RANDOM WALKS

IN RANDOM ENVIRONMENTS

Results and challenges

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A classical model:

One dimensional random walk

$$P(X_n = i+1 | X_{n-1} = i) = p$$
$$P(X_n = i-1 | X_{n-1} = i) = 1-p$$

$$\begin{aligned} E(X_n - X_{n-1}) &= E(X_n - X_{n-1} | X_{n-1}, X_{n-2}, \cdots) \\ &= 2p - 1 := v_{RW(p)} \stackrel{\triangle}{=} \text{drift,speed} \end{aligned}$$

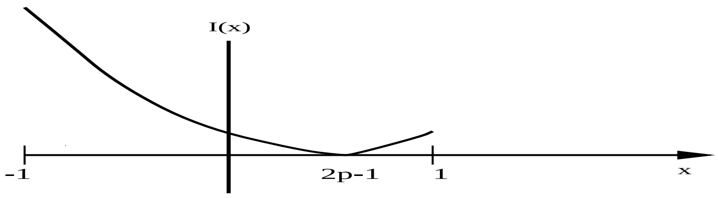
L.L.N.:

$$\frac{X_n}{n} \xrightarrow{\text{a.s.}} 2p-1$$

C.L.T.:

$$\frac{X_n - (2p-1)n}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4p(1-p))$$

Large Deviations



$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log P\left(\frac{X_n}{n} \in (x-\varepsilon, x+\varepsilon)\right) = -I(x)$$

One dimensional random walk
in random environment (RWRE)

<u>Model</u> $\{\omega_i\}$ – i.i.d., law P, supp $P \subset\subset (0,1)$

X_n – Random walk

$$\begin{aligned} P_\omega(X_n = i+1 | X_{n-1} = i) &= \omega_i \\ P_\omega(X_n = i-1 | X_{n-1} = i) &= 1 - \omega_i \end{aligned}$$

<u>*Solomon ('75)*</u>: $\rho_x := \frac{1-\omega_x}{\omega_x}$

$$\begin{aligned} X_n \text{ recurrent} &\iff \langle \log \rho \rangle = 0 \\ X_n \xrightarrow[\text{a.s.}]{} \infty &\iff \langle \log \rho \rangle < 0 \end{aligned}$$

Further,

$$\langle \rho \rangle < 1 \Rightarrow \frac{X_n}{n} \xrightarrow[\text{a.s.}]{} v_P := \frac{1 - \langle \rho \rangle}{1 + \langle \rho \rangle}$$

Proof by looking at $\tau_1 = \inf\{n : X_n = 1\}$

and computing $E_P \tau_1 = 1/v_P$.

Note: $v_P < E\left(v_{RW}(\omega_x)\right)$!!
(and both have same sign)

CLT, etc: Kozlov, Kesten, Spitzer, Sinai, ...

$$\langle \rho^2 \rangle < 1 \;\Rightarrow\; \text{CLT}$$
$$\langle \rho \rangle > 1 \;\Rightarrow\; \exists s \in (0,1) \text{ s.t. } \langle \rho^s \rangle = 1,$$
$$\frac{X_n}{n^s} \xrightarrow{\mathcal{L}} \text{nondegenerate random variable}$$

$$\langle \log \rho \rangle = 0$$
$$\Rightarrow \frac{X_n}{(\log n)^2} \xrightarrow{\mathcal{L}} \text{nondegenerate random variable}$$

All these are due to existence of "traps"!

$$\longrightarrow \quad | \quad \longleftarrow$$
$$+ + + + + + + + + + \;\; + \;\; - - - - - - - - - - - -$$

There is a difference between quenched and annealed behaviors:

$$P_\omega(\text{Time to pass typical trap} > n) \sim c_0^{n/n^{c_1}},$$

while

$$P(\text{Time to pass trap} > n) \sim c_3^{c_4 \log n}.$$

A rich subexponential regime!

$$\boxed{x \in (0, v_P)}$$

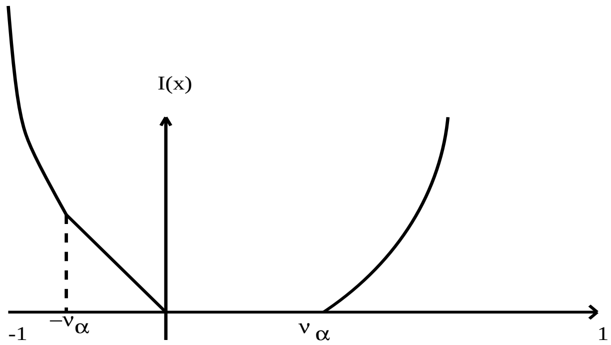
	Annealed	Quenched
		$\forall \delta > 0$,
Case I	$P\left(\frac{X_n}{n} \sim x\right) \sim \frac{1}{n^{s-1}}$	$\frac{1}{n^{1-\frac{1}{s}\pm\delta}} \log P_\omega\left(\frac{X_n}{n} \sim x\right)$
$\leftarrow \longrightarrow$	*Dembo, Peres, Z., '96*	$\longrightarrow \begin{cases} 0 \\ -\infty \end{cases}$
$\langle \rho^s \rangle = 1$		*Gantert, Z. '97*
$s > 1$		(surprising fluctuations?)
Case 0	$P\left(\frac{X_n}{n} \sim x\right)$	$P_\omega\left(\frac{X_n}{n} \sim x\right)$
	$\sim e^{-C^x_\pm n^{1/3}}$	$\sim e^{-c^x_\pm \frac{n}{(\log n)^2}}$
$\cdot \longrightarrow$	*Dembo, Peres, Z. '96*	*Gantert, Z. '97*
	$C^x = C^x_+ = C^x_-$	$c^x = c^x_+ = c^x_-$
	explicit expression	explicit expression
	Povel, Pizstora, Z. '97	*Povel, Pizstora '97*

What about large deviations?

Greven-den Hollander ('94):

Quenched setting (i.e., ω given)

$$\lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{n}\log P_\omega\left(\frac{X_n}{n}\in(x-\delta,x+\delta)\right)=-I(x)\,,P\text{–a.s.}$$



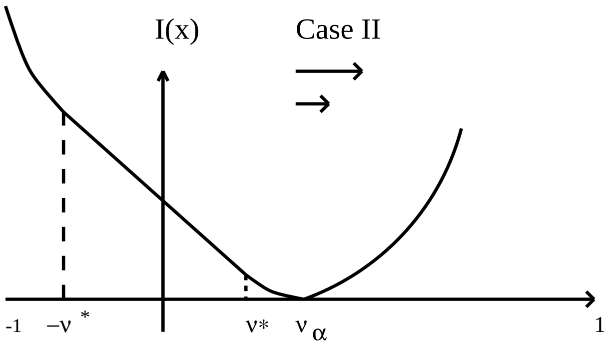
<u>Case 0,I</u>

$\longrightarrow$

$\longleftarrow$

$\longrightarrow$

.



More ambitious: understand LDP?
(Comets, Gantert, Z., '00)

Use recursions:

$$\tau_1 = \mathbf{1}_{\{X_1=1\}} + \mathbf{1}_{\{X_1=-1\}}(1 + \tau^0_{-1} + \tau'_1)$$

$$\varphi(\lambda,\omega) := E_\omega(e^{\lambda\tau_1})$$

$\Rightarrow$

$$\varphi(\lambda,\omega) = \cfrac{\omega_0}{e^{-\lambda} + \cfrac{(1-\omega_0)\omega_{-1}}{e^{-\lambda} + \cfrac{(1+\omega_{-1})\omega_{-2}}{e^{-\lambda}+\cdots}}}$$

$\Rightarrow \quad \frac{1}{n} \log E_\omega(e^{\lambda(\tau_1+\tau_2^1+\cdots+\tau_n^{n-1})}$

$$\longrightarrow \quad E \log \varphi(\lambda,\omega) := \Lambda(\lambda)$$

$\Rightarrow$ By large deviations arguments, relate exponential decay rates for X_n/n to Legendre transform of Λ.

Focus on quenched vs. annealed relation:

Rate function in the annealed case is (Comets-Gantert-Z. '00)

$$I^a(x) = \inf_{\eta \in M_1^{\mathrm{erg}}} \left\{ h(\eta|P)x + I^q_\eta(x) \right\}$$

where $h(\cdot|P)$ is specific relative entropy.

Thus, **Quenched$\neq$ Annealed**

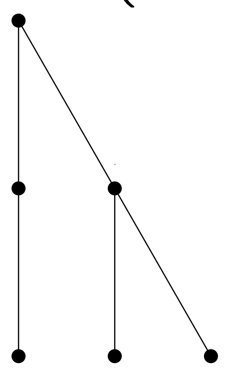
In terms of hitting times,

$$I_a^{\tau,P}(t) = \inf_\eta \left[I_q^{\tau,\eta}(t) + h(\eta|P) \right]$$

Proof is based on the fact that walk **must** traverse changed environment. This is a one dimensional fact.

In high dimension (or on trees!) cost of creating environment may exceed cost of wandering around to find it.

Extreme case: trees (Galton-Watson)



Walker goes to nearest neighbors, uniformly. If X_n denotes where the walker is at time n, then (Lyons, Pemantle, Peres '95)

$$\frac{|X_n|}{n} \to v_T \, , \; a.s.$$

Claim: (Dembo-Gantert-Peres-Z. 01')

For G-W trees, quenched and annealed large deviations behavior coincide:

$$I_q(v) = \lim_{n\to\infty} \frac{1}{n} \log P_\omega\left(\frac{|X_n|}{n} \sim v\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \log P\left(\frac{|X_n|}{n} \sim v\right) = I_a(v)\,,\ P - a.s.$$

Proof here is based on the fact that rate of growth of tree is so large that all important traps occur naturally not too far from root of tree.

Open issue: What happens in $\mathbb{Z}^d$??

Model RWRE in $\mathbb{Z}^d$

Environment: $S -\ 2d$ simplex

$$\Omega = S^{\mathbb{Z}^d},$$

$$\omega = \{\omega(z,e)\}_{z\in\mathbb{Z}^d}, \quad |e|_1 = 1$$

Walk: cd't on ω, Markov chain

$$P_\omega(X_{n+1} = x + e | X_n = x) = \omega(x,e)$$

("quenched")

Q: LLN? CLT? LD?

Simplest model: P, law on Ω, iid.

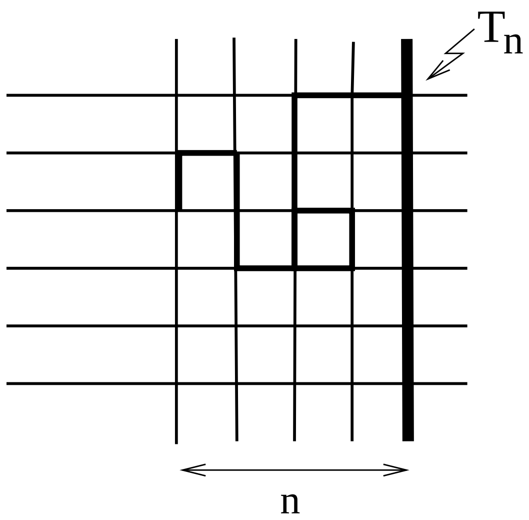
Walk is **ballistic** if $\dfrac{X_n}{n} \underset{n\to\infty}{\longrightarrow} v \neq 0, \quad \mathbb{P}\text{-a.s.}$

($\mathbb{P} = P \times P_\omega$ "annealed")

LD in $\mathbb{Z}^d$

– hitting times do not contain all information (still, under appropriate condition: there is a quenched LDP (Zerner '98))

$$\frac{1}{n} \log E_\omega(e^{\lambda Tn}) \xrightarrow{\text{a.s.}} \Lambda(\lambda)$$



But: annealed LDP? Relation between quenched and annealed?

Is it possible for $d \gg 1$ that $Q = A$?

LLN, CLT in $\mathbb{Z}^d$:

Problem: Cannot compute $E_P \tau_1$

(even for $\bar{\tau}_1 = \inf\{n : X_n \cdot e_1 = 1\}$)

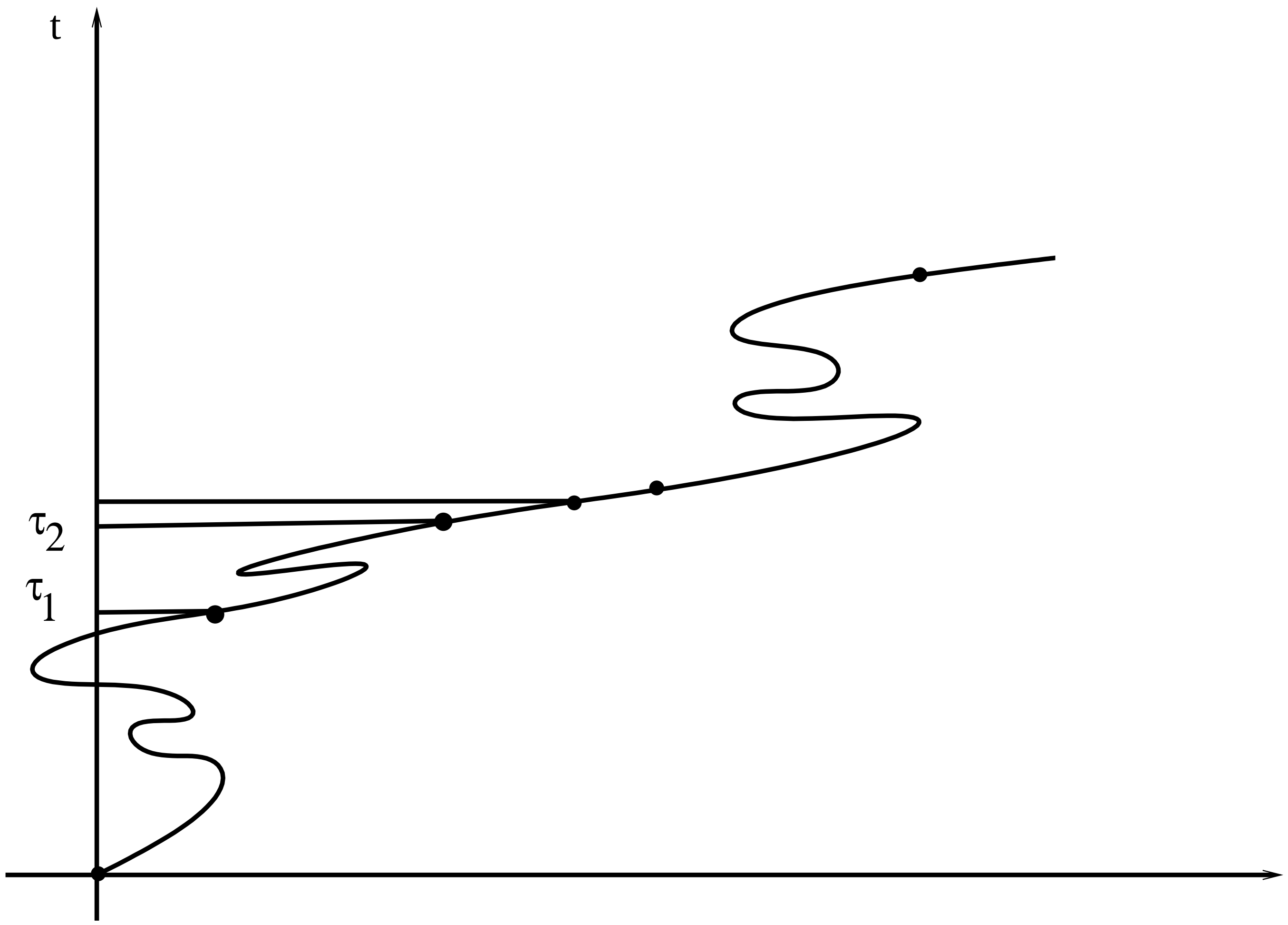
Alternative: introduce regeneration times

For RWRE on trees:

Kesten-Grimnett, ($\sim$ '84)
Lyons-Peres-Pemantle ('95)
Piau ('98)
Dembo-Gantert-Peres-Z. ('01)

For $\mathbb{Z}^d$: fully exploited by Sznitman-Zerner ('99)

Plot $X_n \cdot e_1$:



Fact If P is i.i.d.,

$$\left(\eta_i := (\tau_{i+1} - \tau_i), \left\{X_{\tau_i + n} - X_{\tau_i}\right\}_{n=1,\cdots,\eta_i},\right.$$

$$\left.\{\omega(z,\cdot)\}_{X_{\tau_i}\cdot e_1 \le z\cdot e_1 < X_{\tau_{i+1}}\cdot e_1}\right)$$

are i.i.d. under $\mathbb{P} = P \times P_\omega$ (annealed)

Theorem (Sznitman-Zerner '99)

a) If $E_P(\tau_2 - \tau_1) < \infty$ then walk is ballistic.

b) $E_P(\tau_2 - \tau_1) < \infty$ if Kalikow's condition holds.

Kalikow's condition: $U \subset Z^d, 0 \in U$

$$\widehat{P}_U(x, x+e) = \frac{\mathbb{E}\left(\sum_{n=1}^{\tau_{U^c}} \mathbf{1}_{\{X_n = x\}} \omega(x,e)\right)}{\mathbb{E}\left(\sum_{n=1}^{\tau_{U^c}} \mathbf{1}_{\{X_n = x\}}\right)}$$

$\tau_{U^c} =$ exit time from U

Kalikow's drift: $\hat{d}_U(x) = \sum_{|e|_1 = 1} e \widehat{P}_U(x, x+e)$

Kalikow's condition: $\inf_{U, x \in U} \hat{d}_U(x) \cdot e_1 > 0$

(There are easy to check sufficient conditions).

Improvements Sznitman ('00,'01,'02)
i.i.d. environment:

- CLT + tail estimates under Kalikow's condition.
- milder conditions (T,T'), leading also to CLT + tail estimates.
- criteria for T, T' conditions.

All based on analysis (via renormalization arguments) of tails of $\tau_2 - \tau_1$.

No known examples of sub-diffusive behavior!
No known **criteria** for transience-recurrence.

Conjectures: 1) (T') characterizes ballistic behavior
2) No subdiffusive behavior for $d \geq 2$.

Embarrassing open problem: 0-1 law:

For each direction ℓ, set

$$A_\ell^{\pm} := \{X_n \cdot \ell \to \pm\infty\}$$

Known: $\mathbb{P}(A_\ell^{+} \cup A_\ell^{-}) \in \{0,1\}$ (Kalikow '81)

For $d = 2$, i.i.d. environment:
$\mathbb{P}(A_\ell^{+}) \in \{0,1\}$ (Zerner-Merkl '02).

What about 0-1 law for $d > 2$?

Modifications of regeneration times

What happens for P non i.i.d., mixing?

$$\{\tau_{i+1} - \tau_i\} \text{ not even stationary!}$$

Idea: suppose walk is elliptic; can couple walk with independent coins:

if $\omega(x, x+e) \geq \varepsilon \qquad \forall x, e$, let

ε_n– iid, $Q_\varepsilon(\varepsilon_1 = e) = \varepsilon$, $Q_\varepsilon(\varepsilon_1 = 0) = 1 - 2d\varepsilon$

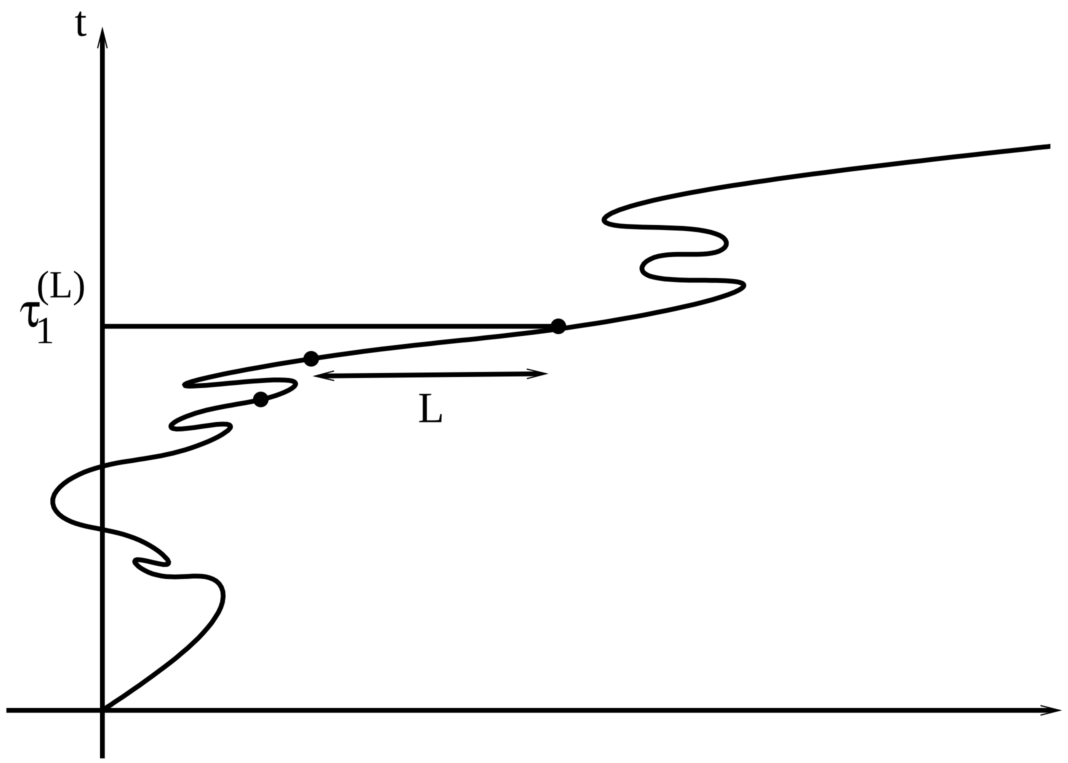
$$\bar{P}_{\omega,\varepsilon}\Big(X_{n+1} = z + e | X_n = z\Big) =$$

$$\mathbf{1}_{\varepsilon_{n+1}=e} + \frac{\mathbf{1}_{\varepsilon_{n+1}} = 0}{1 - 2d\varepsilon}(\omega(z,e) - \varepsilon)$$

Call a point an *L-reg point* if reg point

+ used $L - \varepsilon$ coins just before, in dir. e_1

+ fresh point before those.

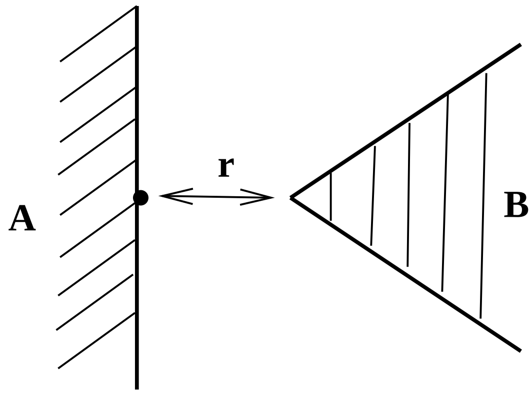


Now, $\{\tau_{i+1}^{(L)} - \tau_i^{(L)}\}$ still not iid, but have better and better (as function of L) mixing properties.

Say environment is ϕ-mixing on a cone if

$$\left|\frac{P(A\cap B)}{P(A)} - P(B)\right| \le \phi(r), \quad \phi(r) \underset{r\to\infty}{\longrightarrow} 0$$

for events with dependence
(e.g., Dobrushin-Shlosman weak mixing)



ϕ exp decay

Theorem (Comets-Z. '02)

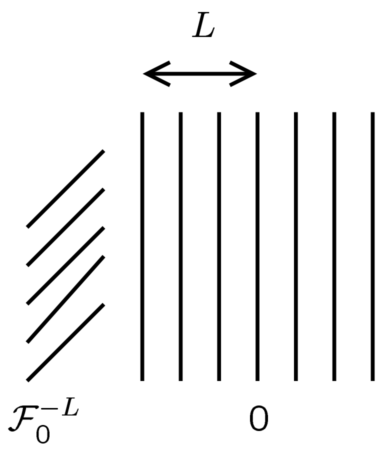
Assume ϕ-mixing on cones

$$P\left(E\left((\tau_1^{(L)}\cdot\varepsilon^L)^\alpha \middle| \mathcal{F}_0^{-L}, \begin{array}{c}\text{no return}\\ \text{left of 0}\end{array}\right) > M\right) = 0$$

where $\phi(L)^{\frac{1}{\alpha'}} M(L)^{\frac{1}{\alpha}} \underset{L\to\infty}{\longrightarrow} 0$

$$\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$$

Then ballistic.



Particular cases

– Finite dependency – take $L >$ dep. length.

– Exponentially mixing on cone $+$ non nestling.

– Some cases of Kalikow environments (strong enough average drift in e_1 dir.)

Proof based on coupling ideas!

Specific result

– Mixing on cones, exponential ϕ
(NN Ising; $\omega(x, x+e) = \omega^{\pm}(e))(\beta, h)$

Assume

$$\begin{aligned} d^{+} &:= \omega^{+}(e_1) - \omega^{+}(-e_1) > 0 \\ -d^{-} &:= \omega^{-}(e_1) - \omega^{-}(-e_1) < 0 \end{aligned}$$

Theorem $\exists h_0, \beta_0(\cdot)$, with

$$\beta_0(h) \underset{h \to \infty}{\longrightarrow} \infty$$

s.t. $\forall h > h_0, \beta < \beta_0(h)$,

$$\frac{1}{n} \sum_{i=1}^{n} X_i \longrightarrow v \neq 0, \quad \mathbb{P}\text{-a.s.}$$

What about non-ballistic?
Very little known!

- Balanced

$$\begin{aligned} & \omega(0,e) = \omega(0,-e) \\ \Rightarrow \quad & X_n \text{ is martingale;} \\ & \text{transient } d \geq 3, \text{ recurrent } d = 2 \\ & + \text{ quenched CLT (Lawler '85)} \end{aligned}$$

Proof by homogenization techniques: show that environment viewed from particle has law absolutely continuous w.r.t. P, and then use that $\theta^{X_n}\omega$ is a Markov chain.

- Small disorder

ω- small disorder $\omega(0,e) = \frac{1}{2d} + \varepsilon k_e$

P i.i.d. invariant under $\mathbb{Z}^d$-rotations.

$\Rightarrow$ Annealed CLT (Bricmont-Kupianien '91)

Proof uses (hard) renormalization techniques

Particular case: P i.i.d., $d = d_1 + d_2$,

$$\omega(z, \pm e_i) = \overline{\omega}_\pm(i), i = 1, \cdots, d_1$$

(i.e. non random in first d_1 coordinates).

Example: $\overline{\omega}_\pm(i) = \dfrac{1}{2d}, \quad i = 1, \cdots, d_1$

Theorem (Bolthausen, Sznitman, Z. '02)

For $d_1 \geq 5$: LLN

$$\text{or} \quad \left.\begin{array}{l} d_1 \geq 7 \ (P \text{ Symmetric }) \\ \\ d_1 \geq 13 \end{array}\right\} \text{CLT}$$

(CLT both quenched and annealed)

Consequence:

There is a P, i.i.d., with

$$E_P(\omega(0,e)\cdot e_d) < 0$$

but

$$\frac{X_n}{n}\cdot e_d \longrightarrow v > 0, \qquad \mathbb{P}-\text{a.s.}$$

(impossible for $d = 1$!)

Example: $\qquad d = d_1 + 1$

$$\omega(z, \pm e_i) = \frac{\varepsilon}{2d_1}, \quad i = 1, \cdots d_1$$

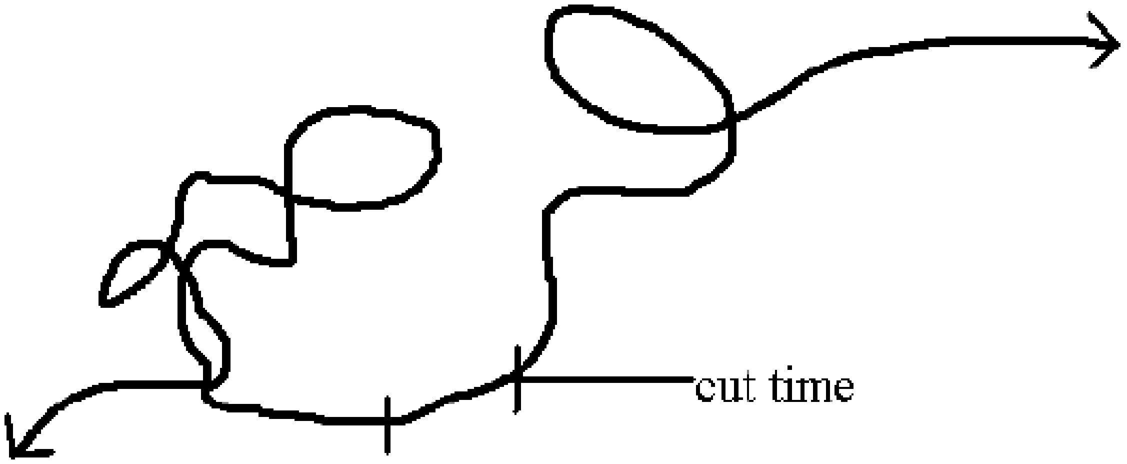
$$E_P(\omega(0,e_d)) > \frac{1}{2}$$

$$E_P\left[\log\frac{\omega(0,-e_d)}{\omega(0,e_d)}\right] > 0 \quad \Rightarrow v_P = 0$$

ε small: essentially $1-d$ RWRE, renewal causes negative drift

ε large: homogenization occurs.

Proof of Theorem based on *cut times* for d_1 random walk:



Theorem (Erdös-Taylor)

For $d > 4$ density of cut times is positive.

(Also, for $d > 6$, $E\tau^1_{\text{cut}} < \infty$)

Write now

$$X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \quad \longleftarrow \; X^1 \text{ is RW component}$$

Law of $\{\tau^i_{\mathsf{cut}}\}$ does not depend on environment, and, conditioned on $\{\tau^i_{\mathsf{cut}}\}$,

$$\left\{ X^2_{\tau^{i+1}_{\mathsf{cut}}} - X^2_{\tau^{i+1}_{\mathsf{cut}}} \right\}$$

are *independent* (not iid).

Analysis of tails of inter-cut times

$$\Rightarrow \text{ LLN, CLT}$$