RANDOM WALKS

IN RANDOM ENVIRONMENTS

Results and challenges

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A classical model:

One dimensional random walk

$$P(X_n = i+1 | X_{n-1} = i) = p$$
$$P(X_n = i-1 | X_{n-1} = i) = 1-p$$

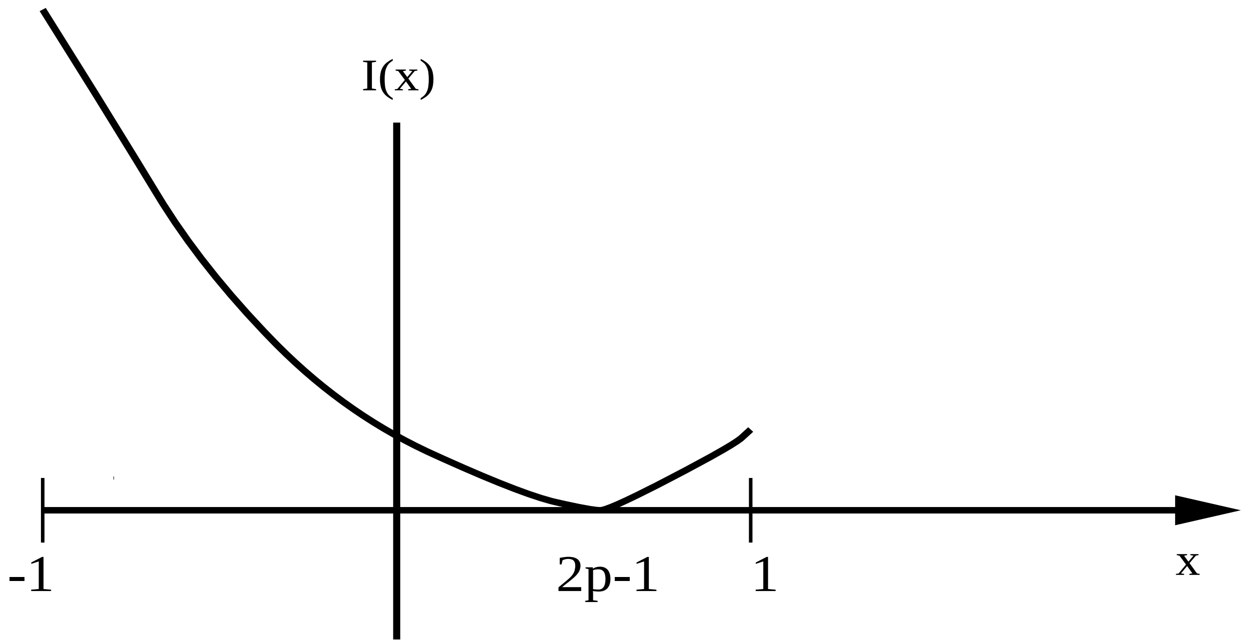
$$\begin{aligned} E(X_n - X_{n-1}) &= E(X_n - X_{n-1} | X_{n-1}, X_{n-2}, \cdots) \\ &= 2p - 1 := v_{RW(p)} \stackrel{\triangle}{=} \text{drift,speed} \end{aligned}$$

L.L.N.:

$$\frac{X_n}{n} \xrightarrow{\text{a.s.}} 2p-1$$

C.L.T.:

$$\frac{X_n-(2p-1)n}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,4p\,(1-p))$$



Large Deviations

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log P\left(\frac{X_n}{n} \in (x-\varepsilon, x+\varepsilon)\right) = -I(x)$$

One dimensional random walk
in random environment (RWRE)

Model $\{\omega_i\}_{i\in\mathbb{Z}}$ – i.i.d., law P, supp $P \subset\subset (0,1)$

X_n – Random walk

$$P_\omega(X_n = i+1|X_{n-1} = i) \;=\; \omega_i$$
$$P_\omega(X_n = i-1|X_{n-1} = i) \;=\; 1-\omega_i$$

Solomon ('75): $\rho_x := \frac{1-\omega_x}{\omega_x}$

$$X_n \text{ recurrent} \iff \langle \log \rho \rangle = 0$$

$$X_n \xrightarrow[\text{a.s.}]{} \infty \iff \langle \log \rho \rangle < 0$$

Further,

$$\langle \rho \rangle < 1 \Rightarrow \frac{X_n}{n} \xrightarrow[\text{a.s.}]{} v_P := \frac{1 - \langle \rho \rangle}{1 + \langle \rho \rangle}$$

Proof by looking at $\tau_1 = \inf\{n : X_n = 1\}$ and computing $E_P(\tau_1) = 1/v_P$.

Note: 1) $0 \le v_P < E_P(v_{RW}(\omega_x))$!

2) $\langle 2\omega_0 - 1 \rangle < 0, v_P > 0$ impossible.

CLT, etc: Kozlov, Kesten, Spitzer, Sinai, ...

$$\langle \rho^2 \rangle < 1 \;\Rightarrow\; \text{CLT}$$

$$\langle \rho \rangle > 1 \;\Rightarrow\; \exists s \in (0,1) \text{ s.t. } \langle \rho^s \rangle = 1,$$

$$\frac{X_n}{n^s} \xrightarrow{\mathcal{L}} \text{nondegenerate random variable}$$

$$\langle \log \rho \rangle = 0$$

$$\Rightarrow \frac{X_n}{(\log n)^2} \xrightarrow{\mathcal{L}} \text{nondegenerate random variable}$$

All these are due to existence of "traps"!

$$\longrightarrow \ | \ \longleftarrow$$

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There is a difference between quenched and annealed behaviors:

$$P_\omega(\text{Time to pass typical trap} > n) \sim c_0^{n/n^{c_1}},$$

while

$$P(\text{Time to pass trap} > n) \sim c_3^{c_4 \log n}.$$

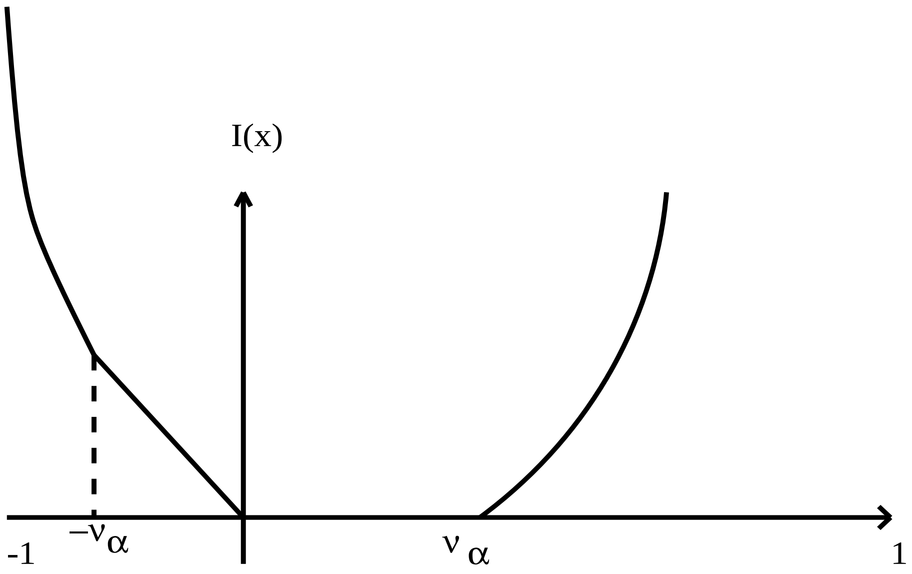
A rich subexponential regime! $x \in (0, v_P)$

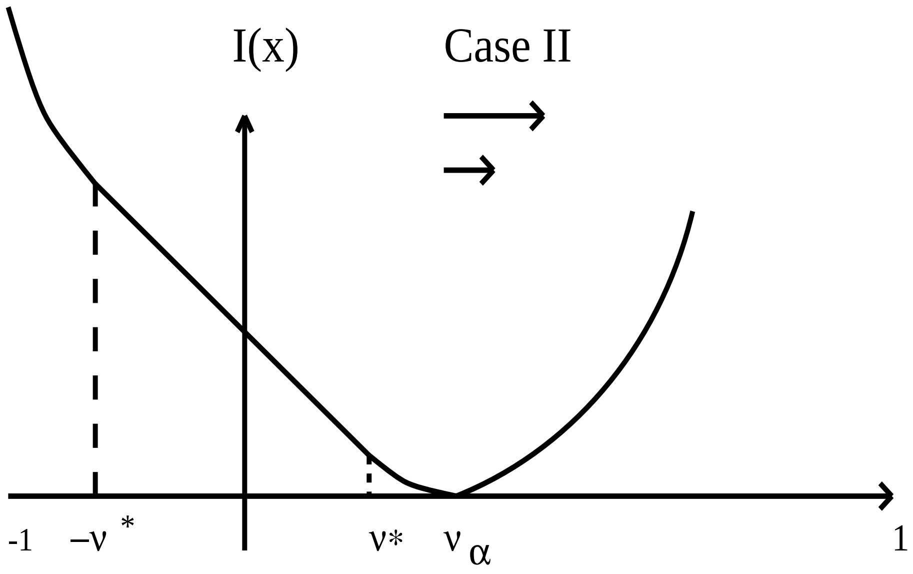
	Annealed	**Quenched**
		$\forall \delta > 0,$
Case I	$P\left(\frac{X_n}{n} \sim x\right) \sim \frac{1}{n^{s-1}}$	$\frac{1}{n^{1-\frac{1}{s}\pm\delta}} \log P_\omega \left(\frac{X_n}{n} \sim x\right)$
$\longrightarrow$ $\leftarrow$	**Dembo, Peres, Z., '96**	$\longrightarrow \begin{cases} 0 \\ -\infty \end{cases}$
$\langle \rho^s \rangle = 1$		**Gantert, Z. '97**
$s > 1$		**(surprising fluctuations?)**
Case 0	$P\left(\frac{X_n}{n} \sim x\right)$	$P_\omega \left(\frac{X_n}{n} \sim x\right)$
	$\sim e^{-C_\pm^x n^{1/3}}$	$\sim e^{-c_\pm^x \frac{n}{(\log n)^2}}$
$\longrightarrow$.	**Dembo, Peres, Z. '96**	**Gantert, Z. '97**
	$C^x = C_+^x = C_-^x$	$c^x = c_+^x = c_-^x$
	explicit expression	**explicit expression**
	Povel, Pizstora, Z. '97	**Povel, Pizstora '97**

What about exponential scale large deviations?

Greven-den Hollander ('94): Quenched setting

$$\lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{n}\log P_\omega\left(\frac{X_n}{n}\in(x-\delta,x+\delta)\right)=-I(x)\,,P\text{–a.s.}$$





Understand LDP? (Comets, Gantert, Z., '00)

Use recursions:

$$\tau_1 = \mathbf{1}_{\{X_1=1\}} + \mathbf{1}_{\{X_1=-1\}}(1 + \tau^0_{-1} + \tau'_1)$$

$$\Rightarrow \quad \varphi(\lambda,\omega) := E_\omega(e^{\lambda\tau_1}) = \cfrac{\omega_0}{e^{-\lambda} + \cfrac{(1-\omega_0)\omega_{-1}}{e^{-\lambda} + \cfrac{(1+\omega_{-1})\omega_{-2}}{e^{-\lambda}+\cdots}}}$$

$$\Rightarrow \quad \frac{1}{n} \log E_\omega(e^{\lambda(\tau_1+\tau_2^1+\cdots+\tau_n^{n-1})})$$

$$\longrightarrow \quad E \log \varphi(\lambda,\omega) := \Lambda(\lambda), \text{ P-a.s.}$$

$\Rightarrow$ By large deviations arguments, relate exponential decay rates for X_n/n to Legendre transform of Λ.

Focus on quenched vs. annealed relation:

Annealed rate function (Comets-Gantert-Z. '00):

$$I_a(x) = \inf_{\eta \in M_1^{\text{erg}}} \left[I_q^\eta(x) + x h(\eta | P) \right]$$

where $h(\cdot|P)$ is specific relative entropy.

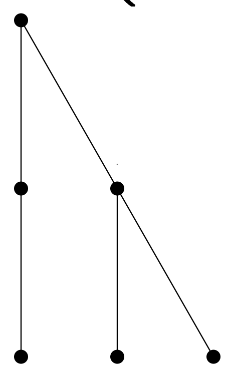
(Modify env., then quenched LDP in new env.)

Results with **Quenched$\neq$ Annealed**

Proof is based on the fact that walk **must** traverse changed environment. This is a **one dimensional** fact.

In high dimension (or on trees!) cost of creating environment may exceed cost of wandering around to find it.

Extreme case: trees (Galton-Watson)



Walker goes to nearest neighbors, uniformly. If X_n denotes where the walker is at time n, then (Lyons, Pemantle, Peres '95) $\frac{|X_n|}{n} \to v_P$, $a.s.$

Claim: (Dembo-Gantert-Peres-Z. 01')
For G-W trees, quenched and annealed large deviations behavior coincide:

$$-I_q(v) = \lim_{n\to\infty} \frac{1}{n} \log P_\omega(\frac{|X_n|}{n} \sim v)$$

$$= \lim_{n\to\infty} \frac{1}{n} \log P(\frac{|X_n|}{n} \sim v) = -I_a(v)\,,\ P - a.s.$$

Proof based on the fact that rate of growth of tree is so large that all relevant traps occur naturally not too far from root of tree.

Open issue: What happens in $\mathbb{Z}^d$??

Model RWRE in $\mathbb{Z}^d$

Environment: S is $2d$-simplex , $\Omega = S^{\mathbb{Z}^d}$,

$\omega = \{\omega(z,e)\}_{z\in\mathbb{Z}^d}$, $|e|_1 = 1$.

Walk is conditional on ω (quenched), Markov:

$$P_\omega(X_{n+1} = x + e | X_n = x) = \omega(x,e)$$

Questions: LLN? CLT? LDP?

Simplest model: P, law on Ω, i.i.d.

Walk is **ballistic** if $\frac{X_n}{n} \xrightarrow[n\to\infty]{} v \neq 0$, $\mathbb{P}$-a.s.

($\mathbb{P} = P \times P_\omega$ "annealed")

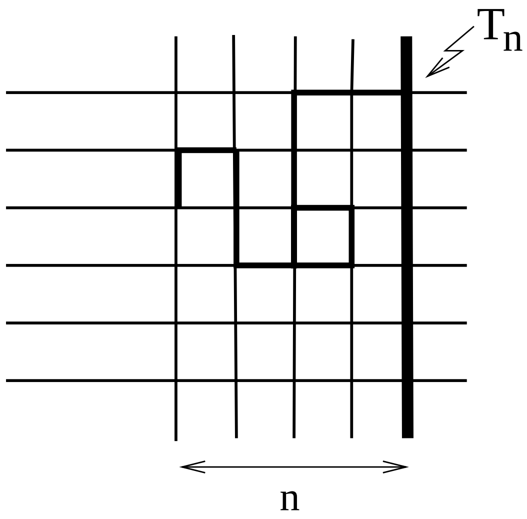
LDP in $\mathbb{Z}^d$: hitting times do not contain all information. Still, under appropriate condition, there is a quenched LDP (Zerner '98):

$$\frac{1}{n} \log E_\omega(e^{\lambda T_n}) \xrightarrow{\text{a.s.}} \Lambda(\lambda)$$

But: annealed LDP? Relation between quenched and annealed?

Is it possible for $d \gg 1$ that $Q = A$ for some range of velocities where rate is nonzero?

Recent annealed results: Varadhan.



Embarrassing open problem: 0-1 law:

For each direction ℓ, set

$$A_\ell^{\pm} := \{X_n \cdot \ell \to \pm\infty\}$$

Prove that $\mathbb{P}(A_\ell^{+}) \in \{0, 1\}$.

Known (P i.i.d.):

$\mathbb{P}(A_\ell^{+} \cup A_\ell^{-}) \in \{0, 1\}$ (Kalikow '81).

True for $d = 2$ (Zerner-Merkl '02).

Not true for certain ergodic environments.

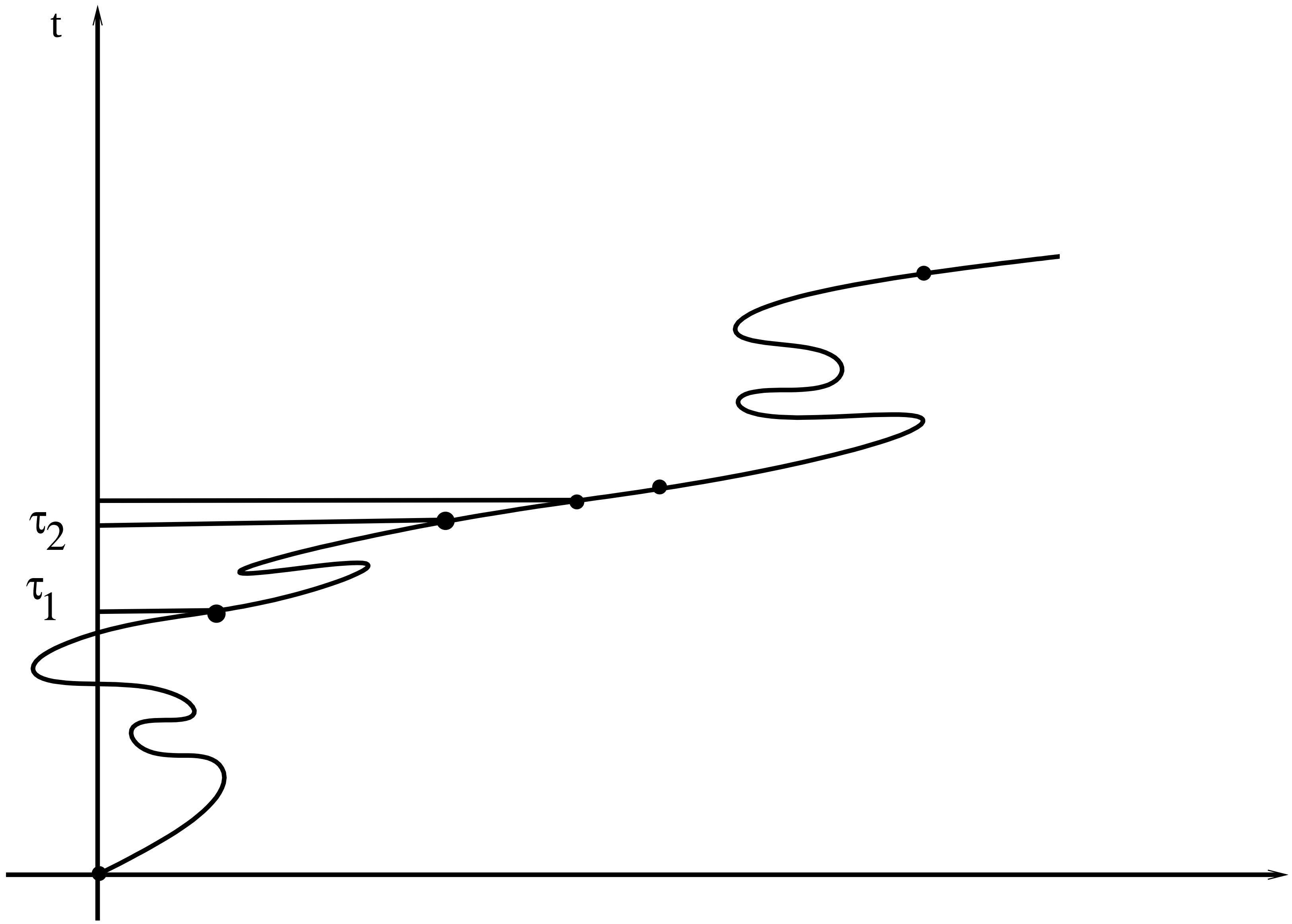
LLN, CLT in $\mathbb{Z}^d$: Cannot compute $E_P(\tau_1)$.
(even for $\bar{\tau}_1 = \inf\{n : X_n \cdot e_1 = 1\}$)

Alternative: introduce regeneration times.

For RWRE on trees: Kesten-Grimmett ($\sim$ '84), Lyons-Peres-Pemantle ('95), Piau ('98), Dembo-Gantert-Peres-Z. ('01).

For $\mathbb{Z}^d$: fully exploited by Sznitman-Zerner ('99)

Plot $X_n \cdot e_1$:



Fact: If P is i.i.d.,

$$\left(\eta_i := (\tau_{i+1} - \tau_i), \left\{X_{\tau_i+n} - X_{\tau_i}\right\}_{n=1,\cdots,\eta_i},\right.$$

$$\left.\{\omega(z,\cdot)\}_{X_{\tau_i}\cdot e_1 \le z\cdot e_1 < X_{\tau_{i+1}}\cdot e_1}\right)$$

are i.i.d. under $\mathbb{P} = P \times P_\omega$ (annealed)

Theorem (Sznitman-Zerner '99)

a) If $E_P(\tau_2 - \tau_1) < \infty$ then walk is ballistic.

b) $E_P(\tau_2 - \tau_1) < \infty$ if Kalikow's condition holds.

Kalikow's condition: $U \subset Z^d, 0 \in U$

$$\widehat{P}_U(x, x+e) = \frac{\mathbb{E}\left(\sum_{n=1}^{\tau_{U^c}} \mathbf{1}_{\{X_n = x\}} \omega(x, e)\right)}{\mathbb{E}\left(\sum_{n=1}^{\tau_{U^c}} \mathbf{1}_{\{X_n = x\}}\right)}$$

τ_{U^c} = exit time from U; $\widehat{P}_U$ and P possess same exit law from U.

Kalikow's drift: $\hat{d}_U(x) = \sum_{|e|_1 = 1} e \widehat{P}_U(x, x+e)$

Kalikow's condition: $\inf_{U, x \in U} \hat{d}_U(x) \cdot e_1 > 0$

(There are easy to check sufficient conditions).

Improvements Sznitman ('00,'01,'02), P- i.i.d.:

- CLT + tail estimates under Kalikow's condition.
- Milder conditions (T,T'), leading also to CLT + tail estimates.
- Criteria for T, T' conditions.

Based on analysis (via renormalization arguments) of tails of $\tau_2 - \tau_1$.

No known examples of sub-diffusive behavior!

No known **criteria** for transience-recurrence.

Conjectures:

1) (T') characterizes ballistic behavior

2) No sub-diffusive behavior for $d \geq 2$.

Modifications of regeneration times

For P non i.i.d., uniformly elliptic:

$$\{\tau_{i+1} - \tau_i\} \text{ not even stationary!}$$

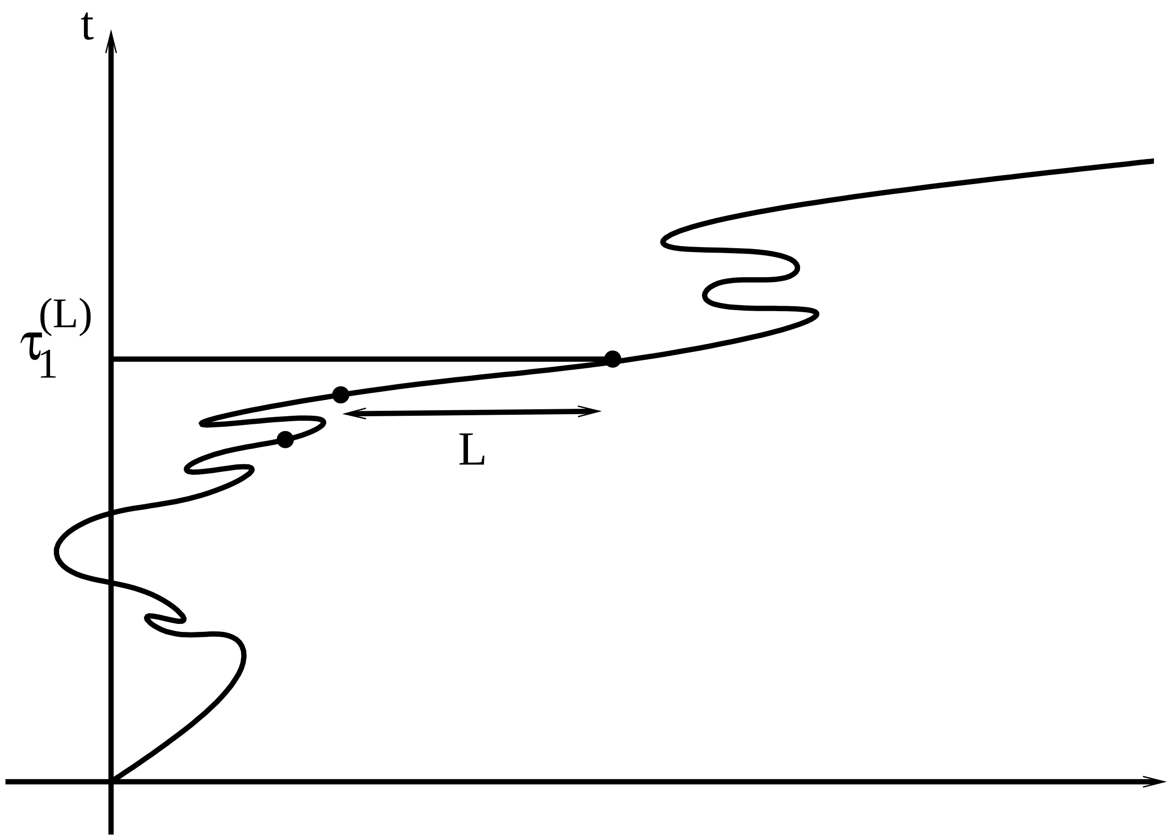
Couple walk with independent coins: if $\omega(x, x+e) \geq \varepsilon$, generate RWRE by ε_n i.i.d., $Q_\varepsilon(\varepsilon_1 = e) = \varepsilon$, $Q_\varepsilon(\varepsilon_1 = 0) = 1 - 2d\varepsilon$,

$$\tilde{P}_{\omega,\varepsilon}\Big(X_{n+1} = z + e | X_n = z\Big) = \mathbf{1}_{\varepsilon_{n+1}=e} + \frac{\mathbf{1}_{\varepsilon_{n+1}=0}}{1 - 2d\varepsilon}\left(\omega(z,e) - \varepsilon\right)$$

Call a point an *L-reg point* if reg point

+ used ε coins L times just before, in dir. e_1

+ fresh point before those.



Now, $\{\tau_{i+1}^{(L)} - \tau_i^{(L)}\}$ still not iid, but have better and better (as function of L) mixing properties.

Theorem (Comets-Z. '02) "good" mixing properties of environment+ tail estimates on $\tau_{i+1}^{(L)} - \tau_i^{(L)}$ (for $L \to \infty$) $\Rightarrow$ ballistic LLN.

Proof based on coupling ideas.

Different approach via environment viewed from particle and regeneration to control law: F. Rassoul-Agha ('02)

What about non-ballistic?

Very little known!

- Balanced ($\omega(0,e) = \omega(0,-e)$):

$\Longrightarrow$ X_n is martingale;

transient $d \geq 3$, recurrent $d = 2$

$+$ quenched CLT (Lawler '85)

Proof by homogenization techniques: $\theta^{X_n}\omega$ is a Markov chain, invariant measure absolutely continuous w.r.t. P.

• Small disorder:

$$\omega(0,e) = \frac{1}{2d} + \varepsilon k(0,e)$$

P i.i.d. invariant under $\mathbb{Z}^d$-rotations.

$\Longrightarrow$ Quenched CLT (Bricmont-Kupianien '91)

Proof uses (hard) renormalization techniques

- Special case: P i.i.d., $d = d_1 + d_2$,

$$\omega(z, \pm e_i) = \overline{\omega}_{\pm}(i), i = 1, \cdots, d_1$$

(i.e. non random in first d_1 coordinates).

Example: $$\overline{\omega}_{\pm}(i) = \frac{1}{2d}, \quad i = 1, \cdots, d_1$$

Theorem (Bolthausen, Sznitman, Z. '02)

For $d_1 \geq 5$: LLN

$$\left.\begin{array}{ll} & d_1 \geq 7 \ (P \text{ Symmetric }) \\ \text{or} & \\ & d_1 \geq 13 \end{array}\right\} \text{CLT}$$

(CLT both quenched and annealed)

Consequence:

There is a P, i.i.d., with

$$E_P(\omega(0,e) \cdot e_d) < 0$$

but

$$\frac{X_n}{n} \cdot e_d \longrightarrow v > 0, \qquad \mathbb{P} - \text{a.s.}$$

(impossible for $d = 1$!)

Example: $d = d_1 + 1$

$$\omega(z, \pm e_i) = \frac{\varepsilon}{2d_1}, \quad i = 1, \cdots d_1$$

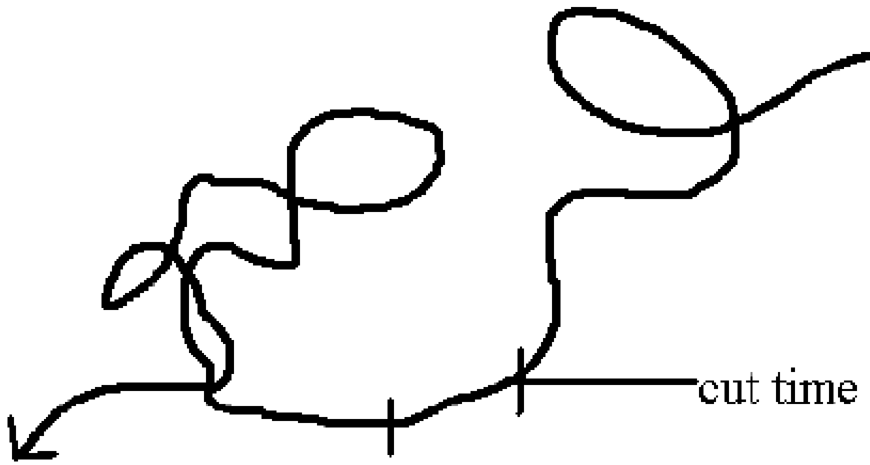
$$E_P(\omega(0, e_d)) > \frac{1}{2}$$

$$E_P\left[\log \frac{\omega(0, -e_d)}{\omega(0, e_d)}\right] > 0 \quad \Rightarrow v_P = 0$$

ε small: essentially $1d$ RWRE, renewal causes negative drift

ε large: homogenization occurs.

Proof of Theorem based on *cut times* for d_1 random walk:



Theorem (Erdös-Taylor)

For $d > 4$ density of cut times is positive.

(Also, for $d > 6$, $E\tau^1_{\text{cut}} < \infty$)

Write now

$$X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \quad \longleftarrow \; X^1 \text{ is RW component}$$

Law of $\{\tau^i_{\text{cut}}\}$ does not depend on environment, and, conditioned on $\{\tau^i_{\text{cut}}\}$,

$$\left\{X^2_{\tau^{i+1}_{\text{cut}}} - X^2_{\tau^{i+1}_{\text{cut}}}\right\}$$

are *independent* (not i.i.d.).

Analysis of tails of inter-cut times

$$\Rightarrow \text{ LLN, CLT}$$

Conclusions: RWRE model exhibits

- Wide range of behaviour.
- Many techniques used in analysis.
- Many questions remain.

Wishing for prosperity, peace, and human rights in my own country, around the world, and here in China.

Please allow me to express my own and other colleagues' concern about government restrictions on access to the internet.

Progress in Science and Society requires openeness.