Concentration of measure

Theorem 1 (Talagrand) *If* $F : [-1,1]^d \to \mathbb{R}$ *is Lipschitz and convex, and*

$$\|F\|_{\mathcal{L}} := \sup \frac{|F(x)-F(y)|}{\|x-y\|} \leq 1 \qquad (*)$$

then, with a product measure μ^d*,*

$$\mu^d(|F - M_F| \geq \delta) \leq 4e^{-\delta^2/16}$$

where M_F *is median of* F*.*

Theorem 2 (Herbst; Ledoux) *If* $F : R^d \to R$ *is Lipschitz with* $(*)$*, and* μ *satisfies the log-Sobolev inequality*

$$\int g^2 \log \frac{g^2}{\langle g^2, \mu\rangle} d\mu \leq 2c \int |\nabla g|^2 d\mu$$

then

$$\mu^d\Big(|F - \langle F, \mu^d\rangle| > \delta) \leq 2e^{-\frac{\delta^2}{2c\| \, \|\nabla F\|^2 \|_\infty}}$$

(no convexity assumptions).

How is it relevant to matrices?

$$(\mathbf{X}_A)_{ij} = \frac{1}{\sqrt{N}} A_{ij}\, \omega_{ij}, \quad \mathbf{X}_A = \mathbf{X}_A^*$$

A_{ij} – deterministic, $\quad |A_{ij}| \leq a$

ω_{ij} – independent, (complex) law $P_{ij} = P_{ij}^R \otimes P_{ij}^I, \quad \omega_{ij} = \omega_{ij}^R + \sqrt{-1}\omega_{ij}^I$

Want: concentration for spectral law of X_A.

Basic Lemma:

(a) *If f is a real valued convex function on $\mathbb{R}$, it holds that $(\omega^R, \omega^I) \mapsto \mathrm{tr}(f(\mathbf{X}_A(\omega)))$ is convex.*

(b) *If f is a Lipschitz function on $\mathbb{R}$, $(\omega^R, \omega^I) \mapsto \mathrm{tr}(f(\mathbf{X}_A(\omega)))$ is a Lipschitz function on $\mathbb{R}^{N^2}$ with Lipschitz constant bounded by $2a|f|_{\mathcal{L}}$.*

(part (a) is “Klein’s Lemma”).

Proof of Lemma (part a)

For any Y, X self-adjoint

$$f(Y) - f(X) = \int_0^1 Df(X + \eta(Y - X))\sharp(Y - X)d\eta$$

where

$$Df(X)\sharp(Y) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}\Big(F(X + \varepsilon Y) - F(X)\Big)\,.$$

For polynomial functions f,

$$Y^p - X^p - = \int_0^1 \left(\sum_{k=0}^{p-1}\Big(X + \eta(Y - X)\Big)^k (Y - X)\Big(X + \eta(Y - X)\Big)^{p-k-1}\right) d\eta\,.$$

Taking the trace:

$$\mathrm{tr}(X^p) - \mathrm{tr}\left(\left(\frac{X+Y}{2}\right)^p\right) = \int_0^1 p\,\mathrm{tr}\left(\left(\frac{X+Y}{2} + \eta\frac{X-Y}{2}\right)^{p-1}\frac{X-Y}{2}\right) d\eta\,,$$

$$\mathrm{tr}(Y^p) - \mathrm{tr}\left(\left(\frac{X+Y}{2}\right)^p\right) = \int_0^1 p\,\mathrm{tr}\left(\left(\frac{X+Y}{2} - \eta\frac{X-Y}{2}\right)^{p-1}\frac{X-Y}{2}\right) d\eta\,.$$

Hence,

$$\Delta \equiv \mathrm{tr}(X^p) + \mathrm{tr}(Y^p) - 2\,\mathrm{tr}\left(\left(\frac{X+Y}{2}\right)^p\right)$$
$$= \frac{p}{2}\sum_{k=0}^{p-2}\int_0^1\int_0^t \eta d\eta d\theta\,\mathrm{tr}\Big((X-Y)Z_{\eta,\theta}^k(X-Y)Z_{\eta,\theta}^{p-2-k}\Big)$$

with

$$Z_{\eta,\theta} \equiv \frac{X+Y}{2} - \eta\frac{X-Y}{2} + \eta\theta(X-Y)\,.$$

Next, for fixed $\eta, \theta \in [0,1]^2, X, Y$, $Z_{\eta,\theta}$ are self-adjoint $\Rightarrow \exists$ a unitary matrix $U_{\eta,\theta}$ and a real diagonal matrix $D_{\eta,\theta}$ so that $Z_{\eta,\theta} = U_{\eta,\theta}D_{\eta,\theta}U^*_{\eta,\theta}$. Let $W_{\eta,\theta} = U^*_{\eta,\theta}(X-Y)U_{\eta,\theta}$. Then,

$$
\begin{aligned}
\Delta &= \frac{p}{2}\sum_{k=0}^{p-2}\int_0^1\int_0^1 \eta d\eta d\theta\, \mathrm{tr}(W_{\eta,\theta}D_{\eta,\theta}^{k}W_{\eta,\theta}D_{\eta,\theta}^{p-2-k}) \\
&= \frac{p}{2}\int_0^1\int_0^1 \eta d\eta d\theta \left(\sum_{k=0}^{p-2}\sum_{1\le i,j\le N}(\lambda_{\eta,\theta}^{i})^{k}(\lambda_{\eta,\theta}^{j})^{p-2-k}|W_{\eta,\theta}^{ij}|^2\right)\,.
\end{aligned}
$$

But

$$
\sum_{k=0}^{p-2}(\lambda_{\eta,\theta}^{i})^{k}(\lambda_{\eta,\theta}^{j})^{p-2-k} = \frac{(\lambda_{\eta,\theta}^{i})^{p-1}-(\lambda_{\eta,\theta}^{j})^{p-1}}{\lambda_{\eta,\theta}^{i}-\lambda_{\eta,\theta}^{j}} = (p-1)\int_0^1\Big(\alpha\lambda_{\eta,\theta}^{j}+(1-\alpha)\lambda_{\eta,\theta}^{i}\Big)^{p-2}d\alpha\,.
$$

Hence,

$$
\Delta = \frac{1}{2}\sum_{1\le i,j\le N}\int_0^1\int_0^1\int_0^1 d\alpha\eta d\eta d\theta |W_{\eta,\theta}^{ij}|^2 f''\Big(\alpha\lambda_{\eta,\theta}^{j}+(1-\alpha)\lambda_{\eta,\theta}^{i}\Big)
$$

for the polynomial function $f(x)=x^p$, and hence for any smooth f.

For any such convex function, $f'' \geq 0$ and

$$\Delta = \frac{1}{2}\,\mathrm{tr}(f(X)) + \frac{1}{2}\,\mathrm{tr}(f(Y)) - \mathrm{tr}\left(f\left(\frac{X+Y}{2}\right)\right) \geq 0\,.$$

$\square$

Proof of Lemma (part b)

$$\partial_{\omega_{ij}^R}\,\mathrm{tr}\,f(\mathbf{X}_A) = \begin{cases} \frac{1}{\sqrt{N}}\Big[a_{ij}f'(\mathbf{X}_A)_{ij} + a_{ji}f'(\mathbf{X}_A)_{ji}\Big], & \text{if } i \neq j \\ \frac{a_{ii}}{\sqrt{N}} f'(\mathbf{X}_A)_{ii}, & \text{if } i = j\,. \end{cases}$$

Hence,

$$\sum_{1\leq i\leq j\leq N}\left(\partial_{\omega_{ij}^R}\,\mathrm{tr}\,f(\mathbf{X}_A)\right)^2 \leq \frac{2}{N}\sum_{1\leq i,j\leq N}|a_{ij}f'(\mathbf{X}_A)_{ij}|^2 \leq \frac{2a^2}{N}\sum_{1\leq i,j\leq N}|f'(\mathbf{X}_A)_{ij}|^2$$

$$= 2a^2\,\mathrm{tr}_N(f'(\mathbf{X}_A)f'(\mathbf{X}_A)^*) \leq \frac{2a^2}{N}\sum_{i=1}^{N} f'(\lambda_i)^2 \leq 2a^2\|f'\|_\infty^2$$

Hence,

$$\left|\mathrm{tr}(f(\mathbf{X}_A(\omega))) - \mathrm{tr}(f(\mathbf{X}_A(\omega')))\right| \leq 2a|f|_{\mathcal{L}}\|\omega - \omega'\| . \qquad \square$$

Hence, for general "band matrices" $\mathbf{X}_A$, we have:

Theorem 3

(a) *Assume $\exists$ a compact set $K \subset \mathbb{C}$ so that for any $1 \leq i \leq j \leq N, P_{ij}(K^c) = 0$. Assume f is convex and Lipschitz. Then, for any $\delta > \delta_0(N) := 8|K|\sqrt{\pi}a|f|_{\mathcal{L}}/N > 0$,*

$$\mathbb{P}^N\left(\left|\mathrm{tr}_N(f(\mathbf{X}_A(\omega))) - \mathbb{E}^N\left[\mathrm{tr}_N(f(\mathbf{X}_A))\right]\right| \geq \delta\right) \leq 4e^{-\frac{1}{16|K|^2a^2|f|^2_{\mathcal{L}}}N^2(\delta-\delta_0(N))^2} .$$

(b) *If the $(P^R_{ij}, P^I_{ij}, 1 \leq i \leq j \leq N)$ satisfy the logarithmic Sobolev inequality with uniform constant c, then for any Lipschitz function f, for any $\delta > 0$,*

$$\mathbb{P}^N\left(\left|\mathrm{tr}_N(f(\mathbf{X}_A(\omega))) - \mathbb{E}^N\left[\mathrm{tr}_N(f(\mathbf{X}_A))\right]\right| \geq \delta\right) \leq 2e^{-\frac{1}{8ca^2|f|^2_{\mathcal{L}}}N^2\delta^2} .$$

Remark Wishart (Generalized Wishart)

Y – iid entries

$$Z = YRY^T, R = \begin{pmatrix} R_{11} & & O \\ & \ddots & \\ O & & R_{NN} \end{pmatrix}$$

Take

$$A = \left(\begin{array}{c|c} O & R^{\frac{1}{2}} \\ \hline R^{\frac{1}{2}} & O \end{array}\right)$$

$$X_A = \left(\begin{array}{c|c} O & YR^{\frac{1}{2}} \\ \hline R^{\frac{1}{2}}Y^T & O \end{array}\right)$$

$$X_A^2 = \left(\begin{array}{c|c} YRY^T & O \\ \hline O & R^{\frac{1}{2}}Y^TYR^{\frac{1}{2}} \end{array}\right)$$

$\Rightarrow$ Spectrum of X_A^2 determines that of Z.

Another application: $\overline{X} = \sqrt{N} X$

$\overline{X}_{ij} - iid$, symmetric, uniformly bounded (by 1).

$F(\overline{\lambda}_1, \cdots, \overline{\lambda}_N) = \max(|\overline{\lambda}_1|)$

F is *convex*, Lipschitz

$$\mu^N(|F - M_F| > \delta) \leq 4e^{-\delta^2/16}$$

But $\lambda_N^* = \max(|\lambda_1|, \cdots, |\lambda_N|)$

$\Rightarrow \quad \mu^N(|\lambda_N^* - c_N| > \delta) = \mu^N(|F - M_F| > \delta\sqrt{N}) \leq 4e^{-N\delta^2/16}$

$\Rightarrow$ Similar results for Prohorov distance: concentration around $E\widehat{\mu}_N$.

$\Rightarrow$ Similar results for spectral norm.

Remark Simple minded argument based on Lidskii's theorem give only concentration at speed N!

References:

A. Guionnet, O. Zeitouni: Concentration of the spectral measure for large matrices, ECP Vol. 5 (2000), pp. 119-136.

N. Alon, M. Krivelevich, V.H. Vu: On the concentration of eigenvalues of random symmetric matrices, preprint.

Non commutative stochastic calculus (after Cabanal-Duvillard & Guionnet)

Hermitian Brownian motion: $(\beta_{ij}, \beta'_{k\ell})_{\substack{1\leq k<\ell\leq N \\ 1\leq i\leq j\leq N}}$ – independent BM

$$H_N^{k\ell} = \begin{cases} \frac{1}{\sqrt{2N}}(\beta_{k,\ell} + i\beta'_{k,\ell}), & k<\ell \\ \frac{1}{\sqrt{2N}}(\beta_{\ell,k} - i\beta'_{\ell,k}), & \ell<k \\ \frac{1}{\sqrt{N}}\beta_{\ell\ell} & k=\ell \end{cases}$$

Stochastic integral: for $Y_{\cdot}$ a *bi-process*,

$$\left(\int_0^t Y_s \sharp dH_N(s)\right)_{i,j} = \sum_{1\leq k,\ell\leq N} \int_0^t Y_{i,k,\ell,j}(s)\, dH_N^{k,\ell}$$

Itô's formula

Non commutative derivatives

Let $C[X]$ denote polynomial functions. Define

$$D_0 : C[X] \to C[X] \otimes C[X], \, X^n \to \sum_{i=0}^{n-1} X^i \otimes X^{n-1-i}$$

$$L_0 : C[X] \to C[X] \otimes C[X], \, X^n \to \sum_{i=1}^{n-1} iX^{i-1} \otimes X^{n-1-i}$$

and extend these formula to smooth functions. ($D_0 f(x,y) = \frac{f(x)-f(y)}{x-y}$ in shorthand notation!)

Itô's formula: $\quad \forall f \in C_b^{2,1}(\mathbb{R} \times [0,1])$,

$$\begin{aligned} f(H_N(t),t) = f(H_N(0),0) &+ \int_0^t D_0 f\Big(H_N(s), H_N(s); s\Big) \sharp dH_N(s) \\ &+ \int_0^t \partial_s f\Big(H_N(s), s\Big) ds \\ &+ \int_0^t (Id_N \otimes \mathrm{tr}_N) \circ (L_0 f)\Big(H_N(s), H_N(s); s\Big) ds \end{aligned}$$

(**Remark:** similar lemma, with small correction, for GOE instead of GUE).

Main outcome:

Let $\widehat{\mu}_t^{(N)}$ denote the spectral measure of $H_N(t)$. Then, the process

$$Q_f^{(N)}(t) = \widehat{\mu}_t^{(N)}\Big(f(x,t)\Big) - \widehat{\mu}_0^{(N)}\Big(f(x,0)\Big) - \int_0^t \widehat{\mu}_s^{(N)}\Big(\partial_s f(x,s)\Big)ds$$
$$- \int_0^t \Big(\widehat{\mu}_s^{(N)} \otimes \widehat{\mu}_s^{(N)}\Big) \circ (L_0 f)(x,y,s)ds$$

is a martingale with bracket

$$\langle Q_f^{(N)}\rangle_t = \frac{1}{N^2}\int_0^t \widehat{\mu}_s^{(N)}\Big((\partial_x f)^2(x,s)\Big)ds$$

Define for such $f \in C_b^{2,1}(\mathbb{R} \times [0,1])$, $\nu \in C([0,1], M_1(\mathbb{R}))$:

$$\overline{S}^{s,t}(\nu, f) = \int f(x,t) d\nu_t(x) - \int f(x,s) d\nu_s(x)$$
$$- \int_s^t \int \partial_u f(x,u) d\nu_u(x) du - \frac{1}{2} \int_s^t \iint \frac{\partial_x f(x,u) - \partial_x f(y,u)}{x-y} d\nu_u(x) d\nu_u(y) du \,,$$

$S^{0,1}$ is given by

$$S^{0,1}(\nu) = \begin{cases} \sup_{f \in C_b^{2,1}(\mathbb{R} \times [0,1])} \left(\overline{S}^{0,1}(\nu, f) - \frac{1}{2} \int_0^1 \int (\partial_x f(x,u))^2 d\nu_u(x) du \right), & \nu_0 = \delta_0 \\ \infty, & \text{otherwise} \end{cases}$$

Then one gets that $S^{0,1}(\nu)$ is a good rate function, and:

Theorem 4 (CD-G) *For every closed subset $F \in \mathcal{C}([0,1], M_1(\mathbb{R}))$,*

$$\limsup_{n\to\infty} \frac{1}{N^2} \ln \mathbb{P}\Big(\widehat{\mu}_{\cdot}^{(N)} \in F\Big) \le - \inf_{\nu\in F} S(\nu)\,.$$

For any open subset $\mathcal{O} \in \mathcal{C}([0,1], M_1(\mathbb{R}))$,

$$\liminf_{n\to\infty} \frac{1}{N^2} \ln \mathbb{P}(\widehat{\mu}_{\cdot}^{(N)} \in \mathcal{O}) \ge - \inf_{\nu\in\mathcal{O}\cap\mathcal{C}^\infty([0,1],M_1(\mathbb{R}))} S(\nu)$$

where $\mathcal{C}^\infty([0,1], M_1(\mathbb{R}))$ is the subset of $\mathcal{C}([0,1], M_1(\mathbb{R}))$ of weak solutions ν such that $\nu_0 = \delta_0$, of the differential equation

$$S^{0,1}(\nu, f) = \int_0^1 \int \partial_x h_u(x) \partial_x f_u(x) d\nu_u(x) du$$

for any $f \in \mathcal{C}_b^{2,1}(\mathbb{R} \times [0,1])$, with $h \in \mathcal{MF}_\infty \subset \mathcal{C}_b^{\infty,1}(\mathbb{R} \times [0,1])$ so that

$$\mathcal{MF}_\infty = \Big\{ h \in \mathcal{C}_b^{\infty,1}\Big(\mathbb{R} \times [0,1]\Big) \cap \mathcal{C}\Big([0,1], L^2(\mathbb{R})\Big); \\ \exists (C,\varepsilon) \in (0,\infty);\ \sup_{t\in[0,1]} |\widehat{h}_t(\lambda)| \le C_e^{-\varepsilon|\lambda|} \Big\}$$

where $\widehat{h}_t$ stands for the Fourier transform of h_t.

Upper bound obtained as follows: for $f \in C_b^{2,1}(\mathbb{R} \times [0,1])$,
$M^f_{\boldsymbol{\cdot}}(\widehat{\mu}^{(N)}) := \exp(N^2 Q_f^{(N)}(t) - \frac{N^2}{2} \langle Q_f^{(N)} \rangle_t)$ is a martingale. Then,

$$\mathbb{P}(\widehat{\mu}^{(N)}_{\boldsymbol{\cdot}} \in F) = \mathbb{E}\left(\mathbf{1}_{\widehat{\mu}^{(N)}_{\boldsymbol{\cdot}} \in F} \frac{M_1^f(\widehat{\mu}^{(N)})}{M_1^f(\widehat{\mu}^{(N)})}\right)$$

$$\leq \sup_{\nu_{\boldsymbol{\cdot}} \in F} \Big(M_1^f(\nu) \Big)^{-1} \cdot E\Big(M_1^f(\widehat{\mu}^{(N)}) \Big)$$

$$= \sup_{\nu_{\boldsymbol{\cdot}} \in F} \Big(M_1^f(\nu) \Big)^{-1} .$$

Lower bound similar: for small ball B,

$$\mathbb{P}(\widehat{\mu}^{(N)}_{\boldsymbol{\cdot}} \in B) \geq \inf_{\nu_{\boldsymbol{\cdot}} \in B} \Big(M_1^f(\nu) \Big)^{-1} E\Big(\mathbf{1}_{\widehat{\mu}^{(N)}_{\boldsymbol{\cdot}} \in B} \, M_1^f(\widehat{\mu}^{(N)}_{\boldsymbol{\cdot}}) \Big) .$$

But $E_f(\cdot) := E(M_1^f(\widehat{\mu}^{(N)}_{\boldsymbol{\cdot}})\cdot)$ is a new measure, and *if* f can be chosen such that $E_f(\mathbf{1}_{\widehat{\mu}^{(N)}_{\boldsymbol{\cdot}} \in B}) \to 1$ then get lower bound.
This can be done for B-balls around points in $C^\infty([0,1], M_1(\mathbb{R}))$.

Improvement of lower bound (Guionnet-Z.):

$$\liminf_{n\to\infty} \frac{1}{N^2} \ln \mathbb{P}\Big(\widehat{\mu}_{\cdot}^{(N)} \in \mathcal{O}\Big) \geq - \inf_{\nu\in\mathcal{O}\cap\mathcal{A}} S(\nu)$$

where

$$\mathcal{A} = \Big\{\nu \in C([0,1], M_1(\mathbb{R})) : \exists\eta > 0, \sup_{t\in[0,1]} \nu_t(|x|^{6+\eta}) < \infty\Big\}$$

(by regularization with Cauchy law).

LARGE DEVIATIONS FOR SPHERICAL INTEGRALS

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(joint work with A. Guionnet – ENS Lyon)

http://www-ee.technion.ac.il/$\sim$zeitouni/ps/ldlbsentcor.ps

Wishart Matrix:

$$X - N \times N \text{ matrix, i.i.d. } N(0, I_N)$$
$$Z = X\,X^T, \text{ eigenvalues } \{\lambda_i\}$$
$$L_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$$

<ins>Questions</ins>: Limits of L_n? Rates? LDP?

Joint distribution of eigenvalues:

$$\prod_{i<j} |\lambda_i - \lambda_j| \prod_i \lambda_i^{-1/2} \cdot \exp\left(-\frac{N}{2} \sum_{i=1}^{N} \lambda_i\right)$$

Naive argument:

$$\simeq \exp\left[-N^2 \left\langle \frac{x}{2}, L_N \right\rangle - N \left\langle \frac{\log x}{2}, L_N \right\rangle + N^2 \left\langle \frac{\log|x-y|}{2} L_N, L_N \right\rangle\right]$$

Here $\langle f(x,y)\mu, \mu\rangle := \iint f(x,y)\mu(dx)\mu(dy)$

Guess then that L_N should converge to *minimizers* of

$$J(\mu) = \left\langle \frac{x}{2}, \mu \right\rangle - \frac{1}{2} \langle \log(x-y)\mu, \mu \rangle$$

and moreover, that LDP (at scale N^2) should hold.

This intuition is *true*, at least for weak topology.

(Result goes back to Pastur-Marchenko, Silverstein, . . . ; LDP follows from technique of Ben-Arous & Guionnet, for this case Hiai & Petz.)

This matrix model has applications in communications theory: If N users transmit N (independent) Gaussian bits each, the performance of the optimal *linear* receiver (in white noise, variance σ^2) is proportional to

$$\frac{1}{N}\,\mathrm{tr}\,(X\,X^* + \sigma^2 I)^{-1} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\lambda_i + \sigma^2}\,.$$

Now, if each user has independent entries but users are correlated, we are led to analyze $X\,X^*$ where columns are independent correlated vectors. Let $a_1, \ldots, a_n$ be the eigenvalues of the covariance matrix of a column of X, the joint distribution of the eigenvalues is now

$$\prod_{i<j}|\lambda_i - \lambda_j|\prod_i \lambda_i^{-1/2}\int_{\mathrm{O}_N}\exp\left[-\frac{N}{2}\mathrm{tr}\,(UD_aU^TD_\lambda)\right]m_n(dU)$$

$$D_a = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_N \end{bmatrix}, \quad , D_\lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix}$$

So, LDP now relates to:

A, B – $N \times N$ matrices, spectral measures $\mu_A^N, \mu_B^N \longrightarrow \mu_A, \mu_B$.

Then

$$\frac{1}{N^2} \log \overbrace{\int_{0_N} \exp\left(-\frac{N}{2}\,\mathrm{tr}\,(UAU^TB)\right) m_N(dU)}^{I_N(A,B)} \xrightarrow{?} J(\mu_A, \mu_B)\,.$$

Our goal: prove limit exists, compute $J(\cdot, \cdot)$.

Remarks

1. $0_N \leftrightarrow U_N$. Then, formula of Harish-Chandra, Itzykson & Zuber:

$$\int_{U_N} \exp\left(-\frac{N}{2}\,\mathrm{tr}\,(UAU^*B)\right) m_N(dU) = \frac{\det\{\exp N a_i b_j\}}{\prod_{i<j}|a_i - a_j| \prod_{i<j}|b_i - b_j|}$$

   Not useful !!??

2. Same asymptotics appear in “matrix models” of physics.

3. One can represent integral as sum over characters of certain Shur polynoms. This has not proven useful (to us).

Main Result

For $f \in C_b^{2,1}(\mathbb{R} \times [0,1]), \mu \in \mu_1(\mathbb{R})$,
$\nu. \in C([0,1];\ \mu_1(\mathbb{R}))$ (path of probability measures)

Define

$$S^{s,t}(\nu, f) = \int f(x,t)\nu_t(dx) - \int f(x,s)\nu_s(dx) - \int_s^t \int \partial_u f(x,u)\nu_u(dx)du$$
$$- \frac{1}{2}\int_s^t \iint \frac{\partial_x f(x,u) - \partial_x f(y,u)}{x-y}\nu_u(dx)\nu_u(dy)du$$
$$- \frac{1}{2}\int_s^t \int |\partial_x f(x,u)|^2 \nu_u(dx)du$$

$$S_\mu(\nu) = \begin{cases} \infty, & \nu_0 \neq \mu \\ \sup\limits_{\substack{0 \le s < t \le 1 \\ f \in C_b^{2,1}(\mathbb{R}\times[0,1])}} S^{s,t}(\nu, f), & \nu_0 = \mu \end{cases}$$

Main Result

$$J(\mu_A, \mu_B) = \frac{1}{2} \inf \{ S_{\mu_A}(\nu.) : \nu_1 = \mu_B \} \qquad (*)$$

Remarks

1. There exists an analogous result for integration over unitary group.

2. Can write *formal* Hamilton-Jacobi equations for (*), agree with physicist's derivation.

3. Can now also write LDP for $X^T X$ from communications example.

4. That correct scale for LDP is N^2 follows from concentration of measure results (apply Talagrand's inequalities).

Steps in Proof

1. Reduction to LDP:

$$\int_{d(L_N^A,\mu_A)<\delta} e^{-\frac{N^2}{2}(L_N^A,x^2)} \prod_{i<j} |a_i - a_j| I_N(A_N, B_N) \prod da_i$$

$$= e^{\frac{N^2}{2}\langle L_N^B, x^2\rangle} \int_{d(L_N^A,\mu_A)<\delta} \int e^{-\frac{N}{2}\mathrm{tr}\,(UA_NU^* - B_N)^2} dm_N(U) \cdot \prod_{i<j} |a_i - a_j|$$

$$= \text{(Known terms) } P(d(L_N^Y(1), \mu_A) < \delta)$$

where $L_N^Y(t)$ is empirical measure of the matrix

$$Y_N(t) = \underset{\text{independent}}{B_N + X_N(t)}$$

where $X_N(\cdot)$ is (symmetric) matrix of Brownian motions.

Thus, main issue: find LDP of

$$Y_N(t) \triangleq B_N + X_N(t) \,.$$

Idea: (Cabanal-Duvillard + Guionnet, $B_N = 0$)

Develop Itô calculus for functions of empirical measure of $Y_N(t)$.

Main Point With test function of (x,t),

$$d\langle L_N^Y, f\rangle = \langle \mathcal{L} f, L_N^Y\rangle dt + M_t + \text{ negligible terms}$$

where M_t is martingale of bracket

$$\int_0^t |\partial_x f|^2 L_N^Y(dx)\; ds$$

$\Rightarrow$ Can construct exponential martingales based on path $L_N^Y(\cdot)$.

A few more details:

(a) Itô Calculus:

$$\overline{S}_N^{s,t}(f) = S^{s,t}(L_N^Y, f) + \frac{1}{2}\int_s^t \int |\partial_x f|^2 L_N^u(dx)du$$

is a Martingale, bracket $\langle f, f\rangle_{s,t} = \int_s^t \int |\partial_x f|^2 L_N^u(dx)du$

(b) Construct exponential martingale:

$$\xi_f^{N,s}(t) = \exp(N^2(\overline{S}_N^{s,t}(L_N^Y, f) - \frac{1}{2}\langle f, f\rangle_{s,t}\rangle)$$

(c) Upper bound:

$$P(L_N^Y(\cdot) \in B(\nu_\cdot, \delta)) = E\left(\mathbf{1}_{L_N^Y(\cdot)\in B(\nu,\delta)} \frac{\xi_f^N}{\xi_f^N}\right)$$
$$\leq \sup_{\nu' \in B(\nu,\delta)} e^{-N^2 S^{s,t}(\nu',f)} \cdot E(\xi_f^N) \simeq e^{-N^2 \overline{S}^{s,t}(\nu',f)}$$

(d) Exponential tightness + compactness-analysis.

(e) Lower Bound: Use change of measure with “drift” f.
For f “smooth”, resulting $L_N^Y \xrightarrow{\omega} \nu_f$.
Then, get lower bound for such ν_f.

(f) Lower bound for general ν by approximation (Highly technical!)

Main Tool: convolution with a Cauchy random variable.

(link with free probability via fact that $\mu \boxplus$ Cauchy $= \mu *$ Cauchy. Hence,

$$Y_N \boxplus \varepsilon \text{ (Cauchy matrix) } \simeq \text{(law of } Y_N) * \varepsilon \text{ Cauchy law.}$$