Big[1 - \mathcal{V}_{k,m,\omega}(0)\Big] = 1.$$

Hence, for $\omega \in T_+$,

$$P^o_\omega(\lim_{n\to\infty} X_n = \infty) = 1\,.$$

– Similarly, for $\omega \in T_- := \{\mathcal{S}_+^c \cap \mathcal{F}_+\}$,

$$P^o_\omega(\lim_{n\to\infty} X_n = -\infty) = 1\,.$$

– Finally, if $\omega \in R := \{\mathcal{S}_+^c \cap \mathcal{F}_+^c\}$ then, for any fixed k,

$$1 - \lim_{m\to\infty} \mathcal{V}_{k,m,\omega}(0) = \lim_{m\to\infty} \mathcal{V}_{m,k,\omega}(0) = 0\,,$$

and hence, for $\omega \in R$,

$$P^o_\omega\Big(-\infty = \liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n = \infty\Big) = 1\,.$$

We observe next that both $\mathcal{S}_+$ and $\mathcal{F}_+$ are invariant events, hence $P(\mathcal{S}_+) \in \{0,1\}$, $P(\mathcal{F}_+) \in \{0,1\}$ by the ergodicity of P. Next, $P(\mathcal{S}_+) = 1 \Rightarrow P(\mathcal{F}_+) = 0$ by the shift-invariance of P. Thus, it is enough to prove that $P(\mathcal{S}_+) = 1$ if and only if $E_P(\log \rho_0) < 0$ and $P(\mathcal{F}_+) = 1$ if and only if $E_P(\log \rho_0) > 0$. We prove the first claim only, the second one possessing a similar proof.

Assume first $c := E_P(\log \rho_0) < 0$. Then, by the ergodic theorem, there exists an $n_0(\omega)$ with $P(n_0(\omega) < \infty) = 1$ such that $\frac{1}{n}\sum_{i=1}^n \log \rho_i \leq c/2 < 0$ for all $n > n_0(\omega)$. But then, for some $C_1(\omega) < \infty$ P-a.s.,

$$\sum_{n=1}^{\infty} \rho_1 \cdots \rho_n \leq C_1(\omega) + \sum_{k=n_0(\omega)+1}^{\infty} e^{kc/2} < \infty, \quad P\text{-.a.s}$$

implying $P(\mathcal{S}_+) = 1$. Conversely, for $\omega \in \mathcal{S}_+$, $\lim_{n\to\infty} \sum_{k=1}^n \log \rho_k = -\infty$. But, $\{Y_i = -\log \rho_i\}$ are stationary, and the claim follows from the following well known:

Lemma 2.1.4 (Kesten[20]) *For any real valued, stationary sequence $\{Y_i\}$, fix $Z_n = \sum_{i=1}^n Y_i$. Then, one has with probability 1 that the event $\{Z_n \to_{n\to\infty} \infty\}$ implies $\liminf_{n\to\infty} Z_n/n > 0$.*

Indeed, Kesten's lemma implies that on $\mathcal{S}_+$,

$$E_P(\log \rho_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \log \rho_k < 0, \quad P - \text{a.s.}, \tag{2.1.5}$$

and $P(\mathcal{S}_+) = 1$ thus implies $E_P(\log \rho_0) < 0$ and completes the proof of Theorem 2.1.1. □

We turn to the law of large numbers. We first note that one cannot apply directly ergodic theorems to the sequence X_n/n: The sequence $\{X_n - X_{n-1}\}$ is not even stationary! Our proof of the LLN involves hitting times. An alternative proof, using the approach of the "environment viewed from the point of view of the particle", is detailed in [42], and briefly sketched in Section 2.2 below.

Introduce the following notations:

$$\overline{S} = \sum_{i=1}^{\infty} \frac{1}{\omega_{(-i)}} \prod_{j=0}^{i-1} \rho_{(-j)} + \frac{1}{\omega_0} \tag{2.1.6}$$

$$\overline{F} = \sum_{i=1}^{\infty} \frac{1}{1-\omega_i} \prod_{j=0}^{i-1} \rho_j^{-1} + \frac{1}{1-\omega_0} \tag{2.1.7}$$

Theorem 2.1.8 (Solomon[32]; Alili[1])

$$
\begin{array}{llll}
\text{(a)} & E_P(\overline{S}) < \infty & \Rightarrow \lim_{n\to\infty} \dfrac{X_n}{n} = \dfrac{1}{E_P(\overline{S})}, & \mathbb{P}^o \ a.s. \\
\text{(b)} & E_P(\overline{F}) < \infty & \Rightarrow \lim_{n\to\infty} \dfrac{X_n}{n} = -\dfrac{1}{E_P(\overline{F})}, & \mathbb{P}^o \ a.s. \\
\text{(c)} & E_P(\overline{S}) = \infty & \textit{and } E_P(\overline{F}) = \infty \Rightarrow \lim_{n\to\infty} \dfrac{X_n}{n} = 0, & \mathbb{P}^o \ a.s.
\end{array}
$$

Remark: In the case that P is i.i.d., (a)–(c) of Theorem 2.1.8 become

$$
\begin{array}{llll}
\text{(a}'\text{)} & E_P(\rho_0) < 1 & \Rightarrow \lim_{n\to\infty} \dfrac{X_n}{n} = \dfrac{1 - E_P(\rho_0)}{1 + E_P(\rho_0)}, & \mathbb{P}^o \text{ a.s.} \\
\text{(b}'\text{)} & E_P(\rho_0^{-1}) < 1 & \Rightarrow \lim_{n\to\infty} \dfrac{X_n}{n} = -\dfrac{1 - E_P\left(\frac{1}{\rho_0}\right)}{1 + E_P\left(\frac{1}{\rho_0}\right)}, & \mathbb{P}^o \text{ a.s.} \\
\text{(c}'\text{)} & \dfrac{1}{E_P(\rho_0)} \le 1 \le E_P(\rho_0^{-1}) & \Rightarrow \lim_{n\to\infty} \dfrac{X_n}{n} = 0, & \mathbb{P}^o \text{ a.s..}
\end{array}
$$

Since $E_P \log \rho_0 \le \log E_P \rho_0$ with a strict inequality whenever P is non-degenerate, it follows that one can find examples where $X_n \to \infty$ $\mathbb{P}^o$-a.s. but $X_n/n \to 0$, $\mathbb{P}^o$-a.s. This does not contradict Lemma 2.1.4 because $\{X_n - X_{n-1}\}$ is not in general a stationary sequence under $\mathbb{P}^o$.

Proof of Theorem 2.1.8 We introduce certain hitting times. Let $T_0 = 0$, and

$$T_n = \min\{k : X_k = n\}$$

with the usual convention that the minimum over an empty set is $+\infty$. Set $\tau_0 = 0$ and

$$\tau_n = T_n - T_{n-1}, \quad n \ge 1\,.$$

Similarly, set

$$T_{-n} = \min\{k : X_k = -n\}$$

and

$$\tau_{-n} = T_{-n} - T_{-n+1}, \quad n \ge 1\,,$$

the convention being that $\tau_{\pm n} = \infty$ if $T_{\pm n} = \infty$.

Lemma 2.1.9 *If* $\limsup_{n\to\infty} X_n = +\infty$, $\mathbb{P}^o$*-a.s., then* $\{\tau_i\}_{i\ge1}$ *is a stationary and ergodic sequence.*

Proof of Lemma 2.1.9 The stationarity of $\{\tau_i\}_{i\ge1}$ follows from the stationarity of the environment. To see the ergodicity, let $\Xi = [0,1]^{\mathbb{N}}$, let U_Ξ denote the measure on Ξ making all coordinates $\{\xi_i\}$ independent and of uniform law on $[0,1]$, and note that $\{X_n\}$ may be constructed by writing

$$X_{n+1} = X_n - \mathbf{1}_{\{\omega_{X_n} < \xi_{n+1}\}} + \mathbf{1}_{\{\omega_{X_n} \ge \xi_{n+1}\}}\,.$$

Suppose $A = A(\omega,\xi) = A(\tau)$ is an event, measurable w.r.t. $\mathcal{G}_n = \sigma\{\tau_i, i \geq 1\}$, which is invariant with respect to the shift $(\theta\tau)_i = \tau_{i+1}$ (we write in the sequel $\theta A = A(\theta\tau)$). We need only show that $P \otimes U_\Xi(A) \in \{0,1\}$. Note however that $\theta^k A$, conditioned on $\sigma\{\omega_i, i \in \mathbb{Z}\}$, is independent of $\xi_1,\ldots,\xi_k$. Thus, since U_Ξ is an i.i.d. law and hence the tail sigma-field of $\{\xi_i\}$ is trival, it follows that $A = \theta^k A$ is, under the above conditioning, independent of $\sigma\{\xi_i, i \geq 1\}$. Thus A depends only on ω. But the shift θ on the sequence $\{\tau_i\}$ induces the usual shift θ on Ω. Thus, $\theta A = A(\theta\omega) = A(\omega)$ and hence $P(A) \in \{0,1\}$. □

Remarks: 1. Note that an attempt to mimick this argument in $\mathbb{Z}^d$, $d \geq 1$, with T_i denoting the hitting times of hyperplanes at distance i from the origin, fails because of the extra information contained in the hitting location.

2. It is not hard to check, see [42], that if P is strongly mixing then so is the sequence $\{\tau_i\}_{i\geq 0}$.

Our strategy consists now of applying the ergodic theorem to the sequence $\{\tau_i\}$. As a first step, we have the

Lemma 2.1.10

$$\text{(a)}\quad E_{\mathbb{P}^o}(\tau_1) = E_P(\overline{S})\,,$$
$$\text{(b)}\quad E_{\mathbb{P}^o}(\tau_{-1}) = E_P(\overline{F})\,.$$

Proof. We prove only (a), the proof of (b) being similar. Decompose, with $X_0 = 0$,

$$\tau_1 = \mathbf{1}_{X_1=1} + \mathbf{1}_{X_1=-1}(1 + \tau_0'' + \tau_1'')\,. \tag{2.1.11}$$

Here, $(1+\tau_0'')$ is the first hitting time of 0 after time 1, and $1 + \tau_0'' + \tau_1''$ is the first hitting time of 1 after time $1+\tau_0''$.

Under P_ω^o, the law of τ_0'' conditioned on the event $\{X_1 = -1\}$ is $P^o_{\theta^{-1}\omega}(\tau_1 \in \cdot)$, while conditioned on the event $\{X_1 = -1\} \cap \{\tau_0'' < \infty\}$, τ_1'' also has law identical to that of τ_1.

Consider first the case $E_{\mathbb{P}^o}(\tau_1) < \infty$. Then, both $E^o_\omega(\tau_1) < \infty$ and $E_{\theta^{-1}\omega}(\tau_1) < \infty$, P-a.s. Taking expectations in (2.1.11), one gets then

$$E^o_\omega(\tau_1) = 1 + (1-\omega_0)E^o_{\theta^{-1}\omega}(\tau_1)\,.$$

Hence,

$$E^o_\omega(\tau_1) = \frac{1}{\omega_0} + \rho_0 E^o_{\theta^{-1}\omega}(\tau_1)\,.$$

Iterating this equation, we get

$$E^o_\omega(\tau_1) = \frac{1}{\omega_0} + \frac{\rho_0}{\omega_{(-1)}} + \frac{\rho_0\rho_{(-1)}}{\omega_{(-2)}} + \cdots + \frac{\prod_{i=0}^{-(m-1)}\rho_{(-i)}}{\omega_{(-m)}} + \left(\prod_{i=0}^{-m+1}\rho_{(-i)}\right) E^o_{\theta^{-m}\omega}(\tau_1)\,. \tag{2.1.12}$$

Omitting the last term, taking expectations on both sides, and then taking $m \to \infty$ using dominated convergence, we get

$$E_{\mathbb{P}^o}(\tau_1) \geq E_P(\overline{S})\,. \tag{2.1.13}$$

To see the reverse inequality, note that by (2.1.11), for any $M < \infty$,

$$E^o_\omega(\tau_1 \mathbf{1}_{\tau_1 < M}) \leq 1 + (1-\omega_0) E^o_\omega(\tau_1 \mathbf{1}_{\tau_1 < M}) + \omega_0 E^o_{\theta^{-1}\omega}(\tau_1 \mathbf{1}_{\tau_1 < M})\,.$$

Iterating, we get that

$$E^o_\omega(\tau_1 \mathbf{1}_{\tau_1 < M}) \leq \overline{S} + M \prod_{i=0}^{-m+1} \rho_{(-i)}\,.$$

Taking expectations, we get that

$$E_{\mathbb{P}^o}(\tau_1 \mathbf{1}_{\tau_1 < M}) \leq E_P(\overline{S}) + M E_P\left(\prod_{i=0}^{-m+1} \rho_{(-i)}\right)\,.$$

Assuming $E_P(\overline{S}) < \infty$ and hence $E_P\left(\prod_{i=0}^{-m+1} \rho_{(-i)}\right) \underset{m\to\infty}{\longrightarrow} 0$, we get that

$$E_{\mathbb{P}^o}(\tau_1 \mathbf{1}_{\tau_1 < M}) \leq E_P(\overline{S})\,.$$

Taking $M \to \infty$ and using monotone convergence we conclude, using also (2.1.13), that

$$E_{\mathbb{P}^o}(\tau_1 \mathbf{1}_{\tau_1 < \infty}) = E_P(\overline{S})\,,$$

completing the proof that $E_{\mathbb{P}^o}(\tau_1) < \infty \Rightarrow E_{\mathbb{P}^o}(\tau_1) = E_P(\overline{S})$.

It thus remains to show that $E_{\mathbb{P}^o}(\tau_1) = \infty \Rightarrow E_P(\overline{S}) = \infty$. Note next that if $E_P(\log \rho_0) \leq 0$, we have by Theorem 2.1.1 that

$$E_{\mathbb{P}^o}(\tau_1 \mathbf{1}_{\tau_1 < \infty}) = E_{\mathbb{P}^o}(\tau_1)$$

hence $E_{\mathbb{P}^o}(\tau_1) = \infty$ implies $E_P(\overline{S}) = \infty$. On the other hand, if $E_P(\log \rho_0) > 0$ then $\prod_{i-1}^{0} \rho_{(-j)} \to_{j\to\infty} \infty$, P-a.s. by the ergodic theorem and hence also $E_P(\overline{S}) = \infty$. This concludes the proof of Lemma 2.1.10. $\square$

Remark: In fact, a similar proof shows that $E^o_\omega(\tau_1) = \overline{S}$, for *every* environment ω.

An application of Lemmas 2.1.9 and 2.1.10 yields that in case (a)

$$\frac{T_n}{n} = \frac{\sum_{i=1}^n \tau_i}{n} \longrightarrow_{n\to\infty} E_{\mathbb{P}^o}(\tau_1) = E_{\mathbb{P}^o}(\overline{S}) < \infty, \quad \mathbb{P}^o\text{-a.s.}\,. \tag{2.1.14}$$

On the other hand, we have the following:

Lemma 2.1.15 *Assume $T_n/n \to \alpha$, for some constant $\alpha < \infty$. Then,*

$$\frac{X_n}{n} \underset{n\to\infty}{\longrightarrow} \frac{1}{\alpha}.$$

Proof of Lemma 2.1.15 Let k_n be the unique (random) integers such that

$$T_{k_n} \le n < T_{k_n+1} \,.$$

Note that $X_n < k_n + 1$ while $X_n \ge k_n - (n - T_{k_n})$. Hence,

$$\frac{k_n}{n} - \left(1 - \frac{T_{k_n}}{n}\right) \le \frac{X_n}{n} \le \frac{k_n + 1}{n} \,.$$

But, $\lim_{n\to\infty} k_n/n = \lim_{n\to\infty} n/T_n$ (due to the existence of the second limit and the definition of k_n). Thus,

$$\frac{1}{\alpha} \ge \limsup_{n\to\infty} \frac{X_n}{n} \ge \liminf_{n\to\infty} \frac{X_n}{n} \ge \frac{1}{\alpha} \,. \qquad \square$$

Lemma 2.1.15 and (2.1.14) complete the proof of Theorem 2.1.8 in case (a). Case (b) is similar, while case (c) is a minor modification of the above argument and is left out. $\square$

2.2 Auxiliary Markov chains: the point of view of the particle and the CLT

Starting from the RWRE X_n, define $\overline{\omega}(n) = \theta^{X_n}\omega$. The sequence $\{\overline{\omega}(n)\}$ is a process with paths in $\Omega^{\mathbb{N}}$. What is maybe more useful is that it is in fact a *Markov* process. More precisely, following Kozlov [23], one notes immediatly:

Lemma 2.2.1 *The process $\{\overline{\omega}(n)\}$ is a Markov process under either P_ω^0 or $\mathbb{P}^0$, with state space Ω and transition kernel*

$$M(\omega, d\omega') = \omega_0 \delta_{\theta\omega=\omega'} + (1-\omega_0)\delta_{\theta^{-1}\omega=\omega'} \,.$$

Our next step is to construct an invariant measure for the transition kernel M. In most of this section we will assume that $E_P \log \rho_0 < 0$, implying, by Theorem 2.1.1, that $T_1 < \infty$, $\mathbb{P}^o$-a.s. Whenever $E_{\mathbb{P}^o}(T_1) < \infty$, define the measures

$$Q(B) = E_{\mathbb{P}^o}\left(\sum_{i=0}^{T_1-1} \mathbf{1}_{\{\overline{\omega}(i)\in B\}}\right), \quad \overline{Q}(B) = \frac{Q(B)}{Q(\Omega)} = \frac{Q(B)}{E_{\mathbb{P}^o}T_1} \,.$$

Using Lemma 2.1.10, one checks that if $E_P(\bar{S}) < \infty$ then $E_{\mathbb{P}^o}T_1 < \infty$, and $\overline{Q}(\cdot)$ in this case is a probability measure.

Lemma 2.2.2 *Assume $E_P(\bar{S}) < \infty$. Then, $Q(\cdot)$ is invariant under the Markov kernel M, that is*

$$Q(B) = \iint \mathbf{1}_{\omega'\in B} M(\omega, d\omega') Q(d\omega) \,.$$

Proof. We have

$$\begin{aligned}
&\iint \mathbf{1}_{\omega' \in B} M(\omega, d\omega') Q(d\omega) \\
&\quad = \sum_{k=0}^{\infty} E_{\mathbb{P}^o}\Big(T_1 > k;\ \mathbf{1}_{\overline{\omega}(k+1)\in B}\Big) \\
&\quad = \sum_{k=0}^{\infty} E_{\mathbb{P}^o}\Big(T_1 = k+1;\ \mathbf{1}_{\overline{\omega}(k+1)\in B}\Big) + \sum_{k=0}^{\infty} E_{\mathbb{P}^o}\Big(T_1 > k+1;\ \mathbf{1}_{\overline{\omega}(k+1)\in B}\Big) \\
&\quad = \mathbb{P}^o(T_1 < \infty; \overline{\omega}(T_1) \in B) + \sum_{k=1}^{\infty} \mathbb{P}^o(T_1 > k; \overline{\omega}(k) \in B)\,.
\end{aligned}$$

But $\mathbb{P}^o(T_1 < \infty) = 1$ while $\mathbb{P}^o(\overline{\omega}(T_1) \in B) = P(\theta\omega \in B) = P(\omega \in B)$, hence

$$= \sum_{k=0}^{\infty} \mathbb{P}^o(T_1 > k; \overline{\omega}(k) \in B) = Q(B)\,. \qquad \square$$

Define next

$$\Lambda(\omega) = \frac{1}{\omega_0}\left[1 + \sum_{i=1}^{\infty}\prod_{j=1}^{i}\rho_j\right]\,.$$

By ellipticity, the condition $E_P(\Lambda(\omega)) < \infty$ is equivalent to $E_P(\overline{S}) < \infty$.

Lemma 2.2.3 *Under the assumptions of Lemma 2.2.2, it holds that*

$$\frac{dQ}{dP} = \Lambda(\omega)\,.$$

Proof. Note first that by Jensen's inequality, $E_P(\Lambda) < \infty$ implies that $E_P(\log\rho_0) < 0$ and hence $X_n \to_{n\to\infty} \infty$, $\mathbb{P}^o$-a.s., by Theorem 2.1.1. Let $f : \Omega \to \mathbb{R}$ be measurable. Then,

$$\int f dQ = E_{\mathbb{P}^o}\left(\sum_{i=0}^{T_1-1} f(\overline{\omega}_i)\right) = E_{\mathbb{P}^o}\left(\sum_{i\le 0} f(\theta^i\omega)N_i\right)$$

where $N_i = \{\#k \in [0, T_1) : X_k = i\}$ (note the difference in the role the index i plays in the two sums!). Using the shift invariance of P, we get

$$\begin{aligned}
\int f dQ &= \sum_{i\le 0} E_P\Big(f(\theta^i\omega)E^o_\omega N_i\Big) \\
&= \sum_{i\le 0} E_P\Big(f(\omega)E^o_{\theta^{-i}\omega}N_i\Big) = E_P\left(f(\omega)\left(\sum_{i\le 0} E^o_{\theta^{-i}\omega}N_i\right)\right)\,.
\end{aligned}$$

Hence,

$$\frac{dQ}{dP} = \sum_{i\le 0} E^o_{\theta^{-i}\omega} N_i \,, \tag{2.2.4}$$

and the right hand side converges, P-a.s.

In order to prove both the convergence in (2.2.4) and the lemma, we turn to evaluate $E^o_\omega N_i$. Define, for $i \le 0$,

$$\begin{aligned}\eta_{i,0} &= \min\{k \le T_1 : \ X_k = i\}\\ \theta_{i,0} &= \min\{\eta_{i,0} < k \le T_1 : \ X_{k-1} = i, X_k = i-1\}\end{aligned}$$

and, for $j \ge 1$,

$$\begin{aligned}\eta_{i,j} &= \min\{\theta_{i,j-1} < k \le T_1 : \ X_k = i\}\\ \theta_{i,j} &= \min\{\eta_{i,j} < k \le T_1 : \ X_{k-1} = i, X_k = i-1\}\end{aligned}$$

(with the usual convention that the minimum over an empty set is $+\infty$). We refer to the time interval $(\theta_{i,j-1}, \eta_{i,j})$ as the j-th excursion from $i-1$ to i. For any $j \ge 0$, any $i \le 0$, define

$$U_{i,j} = \{\#\ell \ge 0 : \ \theta_{i+1,j} < \theta_{i,\ell} < \eta_{i+1,j+1}\} \,.$$

Note that $U_{i,j}$ is the number of steps from i to $i-1$ during the $j+1$-th excursion from i to $i+1$. The Markov property implies that

$$P^o_\omega\Big(U_{i,\ell} = k_\ell \,, \ell = 1, \ldots, L \Big| \{U_{i',j}\}_{i'>i}, \eta_{i+1,L+1} < \infty\Big) = (1-\omega_i)^{\sum_{\ell=1}^L k_\ell} \omega_i^L \tag{2.2.5}$$

Defining $U_i = \sum_j U_{i,j}$, and noting that $\mathbb{P}^o(\{U_i < \infty\}) = 1$ because $X_n \to \infty$, $\mathbb{P}^o$-a.s., (2.2.5) implies that $\{U_i\}$ is under P^o_ω an (inhomogeneous) branching process with geometric offspring distribution of parameter $1-\omega_i$. Further,

$$E^o_\omega(U_i | U_{i+1}, \cdots, U_0) = \rho_i U_{i+1} \tag{2.2.6}$$

and using the relation $N_i = U_i + U_{i+1}$, $\mathbb{P}^o$-a.s., we get

$$E^o_\omega(N_i | U_{i+1}, \ldots, U_0) = E^o_\omega\Big(U_i + U_{i+1} | U_{i+1}, \cdots, U_0\Big) = \frac{1}{\omega_i} U_{i+1} \,.$$

Iterating (2.2.6), one gets

$$E^o_\omega N_i = \frac{1}{\omega_i} \rho_0 \cdots \rho_{i+1} \,.$$

Hence, using (2.2.4), and the assumption,

$$\frac{dQ}{dP} = \frac{1}{\omega_0}\left[1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} \rho_j\right] < \infty, P\text{-a.s.}$$

which completes the proof of the Lemma. □

Remark: Note that $dQ/dP > 0$, P-a.s., and hence under the assumption $E_P(\bar{S}) < \infty$ it holds that $Q \sim P$. This fact is true in greater generality, see the discussion in [33] and in [42].

Corollary 2.2.7 *Under the law induced by $\overline{Q} \otimes P^o_\omega$, the sequence $\{\overline{\omega}(n)\}$ is stationary and ergodic.*

Proof. The stationarity follows from the stationarity of $\overline{Q}$. Let $\overline{\theta}$ denote the shift on $\overline{\Omega} = \Omega^{\mathbb{N}}$, that is, for $\overline{\omega} \in \overline{\Omega}, \overline{\theta}\overline{\omega}(n) = \overline{\omega}(n+1)$. Denote by $\overline{P}_\omega$ the law of the sequence $\{\overline{\omega}(n)\}$ with $\overline{\omega}(0) = \omega$, that is, for any measurable sets $B_i \subset \Omega$,

$$\overline{P}_\omega\Big(\overline{\omega}(i) \in B_i, i = 1, \ldots, \ell\Big) = \int_{B_1} \cdots \int_{B_\ell} M(\omega, d\omega^1) M(\omega^1, d\omega^2) \cdots M(\omega^{\ell-1}, d\omega^\ell)$$

and set $\overline{\mathcal{Q}} = \overline{Q} \otimes \overline{P}_\omega$ (as usual, we also use $\overline{\mathcal{Q}}$ to denote the corresponding marginal induced on $\overline{\Omega}$).

We need to show that for any invariant A, that is $A \in \overline{\Omega}$ such that $\overline{\theta}A = A$, $\overline{\mathcal{Q}}(A) \in \{0,1\}$. Set $\varphi(\omega) = \overline{P}_\omega(A)$, we claim that $\{\varphi(\overline{\omega}(n))\}$ is a martingale with respect to the filtration $\mathcal{G}_n = \sigma(\overline{\omega}(0), \ldots, \overline{\omega}(n))$: indeed,

$$\varphi(\overline{\omega}(n)) = \overline{P}_{\overline{\omega}(n)}(A) = E_{\overline{\mathcal{Q}}}\Big(\mathbf{1}_{\theta^n A} | \mathcal{G}_n\Big) = E_{\overline{\mathcal{Q}}}\Big(\mathbf{1}_A | \mathcal{G}_n\Big)\,,$$

where the second equality is due to the Markov property and the third due to the invariance of A.

Hence, by the martingale convergence theorem,

$$\varphi(\overline{\omega}(n)) \underset{n\to\infty}{\longrightarrow} \mathbf{1}_A, \quad \overline{\mathcal{Q}}\text{-a.s.} \tag{2.2.8}$$

Further, $Q(\varphi(\omega) \notin \{0,1\}) = 0$ because otherwise there exists an interval $[a,b]$ with $\{0\}, \{1\} \notin [a,b]$ and $Q(\varphi(\omega) \in [a,b]) > 0$, while

$$\frac{1}{n} \sum_{0}^{n-1} \mathbf{1}_{\{\varphi(\overline{\omega}(n)) \in [a,b]\}} \to E_{\overline{\mathcal{Q}}}\Big(\mathbf{1}_{\{\varphi(\overline{\omega}(0)) \in [a,b]\}} | \mathcal{I}\Big)\,, \tag{2.2.9}$$

where $\mathcal{I}$ is the invariant σ-field.

Taking expectations in (2.2.9) and using (2.2.8), one concludes that

$$0 = \overline{\mathcal{Q}}\Big(\varphi(\overline{\omega}(0)) \in [a,b]\Big) = \overline{Q}\Big(\varphi(\omega) \in [a,b]\Big),$$

a contradiction. Thus for some measurable $B \subset \Omega$, $\varphi(\omega) = \mathbf{1}_B\,, \overline{Q}$ − a.s.. Further, the Markov property and invariance of A yield that $M\mathbf{1}_B = \mathbf{1}_B$, $\overline{Q}$-a.s. and hence P-a.s. But then,

$$\mathbf{1}_B = M\mathbf{1}_B \geq \omega_0 \mathbf{1}_{\theta B}, P\text{-a.s.}.$$

Since $P(\omega_0^+ = 0) = 0$, it follows that $\mathbf{1}_B \geq \mathbf{1}_{\theta B}$, P-a.s., and then $E_P(\mathbf{1}_B) = E_P(\mathbf{1}_{\theta B})$ implies that $\mathbf{1}_B = \mathbf{1}_{\theta B}$, P-a.s. But then, by ergodicity of P, $P(B) \in \{0,1\}$, and hence $\overline{Q}(B) \in \{0,1\}$. Since $\overline{Q}(A) = E_{\overline{Q}}\varphi(\omega) = \overline{Q}(B)$, the conclusion follows. $\square$

We are now ready to give a:

Second proof of Theorem 2.1.8 We begin with case (a), noting that the proof of case (b) is identical by the transformation $\omega_i \mapsto \hat{\omega}_{-i}$, where $\hat{\omega}_i = 1 - \omega_i$. Set $d(x,\omega) = E_\omega^x(X_1 - x)$. Then

$$X_n = \sum_{i=1}^n (X_i - X_{i-1}) = \sum_{i=1}^n \Big(X_i - X_{i-1} - d(X_{i-1},\omega)\Big) + \sum_{i=1}^n d(X_{i-1},\omega) \tag{2.2.10}$$

$$:= M_n + \sum_{i=1}^n d(X_{i-1},\omega)\,. \tag{2.2.11}$$

But, under P_ω^o, M_n is a martingale, with $|M_{n+1} - M_n| \leq 2$; Hence, with $\mathcal{G}_n = \sigma(M_1, \cdots M_n)$,

$$\begin{aligned} E_\omega^o(e^{\lambda M_n}) &= E_\omega^o\Big(e^{\lambda M_{n-1}} E_\omega^o(e^{\lambda(M_n - M_{n-1})}|\mathcal{G}_{n-1})\Big) \\ &\leq E_\omega^o\Big(e^{\lambda M_{n-1}} e^{2\lambda^2}\Big) \end{aligned}$$

and hence, iterating, $E_\omega^o(e^{\lambda M_n}) \leq e^{2n\lambda^2}$ (this is a version of Azuma's inequality, see [12, Corollary 2.4.7]). Chebycheff's inequality then implies

$$\frac{M_n}{n} \to 0, \quad \mathbb{P}^o\text{-a.s.}$$

(and even with exponential rate). Next, note that

$$\sum_{i=1}^n d(X_{i-1},\omega) = \sum_{i=1}^n d(0,\overline{\omega}(i-1))\,.$$

The ergodicity of $\{\overline{\omega}(i)\}$ under $\overline{Q} \otimes P_\omega^o$ implies that

$$\frac{1}{n}\sum_{i=1}^n d(0,\overline{\omega}(i-1)) \longrightarrow E_{\overline{Q}}(d(0,\overline{\omega}(0))),\ \overline{Q} \otimes P_\omega^o\text{-a.s.}. \tag{2.2.12}$$

But,

$$
\begin{aligned}
&E_{\overline{Q}}(d(0,\overline{\omega}(0)))\\
&\quad= \frac{E_P\left[\Lambda(\omega)(2\omega_0-1)\right]}{E_P(\Lambda(\omega))}\\
&\quad= \frac{1+E_P\left((1-\omega_1)\left[\frac{1}{\omega_1}+\sum_{i=2}^\infty\prod_{j=2}^i\rho_j\right]-(1-\omega_0)\left[\frac{1}{\omega_0}+\sum_{i=1}^\infty\prod_{j=1}^i\rho_j\right]\right)}{E_P(\Lambda(\omega))}\\
&\quad= \frac{1}{E_P(\Lambda(\omega))}=\frac{1}{E_P(\overline{S}(\omega))}\,.
\end{aligned}
$$

Finally, since $E_P(\Lambda(\omega))<\infty$, (2.2.12) holds also $\mathbb{P}^o$-a.s., completing the proof of the theorem in cases (a),(b).

Case (c) is handled by appealing to Lemma 2.1.10. Suppose $\limsup X_n = +\infty, \mathbb{P}^o-$ a.s.. Then, $\tau_1<\infty$, $\mathbb{P}^o$-a.s.. Define $\tau_i^K=\min(\tau_i,K)$. Note that under P_ω^o, the random variables $\{\tau_i^K\}$ are independent and bounded, and hence, with $G_n^K=n^{-1}\sum_{i=1}^n\tau_i^K$, we have

$$|G_n^K-E_\omega^oG_n^K|\to_{n\to\infty}0\,,\quad P_\omega^o-\text{a.s.}$$

But $f(\omega):=E_\omega^o\tau_1^K$ is a bounded, measurable, local function on Ω, and $E_\omega^oG_n^K=n^{-1}\sum_{i=1}^n f(\theta^i\omega)$. Hence, by the ergodic theorem, $E_\omega^oG_n^K\to_{n\to\infty} E_{\mathbb{P}^o}\tau_1^K, P-$ a.s.. Since, by Lemma 2.1.10 we have $E_{\mathbb{P}^o}\tau_1^K\to_{K\to\infty}\infty$, we conclude that

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\tau_i\geq\lim_{K\to\infty}E_{\mathbb{P}^o}\tau_1^K=\infty\,,\mathbb{P}^o-\text{a.s.}$$

This immediately implies $\limsup_{n\to\infty}X_n/n\leq 0,\mathbb{P}^o-$ a.s.. The reverse inequality is proved by considering the sequence $\{\tau_{-i}\}$, yielding part (c) of the Theorem. □

An advantage of the description of the environment viewed from the particle is that it leads immediately to CLT statements. As we will see in the next section, the "technical conditions" described in Theorem 2.2.13 below are essential: non CLT behaviour does occur (for certain measures P, even product) in the one dimensional RWRE model!

Theorem 2.2.13 (Kozlov[23]; Molchanov[26]) *Assume that for some $\varepsilon>0$,*

$$E_Q(\overline{S}^{2+\varepsilon}(\omega)+\overline{S}(\theta^{-1}\omega)^{2+\varepsilon})<\infty\,. \tag{2.2.14}$$

Further assume that

$$\sum_{n\geq1}\sqrt{E_P\left(E_P\left(v_P\overline{S}(\omega)-1\,\Big|\,\sigma(\omega_i,i\leq -n)\right)^2\right)}<\infty, \tag{2.2.15}$$

where $v_P:=1/E_P(\overline{S}(\omega))$. Then, with

$$\sigma_{P,1}^2 := v_P^2 E_Q\Big(\omega_0(\overline{S}(\omega)-1)^2 + (1-\omega_0)(\overline{S}(\theta^{-1}\omega)+1)^2\Big)\,,$$

and

$$\sigma_{P,2}^2 := E_P(v_P\overline{S}(\omega)-1)^2 + 2\sum_{n=1}^{\infty} E_P\left(\left(v_P\overline{S}(\omega)-1\right)\left(v_P\overline{S}(\theta^n\omega)-1\right)\right)\,,$$

we have that

$$\mathbb{P}^o\left(\frac{X_n - nv_P}{\sigma_P\sqrt{n}} > x\right) \longrightarrow_{n\to\infty} \Phi(-x)\,,$$

where

$$\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{\theta^2}{2}}\,d\theta\,,$$

and $\sigma_P^2 = \sigma_{P,1}^2 + v_P\sigma_{P,2}^2$.

Proof. The proof is based on the construction of an appropriate martingale, and then using the Martingale CLT and the CLT for stationary ergodic sequences. We thus begin with recalling the version of these CLT's most useful to us.

Lemma 2.2.16 ([14], pg. 417) *Suppose* $(Z_n, \mathcal{F}_n)_{n\geq 0}$ *is a martingale difference sequence, and let* $V_n = \sum_{1\leq k\leq n} E(Z_k^2|\mathcal{F}_{k-1})$. *Assume that*

$$\text{(a)}\quad \frac{V_n}{n} \to_{n\to\infty} \sigma^2, \text{ in probability.}$$

$$\text{(b)}\quad \frac{1}{n}\sum_{m\leq n} E\Big(Z_m^2 \mathbf{1}_{\{|Z_m|>\varepsilon\sqrt{n}\}}\Big) \to_{n\to\infty} 0\,.$$

Then, $\sum_{i=1}^n Z_i/\sigma\sqrt{n}$ *converges in distribution to a standard Gaussian random variable.*

Lemma 2.2.17 ([14], p. 419) *Suppose* $\{Z_n\}_{n\in\mathbb{Z}}$ *is a stationary, zero mean, ergodic sequence, and set* $\mathcal{F}_n = \sigma(Z_i, i\leq n)$. *Assume that*

$$\sum_{n\geq 0}\sqrt{E\left(E\left(Z_0|\mathcal{F}_{-n}\right)\right)^2} < \infty\,. \tag{2.2.18}$$

Then, $\left\{\sum_{i=1}^{nt} Z_i/\sigma\sqrt{n}\right\}_{t\in[0,1]}$ *converges in distribution to a standard Brownian motion, where*

$$\sigma^2 = EZ_0^2 + 2\sum_{n=1}^{\infty} E(Z_0 Z_n)\,.$$

We next recall that by Theorem 2.1.8,

$$\frac{X_n}{n} \to v_P, \quad \mathbb{P}^o\text{-a.s.},$$

where $v_P := 1/E_P(\overline{S})$. One is tempted to use the martingale M_n appearing in the environment proof of Theorem 2.1.8 (see (2.2.10)), however this strategy is not so successful because of the difficulties associated with separating the fluctuations in M_n and $\sum_{i=1}^n d(X_{i-1}, \omega)$. Instead, write

$$f(x, n, \omega) = x - v_P n + h(x, \omega), \quad x \in \mathbb{Z}\,.$$

We want to make $f(X_n, n, \omega)$ into a martingale w.r.t. $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$ and the law P_ω^o. This is automatic if we can ensure that

$$E_\omega^{X_n} f(X_{n+1}, n+1, \omega) = f(X_n, n, \omega), \quad P_\omega^o\text{-a.s.} \tag{2.2.19}$$

Developing this equality and defining $\Delta(x, \omega) = h(x+1, \omega) - h(x, \omega)$, we get that (2.2.19) holds true if a bounded solution to the equation

$$\Delta(x, \omega) = -\left[\frac{2\omega_x - 1 - v_P}{\omega_x}\right] + \rho_x \Delta(x-1, \omega)$$

exists. One may verify that $\Delta(x, \omega) = -1 + v_P \overline{S}(\theta^x \omega)$ is such a solution.

Fixing $h(0, \omega) = 0$, and defining $\overline{M}_0 = 0$ and $\overline{M}_n = f(X_n, n, \omega)$, one concludes that $\overline{M}_n$ is a martingale, and further

$$\begin{aligned}
&E_\omega^o\left((\overline{M}_{k+1} - \overline{M}_k)^2 | \mathcal{F}_k\right) \\
&\quad = \omega_{X_k} v_P^2 (\overline{S}(\theta^{X_k}\omega) - 1)^2 + (1 - \omega_{X_k}) v_P^2 (\overline{S}(\theta^{X_k - 1}\omega) + 1)^2 \\
&\quad = v_P^2 \left[\overline{\omega}(k)_0 (\overline{S}(\overline{\omega}_k) - 1)^2 + (1 - \overline{\omega}(k)_0)(\overline{S}(\theta^{-1}\overline{\omega}_k) + 1)^2\right].
\end{aligned}$$

Hence,

$$\frac{V_n}{n} = \frac{1}{n} \sum_{k=1}^n E_\omega^o\left((\overline{M}_{k+1} - \overline{M}_k)^2 | \mathcal{F}_k\right) \underset{n\to\infty}{\longrightarrow} \sigma_{P,1}^2\,, \quad \mathbb{P}^o\text{-a.s.},$$

using the machinery developed above. The integrability condition (2.2.14) is enough to apply the Martingale CLT (Lemma 2.2.16), and one concludes that for any $\delta > 0$,

$$P\left(\left|P_\omega^o\left(\frac{\overline{M}_n}{\sigma_{P,1}\sqrt{n}} \geq x\right) - \Phi(-x)\right| > \delta\right) \to_{n\to\infty} 0\,. \tag{2.2.20}$$

Note that since both $P_\omega^o(\overline{M}_n \geq x\sigma_{P,1}\sqrt{n})$ and $\Phi(x)$ are monotone in x, and that $\Phi(\cdot)$ is continuous, the convergence in (2.2.20) actually is uniform on $\mathbb{R}$. Further, note that

$$h(X_n,\omega) = \sum_{j=1}^{X_n-1} \Delta(j,\omega) = \sum_{j=1}^{nv_P} \Delta(j,\omega) + R_n := Z_n + R_n\,.$$

Note that, for every $\delta > 0$ and some $\delta_n \to 0$,

$$\mathbb{P}^o\left(\frac{|R_n|}{\sqrt{n}} \geq \delta\right) \leq \mathbb{P}^o\Big(|X_n - nv_P| \geq \delta_n n\Big)$$
$$+P\left(\max_{j_-,j_+\in(-n\delta_n,n\delta_n)} |\sum_{i=j_-}^{j_+} \frac{\Delta(i,\omega)}{\sqrt{n}}| \geq \delta\right) := P_{1,n}(\delta_n)+P_{2,n}(\delta,\delta_n) \underset{n\to\infty}{\longrightarrow} 0, \tag{2.2.21}$$

where the convergence of the first term is due (choosing an appropriate $\delta_n \to_{n\to\infty} 0$ slowly enough) to Theorem 2.1.8 and that of the second one due to $E_P\Delta(i,\omega) = 0$ and the stationary invariance principle (Lemma 2.2.17), which can be applied, for any $\delta_n \to_{n\to\infty} 0$, due to (2.2.15).

Another application of Lemma 2.2.17 yields that

$$\lim_{n\to\infty} P(Z_n \geq z\sqrt{nv_P}\sigma_{P,2}) = \Phi(-z)\,. \tag{2.2.22}$$

Writing $X_n - nv_P = \overline{M}_n - Z_n - R_n$, and using that $R_n/\sqrt{n} \to_{n\to\infty} 0$ in $\mathbb{P}^o$-probability, one concludes that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}^o(\frac{X_n - nv_P}{\sqrt{n}} > x) &= \lim_{n\to\infty} E_P(P^o_\omega(\overline{M}_n/\sqrt{n} > x + Z_n/\sqrt{n}))\\ &= \lim_{n\to\infty} E_P(\Phi\left(-\frac{x+Z_n/\sqrt{n}}{\sigma_{P,1}}\right))\,,\end{aligned} \tag{2.2.23}$$

where the second equality is due to the uniform convergence in (2.2.20). Combining (2.2.23) with (2.2.22) yields the claim. □

Remarks: 1. The alert reader will have noted that under assumption (2.2.14), and a mild mixing assumption on P which ensures that for any $\delta > 0$ and $\delta_n \to 0$, $P_{2,n}(\delta,\delta_n) \to_{n\to\infty} 0$, c.f. (2.2.21),

$$P^o_\omega\Big(\frac{X_n - v_P n - Z_n}{\sqrt{n}\sigma_{P,1}} > x\Big) \to_{n\to\infty} \Phi(-x)\,.$$

That is, using a random centering one also has a *quenched* CLT.
2. For measures P (even mixing) that are farther removed from the product case, the CLT may fail to hold true. We refer to [5] for a discussion and examples.
3. It is not hard to check that if P is a product measure, then all the conditions in the statement of Theorem 2.2.13 hold true as soon as $E_P\rho_0^2 < 1$. Comparing it with the LLN, it is natural to inquire what happens in the range of parameters $s \leq 2$ where $E_P\rho_0^s = 1$, and in the recurrent case. That is the topic of the next sections.

2.3 Recurrent walks: subdiffusive limits and aging

Throughout this section, define for $|k| > 1$, $\overline{R}_k = k^{-1}\sum_{i=0}^{k-1}\log\rho_i(\text{sign } i)$. We further need the following assumption, that ensures that the RWRE is recurrent and that the potential in (2.1.3) exhibits Brownian scalling. We refer to a model satisfying these assumptions as the "Sinai model".

Assumption 2.3.1 *$E_P\log\rho_0 = 0$. Further, P is strongly mixing, and the functional invariance principle holds for $\sqrt{k}\,\overline{R}_k/\sigma_P$; that is, $\{\sqrt{k}\,\overline{R}_{[kt]}/\sigma_P\}_{t\in\mathbb{R}}$ converges weakly to a Brownian motion for some $\sigma_P > 0$ (sufficient conditions for such convergence are as in Lemma 2.2.16).*

(In the i.i.d. case, note that $\sigma_P^2 = E_P(\log\rho_0)^2$). Define

$$W^n(t) = \frac{1}{\log n}\sum_{i=0}^{\lfloor(\log n)^2 t\rfloor}\log\rho_i\cdot(\text{sign } t)$$

with $t\in\mathbb{R}$. By Assumption 2.3.1, $\{W^n(t)\}_{t\in\mathbb{R}}$ converges weakly to $\{\sigma_P B_t\}$, where $\{B_t\}$ is a two sided Brownian motion.

Next, we call a triple (a,b,c) with $a<b<c$ a valley of the path $\{W^n(\cdot)\}$ if

$$\begin{aligned} W^n(b) &= \min_{a\le t\le c} W^n(t)\,,\\ W^n(a) &= \max_{a\le t\le b} W^n(t)\,,\\ W^n(c) &= \max_{b\le t\le c} W^n(t)\,. \end{aligned}$$

The *depth* of the valley is defined as

$$d_{(a,b,c)} = \min(W^n(a)-W^n(b), W^n(c)-W^n(b))\,.$$

If (a,b,c) is a valley, and $a<d<e<b$ are such that

$$W^n(e)-W^n(d) = \max_{a\le x<y\le b} W^n(y)-W^n(x)$$

then (a,d,e) and (e,b,c) are again valleys, which are obtained from (a,b,c) by a *left refinement*. One defines similarly a *right refinement*. Define

$$\begin{aligned} c_0^n &= \min\{t\ge 0: \quad W^n(t)\ge 1\}\\ a_0^n &= \max\{t\le 0: \quad W^n(t)\ge 1\}\\ W^n(b_0^n) &= \min_{a_0^n\le t\le c_0^n} W^n(t)\,. \end{aligned}$$

(b_0^n is not uniquely defined, however, due to Assumption 2.3.1, with P-probability approaching 1 as $n\to\infty$, all candidates for b_0^n are within distance

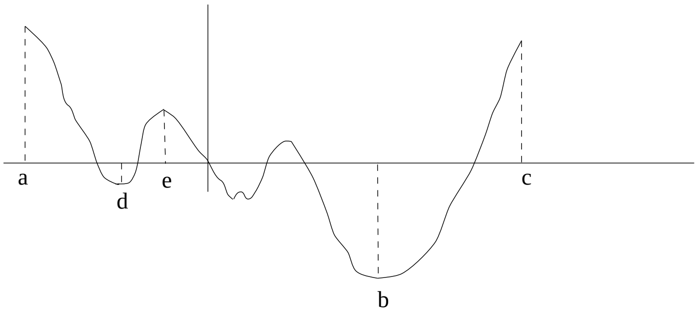

Fig. 2.3.1. Left refinement of (a, b, c)

converging to 0 as $n \to \infty$; we define b_0^n then as the smallest one in absolute value.)

One may now apply a (finite) sequence of refinements to find the *smallest* valley $(\overline{a}^n, \overline{b}^n, \overline{c}^n)$ with $\overline{a}^n < 0 < \overline{c}^n$, while $d_{(\overline{a}^n, \overline{b}^n, \overline{c}^n)} \geq 1$. We define similarly the smallest valley $(\overline{a}_\delta^n, \overline{b}_\delta^n, \overline{c}_\delta^n)$ such that $d_{(\overline{a}_\delta^n, \overline{b}_\delta^n, \overline{c}_\delta^n)} \geq 1 + \delta$. Let

$$A_n^{J,\delta} = \left\{ \begin{array}{c} \omega \in \Omega : \overline{b}^n = \overline{b}_\delta^n, \text{ any refinement } (a,b,c) \text{ of } (\overline{a}_\delta^n, \overline{b}_\delta^n, \overline{c}_\delta^n) \text{ with} \\ b \neq \overline{b}^n \text{ has depth } < 1 - \delta\,, |\overline{a}_\delta^n| + |\overline{c}_\delta^n| \leq J\,, \\ \min_{t \in [\overline{a}^n, \overline{c}^n] \setminus [\overline{b}^n - \delta, \overline{b}^n + \delta]} W^n(t) - W^n(\overline{b}^n) > \delta^3 \end{array} \right\}$$

then it is easy to check by the properties of Brownian motion that

$$\lim_{\delta \to 0} \lim_{J \to \infty} \lim_{n \to \infty} P(A_n^{J,\delta}) = 1\,. \tag{2.3.2}$$

Theorem 2.3.3 (Sinai[31]) *Assume Assumption 2.3.1. For any $\eta > 0$,*

$$\mathbb{P}^o \left(\left| \frac{X_n}{(\log n)^2} - \overline{b}^n \right| > \eta \right) \underset{n \to \infty}{\to} 0\,.$$

Proof. Fix $\delta < \eta/2, J$ and n large enough with $\omega \in A_n^{J,\delta}$. For simplicity of notations, assume in the sequel that ω is such that $\overline{b}^n > 0$. Write $a^n = \overline{a}^n (\log n)^2, b^n = \overline{b}^n (\log n)^2, c^n = \overline{c}^n (\log n)^2$, with similar notations for $a_\delta^n, b_\delta^n, c_\delta^n$. Define

$$\overline{T}_{b,n} = \min\{t \geq 0 : X_t = b^n \text{ or } X_t = a_\delta^n\}\,.$$

By (2.1.3),

$$P_\omega^o \Big(X_{\overline{T}_{b,n}} = a_\delta^n \Big) \leq \frac{1}{1 + \frac{\exp\{(\log n)(W^n(\overline{a}_\delta^n) - W^n(\overline{b}^n))\}}{Jn(\log n)^2}} \leq \frac{J(\log n)^2}{n^\delta}\,. \tag{2.3.4}$$

On the other hand, let $\tilde{T}_{b,n}$ have the law of $\overline{T}_{b,n}$ except that the walk $\{X_\cdot\}$ is reflected at a_δ^n, and define similarly $\tilde{\tau}_1$. Using the same recursions as in (2.1.12), we have that

$$E_\omega^o(\tilde{\tau}_1) = \frac{1}{\omega_0^+} + \frac{\rho_0}{\omega_{(-1)}^+} + \cdots + \frac{\prod_{i=0}^{a_\delta^n+2} \rho_{-i}}{\omega_{a_\delta^n-1}^+} + \prod_{i=0}^{a_\delta^n+1} \rho_{-i}\,.$$

Hence, with $\tilde{\omega}_i = \omega_i$ for $i \neq a_\delta^n$ and $\tilde{\omega}_{a_\delta^n}^+ = 1$, for all n large enough,

$$\begin{aligned} E_\omega^o(\overline{T}_{b,n}) \le E_\omega^o(\tilde{T}_{b,n}) &= \sum_{i=1}^{b^n} \sum_{j=0}^{i-1-a_\delta^n} \frac{\prod_{k=1}^{j} \rho_{i-k}}{\omega_{(i-j-1)}^+} \\ &\le \frac{1}{\varepsilon} \sum_{i=1}^{b^n} \sum_{j=0}^{i-1-a_\delta^n} e^{(\log n)(W^n(i)-W^n(i-j))} \le \frac{2J^2}{\varepsilon} e^{\log n(1-\delta)} \le n^{1-\frac{\delta}{2}}\,. \end{aligned}$$

We thus conclude that

$$P_\omega^o\Big(\overline{T}_{b,n} < n, \quad X_{\overline{T}_{b,n}} = b^n\Big) \underset{n\to\infty}{\longrightarrow} 1$$

implying that

$$P_\omega^o\Big(T_{b^n} < n\Big) \underset{n\to\infty}{\to} 1\,. \tag{2.3.5}$$

Next note that another application of (2.1.3) yields

$$\begin{aligned} &P_\omega^{b^n-1}(X_\cdot \text{ hits } b^n \text{ before } a_\delta^n) \ge 1 - n^{-(1+\frac{\delta}{2})} \\ &P_\omega^{b^n+1}(X_\cdot \text{ hits } b^n \text{ before } c_\delta^n) \ge 1 - n^{-(1+\frac{\delta}{2})}\,. \end{aligned} \tag{2.3.6}$$

On the same probability space, construct a RWRE $\{\tilde{X}_t\}$ with the same transition mechanism as $\{X_t\}$ except that it is reflected at a_δ^n, i.e. replace ω by $\tilde{\omega}$. Then, using (2.3.6),

$$\begin{aligned} P_\omega^o\left(\left|\frac{X_n}{(\log n)^2} - \overline{b}^n\right| > \delta\right) &\le P_\omega^o\Big(T_{b^n} > n\Big) + \max_{t\le n} P_\omega^{b^n}\left(\left|\frac{X_t}{(\log n)^2} - \overline{b}^n\right| > \delta\right) \\ &\le P_\omega^o\Big(T_{b^n} > n\Big) + \Big[1 - (1-n^{-(1+\frac{\delta}{2})})^n\Big] \\ &\quad + \max_{t\le n} P_\omega^{b^n}\left(\left|\frac{\tilde{X}_t}{(\log n)^2} - \overline{b}^n\right| > \delta\right)\,. \end{aligned}$$

Hence, in view of (2.3.2) and (2.3.5), the theorem holds as soon as we show that

$$\sup_{\omega\in A_n^{J,\delta}} \max_{t\le n} P_\omega^{\overline{b}^n}\left(\left|\frac{\tilde{X}_t}{(\log n)^2} - \overline{b}^n\right| > \delta\right) \underset{n\to\infty}{\longrightarrow} 0\,. \tag{2.3.7}$$

To see (2.3.7), define

$$f(z) = \frac{\prod_{a_\delta^n+1\le i<z} \omega_i^+}{\prod_{a_\delta^n+1\le i<z} \omega_{i+1}^-}, \quad \overline{f}(z) = \frac{f(z)}{f(b^n)}$$

(as usual, the product over an empty set of indices is taken as 1. $\overline{f}(\cdot)$ corresponds to the invariant measure for the resistor network corresponding to $\tilde{X}_\cdot$). Next, define the operator

$$(Ag)(z) = \overline{\omega}_{z-1}^+ g(z-1) + \overline{\omega}_{z+1}^- g(z+1) + \overline{\omega}_z^0 g(z) \tag{2.3.8}$$

where $\overline{\omega}_z = \omega_z$ for $z > a_\delta^n$, $\overline{\omega}_{a_\delta^n}^+ = 1, \overline{\omega}_{a_\delta^n-1}^+ = 0$. Note that $A\overline{f} = \overline{f}$, and further that

$$P_\omega^{b^n}(\tilde{X}_t = z) = A^t \mathbf{1}_{b^n}(z)\,.$$

Since $\overline{f}(z) \ge \mathbf{1}_{b^n}(z)$ and A is a positive operator, we conclude that

$$P_\omega^{b^n}(\tilde{X}_t = z) \le \overline{f}(z)\,.$$

But, for z with $|z/(\log n)^2 - \overline{b}^n| > \delta$, it holds that $\overline{f}(z) \le e^{-\delta^3 \log n}$, and hence

$$P_\omega^{b^n}(\tilde{X}_t = z) \le n^{-\delta^3}\,.$$

Thus, for $\omega \in A_n^{J,\delta}$, using the fact that the second inequality in (2.3.6) still applies for $\tilde{X}$,

$$\max_{t\le n} P_\omega^{\overline{b}^n}\left(\left|\frac{\tilde{X}_t}{(\log n)^2} - \overline{b}^n\right| > \delta\right) \le (\overline{b}^n + \delta)(\log n)^2 n^{-\delta^3} + 1 - \left(1 - n^{-(1+\delta/2)}\right)^n,$$

yielding (2.3.7) and completing the proof of the theorem. □

The intuition behind Theorem 2.3.3 is that it takes a time exponential in the depth of a valley for the walker to emerge from it. Thus, the scaling for time of order n corresponds to valleys of depth proportional to $\log n$. By Brownian scaling, such valleys occur naturally in the environment at a distance proportional to $(\log n)^2$, hence the result.

A precise description of the limit law of $\overline{b}^n$ is based on properties of the Brownian motion. We skip it here and refer instead to [21]. As the intuition above suggests, the RWRE is getting stuck in deeper and deeper valleys for longer and longer periods of time. A modern way of stating it is to say that the RWRE exhibits aging, and explicitely:

Theorem 2.3.9 (Golosov[17], Le Doussal, Monthus and Fisher[13]) (Dembo, Guionnet and Zeitouni[10])
Assume Assumption 2.3.1. Then, for $h > 1$,

$$\lim_{\eta\to 0}\lim_{n\to\infty} \mathbb{P}^o\left(\frac{|X_{n^h} - X_n|}{(\log n)^2} < \eta\right) = \frac{1}{h^2}\left[\frac{5}{3} - \frac{2}{3}e^{-(h-1)}\right]. \tag{2.3.10}$$

The proof is detailed in [42].

2.4 Subexponential decay of probabilities, subdiffusive behaviour, and stable limits

It is our goal in this section to further study sub-diffusive behavior, in the *transient* set-up. The following are natural assumptions that will be imposed throughout.

Assumption 2.4.1

(S1) $\text{essinf}\rho_0 =: \rho_{\min} < 1, \quad \text{esssup}\rho_0 =: \rho_{\max} > 1,$ *and* $E_P \log \rho_0 \le 0$.
(S2) P *is* α*-mixing with* $\alpha(n) = \exp(-n(\log n)^{1+\eta})$ *for some* $\eta > 0$*; that is, for any* ℓ*-separated measurable bounded by* 1 *functions* f_1, f_2,

$$E_P\Big(f_1(\omega)(f_2(\omega) - E_P f_2(\omega))\Big) \le \alpha(\ell)\,.$$

(functions f_i *are* ℓ *separated if* f_i *is measurable on* $\sigma(\omega_j, j \in I_i)$ *with* I_i *intervals satisfying* $\text{dist}(I_i, I_k) > \ell$ *for any* $i \ne k$*).*

It is known that ***(S2)*** *implies that* $\overline{R}_k$ *(defined in Section 2.3) satisfies the LDP with good rate function* $J(\cdot)$*. We add the following assumption on* $J(\cdot)$*:*

(S3) $J(0) > 0$.

Condition **(S3)** implies that $E_P(\log \rho_0) < 0$. Define next $s := \min_{y\ge 0} \frac{1}{y} J(y)$. Note that the condition $E_P(\overline{S}) < \infty$ and the existence of a LDP for $\overline{R}_k$ with good rate function $J(\cdot)$ are enough to imply, by Varadhan's lemma, that $0 \ge \sup_y (y - J(y))$, and in particular that $s \ge 1$. (In the case where P is a product measure, we can identify s as satisfying $E_P(\rho_0^s) = 1$, and then $E_P(\overline{S}) < \infty$, which is equivalent to $E_P(\rho_0) < 1$, implies that $s > 1$.)

The significance of the parameter s can be seen by noting that for $s > 2$, a CLT holds true by Theorem 2.2.13 (this is immediate in the case of P being a product measure, and easy to verify otherwise). To gain some intuition into the role s plays, we begin with the following:

Lemma 2.4.2 *For each* $1 \le \kappa < s$*, there exists a constant* $c(\kappa) < \infty$ *such that*

$$\mathbb{E}^o(T_k)^\kappa \le c(\kappa) k^\kappa\,. \tag{2.4.3}$$

Proof of Lemma 2.4.2 Note first that by Minkowski's inequality, for any $k \ge 1$,

$$\mathbb{E}^o(T_k^\kappa) = \mathbb{E}^o\left(\sum_{i=1}^k \tau_i\right)^\kappa \le k^\kappa \mathbb{E}^o \tau_1^\kappa\,.$$

Hence, it will be enough to prove that

$$\mathbb{E}^o(\tau_1^\kappa) < \infty\,. \tag{2.4.4}$$

To prove (2.4.4), we build upon the techniques developed in the course of proving Lemma 2.2.3. Indeed, recall the random variables $U_{i,j}$ and N_i defined there, and note that since $\tau_1 = \sum_{i=-\infty}^o N_i$, it is enough to estimate

$$\mathbb{E}^o\left(\sum_{i=-\infty}^{0} N_i\right)^\kappa = \mathbb{E}^o\left(\sum_{i=-\infty}^{0} U_i + U_{i+1}\right)^\kappa \le C_\varepsilon \mathbb{E}^o\left(\sum_{i=-\infty}^{0} U_i\right)^\kappa . \quad (2.4.5)$$

An important step in the evaluation of the RHS in (2.4.5) involves the computation of moments of U_i. To present the idea, consider first the case $\kappa > 2$, and write

$$U_i = \sum_{j=1}^{U_{i+1}} G_j$$

where, under P^o_ω, the G_j are i.i.d. geometric random variables, independent of $\{U_{i+1}, \cdots U_0\}$, of parameter $(1-\omega_i)$. Hence,

$$\begin{aligned} E^o_\omega(U_i^2) &= E^o_\omega\left(\sum_{j=1}^{U_i+1}(G_j - E^o_\omega G_j) + \sum_{j=1}^{U_{i+1}} E^o_\omega G_j\right)^2 \qquad (2.4.6)\\ &\le c_\delta E^o_\omega\left(\sum_{j=1}^{U_i+1}(G_j - E^o_\omega G_j)\right)^2 + (1+\delta)(E^o_\omega G_j)^2 \cdot E^o_\omega(U_{i+1}^2)\\ &\le c'_\delta E^o_\omega(U_{i+1}) \cdot E^o_\omega(G_j^2) + (1+\delta)\rho_i^2 E^o_\omega(U_{i+1}^2) . \end{aligned}$$

Here, c_δ, c'_δ are constants which depend on δ only. Since $E^o_\omega(G_j^2)$ is uniformly (in ω) bounded, and $E^o_\omega(U_{i+1}) = \rho_i$, we get

$$E^o_\omega(U_i^2) \le c''_\delta \rho_i E^o_\omega U_{i+1} + (1+\delta)\rho_i^2 E^o_\omega(U_{i+1}^2) .$$

Iterating and using (cf. (2.2.6)) that $E^o_\omega U_{i+1} = \prod_{j=i+1}^0 \rho_j$, we conclude the existence of a constant c'''_δ such that

$$E^o_\omega(U_i^2) \le c_\delta{}''' \left(\sum_{j=0}^{|i|}\left(\prod_{k=-j}^{0}\rho_k + \prod_{k=-j}^{0}\left(\rho_k^2(1+\delta)\right)\right)\right) ,$$

and hence

$$\mathbb{E}^o(U_i^2) \le c_\delta{}''' \sum_{j=0}^{|i|}\left(E_P\prod_{k=-j}^{0}\rho_k + E_P\prod_{k=-j}^{0}\left(\rho_k^2(1+\delta)\right)\right) . \qquad (2.4.7)$$

Note that, by Varadhan's lemma (see [12, Theorem 4.3.1]), for any constant β,

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}^o\left(\prod_{k=-n}^{0}\rho_k^\beta\right) = \sup_y\Big(\beta y - J(y)\Big) = \sup_y y\left(\beta - \frac{J(y)}{y}\right) := \beta'(\beta) , \qquad (2.4.8)$$

and $\beta'(\beta) < 0$ as soon as $\beta < s$. Hence, substituting in (2.4.7), and choosing δ such that $\log(1+\delta) < \beta'(\beta)/4$, we obtain that for some constant c_δ'''',

$$\mathbb{E}^o(U_i^2) \le c_\delta'''' e^{-i\beta'(2)/2},$$

implying that

$$\sqrt{\mathbb{E}^o\left(\sum_{i=-\infty}^{0} N_i\right)^2} \le C_\varepsilon^{\frac{1}{2}} \sqrt{\sum_{i=-\infty}^{0} \mathbb{E}^o(U_i^2)} < \infty .$$

A similar argument holds for any integer $\kappa < s$: mimicking the steps leading to (2.4.7), we get that

$$E_\omega^o(U_i^\kappa) \le c_\delta''' \left(\sum_{j=0}^{|i|} \left(\prod_{k=-j}^{0} \rho_k^{\kappa/2} + \prod_{k=-j}^{0} \rho_k^\kappa \right) \right) ,$$

and using (2.4.8) and an induction on lower (integer) moments, we get that $\mathbb{E}^o\left(\sum_{i=-\infty}^0 N_i\right)^\kappa < \infty$ for all $\kappa < s$ integer. Finally, to handle $\lfloor s \rfloor < \kappa < s$, we replace (2.4.6) by

$$\begin{aligned} E_\omega^o(U_i^\kappa) &\le c_\delta' E_\omega^o(U_{i+1}^{\kappa/2 \vee 1}) E_\omega^o(G_j^\kappa) + (1+\delta)\rho_i^\kappa E_\omega^o(U_{i+1}^\kappa) \\ &\le c_\delta''(E_\omega^o U_i^{\lceil \kappa/2 \rceil})^{\frac{\kappa/2}{\lceil \kappa/2 \rceil}} + (1+\delta)\rho_i^\kappa E_\omega^o(U_{i+1}^\kappa) , \end{aligned}$$

and one proceeds as before. □

Lemma 2.4.2 already shows that subdiffusive scaling occurs for $1 < s < 2$. Before giving a precise limit law, we note that tail estimates for probabilities already follow from Lemma 2.4.2. The proofs of the following statements can be found in [42]. Note the difference between annealed and quenched rates of decay, which are caused by the fact that under the annealed law, one may, in order to see a rare event, modify the environment.

Theorem 2.4.9 (Dembo, Peres, Zeitouni[28], Gantert, Zeitouni[16]) Slowdown probabilities *Assume Assumption 2.4.1 and $v_P > 0$ (hence, $s > 1$). Then, for any $v \in (0, v_P)$ and any $\delta > 0$ small enough,*

$$\lim_{n\to\infty} \frac{\log \mathbb{P}^o\left(\frac{X_n}{n} \in (v-\delta, v+\delta)\right)}{\log n} = 1 - s .$$

Further, for any $\eta > 0$,

$$\liminf_{n\to\infty} \frac{1}{n^{1-1/s+\eta}} \log P_\omega^o \left(\frac{X_n}{n} \in (v-\delta, v+\delta) \right) = 0 , \quad P-a.s. \qquad (2.4.10)$$

Finally,

$$\limsup_{n\to\infty} \frac{1}{n^{1-1/s-\eta}} \log P_\omega^o \left(\frac{X_n}{n} < v \right) = -\infty , \quad P-a.s.. \qquad (2.4.11)$$

Remark: : Tail estimates, quenched and annealed, for the case $\rho_{\min} = 1$ are detailed in [27, 28].

We turn next to the description of limit results. At this point, limit theorems for the general ergodic (mixing) case are not available, and severe restrictions on the environment need to be made. We say that a Markov kernel $H(x, \cdot)$ on a state space $\mathcal{S}$ is dominated above and below if there exists an integer m, a probability measure ν and a constant $c_r > 0$ such that for some $m \in \mathbb{N}$,

$$c_r^{-1}\psi(A) < H^m(x, A) < c_r\psi(A) \quad \forall x \in \mathcal{S}, A \subset \mathcal{S} \text{ measurable}. \tag{2.4.12}$$

For $\kappa \in (0, 2)$, denote by $\mathcal{L}_{\kappa,b}$ the distribution function of the stable law with the logarithm of the characteristic function

$$\log \widehat{\mathcal{L}}_{\kappa,b}(t) = -b|t|^\kappa \left(1 + i\frac{t}{|t|} f_\kappa(t)\right), \tag{2.4.13}$$

where $b > 0$, and $f_\kappa(t) = \tan \frac{\pi}{2}\kappa$ if $\kappa \neq 1$, $f_1(t) = 2/\pi \log t$. It can be shown that if $\kappa < 1$, then the law $\mathcal{L}_{\kappa,b}$ is supported on the positive reals. Finally, set

$$\Lambda(\beta) := \lim_{n\to\infty} \frac{1}{n} \log E_P \left(\prod_{i=0}^{n-1} \rho_i^\beta\right),$$

and, assuming parts (S1) and (S3) of Assumption 2.4.1 (implicitly, assuming the existence of a LDP for $\overline{R}_k$), set $s > 0$ such that $\Lambda(s) = 0$.

Theorem 2.4.14 (Kesten,Kozlov, Spitzer[22]) (Mayer-Wolf, Roitershtein, Zeitouni[25]) *Suppose P is either a product measure or an irreducible Markov chain with either a finite state space or dominated above and below Markov kernel. Further assume that the law of* $\log \rho_0$ *is non-arithmetic. Then, for some constant b that depends on P,*

(i) If $s \in (0, 1)$, then $\lim_{n\to\infty} \mathbb{P}\,(n^{-s}X_n \le t) = 1 - \mathcal{L}_{s,b}(t^{-1/s})$,

(ii) If $s = 1$, then $\lim_{n\to\infty} \mathbb{P}\big(n^{-1}(\log n)^2(X_n - \delta(n)) \le t\big) = 1 - \mathcal{L}_{1,b}(-t)$, *for a suitable* $\delta(n) \sim A_1(\log n)^{-1}n$,

(iii) If $s \in (1, 2)$, then $\lim_{n\to\infty} \mathbb{P}\left(n^{-1/s}(X_n - n\mathrm{v}_P) \le t\right) = 1 - \mathcal{L}_{s,b}(-t\mathrm{v}_P^{1+1/s})$.

(iv) If $s = 2$, then $\lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - n\mathrm{v}_P}{\sqrt{n \log n}} \le t\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^t e^{-r^2/2\sigma^2} dr$, *for some $\sigma > 0$.*

We conclude this section by describing a sketch, following [22], of the proof of Theorem 2.4.14, in the case of P being a product measure. First, one notes that the (annealed) limit results above follow from limit results for the sequence $\{T_n\}$. To handle these, the analysis of T_n is best understood in terms of certain regeneration times $\{\nu_n\}$, constructed starting with the variables U_i introduced in Section 2.2: indeed, modifying the $\{U_i\}$ to variables U_i^n counting only excursions up to T_n, and writing $T_n = n + 2\sum_{i=-\infty}^n U_i^n$, one focuses attention to the U_i^n. Since the walk is transient to ∞, $\sum_{i=-\infty}^{0} U_i^n$ is finite $\mathbb{P}$-a.s. and does not affect the asymptotics, and thus one needs only consider the variable $\sum_{i=1}^{n} U_i^n$. This variable can be represented as $\sum_{t=0}^{n-1} Z_t$, where $Z_0 = 0, Z_1, Z_2, \ldots$ forms a (subcritical) branching process in random environment with one immigrant at each unit of time. One now lets ν_i denote the successive times in which $Z_t = 0$. In the case that P is a product measure, the partial sums of the Z_t between these "collapse" times, denoted W_n, form an i.i.d. sequence, and the crux of the proof is to control the tails of these partial sums (when P is Markov with finite state space, one modifies the collapse time by adding the requirement that the Markov chain visits a particular state at that time, recovering the i.i.d. structure of the sequence $\{W_n\}$. In the more general setup, one obtains a 1-dependent sequence, and some extra work is needed to proceed, see [25]).

Thus, all the work has been reduced to computing precise tail estimates for the $\{W_n\}$. One checks, with some pain, that ν_1 possesses exponential tails. Further, definining $S_1 = \sum_{i=1}^{\infty} \prod_{j=0}^{i} \rho_{-j}$, one has that $S_1 < \infty$, $\mathbb{P} - a.s.$, and further, a delicate renewal argument shows that

$$\lim_{t\to\infty} t^s P(S_1 > t) = K \in (0, \infty) .$$

(in the general Markov case, one considers $P(S_1 > t | \sigma(\omega_i, i > 0))$, and the statement is that convergence to a *random* limit holds). Once such tail estimates for S_1 hold, it is non-trivial but routine to transfer them to tail estimates on W_1, and the stable limit results follow from standard limit laws for sums of independent random variables.

Remark: It is worthwhile to note the significance of the collapse times ν_i. Re-working the definitions, one sees that these are time at which the walk crosses an edge on its way to $+\infty$, never to backtrack again. Such regeneration times play a crucial role in the study of RWRE in $\mathbb{Z}^d$.

3 RWRE – $d > 1$

3.1 $0 - 1$ laws and law of large numbers

In this section we turn our attention to the study of RWRE in $\mathbb{Z}^d$. Even if considerable progress was achieved in recent years, the situation here is, unfortunately, much less satisfying than for $d = 1$.

To simplify things, we assume throughout this chapter that P is a uniformly elliptic product measure. In the last section, we indicate some techniques allowing for partially relaxing this assumption.

Fix $\ell \in \mathbb{R}^d \setminus \{0\}$, and consider the events

$$A_{\pm\ell} = \{\lim_{n\to\infty} X_n \cdot \ell = \pm\infty\} .$$

We have the

Theorem 3.1.1 (Kalikow[19])

$$\mathbb{P}^o(A_\ell \cup A_{-\ell}) \in \{0,1\}\,.$$

Proof. We will prove the theorem for $\ell = (1,0\ldots0)$, the general case being similar but requiring more cumbersome notations. Although not strictly needed for the proof of the theorem in the product measure case, the following construction of an extension of our probability space turns out to be very useful. Recall that the RWRE was defined by means of the law $\mathbb{P}^o = P \otimes P_\omega^o$ on $(\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{F} \times \mathcal{G})$. Set $W = \{0\} \cup \{\pm e_i\}_{i=1}^d$ and $\mathcal{W}$ the cylinder σ-algebra on $W^{\mathbb{N}}$. We now define the measure

$$\overline{\mathbb{P}}^o = P \otimes Q_\varepsilon \otimes \overline{P}^o_{\omega,\varepsilon}$$

on

$$\Big(\Omega \times W^{\mathbb{N}} \times (\mathbb{Z}^d)^{\mathbb{N}}, \quad \mathcal{F} \times \mathcal{W} \times \mathcal{G}\Big)$$

in the following way: Q_ε is a product measure, such that with $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$ denoting an element of $W^{\mathbb{N}}$, $Q_\varepsilon(\varepsilon_1 = \pm e_i) = \varepsilon$, $i = 1, \cdots, d$, $Q_\varepsilon(\varepsilon_1 = 0) = 1 - 2\varepsilon d$. For each fixed ω, ε, $\overline{P}^o_{\omega,\varepsilon}$ is the law of the Markov chain $\{X_n\}$ with state space $\mathbb{Z}^d$, such that $X_0 = 0$ and, for each $e \in W$, $e \neq 0$,

$$\overline{P}^o_{\omega,\varepsilon}(X_{n+1} = z + e | X_n = z) = \mathbf{1}_{\{\varepsilon_{n+1} = e\}} + \frac{\mathbf{1}_{\{\varepsilon_{n+1}=0\}}}{1 - 2d\varepsilon}[\omega(z, z+e) - \varepsilon]\,.$$

It is not hard to check that the law of $\{X_n\}$ under $\overline{\mathbb{P}}^o$ coincides with its law under $\mathbb{P}^o$, while its law under $Q_\varepsilon \otimes \overline{P}^o_{\omega,\varepsilon}$ coincides with its law under P^o_ω.

Note that for any $u < v$, the walk cannot visit infinitely often the strip $u \le z \cdot \ell \le v$ without crossing the line $z \cdot \ell = v$. More precisely, with

$$T_v = \inf\{n \ge 0 : \quad X_n \cdot \ell \ge v\}, \tag{3.1.2}$$

we have

$$\mathbb{P}^o\Big(\#\{n > 0 :\ X_n \cdot \ell \ge u\} = \infty, T_v = \infty\Big) = 0\,. \tag{3.1.3}$$

Indeed, note that for any z with $u \le z \cdot \ell \le v$, and any ω,

$$P^z_\omega(X_{v-u} \cdot \ell \ge v) = Q_\varepsilon \otimes P^z_{\omega,\varepsilon}(X_{v-u} \cdot \ell \ge v) \ge \varepsilon^{v-u}\,,$$

yielding (3.1.3) by the strong Markov property.

Assume next that $\mathbb{P}^o(A_\ell) > 0$. Set $D = \inf\{n \ge 0 :\ X_n \cdot \ell < X_0 \cdot \ell\}$. Clearly, $\mathbb{P}^o(D = \infty) > 0$, because if $\mathbb{P}^o(D = \infty) = 0$ then $\mathbb{P}^z(D < \infty) = 1$ $\forall z \in \mathbb{Z}^d$, and thus P-a.s., for all $z \in \mathbb{Z}^d$, $P^z_\omega(D < \infty) = 1$. This implies by the Markov property that

$$\liminf_{n\to\infty} X_n \cdot \ell \le 0, \quad \mathbb{P}^o\text{-a.s.}\,,$$

contradicting $\mathbb{P}^o(A_\ell) > 0$.

Define $\mathcal{O}_\ell$ to be the event that $X_n \cdot \ell$ changes its sign infinitely often. We next show that whenever $\mathbb{P}^o(A_\ell) > 0$, then $\mathbb{P}^o(\mathcal{O}_\ell) = 0$. Set $M = \sup_n X_n \cdot \ell$, fix $v > 0$ and note by (3.1.3) that

$$\mathbb{P}^o(\mathcal{O}_\ell \cap \{M < v\}) = 0\,. \tag{3.1.4}$$

We next prove that if $\mathbb{P}^o(A_\ell) > 0$ then $\mathbb{P}^o(\mathcal{O}_\ell \cap \{M = \infty\}) = 0$, by first noting that

$$\mathbb{P}^o(\mathcal{O}_\ell \cap \{M = \infty\}) = \overline{\mathbb{P}}^o(\mathcal{O}_\ell \cap \{M = \infty\})\,.$$

Then, set $\mathcal{G}_n = \sigma((\varepsilon_i, X_i),\, i \le n)$, fix $L > 0$ (for the proof below, $L = 1$ works), and, setting $S_0 = 0$, define recursively $\mathcal{G}_n$ stopping times as follows:

$$\begin{aligned} R_k &= \inf\Big\{n \ge S_k :\ X_n \cdot \ell < 0\}\,, \\ S_{k+1} &= \inf\{n \ge R_k : X_{n-L} \cdot \ell \\ &\qquad \ge \max\{X_m \cdot \ell :\ m \le n-L\},\, \varepsilon_{n-1} = \varepsilon_{n-2} = \ldots = \varepsilon_{n-L} = e_1\Big\}\,. \end{aligned}$$

On $\mathcal{O}_\ell \cap \{M = \infty\}$, all these stopping times are finite. Now, at each time $S_k - L$ the walk enters a half space it never visited before, and then *due to the action of the ε sequence alone*, it proceeds L steps in the direction e_1. Formally, "events in the σ-algebra $\mathcal{G}_{S_k}$ are L-separated from $\sigma(\omega_z :\ z \cdot \ell \ge X_{S_k} \cdot \ell)$". Note that, using $\mathbb{P}^o(A_\ell) > 0$ in the second inequality,

$$\overline{\mathbb{P}}^o(R_0 < \infty) = \overline{\mathbb{P}}^o(D < \infty) < 1\,,$$

whereas, using θ to denote both time and space shifts as needed from the context,

$$\begin{aligned} \overline{\mathbb{P}}^o(R_1 < \infty) &\le \overline{\mathbb{P}}^o(R_0 < \infty, R_0 \circ \theta_{X_{S_1}} < \infty) \\ &= \sum_{z \in \mathbb{Z}^d} \overline{\mathbb{P}}^o(R_0 < \infty, R_0 \circ \theta_z < \infty, X_{S_1} = z) \\ &= \sum_{z \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}} E_{P \otimes Q_\varepsilon}\Big(\overline{P}^o_{\omega,\varepsilon}(R_0 < \infty, X_{S_1} = z, S_1 = n) \cdot \overline{P}^o_{\theta^z\omega, \theta^n\varepsilon}(R_0 < \infty)\Big)\,. \end{aligned}$$

Note that $\overline{P}^o_{\theta^z\omega,\theta^n\varepsilon}(R_0 < \infty)$ is measurable on $\sigma(\omega_x :\ x \cdot \ell \ge z \cdot \ell) \times \sigma(\varepsilon_i, i > n)$, whereas $\overline{P}^o_{\omega,\varepsilon}(R_0 < \infty, X_{S_1} = z, S_1 = n)$ is measurable on $\sigma(\omega_x : x \cdot \ell \le z \cdot \ell - L) \times \sigma(\varepsilon_i, i \le n)$. Hence, by the product structures of P and Q_ε,

$$\begin{aligned} \overline{\mathbb{P}}^o(R_1 < \infty) &\le \sum_{z \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}} \Big[E_{P \otimes Q_\varepsilon}\Big(\overline{P}^o_{\omega,\varepsilon}(R_0 < \infty, X_{S_1} = z, S_1 = n)\Big) \\ &\qquad\qquad \cdot E_{P \otimes Q_\varepsilon}\Big(\overline{P}^o_{\omega,\varepsilon}(R_0 < \infty)\Big)\Big] \\ &\le (\overline{\mathbb{P}}^o(R_0 < \infty))^2 \le (\overline{\mathbb{P}}^o(D < \infty))^2\,. \end{aligned}$$

Repeating this procedure, we conclude that $\overline{P}^o(\mathcal{O}_\ell \cap \{M = \infty\}) \leq \overline{\mathbb{P}}^o(R_k < \infty) \leq (\mathbb{P}^o(D < \infty))^{k+1}$. Since k is arbitrary, we conclude that $\overline{\mathbb{P}}^o(\mathcal{O}_\ell \cap \{M = \infty\}) = 0$, yielding with the above that $\mathbb{P}^o(\mathcal{O}_\ell) = 0$ as soon as $\mathbb{P}^o(A_\ell) > 0$. In a similar manner one proves that $\mathbb{P}^o(A_{-\ell}) > 0$ also implies $\mathbb{P}^o(\mathcal{O}_\ell) = 0$.

Assume now $1 > \mathbb{P}^o(A_\ell \cup A_{-\ell})$. Then one can find a v such that $\mathbb{P}^o(X_n \cdot \ell \in [-v, v]$ infinitely often$) > 0$. Therefore, $\mathbb{P}^o(\mathcal{O}_\ell) > 0$, implying by the above $\mathbb{P}^o(A_\ell) = \mathbb{P}^o(A_{-\ell}) = 0$. □

In the particular case of $d = 2$, [45] proved that in fact, $\mathbb{P}^o(A_\ell) \in \{0, 1\}$, and provided counter examples to this claim for certain ergodic (but not uniformly elliptic!) environments. Whether or not this $0 - 1$ law holds for uniformly elliptic product measures with $d > 2$ is one of the major open problems in the study of multi-dimensional RWRE. As we are about to see, the resolution of this question is the only obstacle toward proving a general law of large numbers, to which we turn next.

Theorem 3.1.5 (Sznitman-Zerner[40], Zerner[46]) *There exist deterministic $v_\ell, v_{-\ell}$ (possibly zero) such that*

$$\lim_{n\to\infty} \frac{X_n \cdot \ell}{n} = v_\ell \mathbf{1}_{A_\ell} + v_{-\ell} \mathbf{1}_{A_{-\ell}}, \quad \mathbb{P}^o\text{-}a.s. \tag{3.1.6}$$

(See (3.1.13) for some information on v_ℓ. When $v_\ell \neq 0$ we say that the walk is *ballistic* in the direction ℓ).

An immediate corollary, obtained by applying Theorem 3.1.5 d times with respect to the basis $\ell = e_i$, $i = 1, \ldots, d$, is the following:

Corollary 3.1.7 *Assume that $\mathbb{P}^o(A_\ell) \in \{0, 1\}$ for every ℓ. Then there exists a deterministic v (possibly zero) such that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v, \quad \mathbb{P}^o\text{-}a.s.$$

Proof of Theorem 3.1.5: (sketch) As before, to prove (3.1.6) we will only consider $\ell = (1, 0, \cdots, 0)$. First, assume that $\mathbb{P}^o(A_\ell) > 0$ (and hence, by the $0 - 1$ law, $\mathbb{P}^o(A_\ell \cup A_{-\ell}) = 1$). The proof is based on introducing a renewal structure, as follows: Define $\overline{S}_0 = 0, M_0 = \ell \cdot X_0$,

$$\begin{aligned} &\overline{S}_1 = T_{M_0+1} \leq \infty,\ \overline{R}_1 = D \circ \theta_{\bar{S}_1} + \overline{S}_1 \leq \infty, \\ &M_1 = \sup\{\ell \cdot X_m, \quad 0 \leq m \leq \overline{R}_1\} \leq \infty \end{aligned}$$

and by induction, for $k \geq 1$,

$$\begin{aligned} &\overline{S}_{k+1} = T_{M_k+1} \leq \infty,\ \overline{R}_{k+1} = D \circ \theta_{\bar{S}_{k+1}} + \overline{S}_{k+1} \leq \infty, \\ &M_{k+1} = \sup\{\ell \cdot X_m, \quad 0 \leq m \leq \overline{R}_{k+1}\} \leq \infty\,. \end{aligned}$$

The times $\overline{S}_1, \overline{S}_2, \ldots$, are called "fresh times", and the locations $X_{\overline{S}_1}$, $X_{\overline{S}_2}$, $\cdots$, are "fresh points": at the time $\overline{S}_k$, the path $X.$ visits for the first time after $\overline{S}_{k-1}$ and after hitting again the hyperplane $X_{\overline{S}_{k-1}} \cdot \ell - 1$, a fresh part of the environment. Note that $(\overline{S}_i, \overline{R}_i)$ are related to, but differ slightly from, (S_i, R_i) introduced earlier. Clearly,

$$0 = \overline{S}_0 \le \overline{S}_1 \le \overline{R}_1 \le \overline{S}_2 \le \cdots \le \infty$$

and the inequalities are strict if the left member is finite. Define:

$$\begin{aligned} K &= \inf\{k \ge 1 : \overline{S}_k < \infty, \overline{R}_k = \infty\} \le \infty, \\ \tau_1 &= \overline{S}_K \le \infty\,. \end{aligned}$$

τ_1 is called a "regeneration time", because after τ_1, $X_\cdot \cdot \ell$ never falls behind $X_{\tau_1} \cdot \ell$.

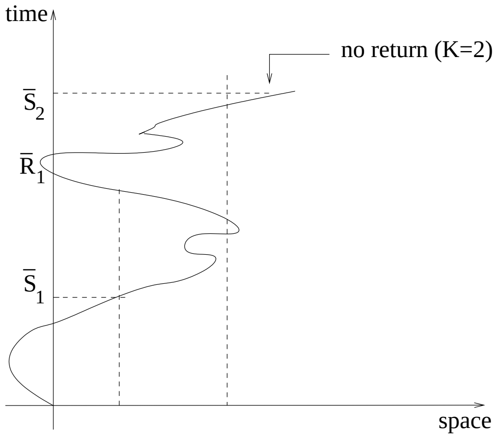

Fig. 3.1.1. Regeneration structure

By the same argument as in the proof of Theorem 3.1.1, $\mathbb{P}^o(\overline{R}_k < \infty) \le \mathbb{P}^o(D < \infty)^k \underset{k\to\infty}{\longrightarrow} 0$ because $\mathbb{P}^o(A_\ell) > 0$ implies $\mathbb{P}^o(D < \infty) < 1$. On the other hand, on A_ℓ, $\overline{R}_k < \infty \Rightarrow S_{k+1} < \infty$, $\mathbb{P}^o$-a.s., and hence

$$\mathbb{P}^o(A_\ell \cap \{K = \infty\}) = \mathbb{P}^o(A_\ell \cap \{\tau_1 = \infty\}) = 0\,.$$

Define now the measure

$$\mathbb{Q}^o(\cdot) = \mathbb{P}^o(\cdot\,|\{\tau_1 < \infty\}) = \mathbb{P}^o(\cdot\,|A_\ell)$$

and set

$$\mathcal{G}_1 = \sigma\left(\tau_1, X_0, \cdots, X_{\tau_1}, \{\omega(y,\cdot)\}_{\ell\cdot y < \ell\cdot X_{\tau_1}}\right)\,.$$

Note that since $\{D = \infty\} \subset \{\tau_1 < \infty\}$, we have that $\{D = \infty\} \in \mathcal{G}_1$. We have the following crucial lemma, whose proof is a simple exercise in the application of the Markov property, and is omitted. It is here that the i.i.d. assumption on the environment plays a crucial role:

Lemma 3.1.8 *For any measurable sets A, B,*

$$\begin{aligned}&\mathbb{Q}^o\Big(\{X_{\tau_1+n}-X_{\tau_1}\}_{n\geq 0}\in A, \{\omega(X_{\tau_1}+y,\cdot)\}_{y\cdot\ell\geq 0}\in B\Big)\\ =&\mathbb{P}^o\Big(\{X_n\}_{n\geq 0}\in A, \{\omega(y,\cdot)\}_{y\cdot\ell\geq 0}\in B|\{D=\infty\}\Big)\,.\end{aligned}$$

In fact,

$$\begin{aligned}&\mathbb{Q}^o\Big(\{X_{\tau_1+n}-X_{\tau_1}\}_{n\geq 0}\in A, \{\omega(X_{\tau_1}+y,\cdot)\}_{y\cdot\ell\geq 0}\in B|\mathcal{G}_1\Big)\\ =&\mathbb{P}^o\Big(\{X_n\}_{n\geq 0}\in A, \{\omega(y,\cdot)\}_{y\cdot\ell\geq 0}\in B|\{D=\infty\}\Big)\,.\end{aligned}\tag{3.1.9}$$

Consider now τ_1 as a function of the path $(X_n)_{n\geq 0}$ and set

$$\tau_{k+1}=\tau_k(X.)+\tau_1(X_{\tau_k+\cdot}-X_{\tau_k})\,,$$

with $\tau_{k+1}=\infty$ on $\{\tau_k=\infty\}$ (the sequence $\{\tau_k\}$ enumerates times such that for all $k<m<n$, $X_k\cdot\ell<X_m\cdot\ell\leq X_n\cdot\ell$). By the definition and the fact that $\mathbb{P}^o(A_\ell\cap\{\tau_1=\infty\})=0$, we have that $\mathbb{P}^o(A_\ell\cap\{\tau_k=\infty\})=0$. Setting

$$\mathcal{G}_k=\sigma\Big(\tau_1,\ldots,\tau_k,\quad X_0,\cdots,X_{\tau_k},\quad \{\omega(y,\cdot)\}_{\ell\cdot y<\ell\cdot X_{\tau_k}}\Big)\,,$$

an obvious restatement of Lemma 3.1.8 yields that

$$\begin{aligned}&\mathbb{Q}^o\Big(\{X_{\tau_k+n}-X_{\tau_k}\}_{n\geq 0}\in A, \{\omega(X_{\tau_k+y},\cdot)\}_{\ell\cdot y\geq 0}\in B|\ \mathcal{G}_k\Big)\\ =&\ \mathbb{P}^o\Big(\{X_n\}_{n\geq 0}\in A, \{\omega(y,\cdot)\}_{\ell\cdot y\geq 0}\in B|\{D=\infty\}\Big)\,.\end{aligned}$$

We thus conclude that under $\mathbb{Q}^o$,

$$(X_{\tau_2}-X_{\tau_1},\tau_2-\tau_1)\,,\cdots,(X_{\tau_{k+1}}-X_{\tau_k},\tau_{k+1}-\tau_k)$$

are i.i.d. pairs of random variables, independent of (X_{τ_1},τ_1), such that

$$\mathbb{Q}^o(X_{\tau_2}-X_{\tau_1}\in C_1,\tau_2-\tau_1\in C_2)=\mathbb{P}^o(X_{\tau_1}\in C_1,\tau_1\in C_2|\{D=\infty\})\,.$$

Next, we have the following lemma, whose proof is deferred:

Lemma 3.1.10 (Zerner)

$$\mathbb{E}^o(X_{\tau_1}\cdot\ell|\{D=\infty\})=\frac{1}{\mathbb{P}^o(D=\infty)}\,.$$

We are now ready to complete the proof of (3.1.6). Assume first that $\mathbb{E}^o(\tau_1|\{D=\infty\})<\infty$. Then, by the law of large numbers, and the renewal structure,

$$\frac{\tau_k}{k} \to_{k\to\infty} \mathbb{E}^o(\tau_1|\{D=\infty\}), \qquad \mathbb{Q}^o\text{-a.s.} \tag{3.1.11}$$

$$\frac{X_{\tau_k}\cdot\ell}{k} \to_{k\to\infty} \mathbb{E}^o(X_{\tau_1}\cdot\ell|\{D=\infty\}), \quad \mathbb{Q}^o\text{-a.s.} \tag{3.1.12}$$

(note that the finiteness of the expression in the right hand side of (3.1.12) is trivial if the right hand side of (3.1.11) is finite, and Lemma 3.1.10 is not needed in this case).

Hence,

$$\frac{X_{\tau_k}\cdot\ell}{k} \longrightarrow_{k\to\infty} \frac{\mathbb{E}^o(X_{\tau_1}\cdot\ell|\{D=\infty\})}{\mathbb{E}^o(\tau_1|\{D=\infty\})} =: v_\ell, \quad \mathbb{Q}\text{-a.s.} \tag{3.1.13}$$

Mimicking now the argument at the end of the proof of Lemma 2.1.4, we conclude that $\frac{X_n\cdot\ell}{n} \to_{n\to\infty} v_\ell, \mathbb{Q}^o$-a.s., in the case $\mathbb{E}^0(\tau_1|\{D=\infty\})<\infty$.

On the other hand, Lemma 3.1.10 implies that (3.1.12) holds true even when $\mathbb{E}^o(\tau_1|\{D=\infty\})=\infty$. But then, $\tau_k/k\to_{k\to\infty}\infty$, $\mathbb{Q}^o$-a.s. With $v_\ell=0$ in this case, we conclude that

$$\frac{X_{\tau_k}\cdot\ell}{\tau_k} \longrightarrow_{k\to\infty} v_\ell=0, \quad \mathbb{Q}^o\text{-a.s.}$$

Finally, setting k_n such that $\tau_{k_n}\le n<\tau_{k_n+1}$, we have that $k_n\to_{n\to\infty}\infty$ and $k_n/n \underset{n\to\infty}{\longrightarrow} 0$, $\mathbb{Q}^o$-a.s. because $n/k_n\ge\tau_{k_n}/k_n$. Thus,

$$\frac{X_n\cdot\ell}{n} \le \frac{X_{\tau_{k_n+1}}\cdot\ell}{k_n+1}\cdot\frac{k_n+1}{n} \underset{n\to\infty}{\longrightarrow} 0,\ \mathbb{Q}^o\text{-a.s.}$$

Since $\liminf_{n\to\infty}\frac{X_n\cdot\ell}{n}\ge 0$, $\mathbb{Q}^o$-a.s., we conclude

$$\frac{X_n\cdot\ell}{n} \underset{n\to\infty}{\longrightarrow} 0, \quad \mathbb{Q}^o\text{-a.s.} \tag{3.1.14}$$

A similar proof works on the event $A_{-\ell}$, and thus all that remains to prove is that if $\mathbb{P}^o(A_\ell\cup A_{-\ell})=0$ then $X_n\cdot\ell/n\to 0$, $\mathbb{P}^o$-a.s. The complete proof for that is given in [46], and we provide next a brief description.

Consider the set of visits to the hyperplane $\mathcal{H}_m:=\{z: z\cdot\ell=m\}$, defining $\tau_m^0=T_m$ and $\tau_m^i=\min\{n>\tau_m^{i-1}: X_n\cdot\ell=m\}$. Fixing an integer L, let

$$h_{m,L}=\sup_{i\ge 0}\{\tau_m^i-\tau_m^0:\tau_m^i<T_{m+L}\}$$

be the diameter of the set of visits to $\mathcal{H}_m$ before T_{m+L}. For any constant $c>0$, let

$$F_{M,L}(c) = \frac{\#\{0 \le m \le M : h_{m,L} \le c\}}{M+1}$$

denote the fraction of m's smaller than M such that the time between the first and last visit to $\mathcal{H}_m$ before T_{m+L} is smaller than c. The first observation, which is a deterministic (combinatorial) computation that we skip, is that for any path with $\liminf_{n\to\infty} X_n \cdot \ell/n > 0$ there exists a constant c such that

$$\inf_{L\ge 1} \limsup_{M\to\infty} F_{M,L}(c) > 0\,,$$

that is, roughly, there is a fraction of m's for which the time between first and last visits of $\mathcal{H}_m$ (before hitting $\mathcal{H}_{m+L}$) is not too large.

Assume now that $\mathbb{P}^o(\limsup X_n \cdot \ell/n > 0) > 0$. Then, by the above observation, there is some $c > 0$ such that

$$\mathbb{P}^o(\limsup_{L\to\infty} \limsup_{M\to\infty} F_{M,L}(c) > 0) > 0\,. \tag{3.1.15}$$

But on the event $\{h_{m,L} \le c\}$, the last point visited in $\mathcal{H}_m$ before hitting $\mathcal{H}_{m+L}$ is at most at distance c from X_{T_m} and has been visited at most c times before T_{m+L}. Thus, there is a $z \in \mathcal{H}_0$ with $|z|_1 \le c$, and an $1 \le r \le c$ such that the r-th visit to $X_{T_m} + z$ occurs before T_{m+L} and the walk does not backtrack from $\mathcal{H}_m$ after this r-th visit. Denoting the last event by $B^1_{m,L}(z,r)$, it follows that

$$F_{M,L}(c) \le \frac{1}{M+1} \sum_{z\in\mathcal{H}_0, |z|_1 \le c} \sum_{r=1}^{c} \sum_{m=0}^{M} \mathbf{1}_{B^1_{m,L}(z,r)}\,.$$

Noting that the summation over r and z is over a finite set, and combining the last inequality with (3.1.15), it follows that for some z and r,

$$\mathbb{P}^o(\limsup_{L\to\infty} \limsup_{M\to\infty} \frac{1}{M+1} \sum_{m=0}^{M} \mathbf{1}_{B^1_{m,L}(z,r)} > 0) > 0\,. \tag{3.1.16}$$

While the events $\{B^1_{m,L}(z,r)\}_m$ are not independent, some independence can be restored in the following way: construct independent (given the environment) copies $Y^y_\cdot$ of the RWRE, starting at y. Define the event $B_{m,L}(z,r)$ as the union of $B^1_{m,L}(z,r)$ with the event that $X_\cdot$ does not hit $X_{T_m} + z$ for the r-th time before T_{m+L}, but $Y_\cdot^{X_{T_m}+z}$ does not backtrack from $\mathcal{H}_m$ before it hits $\mathcal{H}_{m+L}$. An easy computation involving the Markov property shows that for each fixed $i = 0, 1, \ldots, L-1$, the events $\{B_{jL+i,L}(z,r)\}_j$ are independent, with

$$\mathbb{P}^o(B_{jL+i,L}(z,r)) = \mathbb{P}^o(D \ge T_L)\,.$$

(Here and in the sequel, we abuse notations by still using $\mathbb{P}^o$ to denote the annealed law on the enlarged probability space that supports the extra Y^y walks). Hence, since we have from (3.1.16) that

$$\mathbb{P}^o(\limsup_{L\to\infty}\limsup_{M\to\infty}\frac{1}{M+1}\sum_{i=0}^{L-1}\sum_{j=0}^{[M/L]}\mathbf{1}_{B^1_{jL+i,L}(z,r)}>0)>0\,, \tag{3.1.17}$$

it follows, by the standard law of large numbers, that

$$\mathbb{P}^o(D=\infty)=\limsup_{L\to\infty}\mathbb{P}^o(D\geq T_L)>0\,.$$

But from (3.1.3), we have that $\mathbb{P}^o(A_\ell)\geq\mathbb{P}^o(D=\infty)>0$. In particular, this shows that $\mathbb{P}^o(A_\ell)=0$ implies that $\limsup X_n\cdot\ell/n\leq 0$, $\mathbb{P}^o$-a.s. Repeating this argument with $-\ell$ instead of ℓ completes the proof of the theorem. □

Proof of Lemma 3.1.10 Recall that we consider $\ell=(1,0,\ldots,0)$. Then,

$$\begin{aligned}
&\mathbb{Q}^o\Big(\{\exists k:\ X_{\tau_k}\cdot\ell=i\}\Big)\\
&=\frac{\sum_{y\in\mathbb{Z}^{d-1}}E\Big(P^o_\omega(\{\exists k:\ X_{\tau_k}=(i,y)\},A_\ell)\Big)}{\mathbb{P}^o(A_\ell)}\\
&=\frac{\sum_{y\in\mathbb{Z}^{d-1}}E\Big(P^o_\omega(T_i<\infty,X_{T_i}=(i,y),D\circ\theta_{T_i}=\infty)\Big)}{\mathbb{P}^o(A_\ell)}\\
&=\frac{\sum_{y\in\mathbb{Z}^{d-1}}E\Big(P^o_\omega(T_i<\infty,X_{T_i}=(i,y))P^{(i,y)}_\omega(D=\infty)\Big)}{\mathbb{P}^o(A_\ell)}\\
&=\frac{\mathbb{P}^o(T_i<\infty)\mathbb{P}^o(D=\infty)}{\mathbb{P}^o(A_\ell)}\underset{i\to\infty}{\longrightarrow}\mathbb{P}^o(D=\infty)
\end{aligned} \tag{3.1.18}$$

(since $\mathbb{P}^o(A_\ell\cup A_{-\ell})=1$ and $\lim_{i\to\infty}\mathbb{P}^o(\{T_i<\infty\}\cap A_{-\ell})=0$). On the other hand,

$$\lim_{i\to\infty}\mathbb{Q}^o\Big(\{\exists k:\ X_{\tau_k}\cdot\ell=i\}\Big)=\lim_{i\to\infty}\mathbb{Q}^o\Big(\{\exists k\geq 2:\ X_{\tau_k}\cdot\ell=i\}\Big)$$

(because $\mathbb{Q}^o(\tau_k>i)\underset{i\to\infty}{\longrightarrow}0$)

$$\begin{aligned}
&=\lim_{i\to\infty}\sum_{n\geq 1}\mathbb{Q}^o\Big(\{\exists k\geq 2:\ X_{\tau_k}\cdot\ell=i,X_{\tau_1}\cdot\ell=n\}\Big)\\
&=\lim_{i\to\infty}\sum_{n\geq 1}\mathbb{Q}^o\Big(\{\exists k\geq 2:\ (X_{\tau_k}-X_{\tau_1})\cdot\ell=i-n,X_{\tau_1}\cdot\ell=n\}\Big)\\
&=\lim_{i\to\infty}\sum_{n\geq 1}\mathbb{Q}^o(X_{\tau_1}\cdot\ell=n)\cdot\mathbb{Q}^o\Big(\{\exists k\geq 2:(X_{\tau_k}-X_{\tau_1})\cdot\ell=i-n\}\Big)\,.
\end{aligned}$$

But, recall that by the renewal theorem,

$$\mathbb{Q}^o\Big(\exists k\geq 2:\ (X_{\tau_k}-X_{\tau_1})\cdot\ell=i-n\Big)\underset{i\to\infty}{\longrightarrow}\frac{1}{E_{\mathbb{Q}^o}((X_{\tau_2}-X_{\tau_1})\cdot\ell)}$$

and hence, by dominated convergence,

$$\lim_{i\to\infty} \mathbb{Q}^o\Big(\{\exists k:\ X_{\tau_k}\cdot\ell = i\}\Big) = \frac{\sum_{n\geq 1}\mathbb{Q}^o(X_{\tau_1}\cdot\ell = n)}{E_{\mathbb{Q}^o}((X_{\tau_2}-X_{\tau_1})\cdot\ell)} = \frac{1}{E_{\mathbb{Q}^o}((X_{\tau_2}-X_{\tau_1})\cdot\ell)}. \tag{3.1.19}$$

Comparing (3.1.18) and (3.1.19), we conclude that

$$E_{\mathbb{Q}^o}\Big((X_{\tau_2}-X_{\tau_1})\cdot\ell\Big) = \frac{1}{\mathbb{P}^o(D=\infty)} < \infty\,. \qquad \square$$

Remarks:

1. Note that on A_ℓ, $v_\ell > 0$ if $\mathbb{E}^o(\tau_1|\{D=\infty\}) < \infty$ and $v_\ell = 0$ otherwise.
2. Modifying slightly the definition of $(\overline{R}_k, \overline{S}_k)$ by adding an L-safeguard as in the proof of Theorem 3.1.1, the results of this section extend immediately to the case where the environment is K-dependent (i.e., $\{\omega(x,\cdot)\}_{x\cdot\ell\leq 0}$ and $\{\omega(x,\cdot)\}_{x\cdot\ell>K}$ are independent). The extension to a mixing setup is more complicated, and one needs to consider a modification of these times in which the event $D=\infty$ is replaced by the event that the walk does not exit a cone based at the origin, and assume that the environment is "mixing on cones". For more details, see [9].
3. Still discussing mixing environements, some progress has been made using the approach of the environment viewed from the particle. We mention here in particular [29], that uses a-priori estimates concerning regeneration times in the ballistic case to construct an invariant measure for the environment viewed from the particle which is absolutely continuous with respect to P on certain half-spaces, and deduces a LLN using that measure.

3.2 More on the LLN and Kalikow's condition

We introduce in this section a condition on the environment, due to Kalikow, which ensures that the RWRE is ballistic. As before, we let P be a uniformly elliptic product measure. Let U be a strict subset of $\mathbb{Z}^d$, with $0\in U$, and define on $U\cup\partial U$ an auxiliary Markov chain with transition probabilities

$$\hat{P}_U(x,x+e) = \begin{cases} \frac{\mathbb{E}^o\left[\sum_{n=0}^{\tau_{U^c}} \mathbf{1}_{\{X_n=x\}}\omega(x,x+e)\right]}{\mathbb{E}^o\left[\sum_{n=0}^{\tau_{U^c}} \mathbf{1}_{\{X_n=x\}}\right]}, & x\in U, |e|=1 \\ 1 & x\in\partial U, e=0 \end{cases} \tag{3.2.1}$$

where $\tau_{U^c} = \min\{n\geq 0: X_n\in\partial U\}$ (note that the expectations in (3.2.1) are finite due the Markov property and ellipticity). The transition kernel $\hat{P}_U$ weights the transitions $x\mapsto x+e$ according to the occupation time of the vertex x before exiting U. We denote by $\hat{E}_U$ expectations with respect to the measure $\hat{P}_U$.

The following is a basic consequence of the definition of $\hat{P}_U(\cdot,\cdot)$, and hints at its usefuleness:

Lemma 3.2.2 (Kalikow) *Assume* $\hat{P}_U(\tau_{U^c} < \infty) = 1$*. Then,* $\hat{P}_U(X_{\tau_{U^c}} = v) = \mathbb{P}^o(X_{\tau_{U^c}} = v), v \in \partial U$*. In particular,* $\mathbb{P}^o(\tau_{U^c} < \infty) = 1$*.*

Proof of Lemma 3.2.2: Set $g_\omega(x) = E^o_\omega\left(\sum_{n=0}^{\tau_{U^c}} \mathbf{1}_{\{X_n = x\}}\right)$. Then

$$\hat{P}_U(x,y) = \frac{E(g_\omega(x)\omega(x,y))}{E(g_\omega(x))}, \quad x \in U,\ y \in U \cup \partial U\,. \tag{3.2.3}$$

But, due to the Markov property,

$$g_\omega(x) = \mathbf{1}_{\{x=0\}} + \sum_{z \in U} \omega(z,x) g_\omega(z)\,,$$

and hence, using (3.2.3),

$$\sum_{x \in U} \Big(E(g_\omega(x))\Big) \hat{P}_U(x,y) + \mathbf{1}_{\{y=0\}} = E(g_\omega(y))\,.$$

Set $\hat{\pi}_n(y) = \hat{E}_U\left(\sum_{j=0}^{\tau_{U^c}\wedge n} \mathbf{1}_{\{X_j = y\}}\right)$. Then, $\hat{\pi}_0(y) = \mathbf{1}_{\{y=0\}}$ and

$$\hat{\pi}_{n+1}(y) = \mathbf{1}_{\{y=0\}} + \sum_{x \in U} \hat{P}_U(x,y)\hat{\pi}_n(x)\,.$$

Then, for $y \in U \cup \partial U$,

$$E\Big(g_\omega(y)\Big) - \hat{\pi}_{n+1}(y) = \sum_{x \in U} \hat{P}_U(x,y)\Big(E(g_\omega(x)) - \hat{\pi}_n(x)\Big)\,.$$

Since $E(g_\omega(y)) - \hat{\pi}_0(y) \geq 0$, it follows by the positivity of $\hat{P}_U(x,y)$ that for $y \in U \cup \partial U$,

$$\hat{E}_U\left(\sum_{n=0}^{\tau_{U^c}} \mathbf{1}_{\{X_n = y\}}\right) = \lim_{n\to\infty} \hat{\pi}_n(y) \leq E(g_\omega(y))\,.$$

Taking $y \in \partial U$ yields

$$\hat{P}_U(X_{\tau_{U^c}} = y) \leq \mathbb{P}^o(X_{\tau_{U^c}} = y), \quad y \in \partial U\,.$$

On the other hand, $\sum_{y \in \partial U} \hat{P}_U(X_{\tau_{U^c}} = y) = 1$ because $\hat{P}_U(\tau_{U^c} < \infty) = 1$ by assumption. Hence

$$\mathbb{P}^o(X_{\tau_{U^c}} = y) = \hat{P}_U(X_{\tau_{U^c}} = y), \quad \forall y \in \partial U\,. \qquad \square$$

We are now ready to introduce Kalikow's condition. Fix a hyperplane by picking a point $\ell \in \mathbb{R}^d \backslash \{0\}$, $|\ell|_1 \leq 1$. Define

$$\varepsilon_\ell := \inf_{U, x \in U} \sum_{|e|=1} (\ell \cdot e)\hat{P}_U(x, x+e)$$

where the infimum is over all connected strict subsets of $\mathbb{Z}^d$ containing 0. We say that *Kalikow's condition with respect to* ℓ holds if $\varepsilon_\ell > 0$. Note that ε_ℓ acts as a drift in the direction ℓ for the Markov chain $\hat{P}_U$.

A consequence of Lemma 3.2.2 is the following:

Theorem 3.2.4 (Kalikow[19], Sznitman and Zerner[40]) *If Kalikow's condition with respect to* ℓ *holds, then* $\mathbb{P}^o(A_\ell) = 1$*, and* $v_\ell > 0$.

Proof. Fix $U_L = \{z \in \mathbb{Z}^d : \ |z \cdot \ell| \le L\}$. Let $\hat{X}_{n,L}$ denote the Markov chain with $\hat{X}_{0,L} = 0$ and transition law $\hat{P}_{U_L}(x, x+e)$. Set the local drift at x, $\hat{d}(x) = \sum_{|e|=1} e\hat{P}_{U_L}(x, x+e)$, and recall that $\hat{X}_{n,L} - \sum_{i=0}^{n-1} \hat{d}(\hat{X}_{i,L})$ is a martingale, with bounded increments. It follows that for some constant C,

$$\hat{P}_{U_L}\left(\sup_{0\le n\le N} |\hat{X}_{n,L} - \sum_{i=0}^{n-1} \hat{d}(\hat{X}_{i,L})| > \delta N\right) \le Ce^{-C\delta^2 N}. \tag{3.2.5}$$

On the other hand, $\sum_{i=0}^{n-1} \hat{d}(\hat{X}_{i,L}) \cdot \ell \ge \varepsilon_\ell (n \wedge \tau_{U_L^c})$ while $|\hat{X}_{n,L} \cdot \ell| \le L+1$. We thus conclude from (3.2.5) that

$$\hat{P}_{U_L}\left(\tau_{U_L^c} > \frac{L+1}{\varepsilon_\ell} + \frac{\delta}{\varepsilon_\ell} N\right) \le Ce^{-C\delta^2 N/\varepsilon_\ell^2},$$

and hence, for some C_1 independent of L, and all L large,

$$\hat{P}_{U_L}\left(|\hat{X}_{\tau_{U_L^c}} \cdot \ell - L| > 1\right) \le C_1 e^{-C_1 L}.$$

It follows from Lemma 3.2.2 that

$$\mathbb{P}^o\left(|X_{\tau_{U_L^c}} \cdot \ell - L| > 1\right) \le C_1 e^{-C_1 L}.$$

A similar argument shows that $\mathbb{P}^o(D = \infty) > 0$: indeed, take now $U_{L,+} = \{z \in \mathbb{Z}^d : \ 0 \le z \cdot \ell \le L\}$. Arguing as above, one finds that for some $C_2 > 0$ independent of L,

$$\hat{P}_{U_{L,+}}\left(|\hat{X}_{\tau_{U_{L,+}^c}} - L| \le 1\right) > C_2,$$

implying that

$$\mathbb{P}^o\left(|X_{\tau_{U_{L,+}^c}} - L| \le 1\right) > C_2.$$

Thus, $\mathbb{P}^o(D = \infty) > 0$, and then, by an argument as in the proof of Theorem 3.1.1, $\mathbb{P}^o(A_\ell) > 0$. By Theorem 3.1.1, it follows that $\mathbb{P}^o(A_\ell \cup A_{-\ell}) = 1$. Due to (3.2.5), it holds that $\mathbb{P}^o(\limsup_{n\to\infty} X_n \cdot \ell = \infty) = 1$. We thus conclude that $\mathbb{P}^o(A_\ell) = 1$.

To see that if P is i.i.d. then $v_\ell > 0$, recall the regeneration times $\{\tau_i\}_{i\ge 1}$ introduced in Section 3.1. By Lemma 3.1.10, it suffices to prove that $\mathbb{E}^o(\tau_1 | D = \infty) < \infty$. Let $U_{m,k,-} = \{z \in \mathbb{Z}^d : |z| < k, z \cdot \ell < m\}$, and set

$T_{m,k} := T_{U_{m,k,-}}$ with $T_m = \lim_{k\to\infty} T_{m,k} = \min\{n : X_n \cdot \ell \geq m\}$, $m \geq 1$. By Kalikow's condition,

$$\mathbb{E}^o\left(\sum_{n=0}^{T_{m,k}} \mathbf{1}_{\{X_n=x\}} \sum_{e:|e|=1} \omega(x,x+e)\ell\cdot e\right) \geq \varepsilon_\ell \mathbb{E}^o\left(\sum_{n=0}^{T_{m,k}} \mathbf{1}_{\{X_n=x\}}\right),$$

and hence, summing over $x \in U_{m,k,-}$ and recalling that $X_{i+1} - X_i - d(\theta^{X_i}\omega)$ is a martingale difference sequence, one gets

$$1 + m \geq \mathbb{E}^o(X_{T_{m,k}} \cdot \ell) \geq \varepsilon_\ell \mathbb{E}^o(T_{m,k}),$$

and taking $k \to \infty$ one concludes that $m + 1 \geq \varepsilon_\ell \mathbb{E}^o(T_m)$. In particular,

$$\mathbb{E}^o(\liminf_{m\to\infty} T_m/m) \leq \liminf_{m\to\infty} \mathbb{E}^o(T_m/m) \leq 1/\varepsilon_\ell\,. \tag{3.2.6}$$

Since $\tau_i \to_{i\to\infty} \infty$, $\mathbb{P}^o$-a.s., one may find a (random) sequence k_m such that $\tau_{k_m} \leq T_m < \tau_{k_m+1}$. By definition,

$$\ell \cdot X_{\tau_{k_m}} \leq \ell \cdot X_{T_m} \leq \ell \cdot X_{\tau_{k_m+1}},$$

and further,

$$\ell \cdot X_{\tau_k}/k \to_{k\to\infty} \mathbb{E}^o(\ell \cdot X_{\tau_1}|\{D=\infty\}) = \frac{1}{\mathbb{P}^o(D=\infty)} < \infty\,, \mathbb{P}^o - a.s.\,,$$

due to Lemma 3.1.10 and (3.1.12). Thus, $k_m/m \to_{m\to\infty} \mathbb{E}^o(X_{\tau_1} \cdot \ell|\{D = \infty\})^{-1}$, $\mathbb{P}^o$-a.s. But, since $\tau_{k_m}/k_m \to_{m\to\infty} \mathbb{E}^o(\tau_1|\{D=\infty\}) \in [1,\infty]$, $\mathbb{P}^o$-a.s., it follows that

$$\begin{aligned}\liminf_{m\to\infty} \frac{T_m}{m} \geq \liminf_{m\to\infty} \frac{\tau_{k_m}}{k_m}\frac{k_m}{m} = \lim_{m\to\infty} \frac{\tau_{k_m}}{k_m}\frac{k_m}{m} &= \frac{\mathbb{E}^o(\tau_1|\{D=\infty\})}{\mathbb{E}^o(X_{\tau_1}\cdot\ell|\{D=\infty\})} \\ &= \mathbb{E}^o(\tau_1|\{D=\infty\})\mathbb{P}^o(D=\infty)\,.\end{aligned}$$

Since (3.2.6) implies that $\liminf_{m\to\infty} T_m/m < \infty$, we conclude that $\mathbb{E}^o(\tau_1|\{D = \infty\}) < \infty$, and hence $v_\ell > 0$. □

By noting that if Kalikow's condition holds for some ℓ_0 then it holds for all ℓ in a neighborhood of ℓ_0, one gets immediately the

Corollary 3.2.7 *If Kalikow's condition with respect to some ℓ holds, then there exists a non-zero deterministic v such that*

$$\frac{X_n}{n} \to_{n\to\infty} v\,, \quad P - a.s.$$

The following is a sufficient condition for Kalikow's condition to hold true:

Lemma 3.2.8 *Kalikow's condition with respect to ℓ holds if*

$$\inf_{f\in\mathcal{F}} \frac{E\left(\frac{\sum_{e:|e|=1}\omega(0,e)\ell\cdot e}{\sum_{e:|e|=1}\omega(0,e)f(e)}\right)}{E\left(\frac{1}{\sum_{e:|e|=1}\omega(0,e)f(e)}\right)} > 0\,, \tag{3.2.9}$$

where $\mathcal{F}$ denotes the collection of nonzero functions on $\{e : |e| = 1\}$ taking values in $[0,1]$.

Proof. Fix U a strict subset of $\mathbb{Z}^d$, $x \in U$, and let $\tau_x = \min\{n \geq 0 : X_n = x\}$. Define $g(x,y,\omega) := E^y_\omega(\mathbf{1}_{\{\tau_x<\tau_{U^c}\}})$. Note that $g(x,y,\omega)$ is independent of ω_x. Next,

$$\begin{aligned}
&\sum_{|e|=1}(\ell\cdot e)\hat{P}_U(x,x+e)\\
&\quad= \frac{E\left(E^o_\omega\left(\sum_{n=0}^{\tau_{U^c}}\mathbf{1}_{\{X_n=x\}}\sum_{e:|e|=1}\omega(x,x+e)e\cdot\ell\right)\right)}{E\left(E^o_\omega\left(\sum_{n=0}^{\tau_{U^c}}\mathbf{1}_{\{X_n=x\}}\right)\right)}\\
&\quad= \frac{E\left(g(x,0,\omega)E^x_\omega\left(\sum_{n=0}^{\tau_{U^c}}\mathbf{1}_{\{X_n=x\}}\sum_{e:|e|=1}\omega(x,x+e)e\cdot\ell\right)\right)}{E\left(g(x,0,\omega)E^x_\omega\left(\sum_{n=0}^{\tau_{U^c}}\mathbf{1}_{\{X_n=x\}}\right)\right)}\,.
\end{aligned}$$

Under P^x_ω, the process X_n is a Markov chain with Geometric($\sum_{e:|e|=1}\omega(x,x+e)g(x,x+e,\omega)$) number of visits at x. The last equality and the Markov property then imply

$$\begin{aligned}
&\sum_{|e|=1}(\ell\cdot e)\hat{P}_U(x,x+e)\\
&\quad= \frac{E\left(g(x,0,\omega)\frac{\sum_{e:|e|=1}\omega(x,x+e)e\cdot\ell}{\sum_{e:|e|=1}\omega(x,x+e)g(x,x+e,\omega)}\right)}{E\left(g(x,0,\omega)\frac{1}{\sum_{e:|e|=1}\omega(x,x+e)g(x,x+e,\omega)}\right)}\\
&\quad= \frac{E\left(\frac{\sum_{e:|e|=1}\omega(x,e)\ell\cdot e}{\sum_{e:|e|=1}\omega(x,e)g(x,x+e,\omega)/g(x,0,\omega)}\right)}{E\left(\frac{1}{\sum_{e:|e|=1}\omega(x,e)g(x,x+e,\omega)/g(x,0,\omega)}\right)}\\
&\quad\geq \inf_{f\in\mathcal{F}}\frac{E\left(\frac{\sum_{e:|e|=1}\omega(x,e)\ell\cdot e}{\sum_{e:|e|=1}\omega(x,e)f(e)}\right)}{E\left(\frac{1}{\sum_{e:|e|=1}\omega(x,e)f(e)}\right)} = \inf_{f\in\mathcal{F}}\frac{E\left(\frac{\sum_{e:|e|=1}\omega(0,e)\ell\cdot e}{\sum_{e:|e|=1}\omega(0,e)f(e)}\right)}{E\left(\frac{1}{\sum_{e:|e|=1}\omega(0,e)f(e)}\right)} > 0\,,
\end{aligned}$$

where the first inequality is due to the independence of $g(x,x+e,\omega)/g(x,0,\omega)$ in ω_x. $\square$

An easy corollary is the following

Corollary 3.2.10 *If either*
(a) $\mathrm{supp}(P_d) \subset \{z \in \mathbb{R}^d : \ell\cdot z \geq 0\}$ *but* $\mathrm{supp}(P_d) \not\subset \{z \in \mathbb{R}^d : \ell\cdot z = 0\}$,

or
(b) $E((\sum_{e:|e|=1}\omega(0,e)e\cdot\ell)_+) > \frac{1}{\varepsilon}E((\sum_{e:|e|=1}\omega(0,e)e\cdot\ell)_-)$,
then Kalikow's condition with respect to ℓ *holds.*

In particular, non-nestling walks or walks with drift "either neutral or pointing to the right" satisfy Kalikow's condition with respect to an appropriate hyperplane ℓ. Further there exist truly nestling walks which do satisfy Kalikow's condition.
Remark: It is interesting to note that when P is elliptic and i.i.d. and $d=1$, Kalikow's condition is equivalent to $v \neq 0$, i.e. to the walk being "ballistic". For $d>1$, it is not clear yet whether there exist walks with $\mathbb{P}^o(A_\ell)>0$ but with zero speed. Such walks, if they exist, necessarily cannot satisfy Kalikow's condition.

Our next goal is to provide tail estimates on $X_{\tau_1}\cdot\ell$ and on τ_1. Our emphasis here is in providing a (relatively) simple proof and not the sharpest possible result. For the latter we refer to [35]. In this spirit we continue to assume that $\ell=e_1$.

Lemma 3.2.11 *Assume* P *satisfies Kalikow's condition. Then, there exists a constant* c *such that*
$$\mathbb{E}^o(\exp c\,X_{\tau_1}\cdot\ell)<\infty\,.$$

Proof. Recall the notations of Section 3.1 and write

$$\begin{aligned}\mathbb{E}^o(\exp(c\,X_{\tau_1}\cdot\ell)) &= \sum_{k\geq 1}\mathbb{E}^o\Big(\exp(c\,X_{\tau_1}\cdot\ell)\mathbf{1}_{\{K=k\}}\Big)\\ &= \sum_{k\geq 1}\sum_{x\in\mathbb{Z}^d} e^{c\,x\cdot\ell}E\Big(E^o_\omega\left(\mathbf{1}_{\{X_{\overline{S}_k}=x\}}\mathbf{1}_{\{\overline{S}_k<\infty\}}\right)\cdot P^x_\omega(D=\infty)\Big)\\ &= \mathbb{P}^o(D=\infty)E\left(\sum_{k\geq 1}\sum_{x\in\mathbb{Z}^d} e^{cx\cdot\ell}E^o_\omega\Big(\mathbf{1}_{\{X_{\overline{S}_k}=x\}}\mathbf{1}_{\{\overline{S}_k<\infty\}}\Big)\right)\\ &= \mathbb{P}^o(D=\infty)\sum_{k\geq 1}\mathbb{E}^o\Big(\exp(c\,X_{\overline{S}_k}\cdot\ell)\mathbf{1}_{\{\overline{S}_k<\infty\}}\Big)\,. \qquad (3.2.12)\end{aligned}$$

But, using the Markov property,

$$\begin{aligned}&\mathbb{E}^o\Big(\exp(c\,X_{\overline{S}_k}\cdot\ell)\ \mathbf{1}_{\{\overline{S}_k<\infty\}}\Big)\\ &\qquad\leq \mathbb{E}^o\Big(\exp(c\,X_{\overline{S}_{k-l}}\cdot\ell)\ \mathbf{1}_{\{\overline{S}_{k-1}<\infty\}}\Big)\mathbb{E}^o\Big(\exp(c\,M_0\cdot\ell)\ \mathbf{1}_{\{D<\infty\}}\Big)\,.\end{aligned}$$

Hence,

$$\mathbb{E}^o\Big(\exp(c\,X_{\overline{S}_k}\cdot\ell)\mathbf{1}_{\{\overline{S}_k<\infty\}}\Big)\leq\mathbb{P}^o(D=\infty)\sum_{k\geq 0}\Big(\mathbb{E}^o(\exp(c\,M_0\cdot\ell)\mathbf{1}_{\{D<\infty\}}\Big)^k\,.$$

Note, using $\ell=e_1$, that

$$\begin{aligned}
&\mathbb{E}^o\left(e^{cM_1\cdot\ell}\mathbf{1}_{\{D<\infty\}}\right)\\
&\quad=\sum_{k=1}^{\infty}e^{ck}\mathbb{E}^o\left(\mathbf{1}_{\{M_0\cdot\ell=k\}}\mathbf{1}_{\{D<\infty\}}\right)\\
&\quad=\sum_{k=1}^{\infty}e^{ck}\sum_{y\in\mathbb{Z}^{d-1}}\mathbb{E}^o\left(E^o_\omega\left(\mathbf{1}_{\{X_{T_k}=(k,y)\}}E^{(k,y)}_\omega(T_0<T_{k+1})\right)\right). \qquad (3.2.13)
\end{aligned}$$

Using Kalikow's condition and a computation as in Theorem 3.2.4, one has that for some $c_1>0$, $\mathbb{P}^o\left(\sum_{|y|>\frac{2k}{\varepsilon_\ell}}\mathbf{1}_{\{X_{T_k}=(k,y)\}}\right)\le e^{-c_1k}$, while $\mathbb{P}^o(T_{-k}<T_1)\le e^{-c_1k}$. Hence, substituting in (3.2.13), one has

$$\mathbb{E}^o\left(e^{cM_1\cdot\ell}\mathbf{1}_{\{D<\infty\}}\right)\le\sum_{k=1}^{\infty}e^{ck}\left(\left(\frac{4k}{\varepsilon_\ell}\right)^{d-1}+1\right)e^{-c_1k}.$$

Taking $c<c_1$, the lemma follows. □

A direct consequence of Lemma 3.2.11 is that

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}^o(X_{\tau_1}\cdot\ell>vn)\le-\beta(v) \qquad (3.2.14)$$

where $\beta(v)>0$ for $v>0$.

With a proof very similar to that of Lemma 3.2.11, we have in fact the

Lemma 3.2.15 *Assume P satisfies Kalikow's condition. Set $X^*=\sup_{0\le n\le\tau_1}|X_n|$. Then, there exists a constant c' such that*

$$\mathbb{E}^o(\exp c'X^*)<\infty.$$

We next turn to obtaining tail estimates on τ_1. Here, due to the presence of "traps", one cannot in general expect exponential decay as in (3.2.14). We aim at proving the following result.

Theorem 3.2.16 (Sznitman[35]) *Assume P satisfies Kalikow's condition. Then, with $d\ge2$, there exists an $\alpha>1$ such that for all u large,*

$$\mathbb{P}^o(\tau_1>u)\le e^{-(\log u)^\alpha}.$$

In particular, τ_1 possesses all moments.

Proof. Recall that we take $\ell=e_1$ and, for $L>0$, set $c_L=(-L,L)\times\left(-\frac{2L}{\varepsilon_\ell},\frac{2L}{\varepsilon_\ell}\right)^{d-1}$. Note that, with c as in Lemma 3.2.11,

$$\begin{aligned}
\mathbb{P}^o(\tau_1\ge u)&\le\mathbb{P}^o\Big(\tau_1\ge u,X_{\tau_1}\cdot\ell\le L\Big)+\mathbb{P}^o\Big(X_{\tau_1}\cdot\ell\ge L\Big)\\
&\le e^{-cL/2}+\mathbb{P}^o(\tau_L>\tau_{c_L})+\mathbb{P}^o(\tau_{c_L}=\tau_L\ge u)
\end{aligned}$$

where

$$\tau_L = \inf\{t : \ X_t \cdot \ell \geq L\} \text{ and}$$
$$\tau_{c_L} = \inf\{t : \ X_t \notin c_L\}\,.$$

Hence, by Kalikow's condition,

$$\mathbb{P}^o(\tau_1 \geq u) \leq e^{-\overline{c}L/2} + \mathbb{P}^o\Big(\tau_{c_L} = \tau_L \geq u\Big) \tag{3.2.17}$$

for some constant $\overline{c} > 0$.

The heart of the proof of Theorem 3.2.16 lies in the following lemma, whose proof is deferred.

Lemma 3.2.18 *There exist a $\beta < 1$ and $\xi > 1$ such that for any $c > 0$,*

$$\limsup_{L\to\infty} \frac{1}{L^\xi} \log \mathbb{P}\Big(P^o_\omega(X_{\tau_{U_L}} \cdot \ell \geq L) \leq e^{-cL^\beta}\Big) < 0\,,$$

where $U_L = \{z \in \mathbb{Z}^d : |z \cdot \ell| \leq L\}$.

Accepting Lemma 3.2.18, let us complete the proof of Theorem 3.2.16. Toward this end, set $\Delta(u) = a \log u$ with a small enough such that $\varepsilon^{\Delta(u)} > \frac{1}{u^{1/6}}$, and set $L = L(u) = (\log u)^{\overline{\alpha}}$, with $N = L/\Delta = \frac{1}{a}(\log u)^{\overline{\alpha}-1}$ and $2 - \overline{\alpha} = \beta$ where β is as in Lemma 3.2.18. Observe that

$$\begin{aligned}
\mathbb{P}^o(\tau_{c_L} \geq u) &\leq \left(\frac{1}{2}\right)^{(\log u)^{\overline{\alpha}}} + P^o\left(\exists\, x_1 \in c_L, P^{x_1}_\omega\left(\tau_{c_L} > \frac{u}{(\log u)^{\overline{\alpha}}}\right) \geq \frac{1}{2}\right) \\
&=: \left(\frac{1}{2}\right)^{(\log u)^{\overline{\alpha}}} + P(\mathcal{R})\,.
\end{aligned} \tag{3.2.19}$$

Note that

$$\begin{aligned}
&\frac{u}{(\log u)^\alpha} \cdot P^{x_1}_\omega\Big(\tau_{c_L} > \frac{u}{(\log u)^{\overline{\alpha}}}\Big) \\
&\quad \leq E^{x_1}_\omega(\tau_{c_L}) = E^{x_1}_\omega\left(\sum_{i=1}^{\tau_{c_L}} 1\right) = E^x_\omega\left(\sum_{y\in c_L}\sum_{i=1}^{\tau_{c_L}} \mathbf{1}_{\{X_n=y\}}\right) \\
&\quad = \sum_{y\in c_L} \frac{E^x_\omega\Big(\mathbf{1}_{\{\tau_y<\tau_{c_L}\}}\Big)}{E^y_\omega\Big(\mathbf{1}_{\{\tau_y>\tau_{c_L}\}}\Big)} \leq |c_L| \frac{1}{\inf_{y\in c_L} P^y_\omega\Big(\mathbf{1}_{\{\tau_y\}} > \tau_{c_L}\Big)}\,,
\end{aligned}$$

with $\tau_y = \inf\{t : X_t = y\}$. Hence,

$$\inf_{y\in c_L} P^y_\omega(\tau_y > \tau_{c_L}) \leq \frac{|c_L|(\log u)^{\overline{\alpha}}}{u \inf_{x_1\in c_L} P^{x_1}_\omega\left(\tau_{c_L} > \frac{u}{(\log u)^\alpha}\right)}\,.$$

Hence, on $\mathcal{R}$ there exists a y with $\mathbb{P}^y_\omega(\tau_y > \tau_{c_L}) \le \frac{1}{u^{4/5}}$, for all u large enough.

Set $A_i = \{z \in \mathbb{Z}^d : z \cdot \ell = i\Delta\}$. By ellipticity (recall $\varepsilon^{\Delta(u)} > 1/u^{1/6}$), it follows that on the event $\mathcal{R}$, there exists an $i_0 \in [-N+2, N-1]$ and an $x \in A_{i_0}$ such that

$$P^{x_0}_\omega\Big(\tau_{(i_0-1)\Delta} > \tau_{c_L}\Big) \le \frac{1}{\sqrt{u}} \tag{3.2.20}$$

where $\tau_{(i_0-1)\Delta} = \inf\{t : X_t \cdot \ell = (i_0-1)\Delta\}$. Set

$$X_i = -\log \min_{z \in A_i \cap c_L} P^z_\omega\Big(\tau_{(i-1)\Delta} > \tau_{(i+1)\Delta}\Big)\,.$$

Then, the Markov property implies that

$$\begin{aligned} P^x_\omega\Big(\tau_{(i-1)\Delta} > \tau_{c_L}\Big) &\ge \exp\left(-\sum_{j=i}^{N-1} X_j\right)\varepsilon^{\Delta(u)} \\ &\ge \exp\left(-\sum_{j=i}^{N-1} X_j\right)\frac{1}{u^{1/6}}\,. \end{aligned}$$

Thus, using (3.2.20)

$$P(\mathcal{R}) \le P\left(\sum_{i=-N+1}^{N-1} X_j \ge \frac{\log u}{12}\right) \le 2N \sup_{-N+1 \le i \le N-1} P\left(X_i \ge \frac{\log u}{24N}\right)\,. \tag{3.2.21}$$

But, using the shift invariance of P and the definition of $\{X_i\}$,

$$2N \sup_{N+1 \le i \le N-1} P\left(X_i \ge \frac{\log u}{24N}\right) \le |c_L| P\Big(P^o_\omega(\tau_{-\Delta} > \tau_\Delta) \le e^{-(\log u)^{2-\overline{\alpha}} b}\Big)$$

for $b = \frac{a}{24}$. The theorem follows by an application of Lemma 3.2.18. □

Proof of Lemma 3.2.18 The interesting aspect in proving the Lemma is the fact that one constructs *lower* bounds on $P^o_\omega(X_{\tau_{u_L}} \cdot \ell \ge L)$ for many configurations. Toward this end, fix $1 > \beta > \overline{\beta}$, $\gamma \in \left(\frac{1}{2}, 1\right)$, $\chi = \frac{1-\overline{\beta}}{1-\gamma} < \overline{\beta} < 1$ such that $d(\overline{\beta} - \chi) > 1$ (for $\overline{\beta}$ close enough to 1, one may always find a γ close to 1 such that this condition is satisfied, if $d \ge 2$). Set next $L_0 = L^\chi$, $L_1 = L^{\overline{\beta}}$, $N_0 = L_1/L_0$ (for simplicity, assume that L_0, L_1, L_0^γ and N_0 are all integer).

Let R be a rotation of $\mathbb{Z}^d$ such that $\tilde{R}(\ell) = \tilde{R}(e_1) = \frac{v}{|v|}$, and define

$$\begin{aligned} B_1(z) &= \tilde{R}\Big(z + [0, L_0]^d\Big) \cap \mathbb{Z}^d \\ B_2(z) &= \tilde{R}\Big(z + [-L_0^\gamma, L_0 + L_0^\gamma]^d\Big) \cap \mathbb{Z}^d \end{aligned}$$

and $\partial_+ B_2(z) = \partial B_2(z) \cap \left\{x : \, x \cdot \frac{v}{|v|} \geq L_0 + L_0^\gamma\right\}$.

We say that $z \in L_0\mathbb{Z}^d$ is *good* if $\sup_{x\in B_1(z)} P_\omega^x(X_{\tau_{B_2(z)}} \notin \partial_+ B_2(z)) \leq \frac{1}{2}$ and say that it is *bad* otherwise. The following estimate is a direct consequence of Kalikow's condition and Lemma 3.2.15.

Lemma 3.2.22 *For* $\gamma \in (1/2, 1)$,

$$\limsup_{L_0\to\infty} L_0^{1-2\gamma} \log P(0 \textit{ is bad }) < 0\,.$$

Proof of Lemma 3.2.22

Set $u = \max\{\overline{u} : \sup_{x\in B_2(0)} x\cdot\ell \geq \overline{u}\}$ and set $L_u = \sup\{n \geq 0 : X_n\cdot\ell \leq u\}$. Define $\pi(z) = z - \frac{z\cdot v}{|v|^2}v$. Setting $K_n = \sup\{k \geq 0 : \, \tau_k < n\}$, it holds that $n \leq L_u \Rightarrow K_n \leq u$. Setting $w \in \mathbb{R}^d$ with $w \cdot v = 0$, and $|w|_1 = 1$ one has

$$\begin{aligned} X_n \cdot w &= X_{\tau_{K_n}} \cdot w + (X_n - X_{\tau_{K_n}}) \cdot w \\ &\leq X_{\tau_{K_n}} \cdot w + X^* \circ \theta_{\tau_{K_n}}\,. \end{aligned}$$

Hence,

$$\begin{aligned} &\mathbb{P}^o\Big(\sup_{0\leq n\leq L_u} X_n \cdot w \geq u^\gamma\Big) \leq \sum_{0\leq k\leq u} \mathbb{P}^o\Big(X_{\tau_k}\cdot w + X^*\circ\theta_{\tau_k} > u^\gamma\Big) \\ &\leq \sum_{0\leq k\leq u} \mathbb{P}^o\left(\left(X_{\tau_k} - X_{\tau_1}\right)\cdot w > \frac{u^\gamma}{3}\right) + u\,\mathbb{P}^o\left(X_{\tau_1}\cdot w > \frac{u^\gamma}{3}\right) \\ &\quad + u\,\mathbb{P}^o\left(X^* > \frac{u^\gamma}{3} | D = \infty\right) \\ &\leq \sum_{0\leq k\leq u} \mathbb{P}^o\left(\left(X_{\tau_k} - X_{\tau_1}\right)\cdot w > \frac{u^\gamma}{3}\right) + \frac{2u}{\mathbb{P}^o(D=\infty)}\mathbb{P}^o\left(X^* > \frac{u^\gamma}{3}\right)\,. \end{aligned}$$

Note that by Lemma 3.2.15, $\mathbb{P}^o\left(X^* > \frac{u^\gamma}{3}\right) \leq e^{-c_0 u^\gamma}$ while the random variables $(X_{\tau_{i+1}} - X_{\tau_i})\cdot w$ are i.i.d., of zero mean and finite exponential moments. In particular,

$$\mathbb{P}^o\left(\left|\frac{(X_{\tau_k} - X_{\tau_1})\cdot w}{k}\right| > \frac{u^\gamma}{3k}\right) \leq e^{-c_0 k \frac{u^{2\gamma}}{9k^2}} \leq e^{-c_0 u^{2\gamma-1}}$$

by moderate deviations (see e.g., [12, Section 3.7]).

Since $\gamma > 2\gamma - 1$, we conclude that

$$\limsup_{u\to\infty} \frac{1}{u^{2\gamma-1}} \log \mathbb{P}^o\left(\sup_{0\leq n\leq L_u} X_n \cdot w \geq u^\gamma\right) < 0$$

and hence

$$\limsup_{u\to\infty} \frac{1}{u^{2\gamma-1}} \log \mathbb{P}^o\left(\sup_{0\leq n\leq L_u} |\pi(X_n)| \geq u^\gamma\right) < 0\,. \tag{3.2.23}$$

Fix now $x \in B_1(0)$. Then, for some $c = c(d)$,

$$\mathbb{P}^x\Big(X_{\tau_{B_2(0)}} \notin \partial_+ B_2(0)\Big) \le \mathbb{P}^o\Big(\sup_{0\le n\le L_{cu}} \pi(X_n\cdot w) \ge u^\gamma\Big) + \mathbb{P}^o\Big(X_{\tau_{V_u}}\cdot \ell < 0\Big)$$

where $V_u = \left\{z : \frac{-L_0^\gamma}{c} \le z\cdot\ell \le cL_0^\gamma\right\}$ and the conclusion follows from (3.2.23) and Kalikow's condition. □

Construct now the following subsets of U_L:

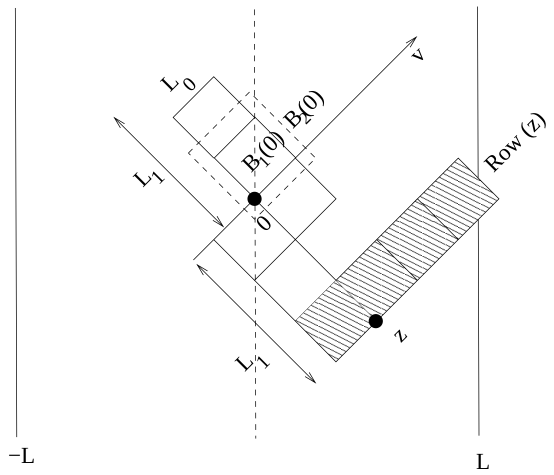

Fig. 3.2.1.

Set $M = \left\{z \in L_0\mathbb{Z}^d, z = (0,\overline{z}), \overline{z} \in \left\{-\frac{L_1}{L_0}, \cdots, 0, \frac{L_1}{L_0}\right\}^{d-1} L_0\right\}$. For $z \in M$, set $\mathrm{Row}(z) = \cup_{j=N_-(z)}^{N_+(z)} B_1(z + jL_0\ell)$ where

$$N_-(z) = \min\{j : B_1(z+jL_0\ell) \cap \{x : x\cdot\ell \ge 0\} \ne \emptyset\} \ge -c\frac{L}{L_0}$$

$$N_+(z) = \max\{j : B_1(z+jL_0\ell) \cap \{x : x\cdot\ell \ge 0\} \ne \emptyset\} \le c\frac{L}{L_0}$$

for some constant c depending on v and uniformly bounded, and set $T = \cup_{j\in\left\{-\frac{L_1}{L_0},\cdots,0,\cdots,\frac{L_1}{L_0}\right\}^{d-1}}\{jL_0\}$.

The idea behind the proof is that if one of the rows $\{R(z)\}_{z\in M}$, say $R(z_0)$, contains mostly good blocks, a good strategy for the event $(X_{\tau_{u_L}}\cdot\ell) \ge L$ is to force the walker started at x to first move to z, then move to the right successively without leaving $\cup_{z\in\mathrm{Row}(z_0)}B_2(z)$ until exiting from u_L. More precisely, let $N(z_0)$ denote the number of bad blocks in $\cup_{z\in\mathrm{Row}(z_0)}B_1(z)$. Then, for some constants c_i, using ellipticity and the definition of good boxes,

$$\begin{aligned}P_\omega^o(X_{\tau_{u_L}}\cdot\ell \geq L) &\geq \varepsilon^{L_1}\left(\frac{1}{2}\,\varepsilon^{2(d-1)}L_0^\gamma\right)^{cL/L_0}(\varepsilon^{L_0})^{N(z_0)}\\ &= e^{-c_4(L^{\overline{\beta}}+L^\chi N(z_0))}\,.\end{aligned}$$

Hence, for an arbitrary constant c_6 and all L large enough ($L > g(c_6, c)$ for some fixed function $g(\cdot)$),

$$\begin{aligned}P\left(P_\omega^o\Big(X_{\tau_{u_L}}\cdot\ell \geq L\Big) \leq e^{-cL^\beta}\right) &\leq P\Big(\{\nexists z_0 \in M\,:\; N(z_0) \leq c_6 L^{\overline{\beta}-\chi}\}\Big)\\ &\leq \Big[P\Big(N(0) \geq c_6 L^{\overline{\beta}-\chi}\Big)\Big]^{\left(\frac{cL_1}{L_0}\right)^{(d-1)}}\end{aligned}$$

using the independence between even rows in Figure 3.2.1. But note that $N(0) = \sum_{i=1}^{L/2L_0}(Y_i + Z_i)$ where $\{Y_i\}$ are i.i.d., $\{Z_i\}$ are i.i.d., $\{0,1\}$ valued, $P(Y_i = 1) = P(B_1(0))$ is a bad block (the division to (Y_i, Z_i) reflects the division to even and odd blocks, which creates independence). Hence,

$$P\Big(N(0) \geq c_6 L^{\overline{\beta}-x}\Big) \leq 2P\left(\sum_{i=1}^{L/2L_0} Y_i \geq \frac{c_6}{2}L^{\overline{\beta}-\chi}\right)\,.$$

Since, by Lemma 3.2.22,

$$\log E(e^{Y_i}) \leq \log\Big(1 + e^{-c_7 L_0^{2\gamma-1}}\Big) \leq e^{-c_7 L_0^{2\gamma-1}}\,,$$

we conclude from the independence of the Y_i's that

$$\begin{aligned}P\Big(N(0) \geq c_6 L^{\overline{\beta}-\chi}\Big) &\leq 2e^{-\frac{c_6}{2}L^{\overline{\beta}-\chi}}e^{e^{-c_7L_0^{2\gamma-1}}\cdot\frac{L}{2L_0}}\\ &\leq e^{-c_8 L^{\overline{\beta}-\chi}}\,.\end{aligned}$$

Hence,

$$P\left(P_\omega^o\Big(X_{\tau_{u_L}}\cdot\ell \geq L\Big) \leq e^{-cL^\beta}\right) \leq e^{-c_9 L^{(\overline{\beta}-\chi)d}} \leq e^{-L^\xi}$$

for some $\xi > 1$, as claimed. □

Remark The restriction to $d \geq 2$ in Theorem 3.2.16 is essential: as we have seen in the case $d = 1$, one may have ballistic walks (and hence, in $d = 1$, satisfying Kalikow's condition) with moments $m_r := \mathbb{E}^o(\tau_1^r)$ of the regeneration time τ_1 being finite only for small enough $r > 1$.

We conclude this section by showing that estimates of the form of Theorem 3.2.16 lead immediately to a CLT. The statement is slightly more general than needed, and does not assume Kalikow's condition but rather some of its consequences.

Theorem 3.2.24 (Sznitman[35]) *Assume that $\mathbb{P}^o(A_\ell) = 1$ and that the regeneration time τ_1 satisfies $\mathbb{E}^o(\tau_1^{(2+\delta)}) < \infty$ for some $\delta > 0$. Then, under the annealed measure $\mathbb{P}^o$,*

$$X_n/n \to_{n\to\infty} v := \frac{\mathbb{E}^o(X_{\tau_2} - X_{\tau_1})}{\mathbb{E}^o(\tau_2 - \tau_2)} \neq 0\,, \quad \mathbb{P}^o - a.s.\,, \tag{3.2.25}$$

and $(X_n - nv)/\sqrt{n}$ converges in law to a centered Gaussian vector.

Proof. The LLN (3.2.25) is a consequence of Theorem 3.1.5 and its proof. To see the CLT, set

$$\xi_i = X_{\tau_{i+1}} - X_{\tau_i} - (\tau_{i+1} - \tau_i)v\,, \quad S_n := \sum_{i=1}^n \xi_i\,,$$

and $\Xi = \mathbb{E}^o(\xi_1 \xi_1^T)$. It is not hard to check that Ξ is non-degenerate, simply because $\mathbb{P}^o(|\xi_1| > K) > 0$ for each $K > 0$. Then S_n is under $\mathbb{P}^o$ a sum of i.i.d. random variables possessing finite $2+\delta$-th moments, and thus $S_{[nt]}/\sqrt{n}$ satisfies the invariance principle, with covariance matrix Ξ. Define

$$\nu_n = \min\left\{ j : \sum_{i=1}^j (\tau_{i+1} - \tau_i) > n \right\}.$$

Note that in $\mathbb{P}^o$ probability, $n/\nu_n \to \mathbb{E}^o(\tau_2 - \tau_1) < \infty$. Hence, by time changing the invariance principle, see e.g. [2, Theorem 14.4], $S_{\nu_n}/\sqrt{\nu_n}$ converges in $\mathbb{P}^o$ probability to a centered Gaussian variable of covariance Ξ. On the other hand, for any positive η,

$$\begin{aligned}\mathbb{P}^o(|S_{\nu_n} - (X_n - nv)| > \eta\sqrt{n}) &\le \mathbb{P}^o(\exists i \le n : (\tau_{i+1} - \tau_i) > \eta\sqrt{n}/2) \\ &\quad + \mathbb{P}^o(\tau_1 > \eta\sqrt{n}/2) \\ &\le \frac{(n+1)\mathbb{P}^o(\tau_1 > \eta\sqrt{n}/2)}{\mathbb{P}^o(D')} \to_{n\to\infty} 0\,,\end{aligned}$$

where we used the moment bounds on $\mathbb{E}^o(\tau_2 - \tau_1)^{2+\delta}$ and the fact that $\mathbb{P}^o(D') > 0$ in the last limit. This yields the conclusion. Further, one observes that the limiting covariance of $X_n/\sqrt{n}$ is $\Xi/(\mathbb{E}^o(\tau_2 - \tau_1))$. □

A direct conclusion of Theorem 3.2.24 is that under Kalikow's condition, $X_n/\sqrt{n}$ satisfies an annealed CLT. In a recent series of papers, Sznitman has shown that many of the conclusions of this section remain valid under a weaker condition, Sznitman's (T) or (T') conditions, see [38, 37, 39].

3.3 Non ballistic walks - partial results

Our goal in this section is to discuss briefly two particular cases of non-ballistic RWRE's in $\mathbb{Z}^d$ with uniformly elliptic, product P, for which law of large numbers and CLT's are available.

Balanced walk and the environment viewed from the particle's point of view We say that an environment is *balanced* (and the resulting RWRE is a balanced walk) if $P(\omega(x, x+e_i) = \omega(x, x-e_i)) = 1$. Note that in the notations of (2.2.10),

$$d(x,\omega) = \sum_{i=1}^{d} e_i \Big[\omega(x, x+e_i) - \omega(x, x-e_i)\Big] = 0\,.$$

Hence, the processes $(X_n(i))_{n\geq 0}, i = 1, \cdots, d$, are martingales, with, denoting $\mathcal{F}_n = \sigma(X_0, \cdots X_n)$,

$$E^o_\omega((X_n(i) - X_{n-1}(i))(X_n(j) - X_{n-1}(j))|\mathcal{F}_{n-1}) = 2\delta_{ij}\omega(X_{n-1}, X_{n-1}+e_i)\,.$$

Since $|\omega(\cdot,\cdot)| \leq 1$ P-a.s., it immediately follows that $X_n/n \underset{n\to\infty}{\longrightarrow} 0$, $\mathbb{P}^o$-a.s. Further, the multi-dimensional CLT (compare with Lemma 2.2.16) yields that if there exists a deterministic vector $\mathbf{a} = (a_1, \cdots, a_d)$ such that

$$\frac{1}{n}\sum_{k=1}^{n} \omega(X_{k-1}, X_{k-1}+e_i) \underset{n\to\infty}{\longrightarrow} \frac{a_i}{2} > 0, \quad \mathbb{P}^o\text{-a.s.}, \tag{3.3.1}$$

then a quenched CLT for $X_n/\sqrt{n}$ holds true (and in fact, so does even a trajectorial version of it, viz. a quenched CLT for $\{X_{[nt]}/\sqrt{t}\}_{t\geq 0})$ holds true.

Consider the environment viewed from the particle. Define $\overline{\omega}(n) = \theta^{X_n}\omega$, and the Markov transition kernel

$$M(\omega, d\omega') = \sum_{e_i}\Big[\omega(0,e_i)\delta_{\theta^{e_i}\omega=\omega'} + \omega(0,-e_i)\delta_{\theta^{-e_i}\omega=\omega'}\Big]\,. \tag{3.3.2}$$

As in Lemma 2.2.1, the process $\overline{\omega}(n)$ is Markov under either P^o_ω or $\mathbb{P}^o$. Mimicking the proof of Corollary 2.2.7, if we can construct a measure Q on Ω which is absolutely continuous with respect to P and such that it is invariant under the Markov transition M, we will conclude, as in Corollary 2.2.7, that $\overline{\omega}(n)$ is stationary and ergodic and hence

$$\frac{1}{n}\sum_{i=1}^{n} \omega(X_{n-1}, X_{n-1}+e_i) = \frac{1}{n}\sum_{i=1}^{n}\overline{\omega}(n)(0,e_i) \underset{n\to\infty}{\longrightarrow} \frac{a_i}{2}$$
$$:= E_Q\overline{\omega}(0,e_i) \geq \varepsilon, \mathbb{P}^o\text{-a.s.}, \tag{3.3.3}$$

yielding (3.3.1). Unlike in the one dimensional case, however, an explicit construction of an invariant measure Q is not available, and much of the work is to obtain properties of such invariant measure indirectly. In particular, a crucial point is to prove that it is absolutely continuous with respect to P.

Theorem 3.3.4 (Lawler[24]) *Assume the RWRE is balanced. Then, there exists a deterministic vector* $\mathbf{a}$ *such that (3.3.1) holds true. Consequently, the quenched CLT holds true.*

Proof of Theorem 3.3.4 Q is constructed from periodic modifications of the RWRE, and requires certain a-priori estimates on harmonic functions, which we state without proof.

The generator of the RWRE, under P_ω, is the operator

$$(L_\omega f)(x) = \sum_{i=1}^{d} \omega(x, x+e_i)\Big[f(x+e_i) + f(x-e_i) - 2f(x)\Big]\,.$$

For any $q > 0$ and $E \subset \mathbb{Z}^d$ bounded, define

$$\|g\|_{E,q} := \Big(\frac{1}{|E|}\sum_{x\in E} |g(x)|^q\Big)^{1/q}\,.$$

Lemma 3.3.5 (Lawler[24]) *(Maximum principle) There exists a constant $C = C(\varepsilon, d)$ such that for any $E \subset \mathbb{Z}^d$ bounded, any functions u and g such that*

$$L_\omega u(x) \geq -g(x), \quad x \in E$$

satisfy

$$\max_{x\in E} u(x) \leq C\mathrm{diam}(E)|E|^{1/d}\|g\|_{E,d} + \max_{x\in\partial E} u^+(x)\,.$$

We now introduce a periodic structure. Set $\Delta_N = \{-N, \cdots, N\}^d \subset \mathbb{Z}^d$ and identify elements of $T_N = \mathbb{Z}^d/(2N+1)\mathbb{Z}^d$ with a point of Δ_N, setting $\pi_N : \mathbb{Z}^d \to T_N$ and $\hat{\pi}_N : \mathbb{Z}^d \to \Delta_N$ to be the canonical projections. Set $\Omega^N = \{\omega \in \Omega : \theta^x\omega = \omega, \forall x \in (2N+1)\mathbb{Z}^d\}$. For any $\omega \in \Omega$, define $\omega^N \in \Omega^N$ by $\omega^N(x) = \omega(\hat{\pi}_N x)$. Note that ω^N is then a well defined function on T_N too.

Due to the ergodicity of P, it holds that in the sense of weak convergence,

$$P_N := \frac{1}{(2N+1)^d}\sum_{x\in\Delta_N} \delta_{\theta^x\omega^N} \underset{N\to\infty}{\longrightarrow} P, \quad P\text{-a.s.} \tag{3.3.6}$$

Let $\Omega_0 \subset \Omega$ denote those environments ω for which the convergence holds in (3.3.6) (clearly, $P(\Omega_0) = 1$).

Fixing $\omega \in \Omega_0$, let $(X_{n,N})_{n\geq 0}$ denote the RWRE on $\mathbb{Z}^d$ with law $P_{\omega^N}^{X_{0,N}}$. Then, $\overline{X}_{n,N} := \pi_N X_{0,N}$ is an irreducible Markov chain with finite state space T_N, and hence it possesses a unique invariant measure $\mu_N = \frac{1}{(2N+1)^d}\sum_{x\in T_N} \phi_N(x)\delta_x$. Setting $\varpi^N(n) := \theta^{X_{n,N}}\omega^N$, it follows that $\varpi^N(n)$ is an irreducible Markov chain with finite state space $S_N := \{\theta^x\omega^N\}_{x\in\Delta_N}$ and transition kernel M. Its unique invariant measure, supported on Ω^N, is then easily checked to be of the form

$$Q_N = \frac{1}{(2N+1)^d}\sum_{x\in\Delta_N} \phi_N(\pi_N x)\delta_{\theta^x\omega^N}\,.$$

Partitioning the state space S_N into finitely many *disjoint* states $\{\omega_\alpha^N\}_{\alpha=1}^K$, set $C_N(\alpha) = \{x \in \Delta_N : \theta^x\omega^N = \omega_\alpha^N\}$. Then,

$$f_N := \frac{dQ_N}{dP_N} = \sum_{\alpha=1}^{K} \mathbf{1}_{\{\omega=\omega_\alpha^N\}} \frac{1}{|C_N(\alpha)|} \sum_{x \in C_N(\alpha)} \phi_N(\pi_N x) .$$

We show below, as a consequence of Lemma 3.3.5, that there exists a constant $C_2 = C_2(\varepsilon, d)$, independent of N, such that

$$\|\phi_N(\pi_N \cdot)\|_{D_{N+1}(0),d/d-1} \leq C_2 . \tag{3.3.7}$$

Thus, using Jensen's inequality in the first inequality and (3.3.7) in the second,

$$\begin{aligned} \int f_N^{d/d-1} dP_N &= \sum_{\alpha=1}^{K} \left[\frac{1}{|C_N(\alpha)|} \sum_{x \in C_N(\alpha)} \phi_N(\pi_N x) \right]^{d/d-1} \frac{|C_N(\alpha)|}{(2N+1)^d} \\ &\leq \sum_{\alpha=1}^{K} \sum_{x \in C_N(\alpha)} \phi_N(\pi_N(x))^{d/d-1} \frac{1}{(2N+1)^d} \\ &= \frac{1}{(2N+1)^d} \sum_{x \in \Delta_N} \phi_N(\pi_N(x))^{d/d-1} \leq C_2^{(d-1)/d} . \end{aligned} \tag{3.3.8}$$

Note that f_N extends to a measurable function on Ω, and the latter is, due to (3.3.8), uniformly integrable with respect to P_N. Thus, any weak limit of Q_N is absolutely continuous with respect to P, and further it is invariant with respect to the Markov kernel M.

Let $E = \{\omega : \frac{dQ}{dP} = 0\}$. By invariance, $E_Q M \mathbf{1}_E = E_Q \mathbf{1}_E = 0$, and hence $M\mathbf{1}_E \leq \mathbf{1}_E$, P-a.s. But, $M\mathbf{1}_E \geq \varepsilon \sum_{i=1}^{d} (\mathbf{1}_E \circ \theta^{e_i} + \mathbf{1}_E \circ \theta^{-e_i})$. Hence, $\mathbf{1}_E \geq \mathbf{1}_E \circ \theta^{\pm e_i}$, P-a.s. Since P is stationary, $\mathbf{1}_E = \mathbf{1}_E \circ \theta^{\pm e_i}$, P-a.s., and hence by ergodicity (considering the invariant event $\cap_{x \in \mathbb{Z}^d} (\theta^x)^{-1} E$) $P(E) \in \{0,1\}$. But $Q \ll P$ implies $P(E) = 0$. Hence, $Q \sim P$, as claimed (further, by (3.3.3), Q is then uniquely defined).

It thus only remains to prove (3.3.7). Fix a function g on T_N, and define the resolvent

$$\begin{aligned} R^{\omega^N} g(x) &:= \sum_{j=0}^{\infty} \left(1 - \frac{1}{N^2}\right)^j E_{\omega^N}^x g(\overline{X}_{j,N}) \\ &= \sum_{j=0}^{\infty} \left(1 - \frac{1}{N^2}\right)^j E_{\omega^N}^x g \circ \pi_N(X_{j,N}), \quad x \in T_N \end{aligned}$$

and the stopping times $\tau_0 = 0, \tau_1 = \tau := \min\{k \geq 1 : |X_{k,N} - X_{0,N}| \geq N\}$ and $\tau_{k+1} = \tau \circ \theta^k + \tau_k$. Since for $x \in \mathbb{Z}^d$ with $|x - X_{0,N}| < N$ it holds that $L_{\omega^N} E_{\omega^N}^x \left(\sum_{j=0}^{\tau-1} g \circ \pi_N(X_{j,N}) \right) = -g(x)$, we have by Lemma 3.3.5 that for some constant $C = C(\varepsilon, d)$,

$$\sup_{|x-X_{0,N}|<N} \left| E^x_{\omega^N} \left(\sum_{j=0}^{\tau-1} g \circ \pi_N(X_{j,N}) \right) \right| \leq CN^2 \|g\|_{D_{N+1}(0),d} \,. \tag{3.3.9}$$

Since $(X_{n,N})_{n\geq 0}$ is a martingale, it follows from Doob's inequality that, for any $K \geq 1$,

$$\begin{aligned} P^o_{\theta^x\omega^N}[\tau \leq K] &\leq 2\sum_{i=1}^d P^o_{\theta^x\omega^N}\left[\sup_{n\leq K} X_n(i) \geq N\right] \\ &\leq \frac{2}{N}\sum_{i=1}^d E^o_{\theta^x\omega^N}\left((X_K(i))_+\right) \leq \frac{2d}{N}\sqrt{K}\,. \end{aligned}$$

Hence, using $K = N^2/8d^2$,

$$E^o_{\theta^x\omega^N}\left(\left(1-\frac{1}{N^2}\right)^\tau\right) \leq \frac{2d}{N}\sqrt{K} + \left(1-\frac{1}{N^2}\right)^K \leq C_3 \tag{3.3.10}$$

where $C_3 = C_3(d) < 1$ is independent of N. Thus, using the strong Markov property, (3.3.9) and (3.3.10),

$$\begin{aligned} |R^{\omega^N} g(x)| &= \sum_{m\geq 0} E^x_{\omega^N}\left(\sum_{\tau_m\leq j<\tau_{m+1}} \left(1-\frac{1}{N^2}\right)^j g\circ\pi_N(X_{j,N})\right) \\ &\leq \sum_{m\geq 0} E^x_{\omega^N}\left(\left(1-\frac{1}{N^2}\right)^{\tau_m} E^{X_{\tau_m,N}}_{\omega^N}\sum_{j=0}^{\tau-1} g\circ\pi_N(X_{j,N})\right) \\ &\leq \sum_{m\geq 0}\left(\sup_{x\in\mathbb{Z}^d} E^x_{\omega^N}\left(\left(1-\frac{1}{N^2}\right)^\tau\right)\right)^m \cdot \sup_{x\in\mathbb{Z}^d} E^x_{\omega^N}\left(\sum_{j=0}^{\tau-1} g\circ\pi_N(X_{j,N})\right) \\ &\leq C_4 N^2 \|g\|_{D_{N+1}(0),d} \end{aligned}$$

where $C_4 = C_4(d,\varepsilon)$. Using the invariance of ϕ_N, we now get

$$\begin{aligned} \|\phi_N(\pi_N\cdot)\|_{D_{N+1}(x_0),d/d-1} &= \|\phi_N(\pi_N\cdot)\|_{D_{N+1}(0),d/d-1} \\ &= \sup_{g:\|g\|_{D_{N+1}(0),d}\leq 1} \frac{1}{|D_{N+1}(0)|}\sum_{y\in D_{N+1}(0)} \phi_N(\pi_N y)g(y) \\ = \frac{1}{N^2}\sup_{g:\|g\|_{D_{N+1}(0),d}\leq 1} &\sum_{k\geq 0}\left(1-\frac{1}{N^2}\right)^k \frac{1}{(2N+1)^d}\sum_{x\in\Delta_N}\phi_N(x)E^x_{\omega_N}(g\circ\pi_N(X_{k,N})) \\ &\leq C_2 \end{aligned}$$

with $C_2 = C_2(d,\varepsilon)$, proving (3.3.7). □

Remark: Stronger forms of Lemma 3.3.5 are available. In particular, a Harnack inequality for L_ω can be proved. Based on this, and some ergodic arguments, one can prove that the balanced RWRE is transient for $d \geq 3$ and recurrent for $d = 2$. We refer to [42] for details.

Cut points and diffusive behaviour We describe briefly another class of walks for which LLN's and CLT's are available, still with P uniformly elliptic and product. Let $d = d_1 + d_2$ with $d_1 \geq 5$, and call the RWRE *RW based* if $\omega(x, x \pm e_i) = c_i$, $i = 1, \ldots, d_1$.

Theorem 3.3.11 (Bolthausen, Sznitman, Zeitouni[3]) *Assume the RWRE X_n is RW based. If $d_1 \geq 5$ then there exists a deterministic constant v (possibly zero) such that $X_n/n \to v$, $\mathbb{P}^o$-a.s. Further, if $d_1 \geq 13$ then $(X_n - nv)/\sqrt{n}$ satisfies both quenched and annealed CLTs.*

Proof. (sketch) Write $X_n = X_n^1 + X_n^2$ where X_n^1 represents in an obvious way the projection of X_n on the first d_1 coordinates. Then, X_n^1 is a random walk in a homogeneous deterministic environment (with holding times), determined by the c_i. Extend X_n^1 to negative indices n. Define $I_0 = 0$ and

$$I_i = \min\{n \geq I_{i-1} : \{X_t^1, t \leq n\} \cap \{X_t^1, t > n\} = \emptyset\}\,.$$

The times $\{I_i\}_{i \geq 1}$ are called *cut times*, and by an adaptation of a celebrated theorem of Erdös and Taylor [15], they are finite almost surely (as soon as $d_1 \geq 5$). The sequence T_i is not stationary (because of the special role that 0 plays!), but it is not hard to define a new probability measure, denoted $\hat{P}$, (essentially, size-biasing I_1) that makes it into a stationary sequence, see Lemma 1.1 in [3]. The crucial observation is now that the increments $X^2_{I_{i+1}} - X^2_{I_i}$ are, conditionally on the sequence $\{I_i\}$, independent. The law of large numbers then follows from stationarity under $\hat{P}$ and the fact that $E_{\hat{P}}(I_{i+1} - I_i) < \infty$. The CLT in the case that P is symmetric under reflections and $d_1 \geq 7$ follows from the fact that in that case, the velocity in the d_2 direction vanishes, and $E_{\hat{P}}(I_{i+1} - I_i)^2 < \infty$. The analysis in the case that P is not symmetric under reflections is subtler, and requires a control on higher moments of $I_{i+1} - I_i$, hence the restriction to $d_1 \geq 13$. We refer to [3] for details. □

Besides the interest in the class of RW based walks, they can be used to demonstrate surprising behaviour of the RWRE. For example, one may construct an product law P with $d \geq 15$ such that the expected local drift does not vanish, that is

$$E_P d(0, \omega) \neq 0\,,$$

but $X_n/n \to 0$ $\mathbb{P}^o$-a.s. and further $X_n/\sqrt{n}$ satisfies a quenched CLT (in fact, one can construct such examples where P is a small perturbation of simple random walk). One also constructs examples, in high dimension, where the expected local drift does vanish but the walk is ballistic.

3.4 Left out topics and open problems

I hope that these notes have succeeded in giving you a glimpse of what's available in the study of the RWRE, and some open questions of interest In this concluding section, I collected some topics I did not touch upon, in which recent progress has been made and in which additional open problems abound. Some of the material here is taken from [43]

Ballistic walks Consider again the case of P a product, uniformly elliptic measure. As we have seen in Section 3.2, Kalikow's condition provides a sufficient condition for the RWRE to be ballistic. In a remarkable sequence of papers [35, 37, 38, 39], Sznitman has introduced a condition, Sznitman's T' condition, in the form of requiring the probability of exit from the left from large slabs (in a direction ℓ) to have "almost exponential" tails, which turns out to be equivalent to the condition that the distance travelled between regenerations has almost exponential tails. He then moves on to provide necessary and sufficient conditions (in his terminology, an *effective criterion*) for T' to hold true, shows that Kalikow's sufficient condition implies T', and provides a class of examples that do not satisfy Kalikow's condition but do satisfy T'. Further, for measures P that are a small perturbation of simple random walk, he shows that if the (small) local drift is strong enough (as a function of the perturbation parameter) then the RWRE is ballistic. Specifically, fix ϵ small and say P is an ϵ perturbation of simple random walk if P is a product measure such that for all $|e| = 1$, $|\omega(0,e) - 1/2d| \leq \epsilon$, P-a.s.. Consider the annealed local drift in the direction e_1, namely

$$\bar{d}_P := E_P\left[\omega(0,e_1) - \omega(0,-e_1)\right] .$$

One has:

Theorem 1 (Sznitman[39]). *For $d \geq 3$ and $\eta > 0$ there exists an $\epsilon_0(d,\eta) > 0$ such that if $\epsilon < \epsilon_0$ and P is an ϵ perturbation of simple random walk with*

$$\bar{d}_P \geq \begin{cases} \epsilon^{2.5-\eta}, & d = 3 \\ \epsilon^{3-\eta}, & d > 3, \end{cases}$$

then the RWRE X_n is ballistic in the direction e_1. In particular,

$$\lim_{n\to\infty} \frac{X_n \cdot e_1}{n} = v_1 > 0 .$$

This theorem stands in sharp constrast with the examples of [3] mentioned at the end of Section 3.3.

Sznitman's work raises the following natural
Open Problem: Do there exist non-ballistic RWRE's for $d > 1$ satisfying that $\mathbb{P}(A_\ell) = 1$ for some ℓ?
For $d = 1$, the answer is affirmative, as we saw, as soon as $E_P \log \rho_0 < 0$ but $s < 1$. For $d > 1$, one suspects that the answer is negative, and in fact one may suspect that $\mathbb{P}(A_\ell) = 1$ implies Sznitman's condition T'. The reason for the striking difference is that for $d > 1$, it is much harder to force the walk to visit large traps.

Large deviations I have not discussed at all large deviations, either quenched or annealed. By these I mean statements of the form

$$-\inf_{x\in A^o} I_q(x) \leq \liminf \frac{1}{n} \log P_\omega^o\left(\frac{X_n}{n} \in A\right)$$

$$\leq \limsup \frac{1}{n} \log P_\omega^o(\frac{X_n}{n} \in A) \leq -\inf_{x \in \bar{A}} I_q(x)\,, P-a.s.,$$

and

$$-\inf_{x \in A^o} I_a(x) \leq \liminf \frac{1}{n} \log \mathbb{P}^o(\frac{X_n}{n} \in A)$$

$$\leq \limsup \frac{1}{n} \log \mathbb{P}^o(\frac{X_n}{n} \in A) \leq -\inf_{x \in \bar{A}} I_a(x)\,,$$

for deterministic functions I_q, I_a that are lower semicontinuous, with compact level sets and, as it turns out, convex, and all measurable sets A. For $d=1$, such statements can be obtained by a careful analysis of the hitting times T_n, and the analysis reveals a variational link between I_a and I_q, see [8]. An interesting observation is that even if one deals only with the case of P product, the analysis of the annealed LDP requires one to consider the quenched LDP for a class of non-product P. Because explicit computations are possible, detailed information of the rate functions I_a, I_q is available. Another approach to the quenched LDP, based on the environment viewed from the point of view of the particle, is presented in [18].

For $d>1$, hitting times have been less succesful, and have yielded only a quenched LDP in the *nestling* situation, that is when the closed convex hull of the support of the law of the local drift $d(x,\omega)$ contains the origin, see [44]. Recently, Varadhan [41] has suggested a different approach toward the LDP, both quenched and annealed, without passing through hitting times. The quenched result is strickingly simple to describe: suppose P is uniformly elliptic and just ergodic. Fix $\epsilon>0$, and define X_n^ϵ to be the RWRE with geometric holding times of parameter $1/\epsilon$. Fix a deterministic v, with $|v|_1<1$, and define

$$g(m,n) = P^0_{\theta^{[mv]}\omega}(X^\epsilon_{m-n} - X^\epsilon_0 = [(n-m)v])\,.$$

Then, $g(0,n+m) \geq g(0,m)g(m,n+m) > 0$ for all $n,m \geq 1$. Consequently, by Liggett's version of Kingman's ergodic sub-additive theorem,

$$\frac{1}{n}\log g(0,n) \to_{n\to\infty} -I^\epsilon(v)\,, P-a.s.,$$

for some deterministic $I^\epsilon(v)$. From this it follows (using ellipticity) (see e.g. [12, Theorem 4.1.11]) that X_n^ϵ/n satisfies the (quenched) LDP with convex, good rate function $I^\epsilon(\cdot)$. Finally, it is easy to check that $I^\epsilon(\cdot) \to_{\epsilon\to 0} I(\cdot)$ (even uniformly on compacts) and that

$$\limsup_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P_\omega^o(|X_n - X_n^\epsilon| > \delta n) = -\infty\,, P-a.s.,$$

from which it follows that X_n/n satisfies the quenched LDP with deterministic, convex, good rate function $I(\cdot)$. Varadhan obtains the annealed LDP for P product by looking at the annealed walk as an evolution of conditional distributions of the environment via Bayes' rule. While the resulting information on the rate function is rather weak, it is strong enough to prove that the rate function can vanish at most on a line passing through the origin.

In recent work, Rassoul-Agha [30] reinterprets Varadhan's work in terms of a class of reinforced random walks and extends his results to a class of mixing environments.

Open Problem Find a variational (or other) relation between the quenched and annealed rate functions in the multi-dimensional case.

Mixing environments Many of the results for $d > 1$ rely on P being a product measure. Recently, some efforts have been invested in trying to get rid of this restriction. Rassoul-Agha [29] considers the environment viewed from the particle and while an invariant measure Q that is absolutely continuous with respect to P may not exist, he shows that under Kalikow's condition and mixing conditions of the Dobrushin-Shlosman type, Q has enough regularity to allow one to conclude a LLN. In a different direction, and under a mixing condition on cones, Comets and Zeitouni [9] give an integrability condition on (modified) regeneration times that ensure the LLN, and check that this condition is satisfied for some environments that satisfy Kalikow's condition. As pointed out in Section 3.2, their approach is particularly simple if the environment P has finite range of dependence.

Open problem Derive generalizations of Sznitman's T' condition in the mixing case.

Open Problem Derive annealed and quenched CLT's for truly (ie, not finitely dependent) mixing environments. That some caution is needed should be clear from reading [4].

Further questions We conclude with the following embarrasing open problems:

Open Problem For P product and uniformly elliptic, and $d > 1$, find criteria for transience.

Open Problem Prove that with P product and uniformly elliptic, RWRE for $d \geq 3$ is always diffusive. Does this remain true when $d = 2$?

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