# LARGE DEVIATIONS ASYMPTOTICS

# IN NONLINEAR FILTERING

With applications to tracking.

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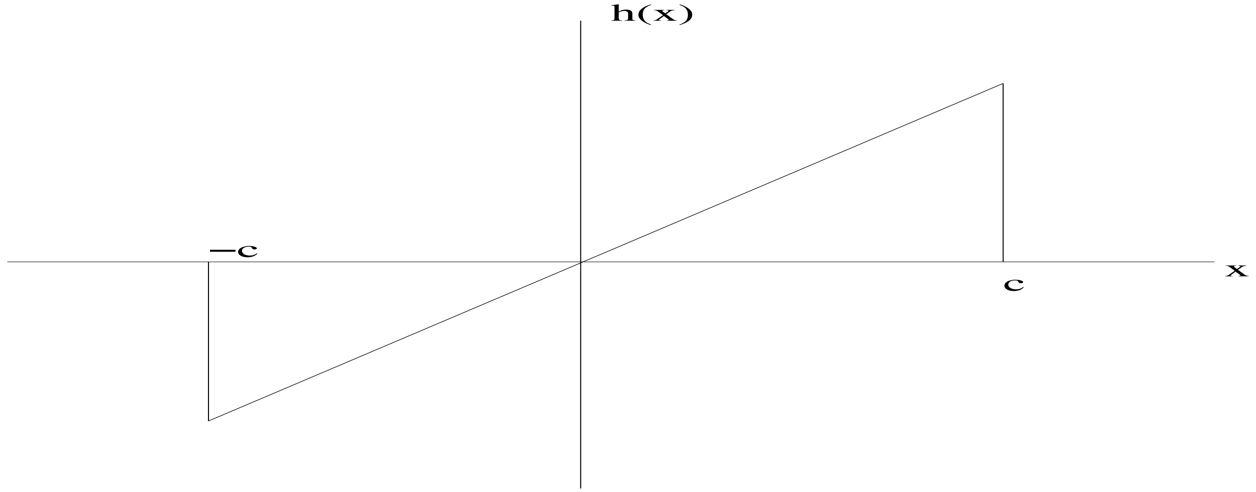
$$
\begin{aligned}
dx_t &= b(x_t)dt + dw_t \\
dy_t &= h(x_t - u_t)dt + \varepsilon d\nu_t \,.
\end{aligned}
$$




Control $u$ adapted to

$$
\mathcal{F}_t^y = \sigma(y_s, 0 \le s \le t) \vee \sigma(u_s, 0 \le s \le t)\,,
$$

such that a solution to SDE's exist.

Objective:

With $T$ given, find control $u$ to maximize

$$P(||x-u||_T \leq c)$$

where

$$||x-u||_T = \sup_{t \leq T} |x_t - u_t|$$

(alternative: maximize expected time until loss of lock).

Theorem: (Zakai & Zeitouni, 1992):

For any admissible control,

$$\liminf_{\epsilon \to 0} \epsilon \log P(||x - u||_T > c) \geq -\frac{c^2 h}{2}.$$

"Linear" controller $d\hat{u}_t = b(\hat{u}_t)dt + \epsilon^{-1} dy_t$ gets

$$\lim_{\epsilon \to 0} \epsilon \log P(||x - \hat{u}||_T > c) = -\frac{c^2 h}{2}.$$

Thus, linear controller is asymptotically optimal.

Strategy of proof: linear controller analysis is Freidlin-Wentzell LD.

Lower bound, on the other hand, uses asymptotics for filtering problem + allowing controller to depend on extra information.

Main estimate works for $h$ linear, standard filtering setup: $u = 0$, Zakai's equation

$$d\rho_t = \mathcal{L}^* \rho_t dt + \frac{1}{\varepsilon} h dy_t \,.$$

Conditional law of $X_t$ given $\mathcal{F}_t^y$ has density

$$p_t = \frac{\rho_t}{\int_{\mathbb{R}} \rho_t(x) dx} \,.$$

Theorem: (Zeitouni, 1988; $h(x) = hx$): $\rho_t$ is *pointwise* dominated between two (unormalized) Gaussians. Further, for any open set $C$, conditionally on $\sigma(X_s, 0 \leq s \leq t)$,

$$\lim_{\epsilon \to 0} \epsilon \log \int_C p_t(x - X_t) dx = - \inf_{z \in C} \frac{hz^2}{2}, a.s.$$

Proof works by comparison with (explicit, Gaussian) linear $b$ solution and hence fails completely for $h$ nonlinear!

Recent news: can get LDP for conditional law (quenched) for $h$ nonlinear.

Assume:

$$b, h, b', h'h'b, h'h, h'', hb \text{ are Lipschitz functions}.$$

$$\lim_{|x|\to\infty} h''(x) = 0\,.$$

$$h'(\cdot) \geq h_0 > 0$$

Define

$$\mathcal{J}(x) = \int_{x_t}^{x} (h(y) - h(x_t))dy\,.$$

Theorem: (Pardoux and Z., 2003) (quenched large deviations):

For any open set $C$, conditionally on $\sigma(X_s, 0 \leq s \leq t)$,

$$\lim_{\epsilon \to 0} \epsilon \log \int_C p_t(x - X_t) dx = - \inf_{z \in C} \mathcal{J}(z) \,.$$

Key to proof is a representation of $\rho_t$, due to Picard: set

$$dM_t = b(M_t)dt + \frac{1}{\epsilon}(dY_t - h(M_t)dt)\,.$$

With $m_s = M_{1-\epsilon s}$, and

$$dZ_s = [-h(Z_s) + m_s h'(Z_s) - \epsilon b(Z_s)]ds + \sqrt{\epsilon}d\tilde{W}_t$$

it holds that

$$\rho_t(x) \sim e^{-F(x,m_0)/\epsilon} E_{\tilde{W}}(\exp(\frac{1}{\varepsilon}\int_0^{1/\varepsilon} g(Z_s, m_s)ds)\,.$$

Problem: Direct large deviations are at scale $\epsilon^2$ (due to small noise+ large time interval).

This precludes direct use of Friedlin-Wentzell theory.

However, it turns out that optimizing path spends most of its time around $X_t$.

Some analysis then reduces study to finite (and large) time interval.

Assumptions on $h$ at infinity needed in this reduction.

## Open questions

- vector case - multiscales!

- $h$ not stabilizing at infinity.

- application to the tracking problem.