REMARKS ON A CONSTRAINED OPTIMIZATION PROBLEM FOR THE GINIBRE ENSEMBLE

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ABSTRACT. We study the limiting distribution of the eigenvalues of the Ginibre ensemble conditioned on the event that a certain proportion lie in a given region of the complex plane. Using an equivalent formulation as an obstacle problem, we describe the optimal distribution and some of its monotonicity properties.

1. Introduction and statement of results

With high probability, a square N-by-N matrix with complex independent, standard Gaussian entries (i.e., the *Ginibre ensemble*) has eigenvalues evenly spread in the ball of radius $\sqrt{N}$. More precisely, after multiplying the matrix by $N^{-1/2}$ and letting $N \to \infty$, one finds that the spectral measure converges (weakly in the sense of measures) to the (suitably normalized) uniform measure on the unit disk $D \subseteq \mathbb{C}$:

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{z_i} \rightharpoonup \mu_0$$

where the z_i's are the eigenvalues of the matrix, δ_z denotes the Dirac mass at $z \in \mathbb{C}$ and

$$d\mu_0 := \frac{1}{\pi}\mathbb{1}_D \, dx. \tag{1.1}$$

This statement is called the *circular law* and goes back to Ginibre [10].

Hiai and Petz [17] proved, in addition, that the law of spectral measure of the eigenvalues obeys a large deviations principle with speed N^2 and rate function

$$\mathcal{I}[\mu] := -\int_{\mathbb{C}\times\mathbb{C}} \log|x-y| \, d\mu(x)\, d\mu(y) + \int_{\mathbb{C}} |x|^2 \, d\mu(x). \tag{1.2}$$

(See Ben Arous and Zeitouni [4] for the case of real Gaussian entries.) Roughly, this means that if $\mathcal{A} \subseteq \mathcal{P}(\mathbb{C})$, then

$$\mathbb{P}\left[\frac{1}{N}\sum_{i=1}^{N} \delta_{z_i} \in \mathcal{A}\right] \simeq \exp\left(-N^2(\min_{\mathcal{A}} \mathcal{I} - \min_{\mathcal{P}(\mathbb{C})} \mathcal{I})\right).$$

Of course, the unique minimizer of $\mathcal{I}$ over the set $\mathcal{P}(\mathbb{C})$ of probability measures on $\mathbb{C}$ is the circular law μ_0. In view of this result, in order to understand the likely arrangement of eigenvalues after conditioning on a certain rare event (e.g., having a "hole" in D without a significant number of eigenvalues) it suffices to find the minimizer of $\mathcal{I}$ on the event.

Related random matrix models are the *Gaussian unitary ensemble* (GUE) and *Gaussian orthogonal ensemble* (GOE), in which the matrices are constrained to be Hermitian and real symmetric,

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respectively, and thus have real eigenvalues. The analogue of the circular law is Wigner's *semicircle law*:

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} \rightharpoonup \nu_0 := \mathbb{1}_{|x|\le 2}\sqrt{4-x^2}\,dx.$$

Here ν_0 minimizes $\mathcal{I}$ over the set $\mathcal{P}(\mathbb{R})$ of probability measures on the real line. The corresponding large deviations principle in this setting was proved in the seminal paper by Ben Arous and Guionnet [3].

In this context, Majumdar, Nadal, Scardicchio and Vivo [14, 15] examined the rare event that a given proportion $p \in [0,1]$ of the eigenvalues must lie on interval $[a,\infty)$. Determining the likely configuration of the eigenvalues as $N \to \infty$, conditioned on this rare event, is equivalent to minimizing $\mathcal{I}$ (over probability measures on $\mathbb{R}$) under the constraint that $\mu([a,\infty)) \ge p$. Explicit formulas are given in [15] for the optimal distribution μ. A nontrivial situation occurs when $p > \nu_0([a,\infty))$, in which case they find that μ has an L^1 density $\varphi(x)$ which tends to infinity as $x \to a+$ and vanishes in an interval $[a-\varepsilon, a)$ for some $\varepsilon > 0$.

In this paper we examine the analogue of this question for the Ginibre ensemble. We consider an open subset $U \subseteq \mathbb{C}$ with $C^{1,1}$ boundary such that the boundary of $U \cap D$ is locally Lipschitz, and we study the unlikely event in which the random matrix has too many eigenvalues in U. To avoid trivial situations we assume that $D \setminus U$ is nonempty, fix $\mu_0(U) < p \le 1$ and minimize $\mathcal{I}$ subject to the constraint that

$$\mu(\overline{U}) \ge p. \tag{1.3}$$

The minimization is not over an empty set, as a convex combination of an appropriate multiple of the uniform measure on U and of $\widehat{\mu}_0$ satisfies (1.3) and has finite $\mathcal{I}$. Due to the convexity of the constraint, the compactness of the level sets of $\mathcal{I}$, and the strict convexity of the functional $\mathcal{I}$, there is a unique minimizer of $\mathcal{I}$ over

$$\mathcal{C} := \{\mu \in \mathcal{P}(\mathbb{C}) \,:\, (1.3) \text{ holds}\},$$

which we denote by $\widehat{\mu}$. Note that if $p = 1$, then $\mathcal{C}$ is the set of probability measures with a "hole" in $\mathbb{C} \setminus \overline{U}$.

In contrast to the results for the one dimensional model studied in [15], our analysis typically does not lead to explicit formulas. Rather, we study certain qualitative properties of the minimizing measure $\widehat{\mu}$. (An exception is the case where $p = 1$ and the set U is a half-space parallel to the imaginary axis; see Section 3 for that setup.)

Here is the statement of the main results. (See Figure 1 for an illustration of the results.)

Theorem 1. *Assume that $U \subseteq \mathbb{C}$ is open, ∂U is locally $C^{1,1}$ and $\partial(U \cap D)$ is locally Lipschitz. Then the measure $\widehat{\mu}$ can be decomposed as*

$$d\widehat{\mu} = \frac{1}{\pi}\mathbb{1}_V\,dx + d\widehat{\mu}_S,$$

where $V \subseteq \mathbb{C}$ is bounded and open and

$$d\widehat{\mu}_S = g\,d\mathcal{H}^1|_{\partial U},$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure and $g \ge 0$ belongs to $L^2(\partial U)$. Moreover,

$$\partial U \cap D \subseteq \operatorname{supp}\widehat{\mu}_S, \tag{1.4}$$

$$\overline{(V \setminus U)} \cap \partial U = \varnothing, \tag{1.5}$$

$$V \setminus U \subseteq D, \quad \text{and moreover, if } U \not\subseteq D \text{ then } \overline{V} \setminus \overline{U} \subseteq D \tag{1.6}$$

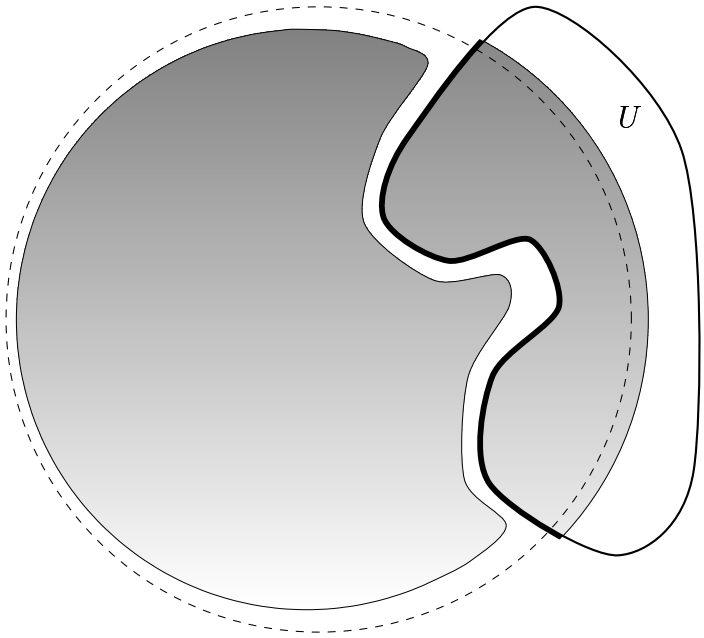

FIGURE 1. An illustration of Theorem 1 in the case p is slightly larger than $\mu_0(U)$ and $U \not\subseteq D$. The dashed line is the unit ball and the shaded area is the support of the measure $\widehat{\mu}$. The thick part of the boundary of ∂U represents the support of $\widehat{\mu}_S$.

and finally

$$\overline{U \cap D} \subseteq V \cup \operatorname{supp} \widehat{\mu}_S. \tag{1.7}$$

The fact that, at least in the case that $\partial U \cap D \neq \emptyset$, the singular part of $\widehat{\mu}$ is nonzero is in contrast to the one-dimensional model [15] which has a minimizing measure absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$. On the other hand, the qualitative picture portrayed by (1.6) and (1.7) – an expansion of the support of the measure in U, the contraction of it in the complement of U and the appearance of a gap on the outside of ∂U along the support of $\widehat{\mu}_S$ – is the same as that of [15].

The proof of Theorem 1 is based on interpreting the minimization problem as an *obstacle problem*. This connection is folklore and was mentioned briefly in [18, Ch. 2]; it was also employed in a context related to this paper in [11, 12]. (For more detail and the case of general potentials and dimensions, see the forthcoming lecture notes [20].) The advantage of writing the problem this way is that it allows us to use the classical free-boundary regularity theory of Caffarelli [5] combined with some maximal principle arguments such as Hopf's Lemma.

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2. Proof of Theorem 1

The proof of Theorem 1 is based on identifying the minimization problem as an obstacle problem for the Newtonian potential of $\widehat{\mu}$. This is the content of subsection 2.1. The actual proof of Theorem 1 is taken up in subsection 2.2, and proceeds by first establishing regularity properties

of $H^{\widehat{\mu}}$ and then applying Hopf's lemma together with Caffarelli's regularity theory for solutions of free boundary problems.

2.1. **Formulation as an obstacle problem.** We first characterize the measure $\widehat{\mu}$ in terms of an obstacle problem. The solution of the obstacle problem turns out to be the Newtonian potential generated by $\widehat{\mu}$, which we denote by

$$H^{\widehat{\mu}}(x) := -\int_{\mathbb{R}^2} \log|x-y|\, d\widehat{\mu}(y). \tag{2.1}$$

(We henceforth identify $\mathbb{R}^2$ with $\mathbb{C}$.) We show in this subsection that $H^{\widehat{\mu}}$ satisfies the variational inequality

$$\forall v \in \mathcal{K}, \qquad \int_{\mathbb{R}^2} \nabla H^{\widehat{\mu}}(y) \cdot \nabla\big(v - H^{\widehat{\mu}}\big)(y)\, dy \geq 0, \tag{2.2}$$

where

$$\mathcal{K} := \left\{ v \in H^1_{\mathrm{loc}}(\mathbb{R}^2) \,:\, v - H^{\widehat{\mu}} \ \text{ has bounded support in } \mathbb{R}^2 \ \text{ and } \ v \geq \psi \text{ q.e. in } \mathbb{R}^2 \right\}$$

and the *obstacle* is the function ψ given by

$$\psi(x) := \begin{cases} \frac{1}{2}(c_1 - |x|^2) & x \in \overline{U} \\ \frac{1}{2}(c_2 - |x|^2) & x \notin \overline{U}, \end{cases} \tag{2.3}$$

where $c_1 > c_2 \geq -\infty$ are constants determined below. Note that (2.2) is equivalent to the statement that, for every $R > 0$, $H^{\widehat{\mu}}$ is the unique solution of the following minimization problem:

$$\min\left\{ \int_{B_R} \left|\nabla v(y)\right|^2 dy \,:\, v \in H^1(B_R) \text{ such that } v - H^{\widehat{\mu}} \in H^1_0(B_R) \text{ and } v \geq \psi \text{ in } B_R \right\}. \tag{2.4}$$

Solutions of the variational inequality (2.2) are said to be solutions of the *obstacle problem*

$$\min\left\{ -\Delta H^{\widehat{\mu}}, H^{\widehat{\mu}} - \psi \right\} = 0 \quad \text{in } \mathbb{R}^2. \tag{2.5}$$

That is, the precise interpretation of (2.5) is that $H^{\widehat{\mu}}$ is a solution of (2.2). The reader can check by an integration by parts that this notion is consistent with the classical interpretation of (2.5) in the case that $H^{\widehat{\mu}}$ is smooth. For general references on the obstacle problem, see [5, 13].

To prove (2.5), we observe that $-\Delta H^{\widehat{\mu}} = 2\pi\widehat{\mu} \geq 0$ and in particular $H^{\widehat{\mu}}$ is superharmonic and harmonic in the complement of the support of $\widehat{\mu}$. It remains to show that $H^{\widehat{\mu}} \geq \psi$ in $\mathbb{R}^2$ and $H^{\widehat{\mu}} = \psi$ on the support of $\widehat{\mu}$. This is the statement of Lemma 2.1 below. In Lemma 2.3, we prove that $H^{\widehat{\mu}}$ satisfies (2.2).

In what follows, "q.e." is short for *quasi-everywhere* which means "except possibly in of a set of capacity zero." A set K is of *capacity zero* if there does not exist a probability measure ν with support in a compact subset of K such that $\iint -\log|x-y|\, d\nu(x)\, d\nu(y)$ is finite (c.f. [18]). In particular, a set of capacity zero has zero Lebesgue measure and we note that a measure μ satisfying $\mathcal{I}[\mu] < +\infty$ does not charge sets of zero capacity.

We first observe that $\widehat{\mu}(\overline{U}) = p$. If, on the contrary, $\widehat{\mu}(\overline{U}) > p$, then for small enough $t > 0$, we have $(1-t)\widehat{\mu} + t\mu_0 \in \mathcal{C}$. The strict convexity of $\mathcal{I}$ then contradicts the minimality of $\widehat{\mu}$:

$$\mathcal{I}[(1-t)\widehat{\mu} + t\mu_0] < (1-t)\mathcal{I}[\widehat{\mu}] + t\mathcal{I}[\mu_0] \leq \mathcal{I}[\widehat{\mu}].$$

Lemma 2.1. *There exist $c_1 > c_2 \geq -\infty$ with $c_2 = -\infty$ if $p = 1$, such that*

$$2H^{\widehat{\mu}}(x) + |x|^2 \geq c_1 \quad \textit{q.e. in } \overline{U} \quad \textit{and} \quad 2H^{\widehat{\mu}}(x) + |x|^2 = c_1 \quad \textit{q.e. in } \operatorname{supp}(\widehat{\mu}) \cap \overline{U} \tag{2.6}$$

and

$$2H^{\widehat{\mu}}(x) + |x|^2 \geq c_2 \quad \textit{q.e. in } \mathbb{C} \setminus \overline{U} \quad \textit{and} \quad 2H^{\widehat{\mu}}(x) + |x|^2 = c_2 \quad \textit{q.e. in } \operatorname{supp}(\widehat{\mu}) \setminus \overline{U}. \tag{2.7}$$

Proof. We argue by the standard method of variations (one could also proceed by computing the convex dual of the optimization problem). Select $\nu \in \mathcal{C}$ such that $\mathcal{I}[\nu] < +\infty$ and observe that, for every $t \in [0,1]$,

$$\mathcal{I}\left[(1-t)\widehat{\mu} + t\nu\right] \geq \mathcal{I}[\widehat{\mu}]. \tag{2.8}$$

Expanding in powers of t and letting $t \to 0$, we discover that

$$\int_{\mathbb{C}} \left(2H^{\widehat{\mu}}(x) + |x|^2\right) d(\nu - \widehat{\mu})(x) \geq 0 \tag{2.9}$$

for every $\nu \in \mathcal{C}$ for which $\mathcal{I}[\nu] < \infty$.

We claim that

$$2H^{\widehat{\mu}}(x) + |x|^2 \geq \frac{1}{p}\int_{\overline{U}} \left(2H^{\widehat{\mu}}(x) + |x|^2\right) d\widehat{\mu}(x) =: c_1 \quad \text{q.e. in } \overline{U}. \tag{2.10}$$

If (2.10) is false, then there exists a compact set $E \subseteq \overline{U}$ of positive capacity on which the opposite inequality holds. By the characterization of positive capacity, there exists a measure ν_E of mass p supported in E with $\iint -\log|x-y|\, d\nu_E(x)\, d\nu_E(y) < \infty$. We obtain a contradiction to (2.9) by taking ν to be the sum of ν_E and the restriction of $\widehat{\mu}$ to $\mathbb{C} \setminus \overline{U}$. This establishes (2.10), and integrating it against $\widehat{\mu}$ (which does not charge sets of zero capacity since it has finite energy) yields (2.6).

If $p = 1$ we stop the reasoning here and all the claims are true since (2.7) holds with $c_2 = -\infty$. If $p < 1$, we obtain (2.7) by a similar argument as above, applying (2.9) to a measure ν which coincides with $\widehat{\mu}$ in $\overline{U}$ and has mass $1-p$ in $\mathbb{C} \setminus \overline{U}$, resulting with

$$c_2 := \frac{1}{1-p}\int_{\mathbb{C}\setminus\overline{U}} \left(2H^{\widehat{\mu}}(x) + |x|^2\right) d\widehat{\mu}(x).$$

It remains to prove that $c_1 > c_2$. By strict convexity of $\mathcal{I}$ and the fact that $\mathcal{I}[\widehat{\mu}] > \mathcal{I}[\mu_0]$, we have

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{I}\left[(1-t)\widehat{\mu} + t\mu_0\right] < 0.$$

This says, by the same computation as above, that

$$\int_{\mathbb{C}} \left(2H^{\widehat{\mu}}(x) + |x|^2\right) d(\mu_0 - \widehat{\mu})(x) < 0.$$

Using (2.6) and (2.7), we obtain

$$c_1\mu_0(U) + c_2\mu_0(\mathbb{C}\setminus U) < pc_1 + (1-p)c_2.$$

Since $p > \mu_0(U)$ by assumption, we deduce that $c_1 > c_2$. □

In the special case $p = 1$, which is considered in Section 3, the result in Lemma 2.1 is actually a characterization.

Lemma 2.2. *Assume $p = 1$ and that $\mu \in \mathcal{C}$ satisfies*

$$2H^{\mu}(x) + |x|^2 \geq c \quad \text{q.e. in } \overline{U} \quad \text{and} \quad 2H^{\mu}(x) + |x|^2 = c \quad \text{q.e. in } \operatorname{supp}(\mu) \cap \overline{U}\,, \tag{2.11}$$

for some $c \in \mathbb{R}$. Then, $\mu = \widehat{\mu}$.

Proof. Assume not, and set, for $t \in (0,1)$,

$$\mu_t := t\mu + (1-t)\widehat{\mu}.$$

We have

$$\begin{aligned}\mathcal{I}[\mu_t] &= t\int (H^{\mu}(x)+|x|^2)\,d\mu_t(x) + (1-t)\int (H^{\widehat{\mu}}(x)+|x|^2)\,d\mu_t(x)\\ &= \frac{t}{2}\int (2H^{\mu}(x)+|x|^2)\,d\mu_t(x) + \frac{(1-t)}{2}\int (2H^{\widehat{\mu}}(x)+|x|^2)d\mu_t(x) + \frac{1}{2}\int |x|^2\,d\mu_t(x)\\ &\geq \frac{t}{2}\left(c+\int |x|^2\,d\mu(x)\right) + \frac{(1-t)}{2}\left(c_1+\int |x|^2\,d\widehat{\mu}(x)\right), \qquad (2.12)\end{aligned}$$

where (2.6) and (2.11) were used in the inequality. On the other hand, integrating (2.6) and (2.11) with respect to μ and $\widehat{\mu}$ gives

$$\mathcal{I}[\mu] = \frac{1}{2}\left(c+\int |x|^2 d\mu(x)\right) \quad \text{and} \quad \mathcal{I}[\widehat{\mu}] = \frac{1}{2}\left(c+\int |x|^2 d\widehat{\mu}(x)\right).$$

Substituting in (2.12) then gives

$$\mathcal{I}[\mu_t] \geq t\mathcal{I}[\mu] + (1-t)\mathcal{I}[\widehat{\mu}],$$

which contradicts the strict convexity of $\mathcal{I}$ unless $\mu = \widehat{\mu}$. □

Lemma 2.3. *The function $H^{\widehat{\mu}}$ is the unique element of $\mathcal{K}$ which satisfies* (2.2).

Proof. First, we show that $\nabla H^{\widehat{\mu}}$ is in $L^2_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$. To see this first note that if ρ is a compactly supported Radon measure on $\mathbb{R}^2$ with $\int d\rho = 0$ satisfying $\iint -\log|x-y|\,d\rho(x)_+\,d\rho(y)_+ < +\infty$, $\iint -\log|x-y|\,d\rho_-(x)\,d\rho_-(y) < +\infty$ and H^{ρ} is the potential generated by ρ, defined as usual by $H^{\rho}(x) := -\int \log|x-y|\,d\rho(y)$, then we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} -\log|x-y|\,d\rho(x)\,d\rho(y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla H^{\rho}(x)|^2\,dx. \qquad (2.13)$$

See the proof of [18, Lemma 1.8] where, more precisely, it is shown that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} -\log|x-y|\,d\rho(x)\,d\rho(y) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{1}{|x-y|}\,d\rho(y)\right)^2 dx.$$

Then (2.13) follows, since $\int_{\mathbb{R}^2}\frac{(x-y)d\rho(y)}{|x-y|^2} = -\nabla H^{\widehat{\mu}}(x)$ in the distributional sense.

Next, consider μ_0 (defined in (1.1)) and its associated potential H^{μ_0}, which belongs to $C^1(\mathbb{R}^2)$ by direct computation. The measure $\rho := \widehat{\mu} - \mu_0$ is a compactly supported Radon measure with $\int d\rho = 0$. In addition, since $\mathcal{I}[\widehat{\mu}] < \infty$ and $\mathcal{I}[\mu_0] < \infty$, it satisfies the desired assumptions. Thus (2.13) holds, which proves that $\nabla H^{\widehat{\mu}} - \nabla H^{\mu_0}$ is in $L^2(\mathbb{R}^2;\mathbb{R}^2)$. Since ∇H^{μ_0} is in $L^{\infty}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$, it follows that $\nabla H^{\widehat{\mu}} \in L^2_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$, as claimed.

We next turn to the proof that $H^{\widehat{\mu}}$ satisfies (2.2). Pick $v \in \mathcal{K}$ and set $\varphi := v - H^{\widehat{\mu}}$. Observe that, in the case that φ is smooth and compactly supported, we have

$$\int_{\mathbb{R}^2} \nabla H^{\widehat{\mu}}\cdot\nabla\varphi = 2\pi\int_{\mathbb{R}^2}\varphi\,d\widehat{\mu} \geq 0. \qquad (2.14)$$

The last inequality of (2.14) is true because $\varphi = v - H^{\widehat{\mu}} \geq 0$ q.e. in $\mathrm{supp}(\widehat{\mu})$ due to the fact that $H^{\widehat{\mu}} = \psi$ q.e. in $\mathrm{supp}(\widehat{\mu})$ and $v \geq \psi$ q.e. in $\mathbb{R}^2$. The conclusion (2.14) follows since $\widehat{\mu}$ does not charge sets of capacity zero, and this proves that (2.2) is satisfied under the additional assumption that φ is smooth and compactly supported.

To obtain (2.2) for general $v \in \mathcal{K}$, we just need to show that the subset of $\mathcal{K}$ consisting of v for which $v - H^{\widehat{\mu}}$ is smooth is dense in $\mathcal{K}$ with respect to the topology of H^1. To see this, we fix $v \in \mathcal{K}$ and select $R > 1$ so large that $v - H^{\widehat{\mu}}$ is supported in $B_{R/2}$. Then we define

$$v_{\varepsilon,\delta} := H^{\widehat{\mu}} + (v - H^{\widehat{\mu}}) * \eta_{\varepsilon} + \delta\chi_R,$$

where $\varepsilon, \delta > 0$, η_ε is the standard mollifier and χ_R is a smooth function supported in B_{2R} with $0 \le \chi_R \le 1$ and $\chi \equiv 1$ in B_R. It is clear that $v_{\varepsilon,\delta} - H^{\widehat{\mu}}$ is smooth. If $\varepsilon > 0$ is sufficiently small relative to δ, then $v_{\varepsilon,\delta} > \psi$ and hence $v_{\varepsilon,\delta} \in \mathcal{K}$. Finally, we note that if $\varepsilon > 0$ is small enough relative to δ, then $\|v_{\varepsilon,\delta} - v\|_{H^1} < C\delta$. This concludes the proof of (2.2).

The uniqueness of the solution to (2.2) is standard: if v is another solution, then first (2.2) holds, and second, testing (2.2) (this time holding for v) with $H^{\widehat{\mu}}$ yields

$$\int_{\mathbb{R}^2} \nabla v \cdot \nabla (H^{\widehat{\mu}} - v) \ge 0,$$

Adding the two relations yields $\int_{\mathbb{R}^2} \nabla(H^{\widehat{\mu}} - v) \cdot \nabla(v - H^{\widehat{\mu}}) \ge 0$ which immediately implies that $H^{\widehat{\mu}} - v = 0$ in $H^1(\mathbb{R}^2)$. □

2.2. **Proof of Theorem 1.** We now present the proof of the main result. It is divided to several steps, with the first four steps devoted to deriving regularity results for $H^{\widehat{\mu}}$ and the last two steps analyzing properties of the minimizer $\widehat{\mu}$ by means of Hopf's lemma.

Proof of Theorem 1. **Step 1: Local regularity of $H^{\widehat{\mu}}$.** In this step, we show that for some $\alpha > 0$,

$$H^{\widehat{\mu}} \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^2). \tag{2.15}$$

The variational characterization provided by Lemma 2.3 asserts that $H^{\widehat{\mu}}$ is a "solution of the obstacle problem with obstacle ψ" (with ψ defined in (2.3)) and allows us to use the classical regularity theory for the obstacle problem. In particular, we see that, away from ∂U, $H^{\widehat{\mu}}$ is locally $C^{1,1}$ since ψ is smooth there. This optimal regularity for the obstacle problem is an old result of Frehse [7]; for a proof, see also [5, Theorem 2].

In neighborhoods of ∂U, in which ψ has a jump discontinuity, we cannot expect $H^{\widehat{\mu}}$ to be so regular, in general. However, a result of Frehse and Mosco [8, Theorem 3.2] implies that $H^{\widehat{\mu}}$ is in fact Hölder continuous in a neighborhood of ∂U. To verify that this result applies, we have to check that ψ verifies the "unilateral Hölder condition" summarized in lines (3.6) and (3.7) of [8], which in our case comes down to a smoothness condition on ∂U. Indeed, it is easy to check that all we need for the obstacle ψ in (2.3) to satisfy this condition is that ∂U satisfies an exterior cone condition, so our assumption that ∂U is $C^{1,1}$ certainly suffices. We conclude that $H^{\widehat{\mu}} \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\alpha > 0$. In particular, $H^{\widehat{\mu}}$ is continuous.

Step 2: Consequences of continuity and existence of a gap between ∂U and $V \setminus U$. Since $H^{\widehat{\mu}}$ is continuous and $\widehat{\mu}$ does not charge sets of capacity zero, we may upgrade Lemma 2.1 by removing the "q.e." qualifier from (2.6) and (2.7). We have:

$$2H^{\widehat{\mu}}(x) + |x|^2 \ge c_1 \quad \text{in } \overline{U} \quad \text{and} \quad 2H^{\widehat{\mu}}(x) + |x|^2 = c_1 \quad \text{in } \operatorname{supp}(\widehat{\mu}) \cap \overline{U}, \tag{2.16}$$

and

$$2H^{\widehat{\mu}}(x) + |x|^2 \ge c_2 \quad \text{in } \mathbb{R}^2 \setminus \overline{U} \quad \text{and} \quad 2H^{\widehat{\mu}}(x) + |x|^2 = c_2 \quad \text{in } \operatorname{supp}(\widehat{\mu}) \setminus \overline{U}. \tag{2.17}$$

We next claim that

$$\widehat{\mu} \text{ is compactly supported.} \tag{2.18}$$

Indeed, $\mathcal{I}[\widehat{\mu}] < \infty$ implies that $\int |x|^2 d\widehat{\mu}(x) < \infty$ and then

$$\begin{aligned} \frac{|x|^2}{2} - \int \log|x-y|\, d\widehat{\mu}(y) &\ge \frac{|x|^2}{2} - \int \log(|x| + |y|)\, d\widehat{\mu}(y) \\ &\ge \frac{|x|^2}{2} - 2\log(|x|+1) - 2\int \log(|y|+1) d\widehat{\mu}(y) \to_{|x|\to\infty} \infty\,. \end{aligned}$$

Together with the equalities in (2.16) and (2.17), this implies (2.18).

Define V to be the interior of the support of $\widehat{\mu}$, which is bounded by (2.18). According to (2.1), (2.16) (2.17) and the fact that $H^{\widehat{\mu}}$ is $C^{1,1}_{\mathrm{loc}} = W^{2,\infty}_{\mathrm{loc}}$ in both U and $\mathbb{R}^2 \setminus \overline{U}$, we find that

$$\widehat{\mu} = -\frac{1}{2\pi}\Delta H^{\widehat{\mu}} = \frac{1}{\pi}\mathbb{1}_V \quad \text{in } U \cup (\mathbb{R}^2 \setminus \overline{U}). \tag{2.19}$$

Let $\widehat{\mu}_{\mathrm{reg}}$ denote the Lebesgue measure on V multiplied by $1/\pi$ and set $\widehat{\mu}_S := \widehat{\mu} - \widehat{\mu}_{\mathrm{reg}}$. It is clear by (2.19) that $\widehat{\mu}_S$ is supported on ∂U.

Since $H^{\widehat{\mu}}$ is continuous and above the obstacle function $\psi = \frac{1}{2}(c_1 - |x|^2)$ in $\overline{U}$, it must be strictly larger than $\frac{1}{2}(c_2 - |x|^2) = \psi$ in a neighborhood of $\overline{U}$. In other words, there is a gap between ∂U and $V \setminus U$.

Step 3: Further local regularity of $H^{\widehat{\mu}}$. In this step, we show that $\nabla H^{\widehat{\mu}} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$, and that $H^{\widehat{\mu}}$ has Neumann trace on each side of ∂U.

We first argue that $H^{\widehat{\mu}} \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^2)$, which amounts to proving that $H^{\widehat{\mu}}$ is Lipschitz continuous in a neighborhood of ∂U. To show this, we introduce a modification $\hat{\psi}$ of ψ which is locally Lipschitz continuous in $\mathbb{R}^2$ and such that $H^{\widehat{\mu}}$ satisfies (2.2) with $\hat{\psi}$ in place of ψ. The Lipschitz continuity of $H^{\widehat{\mu}}$ then follows from [5, Theorem 2(a)]. It is clear from (2.2) or (2.4) that any $\hat{\psi}$ will do, provided that we modify ψ only in the region in which $\{H^{\widehat{\mu}} > \psi\}$ and in such a way that $H^{\widehat{\mu}} \geq \hat{\psi}$ in $\mathbb{R}^2$. We define

$$\hat{\psi}(y) := \begin{cases} h(y) & \text{if } y \in U^\delta \setminus U, \\ \psi(y) & \text{if } y \in U \cup (\mathbb{R}^2 \setminus U^\delta), \end{cases}$$

where $U^\delta := \{z \in \mathbb{R}^2 : \mathrm{dist}(z,U) < \delta\}$, $\delta > 0$ is small enough that ∂U^δ is smooth and $H^{\widehat{\mu}} > \psi$ in $U^\delta \setminus U$, and h is the unique harmonic function in $U^\delta \setminus U$ which is equal to ψ on $\partial(U^\delta \setminus U)$. In view of the fact that $H^{\widehat{\mu}}$ is superharmonic in $\mathbb{R}^2$, it is clear that $\hat{\psi} \leq H^{\widehat{\mu}}$ in $\mathbb{R}^2$ and so ψ has the desired properties. We deduce that $H^{\widehat{\mu}}$ is locally Lipschitz. In particular, we have $\nabla H^{\widehat{\mu}} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ by Rademacher's theorem (c.f. [6, Chapter 3]).

We next show that $H^{\widehat{\mu}}$ has a Neumann trace on each side of ∂U. We mean that $\partial_\nu F$ exists and belongs to $L^2_{\mathrm{loc}}(\partial U)$, where F is either the restriction of $H^{\widehat{\mu}}$ to $\overline{U}$ or to $\mathbb{R}^2 \setminus U$, and ν denotes the outer unit normal to ∂U. We only prove the claim in the case that $F = H^{\widehat{\mu}}|_{\overline{U}}$, since the other case is even easier (due to the gap, (1.5)). We first give the argument in the case that U is bounded. We write $H^{\widehat{\mu}}|_{\overline{U}} = H^{\widehat{\mu}}_1 + H^{\widehat{\mu}}_2$ in $\overline{U}$, where $H^{\widehat{\mu}}_1$ is harmonic in U with $H^{\widehat{\mu}}_1 = H^{\widehat{\mu}}$ on ∂U, and $H^{\widehat{\mu}}_2 := H^{\widehat{\mu}} - H^{\widehat{\mu}}_1$. Note that $H^{\widehat{\mu}} \in W^{1,\infty}(\partial U)$ since $H^{\widehat{\mu}}$ is Lipschitz. Next, we recall that the Dirichlet-to-Neumann map (also called the Poincaré-Steklov operator) with respect to the Laplacian is a continuous linear operator from $H^1(\partial\Omega)$ to $L^2(\partial\Omega)$, for any bounded $C^{1,1}$ domain Ω (c.f. [16, Theorem 4.21]). It follows that $\partial_\nu H^{\widehat{\mu}}_1 \in L^2(\partial U)$. By standard elliptic regularity (c.f. [9, Theorem 9.15]), we have that $H^{\widehat{\mu}}_2 \in W^{2,p}(U) \cap W^{1,p}_0(U)$ for all $1 < p < \infty$. Since $W^{2,p}(U) \hookrightarrow C^{1,\alpha}(\overline{U})$ for $0 < \alpha < 1 - 1/p$ by Sobolev embedding (c.f. [1, Theorem 5.4, Part II]), we deduce that $\partial_\nu H^{\widehat{\mu}}_2 \in C^\alpha(\partial U)$ for all $0 < \alpha < 1$. The claim that $\partial_\nu \left(H^{\widehat{\mu}}|_{\overline{U}}\right) \in L^2(\partial U)$ follows.

For the general case in which U may be unbounded, we fix $R > 0$ and select a $C^{1,1}$ domain $\widetilde{U}_R$ such that $U \cap B_{2R} \subseteq \widetilde{U}_R$ and apply the above argument to $\zeta_R H^{\widehat{\mu}}$, where ζ_R is a smooth cutoff function satisfying $0 \leq \zeta_R \leq 1$, $\zeta_R \equiv 1$ on B_R, and $\zeta_R \equiv 0$ in $\mathbb{R}^d \setminus B_{2R}$.

Step 4: Regularity of $\widehat{\mu}_S$. We show next that $\widehat{\mu}_S$ is absolutely continuous with respect to the one-dimensional Hausdorff measure restricted to ∂U, with a density belonging to $L^2(\partial U)$. Fix a compactly supported smooth function $\zeta \in C^\infty(\mathbb{R}^d)$. Using that $|\nabla H^{\widehat{\mu}}| \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ and the fact proved in the previous paragraph that $H^{\widehat{\mu}}$ has a Neumann trace in $L^2(\partial U)$ from both sides of ∂U,

we may integrate by parts to find

$$\begin{aligned}\int_{\mathbb{R}^2} \nabla H^{\widehat{\mu}}(x)\cdot\nabla\zeta(x)\,dx &= \int_{\overline{U}} \nabla H^{\widehat{\mu}}(x)\cdot\nabla\zeta(x)\,dx + \int_{\mathbb{R}^2\setminus\overline{U}} \nabla H^{\widehat{\mu}}(x)\cdot\nabla\zeta(x)\,dx\\ &= -\int_{\overline{U}} \zeta(x)\Delta H^{\widehat{\mu}}(x)\,dx + \int_{\partial U} \zeta(x)\,\partial_\nu\left(H^{\widehat{\mu}}|_{\overline{U}}\right)(x)\,d\mathcal{H}^1(x)\\ &\qquad -\int_{\mathbb{R}^2\setminus\overline{U}} \zeta(x)\Delta H^{\widehat{\mu}}(x)\,dx - \int_{\partial U} \zeta(x)\,\partial_\nu\left(H^{\widehat{\mu}}|_{\mathbb{R}^2\setminus U}\right)(x)\,d\mathcal{H}^1(x).\end{aligned}$$

It follows from this and (2.19) that $\widehat{\mu}_S$ is equal to the one-dimensional Hausdorff measure, restricted to ∂U, multiplied by $L^2_{\mathrm{loc}}(\partial U)$ function $g := \frac{1}{2\pi}\left(\partial_\nu\left(H^{\widehat{\mu}}|_{\overline{U}}\right) - \partial_\nu\left(H^{\widehat{\mu}}|_{\mathbb{R}^2\setminus U}\right)\right)$. Therefore, g is nonnegative and cannot have mass greater than one. On the other hand, $\widehat{\mu}$ is of compact support due to (2.18), so in fact $g\in L^2(\partial U)$.

Step 5: Potential and comparisons. Having established a-priori regularity for $H^{\widehat{\mu}}$, $\widehat{\mu}$ and its components, we turn to comparisons of $H^{\widehat{\mu}}$ and H^{μ_0}. These comparisons culminate in the relations (2.23)–(2.25).

Let H^{μ_0} be the potential generated by the measure μ_0 given in (1.1). It is also the solution of the variational inequality (2.2) with ψ replaced by $\frac{1}{2}(c_3-|x|^2)$ for some $c_3\in\mathbb{R}$. The reason is that μ_0 is the minimizer of $\mathcal{I}$ over all probability measures (without constraint) and hence the arguments of Lemmas 2.2 and 2.3 apply.

Observe that

$$\left\{H^{\widehat{\mu}} = \tfrac{1}{2}(c_1-|x|^2)\right\} = \overline{V}\cap\overline{U} \quad\text{and}\quad \left\{H^{\widehat{\mu}} = \tfrac{1}{2}(c_2-|x|^2)\right\} = \overline{(V\setminus U)}. \tag{2.20}$$

We show next that

$$H^{\mu_0} - \frac{1}{2}(c_3-c_2) \le H^{\widehat{\mu}} \le H^{\mu_0} - \frac{1}{2}(c_3-c_1). \tag{2.21}$$

We prove only the second inequality, since the first is obtained via a similar argument. The function $G := H^{\mu_0} - \frac{1}{2}(c_3-c_1)$ is larger than $\frac{1}{2}(c_1-|x|^2)$ hence larger than ψ, and thus larger than $H^{\widehat{\mu}}$ in $\{H^{\widehat{\mu}}=\psi\}$. Moreover, G is superharmonic in $\mathbb{R}^2$ and $H^{\widehat{\mu}}$ is harmonic in the complement of $\{H^{\widehat{\mu}}=\psi\}$, and thus $H^{\widehat{\mu}}-G$ is subharmonic in $\mathbb{R}^2\setminus\{H^{\widehat{\mu}}=\psi\}$ and nonpositive on $\{H^{\widehat{\mu}}=\psi\}$. It follows from their definitions that $|H^{\widehat{\mu}}-G|$ is bounded in $\mathbb{R}^2$. We deduce that $H^{\widehat{\mu}}\le G$ in $\mathbb{R}^2$, as desired.

Consider the difference of the two potentials, $w := H^{\mu_0} - H^{\widehat{\mu}} - \frac{1}{2}(c_3-c_1)$, which satisfies

$$-\Delta w \ge 0 \text{ in } \mathbb{R}^2\setminus\overline{V} \qquad\text{and}\qquad -\Delta w \le 0 \text{ in } \mathbb{R}^2\setminus\overline{D} \tag{2.22}$$

and $0\le w\le \frac{1}{2}(c_1-c_2)$.

We claim that:

$$\{w=0\} = \overline{D}\cap\overline{U}, \tag{2.23}$$

$$\text{either}\quad \{w=\tfrac{1}{2}(c_1-c_2)\} = \overline{D}\cap\overline{(V\setminus U)} \tag{2.24}$$

$$\text{or else}\quad \{w=\tfrac{1}{2}(c_1-c_2)\} = (\overline{D}\cap\overline{(V\setminus U)})\cup(\mathbb{R}^2\setminus D)$$

and

$$U\not\subseteq D \qquad\text{implies that}\qquad \{w=\tfrac{1}{2}(c_1-c_2)\} = \overline{D}\cap\overline{(V\setminus U)}. \tag{2.25}$$

In order to prove these, we first observe that, by the monotonicity (2.21) of the obstacle problem,

$$D\cap U\subseteq V\cap U \text{ and } V\setminus U\subseteq D\setminus U. \tag{2.26}$$

Due to (2.20), (2.26), the fact that $H^{\mu_0} \geq \frac{1}{2}(c_3 - |x|^2)$ and $c_2 < c_1$, we have

$$\overline{V} \cap \{w = 0\} \subseteq \overline{V} \cap \overline{D} \cap \overline{U} = \overline{D} \cap \overline{U} \subseteq \{w = 0\} \tag{2.27}$$

and

$$\overline{D} \cap \left\{w = \tfrac{1}{2}(c_1 - c_2)\right\} \subseteq \overline{D} \cap \overline{(V \setminus U)} = \overline{(V \setminus U)} \subseteq \left\{w = \tfrac{1}{2}(c_1 - c_2)\right\}. \tag{2.28}$$

Here comes the argument for (2.23). In view of (2.27), it suffices to show that $\{w = 0\} \subseteq \overline{V}$. Suppose on the contrary that there exists $x \notin \overline{V}$ with $w(x) = 0$. Then, by (2.22) and the strong maximum principle, we have $w \equiv 0$ in $\mathbb{R}^2 \setminus V$. By (2.28), continuity of $H^{\widehat{\mu}}$ and $c_2 \neq c_1$, we deduce that $V \subseteq U$. In particular, $H^{\widehat{\mu}}$ is harmonic in $\mathbb{R}^2 \setminus \overline{U}$ and $w \equiv 0$ in $\mathbb{R}^2 \setminus \overline{U}$, which implies that H^{μ_0} is harmonic in $\mathbb{R}^2 \setminus \overline{U}$. This contradicts the assumption that $D \setminus U$ is nonempty, verifying (2.23).

We continue with the proofs of (2.24) and (2.25). In light of (2.28), if $\{w = \frac{1}{2}(c_1 - c_2)\} \neq \overline{D} \cap \overline{(V \setminus U)}$, then there exists $x \notin D$ such that $w(x) = \frac{1}{2}(c_1 - c_2)$. This implies that $w \equiv \frac{1}{2}(c_1 - c_2)$ in $\mathbb{R}^2 \setminus D$ by the maximum principle and (2.22). This confirms (2.24). Moreover, in this case $H^{\widehat{\mu}}$ is harmonic outside of $\overline{D}$ (since H^{μ_0} is), which implies that $V \subseteq D$. Also, in view of (2.28) and the continuity of w, we deduce that $U \subseteq D$, which completes the proof of (2.25).

Step 6: Completion of the proof of Theorem 1 - Hopf's lemma and free boundary regularity. We have prepared all estimates needed for the conclusion of the proof. The main technical point is that we want to use Hopf's lemma in order to identify the regions where w vanishes. For that, regularity of the free boundary is important, and is provided by Caffarelli's theory.

Turning to the proof, we note first that (1.5) follows from the combination of (2.23), (2.25) and (2.26).

We still need to prove (1.4), (1.6) and (1.7). Each of these follows from a variation of an argument based on Hopf's lemma.

We first prove (1.4), arguing by contradiction. Because of (2.26), if (1.4) fails then there exists $x \in \overline{V} \cap \partial U \cap \overline{D} \setminus \operatorname{supp} \widehat{\mu}_S$. Since $x \notin \widehat{\mu}_S$, ∇w is continuous at x. In fact, we have $\nabla w(x) = 0$ since w vanishes on $\overline{U} \cap \overline{D}$ by (2.27). Due to (2.22) and Hopf's lemma, we deduce that w vanishes identically in the connected component of $\mathbb{R}^2 \setminus V$ containing x. Since $H^{\widehat{\mu}}$ is harmonic in $\mathbb{R}^2 \setminus \overline{V}$ and H^{μ_0} is harmonic nowhere in D, we conclude that the intersection of D and the connected component of $\mathbb{R}^2 \setminus V$ containing x is empty. This contradicts $x \in \overline{D}$, completing the argument for (1.4).

Next we prove (1.6). We need only verify the second claim, since the first claim was obtained above already in (2.26). In the case $U \not\subseteq D$, in view of (2.26), it suffices to prove that $\overline{(V \setminus U)} \cap \partial D$ is empty. Suppose on the contrary that there exists $x \in \overline{(V \setminus U)} \cap \partial D$. We have already demonstrated a gap between $\overline{(V \setminus U)}$ and ∂U and thus $x \notin \partial U$. In particular, $w \in C^{1,1}$ in a neighborhood of x. By (2.24), $w \equiv \frac{1}{2}(c_1 - c_2)$ in $\overline{D} \cap \overline{(V \setminus U)}$ and thus $\nabla w(x) = 0$. In view of Hopf's lemma and the fact that w is subharmonic in $\mathbb{R}^2 \setminus D$ by (2.22), the second alternative must hold in (2.24). This contradicts (2.25) to finish the proof of (1.6).

We conclude with the argument for (1.7), which is a bit more involved due to the fact that we need the best regularity for the boundary of the contact set in Caffarelli's theory, which is "$C^{1,\alpha}$ except for singular points," see [5, Theorem 6]. We proceed again by contradiction and suppose there exists $x \in \overline{U \cap D} \setminus (V \cup \operatorname{supp} \widehat{\mu}_S)$. Since $x \in \overline{D} \setminus \operatorname{supp} \widehat{\mu}_S$, $x \notin \partial U$ by (1.4) and so w is $C^{1,1}$ in a neighborhood of x. Thus $w(x) = 0$ and $\nabla w(x) = 0$. Notice that $x \in \partial V$ and thus it is a point on the boundary of the contact set. In view of (2.23) we have $w > 0$ in $\mathbb{R}^2 \setminus V$. Since $-\Delta w \geq 0$ in $\mathbb{R}^2 \setminus V$ by (2.22), an application of Hopf's lemma gives us a contradiction, completing the proof. However, before we may apply Hopf's lemma, we must check that ∂V is sufficiently smooth in a

neighborhood of x. Indeed, Hopf's lemma holds for $C^{1,\alpha}$ boundaries (for any $\alpha > 0$) but not for domains with boundaries which are merely C^1, see Safonov [19].

Fortunately, the free boundary regularity theory of Caffarelli gives us just what we need: ∂V is $C^{1,\alpha}$ near x, for every $0 < \alpha < 1$. The reason is that x cannot be a singular point (as defined in [5]) of the free boundary, due to the fact that $\partial(U \cap D)$ is Lipschitz and $U \cap D \subseteq V$, which rules out the possibility that ∂V is contained in a "thin strip" near x. See [5, Theorem 6], which asserts that the boundary of V is $C^{1,\alpha}$ near x unless $V \cap B_r(x)$ is contained between two parallel planes separated by a distance of $o(r)$ as $r \to 0$. It follows that ∂V is regular enough in a neighborhood of x to invoke Hopf's lemma. This completes the proof. □

3. An example: real part constraint

We conclude with an explicit example, in which the constraint set U is a half-space and $p = 1$. We show that there is a critical point at which the minimizing measure becomes entirely concentrated (and equal to the semicircle law) on the boundary line.

For each $a \in \mathbb{R}$, we set $U_a := \{z \in \mathbb{C} : \mathrm{Re}(z) < -a\}$, and set $\mathcal{C}_a := \{\mu \in \mathcal{P}(\mathbb{C}) : \mu(\overline{U}_a) = 1\}$. We denote by $L_a = \partial U_a$ the boundary line, by $\widehat{\mu}_a$ the minimizing measure of $I(\cdot)$ on $\mathcal{C}_a$, and by $\widehat{\mu}_{S,a}$ its singular component as in Theorem 1. Note that $\widehat{\mu}_{S,a}(L_a)$ is the total mass of $\widehat{\mu}_{S,a}$. Also denote the semicircle law on $\mathbb{R}$ by

$$\sigma(dy) = \frac{\sqrt{2-y^2}}{\pi} \mathbf{1}_{\{|y|<\sqrt{2}\}}\, dy.$$

The following result asserts that $\widehat{\mu}_a = \widehat{\mu}_{S,a}$ if and only if $a \geq \sqrt{2}$, in which case $\widehat{\mu}_{S,a}$ is the semicircle law on L_a.

Proposition 3.1. *For every $a < \sqrt{2}$, we have $\widehat{\mu}_{S,a}(L_a) < 1$. Conversely, if $a \geq \sqrt{2}$, then $\widehat{\mu}_{S,a}(L_a) = 1$ and, with $z = x + iy$,*

$$\widehat{\mu}_{S,a}(dz) = \delta_a(dx) \times \sigma(dy) =: \sigma_a(dz)\,.$$

Before giving the proof of Proposition 3.1, we recall some preliminaries on the semicircle law. Denote

$$H^\sigma(z) := -\int_{\mathbb{R}} \log|z - iy|\, \sigma(dy).$$

By a direct computation, we have

$$2H^\sigma(x) + x^2 - (1 + (\log 2)/2) \quad \begin{cases} = 0, & |x| \leq \sqrt{2} \\ \geq 0, & |x| \leq \sqrt{2} \end{cases}, \ x \in \mathbb{R}, \quad ([2, \text{Ex. } 2.6.4]). \tag{3.1}$$

$$\mathcal{S}_\sigma(z) := \int \frac{1}{x - z}\sigma(dx) \ = \ -(z - \sqrt{z^2 - 2}), \ z \in \mathbb{C}_+\,, \quad ([2, \text{pg. } 46, (2.4.7)]). \tag{3.2}$$

(We recognize $\mathcal{S}_\sigma(z)$ as the Stieltjes transform of σ.)

Proof of Proposition 3.1. Note first that if $\widehat{\mu}_{S,a}(\overline{U}_a) = 1$ then $\widehat{\mu}_{S,a}(L_a) = 1$ by Theorem 1 and therefore, using that for $z = a + iy$ one has $|z|^2 = a^2 + y^2$, it follows from (3.1) that necessarily in such a situation $\widehat{\mu}_{S,a}(a + idy) = \sigma(dy)$ and

$$2H^{\widehat{\mu}_{S,a}}(a + iy) + |(a + iy)|^2 = a^2 + 1 + \frac{\log 2}{2}. \tag{3.3}$$

Applying Lemma 2.2, we conclude that a necessary and sufficient condition for $\widehat{\mu}_{S,a}(L_a) = 1$ is that

$$2H^{\sigma_a}(b + iy) + |(b + iy)|^2 \geq a^2 + 1 + \frac{\log 2}{2} \quad \text{for all } b > a. \tag{3.4}$$

Suppose $a < \sqrt{2}$. Differentiate the left side of (3.4) with respect to b to obtain, with $\theta \in \mathbb{R}$,

$$\begin{aligned}\frac{\partial}{\partial b}\left(2H^{\sigma_a}(b+iy)+|(b+iy)|^2 i\right) &= 2b - 2\Im\mathcal{S}_\sigma(y+i(b-a)) \\ &= 2b + 2\Im(z-\sqrt{z^2-2}) =: F(a,b,y)\,,\end{aligned}$$

where $z = y + i(b-a)$ and (3.2) was used in the equalities. Noting that

$$\lim_{b \searrow a} F(a,b,0) = 2a - 2\sqrt{2}\,,$$

we conclude that if $a < \sqrt{2}$ the necessary condition for $\widehat{\mu}_{S,a}(L_a) = 1$ is not satisfied. This proves the first statement of the proposition.

For the second statement, we introduce the function

$$G(y,a,b) = y^2 + b^2 + 2H^\sigma(y+i(b-a)) - a^2 - 1 - \frac{\log 2}{2}.$$

Since $G(y,a,a) \geq 0$ by (3.1), it is enough due to Lemma 2.2 to verify that $G(y,a,b) \geq 0$ when $b > a$. By symmetry and convexity, for fixed $a < b$, one has $G(y,a,b) \geq G(0,a,b) =: \overline{G}(a,b)$. Note that

$$\frac{\partial \overline{G}(a,b)}{\partial b} \begin{cases} = 2\big(2b - a - \sqrt{2-(b-a)^2}\big), & 0 \leq b-a < \sqrt{2}\,, \\ > 0, & b-a \geq \sqrt{2}\,, \end{cases}$$

where again we used (3.2). It is straightforward to check that the right side is nonnegative provided that $b \geq a$ and $a \geq \sqrt{2}$. Since $\overline{G}(a,a) \geq 0$, this implies that $\overline{G}(a,b) \geq 0$ for $b > a$. This completes the proof. □

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