Snake representation of a superprocess in random environment

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Abstract

We consider (discrete time) branching particles in a random environment which is i.i.d. in time and possibly spatially correlated. We prove a representation of the limit process by means of a Brownian snake in random environment.

1 Introduction

1.1 Superprocesses in random environments

Superprocesses in random environments were introduced in [10] as the scaling limits of particle systems whose branching are affected by random environments. In particular the limiting behavior of the following model has been studied. At time $t = 0$, $K_n \sim n$ particles are located in $\mathbb{R}^d$. Each of these K_n particles follows the path of an independent Brownian motion until time $t = 1/n$. At time $1/n$ each particle independently of the others either splits into two or dies and then the individual particles in the new population again follow the paths of independent Brownian motions starting at their place of birth, in the interval $[1/n, 2/n)$, and the pattern of alternating branching and spatial spreading continues. Let us describe in details

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the branching mechanism that was suggested in [10]. Let $\{\xi_k(\cdot)\}_{k\geq 0}$ be a sequence of i.i.d. $\mathbb{R}^d$-indexed random fields with mean 0 and covariance

$$g(x,y) = \mathrm{Cov}(\xi_k(x), \xi_k(y)), \quad x, y \in \mathbb{R}^d.$$

At time k/n each particle, independently of the others conditionally on ξ, either splits into two with probability

$$\frac{1}{2} + \frac{1}{2\sqrt{n}}\xi_k(x)$$

or dies with probability

$$\frac{1}{2} - \frac{1}{2\sqrt{n}}\xi_k(x),$$

where x is the location of the particle. That is, the fields $\{\xi_k\}_{k\geq 0}$ create the random environment that affects the branching of the particles. Define the following measure-valued process that describes the evolution of the population:

$$X_t^n(A) = \frac{\text{number of particles in } A \text{ at time } t}{n}, \; A \subset \mathbb{R}^d. \tag{1.1}$$

Before proceeding we introduce some notation. For a locally compact Polish space E, let $\mathcal{M}_F(E)$ (respectively, $\mathcal{M}(E)$) be the space of finite (respectively Radon) non-negative measures on E, equipped with the weak (respectively, vague) topology (see Section 3.1 in [3]). In the case of $E = \mathbb{R}^d$, we will also write $\mathcal{M}_F = \mathcal{M}_F(\mathbb{R}^d)$ and $\mathcal{M} = \mathcal{M}(\mathbb{R}^d)$. Both $\mu(\phi)$ and $\langle \phi, \mu\rangle$ denote the integral of a function ϕ with respect to measure μ. For any metric space E let $D_E = D_E[0,\infty)$ (resp. $C_E = C_E[0,\infty)$) be the space of cadlag (resp. continuous) E-valued functions on $[0,\infty)$ endowed with the Skorohod topology. Let $\mathcal{C}^k(\mathbb{R}^d)$ (resp. $\mathcal{C}_b^k(\mathbb{R}^d)$) be the set of continuous (resp. bounded continous) functions with continuous (resp. bounded continuous) partial derivatives of order k or less. Also we define $\mathcal{B}(\mathbb{R}^d)$ to be the set of bounded measurable functions on $\mathbb{R}^d$.

It was shown in [10], under some additional technical assumptions on ξ, that if

$$X_0^n \Rightarrow X_0 =: \mu\,, \;\; \text{in } \mathcal{M}_F\,,$$

then

$$X^n \Rightarrow X, \;\; \text{in } D_{\mathcal{M}_F}[0,\infty)\,.$$

Here X is a process in $C_{\mathcal{M}_F}[0,\infty)$ which is the unique solution to the following martingale problem: $\forall\ \phi \in \mathcal{C}_b^2(\mathbb{R}^d)$,

$$M_t^\phi \equiv \langle X_t, \phi\rangle - \langle \mu, \phi\rangle - \frac{1}{2}\int_0^t \langle X_s, \Delta\phi\rangle\, ds, \qquad t \geq 0 \tag{1.2}$$

is a continuous martingale with quadratic variation process

$$\begin{aligned} \left\langle M^\phi \right\rangle_t &= \int_0^t \left\langle X_s, \phi^2\right\rangle ds \qquad\qquad (1.3)\\ &\quad + \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x,y)\phi(x)\phi(y)X_s(dx)X_s(dy)ds, \qquad t \geq 0. \end{aligned}$$

In this paper we introduce some minor changes into the above model. Instead of the binary branching we assume that each particle gives birth to a number of particles distributed according to the geometric distribution with parameter $\frac{1}{2} - \frac{1}{4\sqrt{n}}\xi_k(x)$; that is, if N is the number of offspring of the particle located at x at time k/n, then

$$\mathbb{P}(N = m|\xi) = \left(\frac{1}{2} + \frac{1}{4\sqrt{n}}\xi_k(x)\right)^m \left(\frac{1}{2} - \frac{1}{4\sqrt{n}}\xi_k(x)\right), \quad m = 0, 1, 2, \ldots \tag{1.4}$$

In particular, conditioned on the environment ξ, the expected number of offspring of a particle at x at time k/n is

$$\frac{\frac{1}{2} + \frac{1}{4\sqrt{n}}\xi_k(x)}{\frac{1}{2} - \frac{1}{4\sqrt{n}}\xi_k(x)} = 1 + \frac{1}{\sqrt{n}}\xi_k(x) + \frac{1}{2n}\xi_k(x)^2 + o\left(\frac{1}{n}\right). \tag{1.5}$$

Compared with [10], we also allow ξ to be slightly more general, that is, we assume that $\{\xi_k(\cdot)\}_{k\geq 1} = \{\xi_k^n(\cdot)\}_{k\geq 1}$ is a sequence of i.i.d. random fields with mean $\nu/\sqrt{n}$, for some $\nu \in \mathbb{R}$, and covariance

$$g(x, y) = \mathrm{Cov}(\xi_k(x), \xi_k(y)), \quad x, y \in \mathbb{R}^d. \tag{1.6}$$

Let X^n be defined for this model as in (1.1). By the same argument as in [10] one can prove that the limit of $\{X^n\}_{n\geq 1}$ is the solution to the following martingale problem: $\forall\ \phi \in \mathcal{C}_b^2(\mathbb{R}^d)$,

$$M_t^\phi \equiv \langle X_t, \phi\rangle - \langle \mu, \phi\rangle - \frac{1}{2}\int_0^t (\langle X_s, \Delta\phi\rangle + \langle X_s, (\nu + \bar{g}/2)\phi\rangle)\, ds, \qquad t \geq 0 \tag{1.7}$$

is a continuous martingale with quadratic variation process

$$\begin{aligned}\left\langle M^\phi \right\rangle_t &= 2\int_0^t \left\langle X_s, \phi^2\right\rangle ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x, y)\phi(x)\phi(y)X_s(dx)X_s(dy)ds, \qquad t \geq 0\end{aligned} \tag{1.8}$$

where $\bar{g}(x) = g(x, x)$.

1.2 Brownian snake

The main purpose of this paper is to study the Brownian snake representation of the process that solves the above martingale problem (1.7-1.8). For a nice introduction into the topic the reader is referred to [9]. The classical Brownian snake was used to study different properties of super-Brownian motion. Loosely speaking if $\{\mathbb{W}_s\}_{s\geq 0}$ is a Brownian snake then for each $s \geq 0$, $\mathbb{W}_s$ is a stopped Brownian path. To be more precise we call the pair $w = (\mathbf{w}, \zeta) \in C_{\mathbb{R}^d}[0, \infty) \times \mathbb{R}_+$ a stopped path in $\mathbb{R}^d$ if for each $t \geq \zeta$, $\mathbf{w}(t) = \mathbf{w}(\zeta)$. ζ is called the lifetime of the path w and sometimes is denoted by ζ_w or $\zeta(w)$. Let $\mathcal{W}$ denote the space of all stopped paths in $\mathbb{R}^d$ equipped with the distance

$$d(w,w') = \sup_{t\geq 0} |\mathbf{w}(t) - \mathbf{w}'(t)| + |\zeta_w - \zeta_{w'}|.$$

We will also use the notation $\hat{w} = \mathbf{w}(\zeta_w)$ for the terminal point of w. For any $x \in \mathbb{R}^d$ we denote by $\bar{x}$ the path with lifetime 0 constantly equal to x. If $w = (\mathbf{w}, \zeta)$ is a stopped path then with some abuse of notation we will sometimes set $w(s) = \mathbf{w}(s)$ for any $s \geq 0$.

The usual Brownian snake can be thought of as a limit of the so-called discrete snakes that we will now define. Let $\{Y^n_{k/n^2}\}_{k=0,1,\dots}$ be a rescaled simple random walk on $\mathbb{Z}_+/n$ reflected at the origin, that is, the time between the steps is $1/n^2$ and the size of the jump is $\pm 1/n$ with equal probabilities. Explicitely,

$$\begin{aligned}
\mathbb{P}\left(Y^n_{(k+1)/n^2} - Y^n_{k/n^2} = \pm 1/n\right) &= \frac{1}{2}, \text{ if } Y^n_{k/n^2} \geq 1/n,\ k = 0,1,\dots,\\
Y^n_{(k+1)/n^2} &= 1/n, \text{ if } Y^n_{k/n^2} = 0,\ k = 0,1,\dots.
\end{aligned}$$

We also let $Y^n_\cdot$ be constant between the jumps. The process $Y^n_\cdot$ is called the contour or lifetime process of the discrete snake $\mathbb{W}^n$. That is for each $s \geq 0$, the snake $\mathbb{W}^n_s = (\mathbf{W}^n_s, Y^n_s)$ is a stopped path with life time Y^n_s. We next define the paths of the snake. Fix $x \in \mathbb{R}^d$ and set

$$\mathbb{W}^n_0 = \bar{x}.$$

Let $\eta_1, \eta_2, \dots$ be a sequence of independent Brownian paths stopped at time $1/n$, independent of the contour process. Let $\mathbb{W}^n_{k/n^2} = (\mathbf{W}^n_{k/n^2}, Y^n_{k/n^2})$ be the stopped path at time k/n^2 with lifetime $\zeta_{\mathbb{W}^n_{k/n^2}} = Y^n_{k/n^2}$. Then define

$$\mathbf{W}^n_{(k+1)/n^2}(\cdot) = \begin{cases} \mathbf{W}^n_{k/n^2}(\cdot \wedge (Y^n_{k/n^2} - 1/n)), & \text{if } Y^n_{(k+1)/n^2} = Y^n_{k/n^2} - 1/n,\\ \mathbf{W}^n_{k/n^2} \odot \eta_k(\cdot), & \text{if } Y^n_{(k+1)/n^2} = Y^n_{k/n^2} + 1/n, \end{cases} \tag{1.9}$$

where $\eta_1 \odot \eta_2$ denotes the concatenation of two paths η_1 and η_2 in the obvious way. In words, if the lifetime Y^n goes down by $1/n$ we erase the path of the snake from the tip by $1/n$, or to put it differently, we reduce its lifetime by $1/n$. If Y^n goes up by $1/n$ we add the path η to the tip of the snake. Then we define

$$\mathbb{W}^n_{(k+1)/n^2}(\cdot) = (\mathbf{W}^n_{(k+1)/n^2}(\cdot), Y^n_{(k+1)/n^2}(\cdot)).$$

This way we constructed a sequence of discrete snakes. As is the case for $Y^n_\cdot$, we define $\mathbb{W}^n_s(\cdot) = \mathbb{W}^n_{\lfloor sn^2 \rfloor/n}(\cdot)$. The sequence of processes $\mathbb{W}^n$ converges, as $n \to \infty$, to a continuous time Brownian snake (see e.g. Proposition 2.2 in [8]).

We next describe the connection between the snake process and the branching Brownian motion. Define the discrete version of the local time as

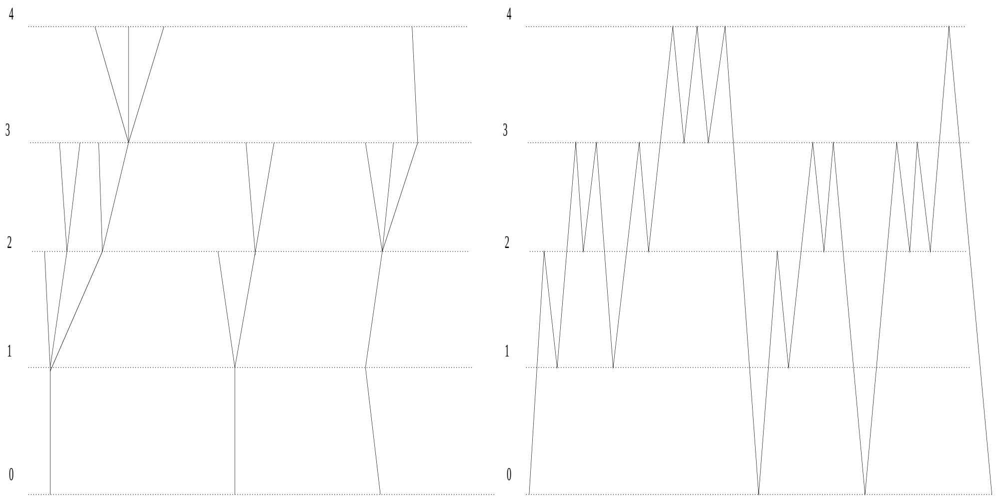

Figure 1: Geneology of the particle system (left) and contour process (right). In this picture, $\tau_1^{1,0} = 23$, $\tau_2^{1,0} = 33$ and $\tau_4^{1,1} = 26$.

the rescaled number of upcrossings of Y^n from the corresponding level:

$$\ell_s^{n,m/n} = n^{-1} \sum_{i=0}^{\lfloor sn^2 \rfloor} 1_{Y^n_{i/n^2}=m/n, Y^n_{(i+1)/n^2}=(m+1)/n}. \tag{1.10}$$

We also define, for $t \geq 0$,

$$\ell_s^{n,t} = \ell_s^{n,\lfloor tn \rfloor/n}. \tag{1.11}$$

Since $s \mapsto \ell_s^{n,t}$ is increasing we define the measure $\ell^{n,t}(ds)$ in an obvious way. In fact this convention will be used throughout the paper: for any non-decreasing function $r \mapsto f_r$ on $\mathbb{R}_+$, $f(dr)$, with a slight abuse of notation, will denote the corresponding measure defined via $f((a,b]) = f_b - f_a$, for any $b \geq a$.

For any $a \geq 0$ introduce the inverse local time at level a as

$$\tau_r^{n,a} = \frac{1}{n^2} \inf\{k : \ell^{n,a}_{k/n^2} > r\}. \tag{1.12}$$

For any $a \geq 0$ and $r_1 < r_2$ define the measure valued process $X_{a,t}^{n,r_1,r_2}$ so that, for $\phi \in \mathcal{B}(\mathbb{R}^d)$

$$X_{a,t}^{n,r_1,r_2}(\phi) \equiv \int_{\tau_{r_1}^{n,a}}^{\tau_{r_2}^{n,a}} \phi(\mathbb{W}_s^n(Y_s^n))\ell^{n,t}(ds)\,,\ t \geq a. \tag{1.13}$$

It is easy to see that $X_{a,k/n}^{n,r_1,r_2}$, $k \geq \lfloor an \rfloor$, is the measure-valued process constructed in the previous section starting at "time" $a_n = \lfloor an \rfloor/n$ such that

$$X_{a,a_n}^{n,r_1,r_2}(1) = X_{a_n,a_n}^{n,r_1,r_2}(1) = r_2 - r_1\,,$$

and therefore

$$X_{a,\cdot}^{n,r_1,r_2} \Rightarrow X_{a,\cdot}^{r_1,r_2}, \tag{1.14}$$

where $X_{a,\cdot}^{r_1,r_2}$ solves the martingale problem starting at time a such that

$$X_{a,a}^{r_1,r_2}(1) = r_2 - r_1$$

and, $\forall\ \phi \in \mathcal{C}_b^2(\mathbb{R}^d)$,

$$M_{a,t}^{\phi} \equiv \left\langle X_{a,t}^{r_1,r_2}, \phi \right\rangle - \left\langle X_{a,a}^{r_1,r_2}, \phi \right\rangle - \frac{1}{2}\int_a^t \left\langle X_{a,s}^{r_1,r_2}, \Delta\phi \right\rangle ds, \qquad t \geq a, \tag{1.15}$$

is a continuous martingale with quadratic variation process

$$\left\langle M_a^{\phi} \right\rangle_t \;=\; 2\int_a^t \left\langle X_{a,s}^{r_1,r_2}, \phi^2 \right\rangle ds, \qquad t \geq a. \tag{1.16}$$

1.3 Our model

We finally define the discrete snake in random environment corresponding to the branching processes in random environment described in Section 1.2. The main difference with the "fixed environment" case is that here the snake cannot be constructed conditionally on the lifetime process. Both processes have to be constructed simultaneously.

The environment $\{\xi_k(\cdot)\}_{k\geq 0} = \{\xi_k^n(\cdot)\}_{k\geq 0}$ is assumed to consist of a sequence of i.i.d. random fields, satisfying $|\xi_k^n(x)| \leq \sqrt{n}/2$ and $\sup_n \mathbb{E}(|\xi_k^n(x)|^2) < \infty$, with mean $\nu/\sqrt{n}$, for some $\nu \in \mathbb{R}$, and covariance $g(x,y)$ as in (1.6), with $\|\bar{g}\|_\infty < \infty$.

Now define the snake with lifetime processes $\mathbb{W}^n = (\mathbf{W}^n, Y^n)$ as follows. Fix a constant $K_1 > 0$. Let $Y_0^n = 0$ and $\mathbb{W}_0^n = \bar{x}$ with $x \in \mathbb{R}^d$. Suppose we are given $(\mathbf{W}_{k/n^2}^n, Y_{k/n^2}^n)$ for some $k \geq 0$. $(\mathbf{W}_{(k+1)/n^2}^n, Y_{(k+1)/n^2}^n)$ will be defined as follows. If $Y_{k/n^2}^n \notin \{0, K_1\}$, then conditionally on ξ and $(\mathbf{W}_{l/n^2}^n, Y_{l/n^2}^n), l \leq k$ we set

$$\mathbb{P}\left(Y_{(k+1)/n^2}^n - Y_{k/n^2}^n = \pm 1/n | \xi, \mathbf{W}_{l/n^2}^n, Y_{l/n^2}^n, l \leq k\right) = \frac{1}{2} \pm \frac{1}{4\sqrt{n}} \xi_{Y_{k/n^2}^n}\left(\hat{\mathbb{W}}_{n^{-2}k}^n\right),$$

where we introduced above the notation for the "tip" of the snake:

$$\hat{\mathbb{W}}_{k/n^2}^n = \mathbf{W}_{k/n^2}^n\left(Y_{k/n^2}^n\right).$$

If $Y_{k/n^2}^n = 0$, then with probability one we set $Y_{(k+1)/n^2}^n = 1/n$. If $Y_{k/n^2}^n = K_1$, then with probability one we set $Y_{(k+1)/n^2}^n = K_1 - 1/n$. (That is, the process is reflected at height K_1; a similar approach of introducing a supercritical branching mechanism via a reflection of the lifetime process was used by J.-F. Delmas in [4].)

Remark 1.1 *With our assumptions, it is easy to see that the m-th moment (for any $m \geq 2$) of the absolute value of the expected (conditioned in the environment) number of offspring minus 1 of a particle at x at time k/n, see (1.5), is bounded by C_m/n, for an appropriate constant C_m. Moreover, the absolute value of the first moment of the number of offspring minus 1 of a particle at x at time i/n, see (1.5), is bounded by C_1/n, for an appropriate constant C_1.*

Let $\eta_1, \eta_2, \ldots$ be a sequence of independent Brownian motions stopped at time $1/n$. Given the evolution of the lifetime process Y^n until time $(k+1)/n^2$, the path of the Brownian snake $\mathbb{W}$ at time $(k+1)/n^2$ is defined exactly as in (1.9).

We next explain the connection between the snake and branching particle system in random environment which is analogous to the connection that exists between the processes in a constant environment. Define the rescaled local time $\ell_s^{n,t}$ for Y^n as in (1.10), (1.11) and the inverse local time as in (1.12). For any $r_1, r_2 > 0, a \geq 0$, we define the measure-valued process in the same way as it is done in (1.13):

$$X_{a,t}^{n,r_1,r_2}(\phi) \equiv \int_{\tau_{r_1}^{n,a}}^{\tau_{r_2}^{n,a}} \phi(\mathbf{W}_s^n(Y_s^n))\ell^{n,t}(ds), \quad t \geq a, \tag{1.17}$$

for all $\phi \in \mathcal{B}(\mathbb{R}^d)$. This process characterizes the branching particle picture in random environment with offspring distribution given by (1.4) and starting with $\lfloor (r_2 - r_1)n \rfloor$ particles at the site $x \in \mathbb{R}^d$ at time $t = a$. In the case of $r_1 = 0, r_2 = r, a = 0$, we will use the notation

$$X_{0,t}^{n,r} \equiv X_{0,t}^{n,r_1,r_2}, \quad t \geq 0, \tag{1.18}$$

for the corresponding process.

The following is our first main result.

Theorem 1.2 *Fix $K_1 > 0$. Then the sequence of processes $\{\mathbb{W}^n\}_{n\geq 1} = \{(\mathbf{W}^n, Y^n)\}_{n\geq 1}$ is C-tight in $D_{\mathcal{W}}$. Let $\mathbb{W} = (\mathbf{W}, Y)$ be an arbitrary limiting point, let ℓ^a be a local time of Y at level a and let $\tau^a(r)$ be the inverse of the local time. Fix an arbitrary $r > 0$. Then*

$$X_t^r(\phi) = \int_0^{\tau_r^0} \phi(\hat{\mathbb{W}}_s)\ell^t(ds), \quad \phi \in \mathcal{B}(\mathbb{R}^d),\ t \in [0, K_1], \tag{1.19}$$

is a measure-valued process satisfying the martingale problem (1.7-1.8) on $[0, K_1]$, with $X_0^r = r\delta_x$.

In the particular case of a spatially "smooth" random environment we can give another description of the snake process. It is easy to check from our assumptions on ξ^n that if we define

$$B_s^n(x) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor sn \rfloor} \xi_i^n(x), \tag{1.20}$$

then

$$B^n \to B,$$

where $\frac{\partial B_t(y)}{\partial t}$ is a Gaussian generalized noise on $\mathbb{R}_+ \times \mathbb{R}^d$, white in time and colored in space, such that

$$\begin{aligned}
\mathbb{E}(B_t(x)) &= t\nu, \quad \forall t \geq 0, \\
\mathrm{Cov}\left(\frac{\partial B_t(x)}{\partial t}, \frac{\partial B_s(y)}{\partial s}\right) &= \delta_0(t-s)g(x,y), \\
B_0 &= 0,
\end{aligned}$$

where $\delta_0(\cdot)$ is the Dirac measure at 0. Given the result on the tightness of $\{\mathbb{W}^n\}_{n\geq 1}$, one can easily deduce that the pair $\{(\mathbb{W}^n, B^n)\}_{n\geq 1}$ is tight. In what follows we assume that $(\mathbb{W}, B)$ is a limit point of the tight sequence $\{(\mathbb{W}^n, B^n)\}_{n\geq 1}$, and we recall that $\mathbb{W} = (\mathbf{W}, Y)$.

Our aim is to introduce a particular functional of the limiting snake that has a simple semimartingale decomposition. The definition of the functional is motivated by the one used by Dhersin and Serlet [5] and also by a functional used to transform Brox's diffusion into a martingale, see [13]. For $w \in \mathcal{W}$, let

$$F(w) = \int_0^{\zeta(w)} e^{-B_r(w(r))} dr.$$

Our second main result is the following.

Theorem 1.3 *Fix $K_1 > 0$. Assume that $B \in C_{\mathcal{C}^2(\mathbb{R}^d)}[0,\infty)$, a.s.. Then there exists a Brownian motion β such that*

$$\begin{aligned}
F(\mathbb{W}_t) &= \int_0^t e^{-B_{Y_s}(\hat{\mathbb{W}}_s)} \left\{ -\frac{1}{2}\Delta B_{Y_s}(\hat{\mathbb{W}}_s) + \frac{1}{2}\sum_{i=1}^d \left(\frac{\partial}{\partial x_i} B_{Y_s}(\hat{\mathbb{W}}_s)\right)^2 \right\} ds \\
&\quad + \ell_t^0 - \int_0^t e^{-B_{K_1}(\hat{\mathbb{W}}_s)}\, \ell^{K_1}(ds) + \int_0^t e^{-B_{Y_s}(\hat{\mathbb{W}}_s)}\, d\beta_s\,. \qquad (1.21)
\end{aligned}$$

Remark 1.4 *The first term on the right side of (1.21) can be written as*

$$\int_0^t \frac{1}{2}\Delta_x e^{-B_{Y_s}(x)}|_{x=\hat{\mathbb{W}}_s}\, ds,$$

and it comes from the fact that $\mathbf{W}_s(\cdot)$ is a Brownian path.

Note that in the case of constant function g, for every s, $B_s(\cdot)$ is a constant function in space, and hence we immediately have the following corollary, which for simplicity we state only in case $\nu = 0$.

Corollary 1.5 *Let $g \equiv 1$ and $\nu = 0$. Then Y is the Brox diffusion reflected at 0 and K_1.*

See the appendix for the definition of the Brox diffusion.

1.4 Structure of the paper

In the next section, we derive some standard estimates on survival probability for branching processes in a random environment. Section 3 is concerned with the proof of tightness of the contour process. (Because of dependence through the environment, natural arguments involving stopping times such as Aldous' tightness criterion cannot be applied directly, and extra care has to be employed in separating dependence on the level of the contour process from dependence on the lifetime of the process.) Most of the work is devoted to proving that large upward jumps of the contour process are unlikely; downward jumps are then handled by a time reversal argument. Section 4 is devoted to the proof of tightness of the snake process and its local time, and a completion of the proof of Theorem 1.2. Section 5 is devoted to the description of the snake provided in Theorem 1.3, while the appendix is devoted to the description of the contour process for environments with no spatial dependence, providing in particular a direct proof of Corollary 1.5, that bypasses the need to consider the Brownian snake.

Notation Throughout, C, K denote generic constants whose values may change from line to line. Numbered constants (such as $K_1, c_0, C_m, \delta_{4.2}$, etc.) are fixed and do not change throughout the paper.

2 Asymptotics for survival probability and useful bounds

We start with a lemma that describes the asymptotics for survival probability for classical branching processes. For any $n \geq 1$ let $\{M_l^n, l = 0, 1, 2, \ldots\}$ be the branching process with geometric offspring distribution with parameter

$$p = 1/2 - b_n/4n$$

for some $b_n \in (-2n, 2n)$. That is if Z^n is the number of offspring in the process M^n, then

$$\mathbb{P}(Z^n = k) = p(1-p)^k, \quad k = 0, 1, 2, \ldots.$$

For $\delta > 0$ define

$$h(b, \delta) = \begin{cases} \frac{b}{1-e^{-b\delta}}, & b \neq 0, \\ \frac{1}{\delta}, & b = 0. \end{cases}$$

Lemma 2.1 *Assume*

$$\lim_{n \to \infty} b_n = b,$$

and $M_0^n = 1$ for all $n \geq 1$. Then for any $\delta > 0$,

$$\lim_{n \to \infty} n\mathbb{P}(M_{[n\delta]}^n > 0) = h(b, \delta).$$

Proof: For $b=0$ the result is well-known (see e.g. [11, Theorem II.1.1] for a more general result). While we believe that the result is also known for $b>0$, we were unable to locate a reference and thus for the sake of completeness we provide a proof.

Let $f(s)$ be the generating function of Z^n, that is

$$f(s)=\frac{1/2-b_n/4n}{1-(1/2+b_n/4n)s},\quad 0\le s\le 1.$$

Define $f_0(s)=s,\ f_1(s)=f(f_0(s))=f(s)$ and in general

$$f_k(s)=f(f_{k-1}(s)),\quad 0\le s.$$

Then by the branching property,

$$E[s^{M_k^n}|M_0^n=1]=f_k(s).$$

Therefore,

$$\begin{aligned}\mathbb{P}(M_k^n=0|M_0^n=1) &= f_k(0)=f(f_{k-1}(0))\\ &= \frac{1/2-b_n/4n}{1-(1/2+b_n/4n)f_{k-1}(0)}.\end{aligned}$$

Fix $k=\lfloor n\delta\rfloor$ and define

$$y_l=k(1-f_l(0)),\quad l=0,1,\ldots,k.$$

One has

$$y_l = \frac{(1/2+b_n/4n)y_{l-1}}{1-(1/2+b_n/4n)(1-y_{l-1}/k)}. \tag{2.1}$$

Let

$$z_l=1/y_l\,,l=0,\ldots,k\,.$$

Then $z_0=1/k$ and

$$z_l=z_{l-1}d_n+1/k,\quad l=1,\ldots,k,$$

with

$$d_n = \frac{1/2-b_n/4n}{1/2+b_n/4n}. \tag{2.2}$$

By iterating we get

$$z_k = \frac{1}{k}\left(d_n^k+\frac{1-d_n^k}{1-d_n}\right). \tag{2.3}$$

Now as $n\to\infty$ we have

$$1-d_n=\frac{b_n}{n(1+\frac{b_n}{2n})}\sim\frac{b_n}{n}.$$

Also recall that $k = \lfloor n\delta \rfloor$ and hence

$$d_n^k = \left(\frac{1/2 - b_n/4n}{1/2 + b_n/4n}\right)^{\lfloor n\delta \rfloor} \sim (1 - b_n/n)^{n\delta} \sim e^{-b_n\delta}.$$

Therefore

$$\lim_{n\to\infty} z_{\lfloor n\delta \rfloor} = \lim_{n\to\infty} \frac{e^{-b_n\delta} + \frac{1-e^{-b_n\delta}}{b_n/n}}{n\delta} = \frac{1-e^{-b\delta}}{b\delta}.$$

Since $n\mathbb{P}(M_{\lfloor n\delta \rfloor]} > 0) \sim \frac{y_{\lfloor n\delta \rfloor}}{\delta} = \frac{1}{\delta z_{\lfloor n\delta \rfloor}}$, this concludes the proof. ∎

Returning to the random environment case, let $\tilde{M}^n$ denote the total mass of the branching Brownian motion in random environment X^n (with geometric offspring distribution) defined in Section 1.1 (that is, $n^{-1}\tilde{M}_k^n = \langle X_{k/n}^n, 1\rangle$) and let M^n be as above with

$$b = \nu + \|\bar{g}\|_\infty/2. \tag{2.4}$$

Lemma 2.2 *Let $\tilde{M}_0^n = 1$ for all $n \geq 1$. Then*

$$\limsup_{n\to\infty} n\mathbb{P}(\tilde{M}_{\lfloor n\delta \rfloor}^n > 0) \leq h(b,\delta).$$

Proof: For $i = 1, 2, \ldots, \tilde{M}_k^n$ we denote by $\mathcal{U}_{i,k}(t), t \in [k/n, (k+1)/n]$, the position at time t of the i-th particle that was born at time k/n. That is we have

$$X_{k/n}^n = \frac{1}{n}\sum_{i=1}^{\tilde{M}_k^n} \delta_{\mathcal{U}_{i,k}(k/n)}\,.$$

Moreover if $Z_{i,k+1}^n$ is the number of offspring at time $(k+1)/n$ of the i-th particle that was born at time k/n, then we also have

$$X_{k/n}^n = \frac{1}{n}\sum_{i=1}^{\tilde{M}_{k-1}^n} Z_{i,k}^n \delta_{\mathcal{U}_{i,k-1}(k/n)}\,.$$

We write for simplicity $\xi_{i,k} = \xi_k(\mathcal{U}_{i,k-1}(k/n))$ and denote by $\mathcal{F}_k^\xi$ the sigma-algebra generated by the environment $\{\xi_j(\cdot), j \leq k\}$. We have, for $s \in \mathbb{R}_+$,

$$\begin{aligned}
\mathbb{E}\left(s^{\tilde{M}_k^n}\right) &= \mathbb{E}\left(\Pi_{i=1}^{\tilde{M}_{k-1}^n} s^{Z_{i,k}^n}\right)\\
&= \mathbb{E}\left(\Pi_{i=1}^{\tilde{M}_{k-1}^n}\mathbb{E}\left(s^{Z_{i,k}^n}\,\middle|\, X_{\frac{k}{n}-}^n, \mathcal{F}_k^\xi\right)\right)\\
&= \mathbb{E}\left(\left(\frac{1}{2-s}\right)^{\tilde{M}_{k-1}^n}\Pi_{i=1}^{\tilde{M}_{k-1}^n}\frac{1-\frac{\xi_{i,k}}{2\sqrt{n}}}{1-\frac{\xi_{i,k}s}{2(2-s)\sqrt{n}}}\right).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathbb{E}\left(s^{\tilde{M}_k^n}\right) &= \mathbb{E}\left(\left(\frac{1}{2-s}\right)^{\tilde{M}_{k-1}^n}\mathbb{E}\left(\Pi_{i=1}^{\tilde{M}_{k-1}^n}\frac{1-\frac{\xi_{i,k}}{2\sqrt{n}}}{1-\frac{\xi_{i,k}s}{2(2-s)\sqrt{n}}}\middle| X^n_{\frac{k}{n}-}\right)\right) \qquad (2.5)\\
&\geq \mathbb{E}\left(\left(\frac{1}{2-s}\right)^{\tilde{M}_{k-1}^n}\exp\left(\mathbb{E}\left(\sum_{i=1}^{\tilde{M}_{k-1}^n}\log\left(\frac{1-\frac{\xi_{i,k}}{2\sqrt{n}}}{1-\frac{\xi_{i,k}s}{2(2-s)\sqrt{n}}}\right)\middle| X^n_{\frac{k}{n}-}\right]\right)\right),
\end{aligned}
$$

where the last inequality follows by Jensen inequality. Since $|\xi_{i,k}| < \sqrt{n}/2$ we get by trivial estimates that for n large enough

$$
\begin{aligned}
\frac{1-\frac{\xi_{i,k}}{2\sqrt{n}}}{1-\frac{\xi_{i,k}s}{2(2-s)\sqrt{n}}} &\geq \left(1-\frac{\xi_{i,k}}{2\sqrt{n}}\right)\left(1+\frac{\xi_{i,k}s}{2(2-s)\sqrt{n}}+\left(\frac{\xi_{i,k}s}{2(2-s)\sqrt{n}}\right)^2+\left(\frac{\xi_{i,k}s}{2(2-s)\sqrt{n}}\right)^3\right)\\
&= 1+\frac{\xi_{i,k}}{2\sqrt{n}}\left(\frac{s}{2-s}-1\right)+\frac{\xi_{i,k}^2 s}{4(2-s)n}\left(\frac{s}{2-s}-1\right)\\
&\quad +\left(\frac{\xi_{i,k}s}{2(2-s)\sqrt{n}}\right)^2\frac{\xi_{i,k}}{2\sqrt{n}}\left(\frac{s}{2-s}-1\right)\\
&\geq 1-\frac{\xi_{i,k}(1-s)}{(2-s)\sqrt{n}}-\frac{\xi_{i,k}^2 s(1-s)}{2(2-s)^2 n}-c_{2.6}(1-s)n^{-3/2}|\xi_{i,k}|^3. \qquad (2.6)
\end{aligned}
$$

Again by trivial estimate on the logarithmic function we get

$$
\begin{aligned}
\log\left(\frac{1-\frac{\xi_{i,k}}{2\sqrt{n}}}{1-\frac{\xi_{i,k}s}{2(2-s)\sqrt{n}}}\right) &\geq \log\left(1-\frac{\xi_{i,k}(1-s)}{(2-s)\sqrt{n}}-\frac{\xi_{i,k}^2 s(1-s)}{2(2-s)^2 n}-c_{2.6}(1-s)n^{-3/2}|\xi_{i,k}|^3\right)\\
&\geq -\frac{\xi_{i,k}(1-s)}{(2-s)\sqrt{n}}-\frac{\xi_{i,k}^2 s(1-s)}{2(2-s)^2 n}\\
&\quad -\frac{\xi_{i,k-1}^2(1-s)^2}{2(2-s)^2 n}-c_{2.7}(1-s)n^{-3/2}|\xi_{i,k}|^3\\
&= -\frac{\xi_{i,k}(1-s)}{(2-s)\sqrt{n}}-\frac{\xi_{i,k}^2(1-s)}{2(2-s)^2 n}-c_{2.7}(1-s)n^{-3/2}|\xi_{i,k}|^3, \qquad (2.7)
\end{aligned}
$$

for all n sufficiently large. Take an expectation to get

$$
\begin{aligned}
&\mathbb{E}\left(-\frac{\xi_{i,k}(1-s)}{(2-s)\sqrt{n}}-\frac{\xi_{i,k}^2(1-s)}{2(2-s)^2 n}-c_{2.7}(1-s)n^{-3/2}|\xi_{i,k}|^3\middle| X^n_{\frac{k}{n}-}\right)\\
&= -\frac{\nu(1-s)}{(2-s)n}-\frac{(\nu^2/n+\bar{g}(x_{i,k-1}(k/n)))(1-s)}{2(2-s)^2 n}-c_{2.8}(1-s)n^{-3/2}\\
&\geq -\frac{(1-s)}{(2-s)n}\left(\nu+\frac{\|\bar{g}\|_\infty}{2}+\nu^2/n+2c_{2.7}n^{-1/2}\right)\\
&\geq -\frac{(1-s)}{(2-s)n}\left(\nu+\frac{\|\bar{g}\|_\infty}{2}+c_{2.8}n^{-1/2}\right). \qquad (2.8)
\end{aligned}
$$

Substituting in (2.6) we get

$$\begin{aligned}
\mathbb{E}\left(s^{\tilde{M}^n_k}\right) &\geq \mathbb{E}\left(\left(\frac{1}{2-s}\right)^{\tilde{M}^n_{k-1}} \exp\left\{-\tilde{M}^n_{k-1}\frac{(1-s)}{(2-s)n}\left(\nu+\frac{\|\bar{g}\|_\infty}{2}+c_{2.8}n^{-1/2}\right)\right\}\right) \\
&\geq \mathbb{E}\left(\left(\frac{1}{2-s}-\frac{(1-s)}{(2-s)^2 n}\left(\nu+\frac{\|\bar{g}\|_\infty}{2}+c_{2.8}n^{-1/2}\right)\right)^{\tilde{M}^n_{k-1}}\right) \\
&=: \mathbb{E}\left(\tilde{f}(s)^{\tilde{M}^n_{k-1}}\right). \qquad (2.9)
\end{aligned}$$

Let $f(s)$ be the generating function of the geometric distribution with parameter $p=\frac{1}{2}-\frac{b_n}{4n}$, then

$$\begin{aligned}
f(s) &= \frac{1/2-b_n/4n}{1-(1/2+b_n/4n)s} \\
&= \frac{1}{2-s}\left(1-\frac{b_n(1-s)}{(2-s)n(1-\frac{bs}{2n(2-s)})}\right) \\
&\leq \frac{1}{2-s}\left(1-\frac{b_n(1-s)}{(2-s)n}\right). \qquad (2.10)
\end{aligned}$$

If one takes $b_n=\nu+\frac{\|\bar{g}\|_\infty}{2}+c_{2.8}n^{-1/2}$ then we get that

$$\tilde{f}(s)\geq f(s),\;\; 0\leq s\leq 1,$$

and hence by iterating (2.9) we get

$$\mathbb{E}\left(s^{\tilde{M}^n_k}\right) \;\geq\; \mathbb{E}\left(s^{M^n_k}\right),\;\; 0\leq s\leq 1.$$

Therefore

$$\mathbb{P}(\tilde{M}^n_k>0)\leq \mathbb{P}(M^n_k>0),\;\; \forall k\geq 1, \qquad (2.11)$$

and hence by Lemma 2.1 we get that

$$\limsup_{n\to\infty} n\mathbb{P}(\tilde{M}^n_{\lfloor n\delta\rfloor}>0)\leq \limsup_{n\to\infty} n\mathbb{P}(M^n_{\lfloor n\delta\rfloor}>0)\leq h(b,\delta). \qquad (2.12)$$

■

Lemma 2.3 *Let $\tilde{M}^n, X^n$ be as above.*

(a) *For any $\delta>0$,*

$$\limsup_{n\to\infty}\mathbb{E}\left(\tilde{M}^n_{\lfloor n\delta\rfloor}\right)\leq \limsup_{n\to\infty}\tilde{M}^n_0 e^{b\delta},$$

and hence,

$$\limsup_{n\to\infty}\mathbb{E}\left(X^n_{\lfloor n\delta\rfloor/n}(1)\right)\leq \limsup_{n\to\infty} X^n_0(1)e^{b\delta}.$$

(b) *For any $\delta, a>0$,*

$$\limsup_{n\to\infty}\mathbb{P}\left(\sup_{k\leq\lfloor n\delta\rfloor} X^n_{k/n}(1)\geq a\right)\leq \frac{\limsup_{n\to\infty}X^n_0(1)\left(e^{b\delta}\vee 1\right)}{a}.$$

Proof: **(a)** The proof of goes along the similar lines as the proof of the previous lemma. First recall that

$$\mathbb{E}\left(Z^n_{i,k}|X^n_{(k-1)/n}\right) \quad \leq \quad 1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}, \tag{2.13}$$

for all i,k,n. Hence, by iteration, we get

$$\mathbb{E}\left(\tilde{M}^n_{\lfloor n\delta\rfloor}\right) \quad \leq \quad \tilde{M}^n_0\left(1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}\right)^{\lfloor n\delta\rfloor}, \tag{2.14}$$

and the result follows.

(b) For all $k\geq 0$, define

$$V^n_k = X^n_{k/n}(1)\left(1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}\right)^{-k}.$$

Then using (2.13) it is easy to check that $\{V^n_k\}_{k\geq 0}$ is a nonnegative $\{\mathcal{F}^{X^n}_k\}_{k\geq 0}$-supermartingale. Therefore by maximal inequalities for non-negative supermartingales we get

$$\mathbb{P}\left(\sup_{k\leq\lfloor n\delta\rfloor} V^n_k\geq a\right) \quad \leq \quad \frac{\mathbb{E}\left(V^n_0\right)}{a}=\frac{X^n_0(1)}{a}. \tag{2.15}$$

To prove the result we consider the cases $\nu+\frac{\|\bar{g}\|_\infty}{2}<0$ and $\nu+\frac{\|\bar{g}\|_\infty}{2}\geq 0$ separately. First suppose that $\nu+\frac{\|\bar{g}\|_\infty}{2}<0$. Recall the definition of V^n_k to get that, in this case,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{k\leq\lfloor n\delta\rfloor} V^n_k\geq a\right) \qquad\qquad (2.16)\\
&= \mathbb{P}\left(\sup_{k\leq\lfloor n\delta\rfloor} X^n_{k/n}(1)\left(1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}\right)^{-k}\geq a\right)\\
&\geq \mathbb{P}\left(\sup_{k\leq\lfloor n\delta\rfloor} X^n_{k/n}(1)\geq a\right).
\end{aligned}$$

By putting (2.15), (2.16) together we get that

$$\mathbb{P}\left(\sup_{k\leq\lfloor n\delta\rfloor} X^n_{k/n}(1)\geq a\right) \quad \leq \quad \frac{X^n_0(1)}{a},\ \text{ for } \nu+\frac{\|\bar{g}\|_\infty}{2}<0. \tag{2.17}$$

Now let $\nu+\frac{\|\bar{g}\|_\infty}{2}\geq 0$. Then we get

$$\begin{aligned}
&\mathbb{P}\left(\sup_{k\leq\lfloor n\delta\rfloor} V^n_k\geq a\left(1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}\right)^{-\lfloor n\delta\rfloor}\right) \qquad\qquad (2.18)\\
&= \mathbb{P}\left(\sup_{k\leq\lfloor n\delta\rfloor} X^n_{k/n}(1)\left(1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}\right)^{-k}\geq a\left(1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}\right)^{-\lfloor n\delta\rfloor}\right)\\
&\geq \mathbb{P}\left(\sup_{k\leq\lfloor n\delta\rfloor} X^n_{k/n}(1)\geq a\right)
\end{aligned}$$

Apply this and (2.13) with $a\left(1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}\right)^{-n}$ instead of a to get

$$\begin{aligned}&\mathbb{P}\left(\sup_{k\le\lfloor n\delta\rfloor} X^n_{k/n}(1)\ge a\right) \qquad (2.19)\\ &\le \frac{X^n_0(1)\left(1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}\right)^{\lfloor n\delta\rfloor}}{a}, \quad \text{for } \nu+\frac{\|\bar{g}\|_\infty}{2}\ge 0.\end{aligned}$$

By combining (2.17), (2.19), we get

$$\mathbb{P}\left(\sup_{k\le\lfloor n\delta\rfloor} X^n_{k/n}(1)\ge a\right) \le \frac{X^n_0(1)\left(\left(1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}\right)^{\lfloor n\delta\rfloor}\vee 1\right)}{a}, \qquad (2.20)$$

and by letting $n\to\infty$ we are done. ∎

The next result generalizes the previous lemma.

Lemma 2.4 *Let f be a bounded non-negative measurable function on $\mathbb{R}^d$. Then, for any $\delta>0$,*

$$\mathbb{E}\left(X^n_{\lfloor n\delta\rfloor/n}(f)\right)\le\left(1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}\right)^{\lfloor n\delta\rfloor} X^n_0\left(S_\delta f\right),$$

where $\{S_t\}_{t\ge 0}$ is the semigroup of the Brownin motion.

Proof: The proof goes along the similar lines as the proof of the previous lemma. For any $k\ge 1$ we have

$$\begin{aligned}\mathbb{E}\left(X^n_{k/n}(f)|X^n_{(k-1)/n}\right) &= \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{\tilde{M}^n_{k-1}} Z^n_{i,k} f(\mathcal{U}_{i,k-1}(k/n))|X^n_{(k-1)/n}\right)\\ &= \frac{1}{n}\sum_{i=1}^{\tilde{M}^n_{k-1}}\mathbb{E}\left(\mathbb{E}\left(Z^n_{i,k}f(\mathcal{U}_{i,k-1}(k/n))\middle|\, X^n_{\frac{k}{n}-}\right)\middle|\, X^n_{(k-1)/n}\right)\\ &= \frac{1}{n}\sum_{i=1}^{\tilde{M}^n_{k-1}}\mathbb{E}\left(f(\mathcal{U}_{i,k-1}(k/n))\mathbb{E}\left(Z^n_{i,k}\middle|\, X^n_{\frac{k}{n}-}\right)\middle|\, X^n_{(k-1)/n}\right)\\ &\le \left(1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}\right)\frac{1}{n}\sum_{i=1}^{\tilde{M}^n_{k-1}}\mathbb{E}\left(f(\mathcal{U}_{i,k-1}(k/n))\middle|\, X^n_{(k-1)/n}\right)\\ &= \left(1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}\right)\frac{1}{n}\sum_{i=1}^{\tilde{M}^n_{k-1}} S_{1/n}f(\mathcal{U}_{i,k-1}((k-1)/n))\\ &= \left(1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}\right)X^n_{(k-1)/n}\left(S_{1/n}f\right),\end{aligned}$$

for all k, n and hence by iteration

$$\mathbb{E}\left(X^n_{\lfloor n\delta\rfloor/n}(f)\right) \quad \leq \quad \left(1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}\right)^{\lfloor n\delta\rfloor} X^n_0\left(S_{\lfloor n\delta\rfloor/n}f\right), \quad (2.21)$$

and the result follows. ∎

3 Tightness of the contour process

In this section we will prove the tightness of the sequence of the contour processes $\{Y^n\}_{n\geq 1}$. The following proposition is the main result of this section.

Proposition 3.1 (Tightness of $\{Y^n\}_{n\geq 1}$) *For any $\delta > 0$, $T > 0$,*

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{0\leq t\leq T}\sup_{0\leq s\leq\epsilon}|Y^n_{t+s}-Y^n_t|>\delta\right)=0, \qquad (3.1)$$

that is, $\{Y^n\}_{n\geq 1}$ is C-tight in $D_{\mathbb{R}}[0,\infty)$.

The proof of the proposition will be given in this section. Recall the definition of the discrete version of the local time for Y^n and its inverse (see (1.10), (1.11) and (1.12) for the same definitions in the case without environment). Fix an arbitrary $c_0 > 0$. We will first handle the tightness on the time interval

$$t \in [0, \tau^{n,0}_{c_0}],$$

and we start with the following proposition.

Proposition 3.2 (Tightness of $\{Y^n\}_{n\geq 1}$ — no jumps up) *For any $\delta >$ 0,*

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{0\leq t\leq \tau^{n,0}_{c_0}}\sup_{0\leq s\leq\epsilon}(Y^n_{t+s}-Y^n_t)_+>\delta\right)=0. \qquad (3.2)$$

The proof of the proposition will be given after we present several preliminary lemmas. For any $a \geq 0$ and $r_1 < r_2$, recall the measure-valued process $X^{n,r_1,r_2}_{a_n,a_n+t}$, see (1.17). Fix an arbitrary $\delta > 0$. Recall that b was defined in (2.4). We have the following lemma.

Lemma 3.3 *For any $\delta > 0, r > 0, \epsilon' > 0$,*

$$\limsup_{n\to\infty}\mathbb{P}\left(X^{n,r,r+\epsilon'}_{a,a_n+\delta}(1)>0\right) \quad \leq \quad \epsilon' h(b,\delta). \qquad (3.3)$$

Proof:

$$\mathbb{P}\left(X^{n,r,r+\epsilon'}_{a,a_n+\delta}(1)>0\right) \quad\leq\quad \sum_{i=0}^{\lfloor n\epsilon'\rfloor}\mathbb{P}\left(X^{n,r+i/n,r+(i+1)/n}_{a,a_n+\delta}(1)>0\right). \qquad (3.4)$$

Since, by Lemma 2.2,

$$\limsup_{n\to\infty} n\mathbb{P}\left(X^{n,r+i/n,r+(i+1)/n}_{a,a_n+\delta}(1)>0\right)\leq h(b,\delta),$$

the result follows. ∎

The following corollary is immediate.

Corollary 3.4 *For any $\delta>0, r>0, \epsilon'>0$,*

$$\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{\tau^{n,a}_r\leq t\leq\tau^{n,a}_{r+\epsilon'}}(Y^n_t-Y^n_{\tau^{n,a}_r})_+>\delta\Big) \quad\leq\quad \epsilon' h(b,\delta). \qquad (3.5)$$

The next corollary gives a bound on the positive increment of Y^n.

Corollary 3.5 *For any $\delta>0, r>0$,*

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{\tau^{n,a}_r\leq t\leq\tau^{n,a}_r+\epsilon}(Y^n_t-Y^n_{\tau^{n,a}_r})_+>\delta\Big) \quad=\quad 0. \qquad (3.6)$$

Proof: We first prove that for any $\epsilon'>0$,

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\left(\tau^{n,a}_{r+\epsilon'}\leq\tau^{n,a}_r+\epsilon\right)=0. \qquad (3.7)$$

Suppose that there exist deterministic $\epsilon'>0, \delta''\in(0,1/2)$ and subsequences $n_k\to\infty, \epsilon_k\downarrow 0$ such that

$$\lim_{k\to\infty}\mathbb{P}\left(\tau^{n_k,a}_{r+\epsilon'}\leq\tau^{n_k,a}_r+\epsilon_k\right)\geq\delta''. \qquad (3.8)$$

To avoid cumbersome notation, for the rest of the proof we write n and ϵ_n for n_k and ϵ_k respectively. Note that it follows from (3.7) that

$$\ell^{n,a_n}_{\tau^{n,a}_{r+\epsilon'}}-\ell^{n,a_n}_{\tau^{n,a}_r}=\epsilon'. \qquad (3.9)$$

Then as in Lemma 3.3, we may define the sequence of measure-valued processes X^n with total mass

$$X^n_{a,a_n+s}(1) \quad=\quad \ell^{n,a_n+s}_{\tau^{n,a}_{r+\epsilon'}}-\ell^{n,a_n+s}_{\tau^{n,a}_r}. \qquad (3.10)$$

This process starts at the total mass

$$X^n_{a,a_n}(1)=\epsilon',$$

and, appealing to [10], as $n\to\infty$, it converges weakly in $D_{\mathbb{R}}[0,\infty)$ to the continuous process $s\mapsto X_{a,a+s}(1)$ starting at ϵ'. Therefore, by the weak convergence properties, there exists $\bar\delta>0$ such that

$$\liminf_{n\to\infty}\mathbb{P}\left(\inf_{s\le\bar\delta}\left(\ell^{n,a_n+s}_{\tau^{n,a}_{r+\epsilon'}}-\ell^{n,a_n+s}_{\tau^{n,a}_{r}}\right)\ge\epsilon'/2\right)\ge 1-\delta''. \tag{3.11}$$

Note that on the event in (3.11), we have

$$\frac{\epsilon'}{2}\le\ell^{n,a_n+s}_{\tau^{n,a}_{r+\epsilon'}}-\ell^{n,a_n+s}_{\tau^{n,a}_{r}}\le n^{-1}\sum_{l:\,\tau^{n,a}_r<l/n^2\le\tau^{n,a}_{r+\epsilon'}}1_{Y^n_{l/n^2}=a_n+s}.$$

Summing over $s=\frac{i}{n}\le\bar\delta$, we get that with probability greater than $1-2\delta''>0$, the occupation time of Y^n on the time interval $(\tau^{n,a}_r,\tau^{n,a}_{r+\epsilon'}]$ is bounded from below by

$$\frac{1}{n^2}\sum_{l:\,\tau^{n,a}_r<l/n^2\le\tau^{n,a}_{r+\epsilon'}}1_{Y^n_{l/n^2}\ge a_n}\ge\frac{1}{2}\bar\delta\epsilon'>0. \tag{3.12}$$

On the other hand the total occupation time of Y^n on the interval

$$[\tau^{n,a}_r,\tau^{n,a}_{r+\epsilon'}]\subset[\tau^{n,a}_r,\tau^{n,a}_r+\epsilon_n]$$

is bounded by $\epsilon_n\downarrow 0$, which contradicts (3.12). Hence (3.7) follows.

Continuing with the proof of the lemma, we have from (3.7) that for any $\epsilon'>0$,

$$\begin{aligned}&\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{\tau^{n,a}_r\le t\le\tau^{n,a}_r+\epsilon}(Y^n_t-Y^n_{\tau^{n,a}_r})_+\ge\delta\right) &(3.13)\\ &\quad\le\ \lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{\tau^{n,a}_r\le t\le\tau^{n,a}_{r+\epsilon'}}(Y^n_t-Y^n_{\tau^{n,a}_r})_+\ge\delta\right)\\ &\qquad+\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\left(\tau^{n,a}_{r+\epsilon'}\le\tau^{n,a}_r+\epsilon\right)\le\epsilon' h(b,\delta),\end{aligned}$$

where the last inequality follows by (3.7) and Corollary 3.4. Since ϵ' was arbitrary we are done. ∎

We now introduce further notation. Let

$$\bar\ell^{n,a}\ \equiv\ \ell^{n,a}_{\tau^{n,0}_{c_0}}. \tag{3.14}$$

We will prove the following lemma.

Lemma 3.6 *For any $\delta>0$,*

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{0\le r\le\bar\ell^{n,a}}\sup_{0\le s\le\epsilon}(Y^n_{\tau^{n,a}_r+s}-Y^n_{\tau^{n,a}_r})_+>\delta\right)=0. \tag{3.15}$$

Proof: Now we will need some further notation. Denote

$$\begin{aligned} N_s^{a,n} &= \{\text{number of excursions of } Y^n \text{ starting at level } a_n \\ &\quad \text{above the level } a_n+s \text{ on the time interval } [0,\tau_{c_0}^{n,0}]\} \\ &= \{\text{number of particles in the original branching particle system} \\ &\quad \text{at time } a_n n \text{ whose descendants survive till time } (a_n+s)n\} \end{aligned}$$

By Lemmas 2.2, 2.3 and the Markov property of the branching system we immediately get

$$\begin{aligned} \limsup_{n\to\infty} \mathbb{E}\left(N_s^{a,n}\right) &\le \limsup_{n\to\infty} \mathbb{E}\left(X_a^n(1)\right) n\mathbb{P}(\tilde{M}_{\lfloor ns\rfloor}^n > 0|\tilde{M}_0^n = 1) \\ &\le c_0 e^{ba} h(b,s). \end{aligned} \tag{3.16}$$

For $i=1,\dots N_{\delta/2}^{a,n}$ define

$$\sigma_i^n = \inf\{t > \hat{\tau}_{i-1}^n :\ Y_t^n \ge a_n + \delta/2\},$$

where

$$\begin{aligned} \hat{\tau}_0^n &= 0, \\ \hat{\tau}_i^n &= \inf\{t > \sigma_i^n :\ Y_t^n = a_n\}. \end{aligned}$$

That is, σ_i^n are the times when successful excursions of Y^n reach the level $a_n+\delta/2$. Then we have, for any fixed integer m,

$$\begin{aligned} &\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty} \mathbb{P}\left(\sup_{0\le r\le \bar{\ell}^{n,a}} \sup_{0\le s\le\epsilon} (Y^n_{\tau_r^{n,a}+s} - Y^n_{\tau_r^{n,a}})_+ > \delta\right) \qquad (3.17) \\ &\le \lim_{\epsilon\downarrow 0}\limsup_{n\to\infty} \mathbb{E}\left(\sum_{i=1}^{N_{\delta/2}^{a,n}} \mathbb{P}\left(\sup_{0\le s\le\epsilon} (Y^n_{\sigma_i^n+s} - Y^n_{\sigma_i^n})_+ > \delta/2 | X^n_{a_n+\delta/2}\right); N_{\delta/2}^{a,n}\le m\right) \\ &\quad + \lim_{\epsilon\downarrow 0}\limsup_{n\to\infty} \mathbb{P}\left(N_{\delta/2}^{a,n} > m\right). \end{aligned}$$

By an argument similar to the one in Corollary 3.5 we get that

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty} \mathbb{P}\left(\sup_{0\le s\le\epsilon} (Y^n_{\sigma_i^n+s} - Y^n_{\sigma_i^n})_+ > \delta/2 | X^n_{a_n+\delta/2}\right) = 0. \tag{3.18}$$

This implies that

$$\begin{aligned} &\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty} \mathbb{P}\left(\sup_{0\le r\le \bar{\ell}^{n,a}} \sup_{0\le s\le\epsilon} (Y^n_{\tau_r^{n,a}+s} - Y^n_{\tau_r^{n,a}})_+ > \delta\right) \\ &\qquad\qquad \le \lim_{\epsilon\downarrow 0}\limsup_{n\to\infty} \mathbb{P}\left(N_{\delta/2}^{a,n} > m\right) \le \frac{c_0 e^{ba} h(b,\delta/2)}{m}, \end{aligned}$$

where the last inequality follows by the Markov inequality and (3.16). Since m was arbitrary we are done. ∎

We can now complete the proof of Proposition 3.2.

Proof of Proposition 3.2: For $\delta > 0$ let

$$\mathcal{T}_n^{i,\delta} \quad = \quad \{t \le \tau_{c_0}^{n,0} : \; Y_t^n \in [i\delta/2, (i+1)\delta/2]\}. \tag{3.19}$$

Then

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0 \le t \le \tau_{c_0}^{n,0}} \sup_{0 \le s \le \epsilon} (Y_{t+s}^n - Y_t^n)_+ > \delta \right) \\
&\le \sum_{i=0}^{\lfloor 2K_1/\delta \rfloor} \mathbb{P}\left(\sup_{t \in \mathcal{T}_n^{i,\delta}} \sup_{0 \le s \le \epsilon} (Y_{t+s}^n - Y_t^n)_+ > \delta \right) \\
&\le \sum_{i=1}^{\lfloor 2K_1/\delta \rfloor + 1} \mathbb{P}\left(\sup_{r \le \bar{\ell}^{n,i\delta/2}} \sup_{0 \le s \le \epsilon} (Y^n_{\tau_r^{n,i\delta/2}+s} - Y^n_{\tau_r^{n,i\delta/2}})_+ > \delta/2 \right). \qquad (3.20)
\end{aligned}$$

However by Lemma 3.6 we get that, for every i,

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \mathbb{P}\left(\sup_{r \le \bar{\ell}^{n,i\delta/2}} \sup_{0 \le s \le \epsilon} (Y^n_{\tau_r^{n,i\delta/2}+s} - Y^n_{\tau_r^{n,i\delta/2}})_+ > \delta/2 \right) = 0,$$

and this finishes the proof of Proposition 3.2. ∎

To handle downward jumps, we need the following proposition.

Proposition 3.7 (Tightness of $\{Y^n\}_{n \ge 1}$ — no jumps down) *For any $\delta > 0$,*

$$\lim_{\epsilon \downarrow 0} \limsup_{n \to \infty} \mathbb{P}\left(\sup_{0 \le t \le \tau_{c_0}^{n,0}} \sup_{0 \le s \le \epsilon} (Y_{t+s}^n - Y_t^n)_- > \delta \right) = 0. \tag{3.21}$$

Proof: In fact the proof is easy if one considers the process Y^n reversed in time, that is the process $\tilde{Y}_t^n = Y^n_{\tau_{c_0}^{n,0} - t}$, which is easily seen (see the explicit argument below) to possess the same law (with $0 \le t \le \tau_{c_0}^{n,0}$) as the original process $Y_\cdot^n$. Since any jump down for Y^n becomes a jump up for $\tilde{Y}^n$, the claim (3.21) follows from Proposition 3.2 applied to $\tilde{Y}^n$.

To see the reversibility claim, we introduce a sequence of path transformations $\{T_z\}_{z=0,1/n,\ldots,K_1-1/n}$ on $\{(\mathbf{W}_t^n, Y_t^n)\}_{t=0,1/n^2,\ldots,\tau_{c_0}^{n,0}}$, each of which is measure preserving and preserves $\tau_{c_0}^{n,0}$, such that

$$\{(\tilde{\mathbf{W}}_t^n, \tilde{Y}_t^n)\}_{t=0,1/n^2,\ldots,\tau_{c_0}^{n,0}} = T_{K_1-1/n} \circ T_{K_1-2/n} \circ \ldots \circ T_0 \{(\mathbf{W}_t^n, Y_t^n)\}_{t=0,1/n^2,\ldots,\tau_{c_0}^{n,0}},$$

where $(\tilde{\mathbf{W}}_\cdot^n, \tilde{Y}_\cdot^n)$ denotes the image of $(\mathbf{W}_\cdot^n, Y_\cdot^n)$ under the transformations. This will prove the claim.

To avoid cumbersome notation, we consider the case of $n = 1$ only, and we omit the index n. The general n can be treated the same way with proper scaling. For $z = 0$, the transformation T_0 is obtained as follows. If $t = \tau_j^0$ for some integer $j \in \{0, 1, \ldots, c_0\}$, that is t is a return time of $Y_\cdot$ to 0, then define $t'(t) = \tau_{c_0}^0 - \tau_j^0$. If $t \in (\tau_{j-1}^0, \tau_j^0)$ for some $j \in \{1, \ldots, c_0\}$, that is t belongs to the jth excursion of $Y_\cdot$ from 0, then define $t'(t) = t'(\tau_j^0) + t - \tau_{j-1}^0$. Then,

$$T_0(\mathbf{W}_\cdot, Y_\cdot)(t) = (\mathbf{W}_{t'(t)}, Y_{t'(t)}).$$

(In words, T_0 reverses the order of the excursions from 0 but keeps the time orientation of each excursion intact; Thus, the total length of the excursions is preserved.) It is straightforward to check that the law of $T_0(\mathbf{W}_\cdot, Y_\cdot)$ is the same as that of $(\mathbf{W}_\cdot, Y_\cdot)$.

For $z = 1$, let

$$t_z(1) = \min\{k > 0 : Y_k = z\}, s_z(1) = \min\{k > t_z(1) : Y_k = z, Y_{k+1} = z - 1\},$$

and for $j \geq 1$,

$$\begin{aligned} t_z(j+1) &= \min\{k > s_z(j) : Y_k = z\}, \\ s_z(j+1) &= \min\{k > t_z(j+1) : Y_k = z, Y_{k+1} = z - 1\}. \end{aligned}$$

T_1 is then defined as T_0 applied to the excursions of the path *from level* z. Explicitly, let $j_z = \max\{j : t_z(j) < \tau_{c_0}^0\}$. For $t \in \left(\cup_{j=1}^{j_z} [t_z(j), s_z(j)]\right)^c$, set $t'(t) = t$. For each j, let $\bar{t}_z(j, 0) = t_z(j)$, $\bar{t}_z(j, \ell) = \min\{t > \bar{t}_z(j, \ell) : Y_t = z\}$, and $\ell_z(j) = \max\{\ell : \bar{t}_z(j, \ell) = s_z(j)\}$. Let t' be defined on the interval $[t_z(j), s_z(j))$ in the same way as the case of $z = 0$ with τ_j^0, $j = 0, 1, \cdots, c_0$ replaced by $\bar{t}_z(j, \ell)$, $\ell = 0, 1, \cdots, \ell_z(j)$. Then,

$$T_z(\mathbf{W}_\cdot, Y_\cdot)(t) = (\mathbf{W}_{t'(t)}, Y_{t'(t)}).$$

Again, in words, T_z reverses the order of the excursions from z but keeps the time orientation of each excursion intact; Thus, the total length of the excursions is preserved.) It is straightforward to check that the law of $T_z(\mathbf{W}_\cdot, Y_\cdot)$ is the same as that of $(\mathbf{W}_\cdot, Y_\cdot)$. We can continue this procedure for $z = 2, 3, \cdots, K_1 - 1$. As explained above, this completes the proof. ∎

To finish the proof of the Proposition 3.1 we need the folowing lemmas that describe the limiting behavior of $\{\bar{\ell}^{n,\cdot}\}_{n\geq 1}$ and $\{\tau^{n,0}\}_{n\geq 1}$ (recall that $\bar{\ell}^{n,\cdot} = \ell^{n,\cdot}_{\tau^{n,0}_{c_0}}$ was introduced in (3.14)).

Lemma 3.8 *For any $c_0 > 0$, the sequence of processes $\{\bar{\ell}^{n,\cdot}\}_{n\geq 1}$ is C-tight in $D_{\mathbb{R}}$.*

Proof: First recall from (3.14) and (1.18), that $X_{0,a}^{n,c_0}(1) = \bar{\ell}^{n,a}$ is the total mass at time a of the measure-valued process $X_{0,\cdot}^{n,c_0}$ defined in the introduction. Since the sequence of measure-valued processes $\{X_{0,\cdot}^{n,c_0}\}_{n\geq 1}$ is

C-tight in $D_{\mathcal{M}_F}$ (see [10] and the comments leading to (1.15)), we get the desired result. ∎

The next lemma studies the limiting behavior of $\{\tau^{n,0}\}_{n\geq 1}$. Toward this end, recall that according to our conventions introduced after (1.11), we use the same notation for an increasing function and the corresponding measure.

Lemma 3.9 **(a)** *For any $r>0$, the sequence of random variables $\{\tau_r^{n,0}\}_{n\geq 1}$ is tight and any limit point τ_r^0 satisfies*

$$\mathbb{P}(\tau_r^0 = 0) = 0.$$

(b) *For any $\epsilon > 0$, $A > 0$, there exists $R > 0$, such that*

$$\liminf_{n\to\infty} \mathbb{P}\left((\tau_R^{n,0} > A \right) \geq 1 - \epsilon.$$

(c) *The sequence $\{\tau^{n,0}\}_{n\geq 1}$ is tight in $\mathcal{M}(\mathbb{R}_+)$.*

(d) *Let $\tau^0 \in \mathcal{M}(\mathbb{R}_+)$ be an arbitrary limit point of $\{\tau^{n,0}\}_{n\geq 1}$. Then for any fixed $r \in \mathbb{R}_+$, τ_t^0 is continuous at $t = r$ with probability 1.*

Proof: **(a)** Define

$$T_t^n(y) = \int_0^y \ell_t^{n,z} dz = n^{-2} \sum_{i=0}^{\lfloor n^2 t \rfloor} \mathbf{1}_{Y^n_{n^{-2}i} \leq y}.$$

Note that

$$\tau_r^{n,0} = T^n_{\tau_r^{n,0}}(K_1). \tag{3.22}$$

On the other hand

$$T^n_{\tau_r^{n,0}}(K_1) = \int_0^{K_1} \ell^{n,z}_{\tau_r^{n,0}} dz \leq K_1 \sup_{s\leq K_1} \ell^{n,s}_{\tau_r^{n,0}},$$

and since by Lemma 3.8, $\{\ell^{n,\cdot}_{\tau_r^{n,0}}\}_{n\geq 1}$ is tight, by (3.22) we get the tightness of $\{\tau_r^{n,0}\}_{n\geq 1}$.

Similarly, since $\{\ell^{n,\cdot}_{\tau_r^{n,0}}\}_{n\geq 1}$ is C-tight for any $\epsilon > 0$ we can fix δ such that

$$\mathbb{P}(\inf_{s\leq\delta} \ell^{n,s}_{\tau_r^{n,0}} \geq c_0/2) \geq 1 - \epsilon$$

for all n sufficiently large. Using this, (3.22) and definition of T^n we get

$$\tau_r^{n,0} = T^n_{\tau_r^{n,0}}(K_1) \geq \int_0^{\delta} \ell^{n,s}_{\tau_r^{n,0}} ds \geq \frac{r}{2}\delta$$

with probability at least $1-\epsilon$ for all n sufficiently large. Since ϵ was arbitrary we get that any limit point of $\tau_r^{n,0}$ is greater than 0 with probability 1.
(b) For any $K>0$ we can represent

$$\tau_{Kr}^{n,0} = \sum_{i=1}^{K} \tau_{i,r}^{n,0},$$

where, for each i, $\tau_{i,r}^{n,0}$ is distributed as $\tau_r^{n,0}$. Fix arbitrary $\epsilon, A>0$. Since, by part **(a)** of the lemma, any limit point of $\tau_{i,r}^{n,0}$ is strictly greater than 0 with probability one, we can easily choose K sufficiently large such that $\tau_{Kr}^{n,0} = \sum_{i=1}^{K} \tau_{i,r}^{n,0} > A$ with probability at least $1-\epsilon$, for all n sufficiently large.
(c) Immediate from **(a)**.
(d) Let $\tau^0 \in \mathcal{M}(\mathbb{R}_+)$ be a limiting point $\{\tau^{n,0}\}_{n\geq 1}$. To prove this part of the lemma we have to show that, for any $\epsilon>0$, there exists $\delta>0$, such that

$$\mathbb{P}\left(\tau_{r+\delta}^{0} - \tau_{r-\delta}^{0} > \epsilon\right) \leq \epsilon. \tag{3.23}$$

Similarly to what we have done in **(a)** define,

$$T_{s,t}^{n,r}(y) \;=\; \int_0^y (\ell_t^{n,z} - \ell_s^{n,z})dz, \;\; 0\leq s\leq t. \tag{3.24}$$

Then we have

$$\begin{aligned}
\tau_{r+\delta}^{n,0} - \tau_{r-\delta}^{n,0} &= T^{n,r}_{\tau_{r-\delta}^{n,0},\tau_{r+\delta}^{n,0}}(K_1) = \int_0^{K_1} (\ell^{n,z}_{\tau_{r+\delta}^{n,0}} - \ell^{n,z}_{\tau_{r-\delta}^{n,0}})dz \\
&= \int_0^{K_1} X_s^{n,r-\delta,r+\delta}(1)ds \leq K_1 \sup_{s\leq K_1} X_{0,s}^{n,r-\delta,r+\delta}(1),
\end{aligned} \tag{3.25}$$

where recall that $X_{0,s}^{n,r-\delta,r+\delta}$ is the measure-valued process corresponding to the branching particle system in random environment, constructed in Section 1 (see (1.17)), that starts at time $s=0$ with initial mass 2δ. By Lemma 2.3(b)

$$\begin{aligned}
\mathbb{P}\left(\sup_{s\leq K_1} X_s^{n,r-\delta,r+\delta}(1) > \epsilon\right) &\leq \frac{2X_0^{n,r-\delta,r+\delta}(1)(e^{bK_1}\vee 1)}{\epsilon} \\
&= \frac{4\delta(e^{bK_1}\vee 1)}{\epsilon},
\end{aligned} \tag{3.26}$$

for all n sufficiently large. We can take δ sufficiently small such that the right hand side of (3.26) is less than $\epsilon/2$, and this together with (3.25) implies that

$$\mathbb{P}\left(\tau_{r+\delta}^{n,0} - \tau_{r-\delta}^{n,0} > \epsilon\right) \leq \epsilon/2. \tag{3.27}$$

for all n sufficiently large. Therefore (3.23) follows for any limit point of $\{\tau^{n,0}\}_{n\geq 1}$. ∎

Now we are ready to complete the proof of Proposition 3.1.

Proof of Proposition 3.1: Proposition 3.1 follows immediately from Propositions 3.2, 3.7, Lemma 3.9**(b)**, and the fact that c_0 was arbitrary. ∎

4 Tightness of $\{(\mathbb{W}^n, \ell^n)\}_{n\geq 1}$ and proof of Theorem 1.2

The bulk of this section is devoted to the proof of the following proposition.

Proposition 4.1 *The sequence $\{(\mathbb{W}^n, \ell^n, \tau^{n,0})\}_{n\geq 1}$ is tight in $D_{\mathcal{W}\times\mathcal{M}(\mathbb{R}_+)} \times \mathcal{M}(\mathbb{R}_+)$. Let $(\mathbb{W}, \ell, \tau^0)$ be its arbitrary limiting point. Then $(\mathbb{W}, \ell, \tau^0)$ belongs to $C_{\mathcal{W}\times\mathcal{M}(\mathbb{R}_+)} \times \mathcal{M}(\mathbb{R}_+)$. Moreover, ℓ is the local time of Y (Y is the lifetime of $\mathbb{W}$), that is,*

$$\int_0^t \mathbf{1}_{Y_s\leq a}\, ds = \int_0^a \ell_t^r\, dr, \ \ \forall a \geq 0,\ t \geq 0. \tag{4.1}$$

Note that following our conventions, we denote by $\ell^r(dt)$ the measure and by $\ell_t^r = l^r([0,t])$ the corresponding increasing distribution function corresponding to ℓ.

The proof of Proposition 4.1 is long and we indicate the main steps. We will first prove the tightness of the sequence of processes $\{\mathbb{W}^n\}_{n\geq 1}$, based on the tightness of the contour process established in Section 3. This will be obtained in Lemma 4.6, after going through a fair amount of preliminary material. The tightness of the sequence of the local time process $\{\ell^n\}_{n\geq 1}$ is then obtained in Lemma 4.7, thus completing the proof of Proposition 4.1. The rest of the section is devoted to the identification of the limiting snake representation. Here we have to identify a limit point of the sequence of the local times $\{\ell^n\}_{n\geq 1}$ as the local time of the limiting contour process, and this is done in Lemma 4.11. Additionally, in Lemma 4.14 we verify that a limiting point of $\{\tau_{c_0}^{n,0}\}_{n\geq 1}$ is indeed the value at c_0 of the inverse function of the limiting local time. The proof of Theorem 1.2 is an immediate corollary of these facts, and is presented at the end of the section.

As in the previous section, where the tightness of the contour processes $\{Y^n\}_{n\geq 1}$ was obtained, we first handle tightness on the time interval $[0, \tau_{c_0}^{n,0}]$. Fix an arbitrary $a \in [0, K_1)$ and recall that $\{X_{0,t}^{n,c_0}\}_{t\geq 0}$ (see (1.18)) is the measure-valued process characterising the branching particle picture, and in particular, $nX_{0,a_n}^{n,c_0}(1)$ is the number of particles alive at time $a_n = \lfloor an \rfloor/n$. First we derive a bound on the maximal displacement of the offsprings from the ancestors during the time interval $[a_n\,, a_n + \delta]$. This estimate will be crucial for proving tightness of paths of the Brownian snake in random environment.

Fix $\eta \in (0, 1/4)$ arbitrary small. Define

$$\begin{aligned} Z^{n,\eta}_{a_n,\delta} &= n \int_0^{\tau^{n,0}_{c_0}} \mathbf{1}_{(|\hat{\mathbb{W}}^n_s - \mathbf{W}^n_s(a_n)| > \delta^{1/2-\eta})} \ell^{a_n+\delta}(ds) \qquad (4.2) \\ &= \#\{\text{particles alive at time } a_n+\delta \text{ that are displaced by more than} \\ &\qquad \delta^{1/2-\eta} \text{ from the ancestor at time } a_n\}. \end{aligned}$$

Lemma 4.2 *There exists $\delta_{4.2} > 0$, such that*

$$\mathbb{P}(Z^{n,3\eta/2}_{a_n,\delta} > 0) \le e^{-\delta^{-\eta}}, \quad \forall \delta \le \delta_{4.2}\,, \forall n. \qquad (4.3)$$

We postpone the proof of Lemma 4.2, and prepare some preliminary estimates. Introduce the event

$$\mathcal{W}_{n,a,\delta,k,\eta,s} = \{|\mathbf{W}^n_s(a_n+\delta-\delta 2^{-k}) - \mathbf{W}^n_s(a_n+\delta-\delta 2^{-(k-1)})| > \delta^{1/2-\eta}2^{-k/4}\}\,,$$

and define

$$\widetilde{Z}^k_{a_n,\delta} = n\int_0^{\tau^{n,0}_{c_0}} \mathbf{1}_{\mathcal{W}_{n,a,\delta,k,\eta,s}} \ell^{n,a_n+\delta-\delta 2^{-k-1}}(ds)\,,$$

which gives the number of particles alive at time $a_n+\delta-\delta 2^{-k-1}$ whose historical paths were displaced by distance more than $\delta^{1/2-\eta}2^{-k/4}$ on the time interval $[a_n+\delta-\delta 2^{-(k-1)}, a_n+\delta-\delta 2^{-k}]$.

Lemma 4.3 *There exist $C = C(K_1)$ and $\delta_{4.3}$ such that, for all n sufficiently large,*

$$\mathbb{P}\left(\widetilde{Z}^k_{a_n,\delta} > 0\right) \le Cc_0 e^{-\delta^{-\eta}2^{k/2}}, \quad \forall \delta \le \delta_{4.3}. \qquad (4.4)$$

Proof: Let

$$\widehat{Z}^k_{a_n,\delta} = n\int_0^{\tau^{n,0}_{c_0}} \mathbf{1}_{\mathcal{W}_{n,a,\delta,k,\eta,s}} \ell^{n,a_n+\delta-\delta 2^{-k}}(ds),$$

that is, $\widehat{Z}^k_{a_n,\delta}$ is the total number of particles that are alive at time $a_n+\delta-\delta 2^{-k}$ and whose historical paths were displaced by distance more than $\delta^{1/2-\eta}2^{-k/4}$ on the time interval $[a_n+\delta-\delta 2^{-(k-1)}, a_n+\delta-\delta 2^{-k}]$. We enumerate these particles by $i = 1, \ldots, \widehat{Z}^k_{a_n,\delta}$ and let $\widehat{Z}^{i,k}_{a_n,\delta}$ be the number of living descendents of the particle i ($i = 1, \ldots, \widehat{Z}^k_{a_n,\delta}$) at time $a_n+\delta-\delta 2^{-k-1}$. Then clearly

$$\widetilde{Z}^k_{a_n,\delta} = \sum_{i=1}^{\widehat{Z}^k_{a_n,\delta}} \widehat{Z}^{i,k}_{a_n,\delta}. \qquad (4.5)$$

Lemma 2.2 and (4.5) imply that for all n sufficiently large

$$\begin{aligned} \mathbb{P}\left(\widetilde{Z}^k_{a_n,\delta} > 0 \Big| X^{n,c_0}_{0,a_n+\delta(1-2^{-k})}\right) &\le \sum_{i=1}^{\widehat{Z}^k_{a_n,\delta}} \mathbb{P}\left(\widehat{Z}^{i,k}_{a_n,\delta} > 0 \Big| X^{n,c_0}_{0,a_n+\delta(1-2^{-k})}\right) \\ &\le \widehat{Z}^k_{a_n,\delta} 2h(b,\delta 2^{-k-1})/n \le \frac{4\widehat{Z}^k_{a_n,\delta}}{\delta 2^{-k-1}n}, \end{aligned}$$

where the last inequality follows, for all δ sufficiently small, from the definition of h. Therefore,

$$\mathbb{P}\left(\widetilde{Z}^k_{a_n,\delta} > 0\right) = \mathbb{E}\left(\mathbb{P}\left(\widetilde{Z}^k_{a_n,\delta} > 0 \middle| X^{n,c_0}_{0,a_n+\delta(1-2^{-k})}\right)\right) \le \frac{4}{\delta 2^{-k-1} n}\mathbb{E}\left(\widehat{Z}^k_{a_n,\delta}\right).$$

We next represent the measure $X^{n,c_0}_{0,a_n+\delta(1-2^{-(k-1)})}$ as

$$X^{n,c_0}_{0,a_n+\delta(1-2^{-(k-1)})} = \frac{1}{n}\sum_{i=1}^{nX^{n,c_0}_{0,a_n+\delta(1-2^{-(k-1)})}(1)} \delta_{\mathcal{U}_i} \tag{4.6}$$

where $\mathcal{U}_i$ are the positions of the particles alive at time $a_n + \delta(1-2^{-(k-1)})$. For the rest of the proof of the lemma we call the particle that is located at $\mathcal{U}_i$ at time $a_n + \delta(1-2^{-(k-1)})$ — the i-th particle. Let $\tilde{X}^{n,i}$ be the measure describing the positions of the living descendents of the i-th particle at time $a_n + \delta(1-2^{-k})$ and similarly to (4.6) we can write

$$\tilde{X}^{n,i} = \frac{1}{n}\sum_{i=1}^{n\tilde{X}^{n,i}(1)} \delta_{\mathcal{U}_{i,k}} \tag{4.7}$$

where $\mathcal{U}_{i,k}$ is the position of the k-th descendent of the i-th particle at time $a_n + \delta(1-2^{-k})$. Then we get that

$$X^{n,c_0}_{0,a_n+\delta(1-2^{-k})} = \sum_{i=1}^{nX^{n,c_0}_{0,a_n+\delta(1-2^{-(k-1)})}(1)} \tilde{X}^{n,i}.$$

Define

$$f_z(x) = \mathbf{1}_{|x-z|>\delta^{1/2-\eta}2^{-k/4}}\,, \quad x,z\in\mathbb{R}^d.$$

Then,

$$\widehat{Z}^k_{a_n,\delta} = \sum_{i=1}^{nX^{n,c_0}_{0,a_n+\delta(1-2^{-(k-1)})}(1)} \tilde{X}^i_{a_n+\delta(1-2^{-(k-1)}),a_n+\delta(1-2^{-k})}(f_{\mathcal{U}_i}).$$

Hence, using Lemma 2.4 in the first inequality, there exists $\delta_{4.3}$ sufficiently small such that

$$\begin{aligned}
&\mathbb{E}\left(\widehat{Z}^k_{a_n,\delta}\middle| X^{n,c_0}_{0,a_n+\delta(1-2^{-(k-1)})}\right)\\
&\le \sum_{i=1}^{nX^{n,c_0}_{0,a_n+\delta(1-2^{-(k-1)})}(1)} \left(1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}\right)^{n\delta 2^{-k}+1}\frac{1}{n}\mathbb{P}_{\mathcal{U}_i}(|B_{\delta 2^{-k}}-\mathcal{U}_i| > \delta^{1/2-\eta}2^{-k/4})\\
&\le X^{n,c_0}_{0,a_n+\delta(1-2^{-(k-1)})}(1)\left(1+\frac{\nu}{n}+\frac{\|\bar{g}\|_\infty}{2n}\right)^{n\delta 2^{-k}+1} e^{-\delta^{-\eta}2^{k/2}}, \quad \forall \delta\le\delta_{4.3}\,,
\end{aligned}$$

where $\mathbb{P}_x$ is the law of the standard Brownian motion starting at x. By taking the expectation we conclude that for all n sufficiently large,

$$\mathbb{E}\left(\widehat{Z}^k_{a_n,\delta}\right) \leq Cc_0 e^{-\delta^{-\eta}2^{k/2}}, \ \forall a \leq K_1, \delta \leq \delta_{4.3},$$

where $C = C(K_1)$, and we are done. ∎

Proof of Lemma 4.2: Fix δ_0 sufficiently small such that $10^3\delta_0^{\eta/2} \leq 1$. Let $\delta \leq \delta_0$. If $\widetilde{Z}^k_{a_n,\delta} = 0$ for each $k \geq 1$ then the maximal displacement of the path of any particle on the time interval $[a_n, a_n+\delta]$ is bounded by

$$\sum_{k=1}^{\infty} \delta^{1/2-\eta}2^{-k/4} \leq \delta^{1/2-\eta}\frac{1}{2^{1/4}-1} \leq \delta^{1/2-1.5\eta}.$$

Hence by Lemma 4.3 we get that for $\delta \leq (\delta_0 \wedge \delta_{4.3})$,

$$\mathbb{P}(Z^{n,3\eta/2}_{a_n,\delta} > 0) \leq \sum_{k=1}^{\infty}\mathbb{P}(\widetilde{Z}^k_{a_n,\delta} > 0) \leq Cc_0\sum_{k=1}^{\infty} e^{-\delta^{-\eta}2^{k/2}}.$$

Now take $\delta_{4.2} \leq (\delta_0 \wedge \delta_{4.3})$ sufficiently small so that for any $\delta \leq \delta_{4.2}$

$$Cc_0\sum_{k=1}^{\infty} e^{-\delta^{-\eta}2^{k/2}} \leq e^{-\delta^{-\eta}},$$

and we are done. ∎

Lemma 4.4 *For any $\epsilon > 0$, there exists $\delta_1 > 0$ such that*

$$\limsup_{n\to\infty}\mathbb{P}\left(\sup_{a\leq K_1}\sup_{\delta\leq\delta_1} Z^{n,2\eta}_{a,\delta} > 0\right) \leq \epsilon. \tag{4.8}$$

Proof: For any $m_0 > 0$ we have by Lemma 4.2 that

$$\begin{aligned} A_{m_0} &:= \mathbb{P}\left(Z^{n,3\eta/2}_{i2^{-m},2^{-m}} > 0, \text{for some } i \leq K_1 2^m, \ m \geq m_0\right) \\ &\leq \sum_{m=m_0}^{\infty}\sum_{i=0}^{K_1 2^m}\mathbb{P}\left(Z^{n,3\eta/2}_{i2^{-m},2^{-m}} > 0\right) \leq \sum_{m=m_0}^{\infty} K_1 2^m e^{-2^{m\eta}}. \end{aligned}$$

Choose m_0 large enough so that $2^{-m_0} \leq \delta_{4.2}$, $A_{m_0} \leq e^{-2^{m_0\eta/2}} \leq \epsilon$, and

$$10\cdot(2\cdot 2^{-m_0})^{1/2-3\eta/2} \leq \left(2^{-m_0}\right)^{1/2-2\eta}. \tag{4.9}$$

Define

$$C(K_1, m_0) = \{\omega : \ Z^{n,3\eta/2}_{i2^{-m},2^{-m}} = 0, \ \forall m > m_0, i \leq K_1 2^m\}.$$

Then

$$\mathbb{P}(C(K_1, m_0)) \geq 1 - e^{-2^{m_0\eta/2}} \geq 1 - \epsilon.$$

Fix $\omega \in C(K_1, m_0)$. Fix arbitrary $a \leq K_1$ and $\delta \leq 2^{-m_0}$. Then there exists $m \geq m_0$ such that

$$2^{-m-1} \leq \delta \leq 2^{-m}. \tag{4.10}$$

For $j \geq m_0$ let $\tilde{a}_j$ denote the smallest integer multiple of 2^{-j} that is larger than a and, with $b = a + \delta$, let $\tilde{b}_j$ denote the largest integer multiple of 2^{-j} that is smaller than b. Let s be any time such that $Y^n_s = a + \delta$. Then since $\delta \leq 2^{-m_0} \leq \delta_{4.2}$ and $\omega \in C(K_1, m_0)$, we have by (4.10) and the continuity of $\mathbf{W}^n_s(\cdot)$ that

$$\begin{aligned}
\left|\hat{\mathbb{W}}^n_s - \mathbf{W}^n_s(a)\right| &\leq \left|\mathbf{W}^n_s(\tilde{b}_m) - \mathbf{W}^n_s(\tilde{a}_m)\right| + \sum_{l=m+1} \left|\mathbf{W}^n_s(\tilde{a}_l) - \mathbf{W}^n_s(\tilde{a}_{l-1})\right| \\
&\quad + \sum_{l=m+1} \left|\mathbf{W}^n_s(\tilde{b}_l) - \mathbf{W}^n_s(\tilde{b}_{l-1})\right| \\
&\leq 10 \cdot 2^{-(1/2-3\eta/2)m} \leq 10 \cdot (2\delta)^{1/2-3\eta/2} \leq \delta^{1/2-2\eta},
\end{aligned}$$

where the last inequality holds by (4.9). By setting $\delta_1 = 2^{-m_0}$ we are done. ■

The following corollary is immediate.

Corollary 4.5 *For any $\epsilon > 0$ there exists $\delta_1 > 0$ such that,*

$$\mathbb{P}\left(\sup_{s\leq\tau^{n,0}_{c_0}} \sup_{\delta\leq\delta_1} \sup_{a\leq(Y^n_s-\delta)_+} \left|\mathbf{W}^n_s(a+\delta) - \mathbf{W}^n_s(a)\right| > \delta_1^{1/2-2\eta}\right) \leq \epsilon.$$

We have made all the preparation for the proof of the following lemma, concerning the tightness of the sequence $\{\mathbb{W}^n\}_{n\geq1}$.

Lemma 4.6 *The sequence of processes $\{\mathbb{W}^n\}_{n\geq1}$ is C-tight in $D_{\mathcal{W}}$.*

Proof: Recall that the C-tightness of the sequence of the contour processes $\{Y^n\}_{n\geq1}$ was proved in Section 3 (see Proposition 3.1). Fix arbitrary $\beta > 0$ and $\alpha = \beta^{1/2-2\eta}$. Then for any $\delta_1 > 0$, we have the following inclusion

$$\begin{aligned}
&\{\sup_{s\leq\tau^{n,0}_{c_0}} \sup_{\delta\leq\delta_1} \sup_{u\geq0} |\mathbf{W}^n_{s+\delta}(u) - \mathbf{W}^n_s(u)| \geq \alpha\} \subset \\
&\{\sup_{s\leq\tau^{n,0}_{c_0}} \sup_{\delta\leq\delta_1} |Y^n_{s+\delta} - Y^n_s| \geq \beta\} \\
&\quad \bigcup \{\sup_{s\leq\tau^{n,0}_{c_0}} \sup_{\delta\leq\beta} \sup_{a\leq(Y_s-\delta)_+} |\mathbf{W}^n_s(a+\delta) - \mathbf{W}^n_s(a)| \geq \beta^{1/2-2\eta}\}.
\end{aligned}$$

The C-tightness of the sequence $\{\mathbb{W}^n\}_{n\geq1}$ now follows from this inclusion together with Proposition 3.1, Corollary 4.5, and Lemma 3.9**(b)**. ■

We next turn to the local time processes $\ell^n, n \geq 1$.

Lemma 4.7 *The sequence of processes $\{\ell^{n,\cdot}\}_{n\geq 1}$ is C-tight in $D_{\mathcal{M}(\mathbb{R}_+)}$.*

Proof: Fix an arbitrary $c_0 > 0$, and define

$$\tilde{\ell}_t^{n,s} \equiv \ell^{n,s}_{t\wedge\tau^{n,0}_{c_0}},\ s,t\geq 0,$$

with $\tilde{\ell}^{n,s}(dt)$ being as usual the corresponding measure. Note that since c_0 is arbitrary, it is enough to show the C-tightness of $\{\tilde{\ell}^{n,\cdot}\}_{n\geq 1}$ in $D_{\mathcal{M}_F}[0,\infty)$ and then the result follows immediately from Lemma 3.9(b) (recall the properties of convergence in vague topology). Since for each t,n, the function $s\mapsto \tilde{\ell}_s^{n,t}$ is non-decreasing, to show the C-tightness of $\{\tilde{\ell}^{n,\cdot}\}_{n\geq 1}$ in $D_{\mathcal{M}_F}[0,\infty)$, it is sufficient to prove the tightness of $\{\tilde{\ell}^{n,\cdot}_t\}_{n\geq 1}$ for each fixed t. That is, in view of Lemma 3.9, we need to prove that for any constant C,

$$\limsup_{h\to 0}\limsup_{n\to\infty}\mathbb{P}(\sup_{0\leq r\leq C}|\tilde{\ell}_t^{n,r+h}-\tilde{\ell}_t^{n,r}|>\epsilon)=0\,. \tag{4.11}$$

The proof requires some care since introducing the time t prevents one from directly exploiting martingale properties and the tightness results in [10].

We use the inverse local times $\tau_r^{n,a}, a\geq 0, r\geq 0$ to define the collection of processes

$$\bar{X}_s^{i,j,\delta}=\tilde{\ell}^{n,i\delta+s}_{\tau^{n,i\delta}_{(j+1)\delta}}-\tilde{\ell}^{n,i\delta}_{\tau^{n,i\delta}_{j\delta}},\quad s\geq 0.$$

Note that $\bar{X}_s^{i,j,\delta}$ represents the total mass of the branching process in random environment $X^{n,j\delta,(j+1)\delta}_{i\delta,i\delta+s}$, defined by (1.17), which starts at "time" $i\delta$, such that

$$\bar{X}_0^{i,j,\delta}=X^{n,j\delta,(j+1)\delta}_{i\delta,i\delta}(1)=\delta.$$

We also denote by $\mathcal{F}_l^{i,j,\delta}$ the filtration generated by the process $X^{n,j\delta,(j+1)\delta}_{i\delta,i\delta+\cdot}$ and its environment by time l/n.

On the event $t<\tau^{n,0}_{c_0}$ we have, for any $T>0$,

$$\begin{aligned}
&\sup_{0\leq r\leq T}|\tilde{\ell}_t^{n,r+h}-\tilde{\ell}_t^{n,r}|\\
&\leq \sup_{i\delta\leq T,j\delta\leq c_0}\sup_{v\in[0,\delta]}|\tilde{\ell}^{i\delta+v+h}_{\tau^{n,i\delta}_{j\delta}}-\tilde{\ell}^{i\delta+v}_{\tau^{n,i\delta}_{j\delta}}|+\sup_{i\delta\leq T,j\delta\leq c_0}\sup_{s\leq\delta}\bar{X}_s^{i,j,\delta}\\
&=:\sup_{i\delta\leq T,j\delta\leq c_0}A_{i,j}+\sup_{i\delta\leq T,j\delta\leq c_0}B_{i,j}\,.
\end{aligned}\tag{4.12}$$

By the C-tightness of the sequence $\{s\mapsto \tilde{\ell}^{i\delta+s}_{\tau^{n,i\delta}_{j\delta}}\}_{n\geq 1}$, see e.g. [10], Theorem 4.2 (proved there for the binary branching but valid, with similar proof, for the geometric case under consideration here), we have that for each fixed δ and each fixed $i\leq T/\delta, j\leq c_0/\delta$,

$$\lim_{h\to 0}\limsup_{n\to\infty}\mathbb{P}(A_{i,j}>\epsilon)=0\,.$$

In particular, for any $\delta>0$ fixed,

$$\lim_{h\to 0}\limsup_{n\to\infty}\mathbb{P}(\sup_{i\delta\leq T,j\delta\leq c_0}A_{i,j}>\epsilon)=0\,. \tag{4.13}$$

To control $B_{i,j}$, we use the following lemma.

Lemma 4.8 *For some universal constant c and all n large,*

$$E \sup_{0\le s\le\delta}((\bar{X}_s^{i,j,\delta})^4) \le c\delta^4\,, \quad \text{for all } \delta\le 1$$

Indeed, Lemma 4.8 and Chebychev's inequality imply that

$$\mathbb{P}(\sup_{i\delta\le T, j\delta\le c_0} B_{i,j} > \epsilon) \le Tc_0\delta^{-2}\delta^4\,.$$

Together with (4.13), this yields the proof of Lemma 4.7, once we complete the proof of Lemma 4.8. ∎

In the proof of Lemma 4.8 we will frequently use the following lemma, whose immediate proof (using iterations) is omitted.

Lemma 4.9 *Let $c_1\,, c_2 > 0$ and suppose $z_i\,, i=1,2,\ldots$ satisfies the following inequalities*

$$z_i \le \frac{c_1}{n} + (1+\frac{c_2}{n})z_{i-1}\,,\ i=1,2,\ldots.$$

Then there exists $\bar{c}>0$ such that for any $\delta\in[0,1]$

$$z_i \le \bar{c}(\frac{c_1}{c_2}\delta + z_0),\ \ \forall i\le\lfloor n\delta\rfloor.$$

Proof of Lemma 4.8: The argument uses computations similar to those in Section 2. Throughout the proof, $\bar{c}$ denotes a constant whose value may change from line to line, but is independent of n or δ. Note that the estimates on $\bar{X}_s^{i,j,\delta}$ that we get throughout the proof below are uniform in i,j and thus we may and will just consider $i=j=1$ and write $\bar{X}_s = \bar{X}_s^{1,1,\delta}$ and $\mathcal{F}_l = \mathcal{F}_l^{1,1,\delta}\,, l=0,1,2,\ldots$. Note that $\bar{X}_s$ is the local time at level s accumulated by the random walk during its first $\lfloor n\delta\rfloor$ excursions from 0. We have the representation

$$\bar{X}_{(m+1)/n} = n^{-1}\sum_{k=1}^{n\bar{X}_{m/n}} Z_{k,m+1}\,,$$

where the $Z_{k,m+1}$ is the number of offspring of the k-th particle at time $(m+1)/n$. Recall that $Z_{k,m}\,, k=1,2,\ldots,$ are conditionally indpendent given $\mathcal{F}_m$, and for each k, $Z_{k,m}$ is geometrically distributed with parameter $1/2-\xi_{k,m}/4\sqrt{n}$. Here with some abuse of notation,

$$\xi_{k,m} = \xi_{m/n}(x_{k,m}),$$

ξ is as in Section 1.1, and $x_{k,m}$ is the position of k-th particle at time m. Note that by (1.5) and our moment assumptions on ξ we have that

$$\alpha_{k,m+1} := \mathbb{E}(Z_{k,m+1}|\mathcal{F}_m) \le 1+\bar{c}/n.$$

Because the mean of $Z_{k,m}$ is close to 1, the sequence $\bar{X}_{(i+1)/n}$ is almost a martingale. To make it into a martingale, introduce the variables, $M_0 = \delta$,

$$M_i = \frac{M_{i-1}}{\bar{X}_{(i-1)/n}} \frac{1}{n} \sum_{k=1}^{n\bar{X}_{(i-1)/n}} \frac{Z_{k,i}}{\alpha_{k,i}}, \quad i \geq 1.$$

Note that

$$\bar{X}_{i/n}/M_i \leq (1 + \bar{c}/n)^i, \; i \geq 1. \tag{4.14}$$

On the other hand, $i \mapsto M_i$ is a discrete martingale, and hence by the Doob-Burkholder-Gundy inequality, we have that

$$\mathbb{E}(\sup_{0\leq i\leq \delta n} M_i^4) \leq \bar{c}E\langle M\rangle_{\delta n}^2 = \mathbb{E}(\sum_{i=1}^{n\delta} \langle \Delta M\rangle_i)^2 \,, \tag{4.15}$$

where

$$\langle \Delta M\rangle_i = \mathbb{E}((M_i - M_{i-1})^2 | \mathcal{F}_{i-1}) \,.$$

We prepare next some estimates. First recall (1.5), our moment assumptions on ξ and its covariance structure to get the following bound on the correlation between the $\{Z_{k,i+1}\}$:

$$|E[(Z_{k,i+1}/\alpha_{k,i+1} - 1)(Z_{k',i+1}/\alpha_{k',i+1} - 1)|\mathcal{F}_i]| \leq \bar{c}/n \,, \;\; \forall k \neq k'.$$

Then we easily get,

$$\begin{aligned}
\langle \Delta M\rangle_{i+1} &= M_i^2 \mathbb{E}\left(\left(\frac{1}{n\bar{X}_{i/n}} \sum_{k=1}^{n\bar{X}_{i/n}} \left(\frac{Z_{k,i+1}}{\alpha_{k,i+1}} - 1 \right) \right)^2 |\mathcal{F}_i \right) \\
\leq& \;\; \bar{c}M_i^2 \frac{1}{n\bar{X}_{i/n}} + M_i^2 \max_{k\neq k', k,k'\leq n\bar{X}_{i/n}} E[(Z_{k,i+1}/\alpha_{k,i+1} - 1)(Z_{k',i+1}/\alpha_{k',i+1} - 1)|\mathcal{F}_i] \\
\leq& \;\; \bar{c}\frac{M_i}{n} + \bar{c}\frac{M_i^2}{n} \,,
\end{aligned} \tag{4.16}$$

Note that $EM_i = EM_0 = \delta$, and hence to control the right hand side of (4.16) we need to bound $\mathbb{E}(M_i^2)$. M_i is a martingale and hence with $B_{1,i} = \mathbb{E}(M_i^2)$ we use (4.16) to get

$$B_{1,i} \leq B_{i-1} + \bar{c}\mathbb{E}(\frac{M_{i-1}}{n}) + \bar{c}\mathbb{E}(\frac{M_{i-1}^2}{n}) \leq (1 + \frac{\bar{c}}{n})B_{1,i-1} + \frac{\bar{c}\delta}{n}.$$

By Lemma 4.9 we get

$$\mathbb{E}(M_i^2) = B_{1,j} \leq \bar{c}(\delta^2 + M_0^2) \leq \bar{c}\delta^2, \;\; i \leq \lfloor n\delta \rfloor.$$

Now recall again that $EM_i = EM_0 = \delta$ and use the above and (4.16) to obtain that

$$E\langle M\rangle_i \leq \bar{c}\delta^2, \;\; i \leq \lfloor n\delta \rfloor.$$

A similar computation, using Remark 1.1, gives

$$\mathbb{E}((M_{i+1}-M_i)^3|\mathcal{F}_i) \le \bar{c}n^{-2}M_i + \bar{c}n^{-3/2}M_i^2 + \bar{c}n^{-1}M_i^3.$$

With $B_{2,j} = EM_j^3$ one then obtains the recursions

$$\begin{aligned} B_{2,j+1} &\le \mathbb{E}(M_j^3) + \mathbb{E}((M_{i+1}-M_i)^3) + \bar{c}\mathbb{E}(\mathbb{E}(M_{i+1}-M_i)^2|\mathcal{F}_i)M_i) \\ &\le (1+\frac{\bar{c}}{n})B_{2,j} + \mathbb{E}(M_j^2)(\bar{c}n^{-3/2}+\bar{c}n^{-1}) + \mathbb{E}(M_j)n^{-2} \\ &\le (1+\frac{\bar{c}}{n})B_{2,j} + \bar{c}\delta^2 n^{-1}\,, \end{aligned}$$

for n sufficiently large ($n \ge \delta^{-1}$), and therefore by Lemma 4.9 we have

$$B_{2,j} \le \bar{c}(\delta^3 + M_0^3) \le \bar{c}\delta^3, \;\; i \le \lfloor n\delta \rfloor. \tag{4.17}$$

Repeating this computation for the fourth moment, one obtains that with $B_{3,j} = EM_j^4$,

$$B_{3,j} \le \bar{c}\delta^4, \;\; i \le \lfloor n\delta \rfloor, \tag{4.18}$$

for all n sufficiently large. Substituting (4.16) into (4.15) and using the last estimates, one gets

$$\mathbb{E}(\sup_{0\le i\le \delta n} M_i^4) \le \bar{c}\delta^4\,, \tag{4.19}$$

for all n suffciently large. Since, by (4.14),

$$\sup_{0\le s\le\delta} \bar{X}_s^4 \le \left(1+\frac{\bar{c}}{n}\right)^{\delta n} \sup_{0\le i\le\delta n} M_i^4\,,$$

this completes the proof of Lemma 4.8. ∎

Corollary 4.10 *$\{(\mathbb{W}^n,\ell^n)\}_{n\ge1}$ is C-tight in $D_{\mathcal{W}\times\mathcal{M}(\mathbb{R}_+)}$.*

Proof: Immediately from Lemma 4.7 and Lemma 4.6. ∎

In what follows let $(\mathbf{W},Y,\ell,\tau^0)$ be a limiting point of $\{(\mathbf{W}^n,Y^n,\ell,\tau^{n,0})\}_{n\ge1}$. To simplify the notation we omit subsequences and simply assume that $\{(\mathbf{W}^n,Y^n,\ell^n,\tau^{n,0})\}_{n\ge1}$ converges to $(\mathbf{W},Y,\ell,\tau^0)$. We also switch (by Skorohod's theorem) to some probability space where the convergence holds a.s.. Recall again that we write ℓ_t^n and ℓ_t for $\ell^n([0,t])$ and $\ell([0,t])$ respectively.

Lemma 4.11 *ℓ is the local time of Y.*

Proof: First note that by properties of weak convergence of measures, for any $a \geq 0$

$$\ell_t^{n,a} \to \ell_t^a \tag{4.20}$$

for any point of continuity of function $t \mapsto \ell_t^a$. However by a limiting argument and the convergence of Y^n to Y, it is easy to derive that if $Y_s \neq a$ then s is a point of continuity of $t \mapsto \ell_t^a$. Therefore, for all a, t such that $Y_t \neq a$, (4.20) follows. Note that

$$T_t^n(a) = \frac{1}{n^2} \sum_{i=0}^{\lfloor n^2 t \rfloor} \mathbf{1}_{Y^n_{n^{-2}i} \leq a} = \int_0^{\lfloor n^2 t \rfloor / n^2} \mathbf{1}_{Y^n_s \leq a}\, ds, \quad t \geq 0.$$

Also for any $a \geq 0$ and $\delta > 0$ we have

$$\int_0^t \mathbf{1}_{a-\delta \leq Y^n_s \leq a+\delta}\, ds = \int_{a-\delta}^{a+\delta} \ell_t^{n,s}\, ds \leq 2\delta \sup_{s \leq K_1} \ell_t^{n,s}.$$

Since $\{\ell_t^{n,s}\}_{n\geq 1}$ is tight and δ was arbitrary we can make the left side arbitrarily small by taking $\delta > 0$ sufficiently small with probability as close to 1 as we wish uniformly in n. This, by a standard argument, that also uses the convergence of $\{Y^n\}_{n\geq 1}$, implies that

$$\int_0^{\lfloor n^2 t \rfloor / n^2} \mathbf{1}_{Y^n_s \leq a}\, ds \to \int_0^t \mathbf{1}_{Y_s \leq a}\, ds \tag{4.21}$$

for any $a \geq 0, t \geq 0$. On the other hand

$$T_t^n(a) = \int_0^a \ell_t^{n,r}\, dr \to \int_0^a \ell_t^r\, dr, t \geq 0,$$

where the last convergence follows by convergence of $\ell_t^{n,r}$ at all the points r, t such that $Y_t \neq r$ (there is just one level r such that $Y_t = r$). This and (4.21) yield

$$\int_0^t \mathbf{1}_{Y_s \leq a}\, ds = \int_0^a \ell_t^r\, dr, \quad t \geq 0, \tag{4.22}$$

for all a, r, and hence ℓ_t^r is indeed the local time of Y, for any $t \geq 0$. ■

Remark 4.12 *The above lemma and Corollary 4.10 finish the proof of Proposition 4.1.*

The next two lemmas are essential for the proof of the "charaterization of the limit points" part of Theorem 1.2. First we prove the continuity of the local time at the level zero.

Lemma 4.13 *$t \mapsto \ell_t^0$ is continuous.*

Proof: It is enough to show that for arbitrary $c_0 > 0$, $\left\{\ell^{n,0}_{\cdot\wedge\tau^{n,0}_{c_0}}\right\}_{n\geq 1}$ is C-tight in $D_{\mathbb{R}}[0,\infty)$, that is, for any $\epsilon > 0$

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{t\leq\tau^{n,0}_{c_0}}\ell^{n,0}_t-\ell^{n,0}_{t-\delta}\geq\epsilon\right)=0. \tag{4.23}$$

Suppose (4.23) does not hold, that is, there exist $\epsilon,\epsilon_1 > 0$, such that for all $\delta > 0$

$$\mathbb{P}\left(\sup_{t\leq\tau^{n,0}_{c_0}}\ell^{n,0}_t-\ell^{n,0}_{t-\delta}\geq\epsilon\right)\geq\epsilon_1. \tag{4.24}$$

Fix such $\epsilon,\epsilon_1 > 0$; we have the inclusion

$$\left\{\sup_{t\leq\tau^{n,0}_{c_0}}\ell^{n,0}_t-\ell^{n,0}_{t-\delta}\geq\epsilon\right\}\subset\left\{\exists i=1,\ldots,\left\lfloor\frac{2c_0}{\epsilon}\right\rfloor:\ \tau^{n,0}_{\frac{(i+1)\epsilon}{2}}-\tau^{n,0}_{\frac{i\epsilon}{2}}<\delta\right\}.$$

Since $\tau^{n,0}_{\frac{(i+1)\epsilon}{2}}-\tau^{n,0}_{\frac{i\epsilon}{2}}, i=1,\ldots,\left\lfloor\frac{2c_0}{\epsilon}\right\rfloor$ are identically distributed we get

$$\mathbb{P}\left(\exists i=1,\ldots,\left\lfloor\frac{2c_0}{\epsilon}\right\rfloor:\ \tau^{n,0}_{\frac{(i+1)\epsilon}{2}}-\tau^{n,0}_{\frac{i\epsilon}{2}}<\delta\right)\leq\left(\left\lfloor\frac{2c_0}{\epsilon}\right\rfloor+1\right)\mathbb{P}\left(\tau^{n,0}_{\epsilon/2}<\delta\right).$$

By Lemma 3.9(a), we can choose δ sufficiently small such that

$$\mathbb{P}\left(\tau^{n,0}_{\epsilon/2}<\delta\right)\leq\frac{\epsilon_1}{2\left(\left\lfloor\frac{2c_0}{\epsilon}\right\rfloor+1\right)}$$

for all n sufficiently large, and hence

$$\mathbb{P}\left(\sup_{t\leq\tau^{n,0}_{c_0}}\ell^{n,0}_t-\ell^{n,0}_{t-\delta}\geq\epsilon\right)\leq\frac{\epsilon_1}{2}$$

and we get a contradiction with (4.24). ∎

Lemma 4.14 *For any fixed $r > 0$, τ^0_r equals, with probability one, to the value of the inverse function of $\ell^0_\cdot$ at r, that is,*

$$\tau^0_r=\inf\{s>0:\ \ell^0_s>r\},\quad a.s..$$

Proof: Recall that we assume that we are considering the probability space where $\ell^{n,0},\tau^{n,0}\to(\ell^0,\tau^0)$ in $D_{\mathbb{R}_+}[0,\infty)\times\mathcal{M}(\mathbb{R}_+)$, $\mathbb{P}$-a.s.. Moreover we know that for any fixed r, $\tau^0(\cdot)$ is continuous at the point r. This, by properties of convergence in $\mathcal{M}$, implies that for any fixed r, $\tau_r^{n,0} \to \tau_r^0$, $\mathbb{P}$-a.s.. Fix arbitrary $c_0, \delta > 0$. Then, by definition of the local time, we get,

$$\ell^{n,0}_{\tau^{n,0}_{c_0+\delta}} \geq c_0 + \delta. \tag{4.25}$$

Since $\ell^{n,0}_{\cdot}$ converges to the continous limit, the convergence is uniform on the compacts. This and the convergence $\tau^{n,0}_{c_0+\delta} \to \tau^0_{c_0+\delta}$ imply, that by passing to the limit in (4.25) we get

$$\ell^0_{\tau^0_{c_0+\delta}} \geq c_0 + \delta, \tag{4.26}$$

and hence

$$\inf\{s > 0 : \ \ell^0_s > c_0\} \leq \tau^0_{c_0+\delta}. \tag{4.27}$$

Similarly we can show that

$$\inf\{s > 0 : \ \ell^0_s > c_0\} \geq \tau^0_{c_0-\delta}. \tag{4.28}$$

Since δ was arbitrary, and by the continuity of τ^0 at c_0 (see Lemma 3.9**(d)**) we get

$$\inf\{s > 0 : \ \ell^0_s > c_0\} = \tau^0_{c_0}. \tag{4.29}$$

and we are done.

∎

Lemma 4.15 *For any $\phi \in \mathcal{C}_b(\mathbb{R}^d)$ and fixed $c_0 > 0$,*

$$\int_0^{\tau^{n,0}_{c_0}} \phi(\hat{\mathbb{W}}^n_s)\ell^{n,t}(ds) \to \int_0^{\tau^0_{c_0}} \phi(\hat{\mathbb{W}}_s)\ell^t(ds), \ \ \forall t \geq 0, \ \mathbb{P}-\text{a.s.}, \tag{4.30}$$

as $n \to \infty$, where

$$\tau^0_{c_0} = \inf\{r > 0 : \ \ell^0_r > c_0\}. \tag{4.31}$$

Proof: $\tau^{n,0}_{c_0} \to \tau^0_{c_0}$, where by Lemma 4.14 $\tau^0_{c_0}$ is defined by (4.31). Moreover, by Lemma 4.13, $\ell^0_{\cdot}$ is continuous at $\tau^0_{c_0}$, therefore by elementary properties of weak convergence, for any continnuous function $f(s)$

$$\int_0^{\tau^{n,0}_{c_0}} f(s)\ell^{n,0}(ds) \to \int_0^{\tau^0_{c_0}} f(s)\ell^0(ds), \ \mathbb{P}-\text{a.s.}, \ \ \text{as } n \to \infty.$$

Now the result for $t = 0$, follows by uniform on the compacts convergence of $\hat{\mathbb{W}}^n$ to $\hat{\mathbb{W}}$. The convergence of the integral for $t > 0$ follows immediately since, by the continuity of Y, the $\ell^t(ds)$ does not charge the point $s = \tau^0_{c_0}$ for every $t > 0$. ∎

Proof of Theorem 1.2: The tightness statement was proved in Proposition 4.1. To finish the proof we need to derive the characterization of the limit points. Fix arbitrary $c_0 > 0$ and let $X^{n,c_0}_{0,t}$ be the measure-valued process defined as in (1.18), that is,

$$X^{n,c_0}_{0,t}(\phi) \equiv \int_0^{\tau^{n,0}_{c_0}} \phi(\hat{\mathbb{W}}^n_s)\ell^{n,t}(ds), \quad t \in [0, K_1], \tag{4.32}$$

for all $\phi \in \mathcal{B}(\mathbb{R}^d)$. Let $(\mathbf{W}, Y, \ell, \tau^0_{c_0})$ be an arbirary limit point of $\{(\mathbf{W}^n, Y^n, \ell^n, \tau^{n,0}_{c_0})\}_{n\geq 1}$. Fix arbitrary $\phi \in \mathcal{C}_b(\mathbb{R}^d)$. As we have mentioned already, due to results in [10], the sequence of process $\{X^{n,c_0}_{0,\cdot}\}_{n\geq 1}$ converges weakly in $D_{\mathcal{M}_F}[0, K_1]$ to the process $X^{c_0} \in C_{\mathcal{M}_F}[0, K_1]$ satisfying the martingale problem (1.7-1.8) on $[0, K_1]$, with $X^{c_0}_0 = c_0\delta_x$, and hence the left hand side of (4.32) converges to $X^{c_0}_t(\phi)$ for any $t \in [0, K_1]$. As for the right hand side of (4.32), due to Proposition 4.1 and Lemma 4.15 it converges, along an appropriate subsequence, to $\int_0^{\tau^0_{c_0}} \phi(\hat{\mathbb{W}}_s)\ell^t(ds)$ for $t \in [0, K_1]$, where ℓ is the local time Y. This gives us (1.19) for any $\phi \in \mathcal{C}_b(\mathbb{R}^d)$. The extension of the equality to any $\phi \in \mathcal{B}(\mathbb{R}^d)$ is trivial. ∎

5 Proof of Theorem 1.3

The proof of the result is based on convergence of approximations. For simplicity, as before, we assume that $(\mathbb{W}^n, B^n, \ell^n) = (\mathbf{W}^n, Y^n, B^n, \ell^n) \to (\mathbf{W}, Y, B, \ell) = (\mathbb{W}, Y, B, \ell)$ a.s. (based on Proposition 4.1 we can always get it by Skorohod's theorem via an appropriate subsequence) and

$$\frac{\xi_j\,(y)}{\sqrt{n}} = B_{\frac{j}{n}}(y) - B_{\frac{j-1}{n}}(y).$$

On the level of nth approximation we will be dealing with the following approximating functional:

$$F_n\left(\mathbb{W}^n_{n^{-2}k}\right) \equiv \frac{1}{n}\sum_{l=1}^{nY^n_{k/n^2}-1} e^{-\frac{1}{\sqrt{n}}\sum_{l'=1}^{l}\xi_{l'}\left(\mathbf{W}^n_{k/n^2}\left(\frac{l+1}{n}\right)\right)}.$$

Note that

$$F_n\left(\mathbb{W}^n_{n^{-2}(k+1)}\right) = \begin{cases} \frac{1}{n}\sum_{l=1}^{nY^n_{k/n^2}-2} e^{-\frac{1}{\sqrt{n}}\sum_{l'=1}^{l}\xi_{l'}\left(\mathbf{W}^n_{k/n^2}\left(\frac{l+1}{n}\right)\right)}, & \text{if } Y^n_{(k+1)/n^2} < Y^n_{k/n^2}, \\ \frac{1}{n}\sum_{l=1}^{nY^n_{k/n^2}-1} e^{-\frac{1}{\sqrt{n}}\sum_{l'=1}^{l}\xi_{l'}\left(\mathbf{W}^n_{k/n^2}\left(\frac{l+1}{n}\right)\right)} & \\ \quad + \frac{1}{n}e^{-\frac{1}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}}\xi_{l'}\left(\mathbf{W}^n_{(k+1)/n^2}(Y^n_{k/n^2}+1/n)\right)}, & \text{if } Y^n_{(k+1)/n^2} > Y^n_{k/n^2}. \end{cases}$$

Further, if $Y^n_{(k+1)/n^2} > Y^n_{k/n^2}$, then

$$\mathbf{W}^n_{(k+1)/n^2}(Y^n_{k/n^2} + 1/n) = \hat{\mathbb{W}}^n_{(k+1)/n^2} = \hat{\mathbb{W}}^n_{k/n^2} + \eta_{1/n},$$

where η is a Brownian path independent of $\mathbb{W}^n_{k/n^2}$. Let

$$\mathcal{F}_k \quad = \quad \sigma\left\{\mathbb{W}^n_{l/n^2}\,,\ l \le k\right\} \vee \sigma\left\{\xi_l\,,\ l = 0,1,2,\ldots\right\}.$$

Define

$$V_k = F\left(\mathbb{W}^n_{k/n^2}\right) - F\left(\mathbb{W}^n_{(k-1)/n^2}\right),\quad k = 1,2,\ldots.$$

Then by the standard decomposition of $F\left(\mathbb{W}^n_{m/n^2}\right)$ we get that

$$F\left(\mathbb{W}_{m/n^2}\right) = M^n_m + A^n_m,\quad m = 1,2,\ldots,$$

where $M^n_m\,, m = 1,2,\ldots$, is the $\{\mathcal{F}_m\}_{m\ge 1}$-martingale given by

$$M^n_m \quad = \quad \sum_{k=0}^{m-1}\left(V_{k+1} - \mathbb{E}\left(V_{k+1}|\mathcal{F}_k\right)\right),\quad m = 1,2,\ldots$$

and

$$A^n_m \quad = \quad \sum_{k=0}^{m-1}\mathbb{E}\left(V_{k+1}|\mathcal{F}_k\right),\quad m = 1,2,\ldots.$$

We first study the limiting behavior of A^n.

Lemma 5.1

$$\begin{aligned} A^n_{\lfloor n^2 t\rfloor} \quad \to \quad & \int_0^t e^{-B_{Y_s}(\hat{\mathbb{W}}_s)}\left\{-\frac{1}{2}\Delta B_{Y_s}(\hat{\mathbb{W}}_s) + \frac{1}{2}\sum_{i=1}^d\left(\frac{\partial}{\partial x_i}B_{Y_s}(\hat{\mathbb{W}}_s)\right)^2\right\} ds \\ & + \ell^0_t - \int_0^t e^{-B_{K_1}(\hat{\mathbb{W}}_s)}\,\ell^{K_1}(ds),\quad \text{as } n\to\infty. \end{aligned} \tag{5.1}$$

Proof: Using $\mathbb{E}_\eta$ to denote expectation with respect to the Brownian path $\eta.$, we have

$$\begin{aligned} &\mathbb{E}\left(V_{k+1}|\mathcal{F}_k\right) \\ &\quad = \mathbb{P}\left(Y^n_{(k+1)/n^2} < Y^n_{k/n^2}|\mathcal{F}_k\right) \\ &\qquad \times \mathbb{E}\left(-\frac{1}{n}e^{-\frac{1}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}-1}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k})}\middle| Y^n_{(k+1)/n^2} < Y^n_{k/n^2}\,,\ \mathcal{F}_k\right) \\ &\qquad + \mathbb{P}\left(Y^n_{(k+1)/n^2} > Y^n_{k/n^2}|\mathcal{F}_k\right) \\ &\qquad \times \mathbb{E}\left(\frac{1}{n}e^{-\frac{1}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}}\xi_{l'}\left(\mathbf{W}^n_{n^{-2}(k+1)}\left(Y^n_{k/n^2}+1/n\right)\right)}\middle| Y^n_{(k+1)/n^2} > Y^n_{k/n^2}\,,\ \mathcal{F}_k\right). \end{aligned}$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\left(V_{k+1}|\mathcal{F}_k\right)\\
= \;& -\left(\frac{1}{2}-\frac{1}{4\sqrt{n}}\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}^n_{n^{-2}k}\right)\right)\frac{1}{n}e^{-\frac{1}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}-1}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k})}\\
&+\left(\frac{1}{2}+\frac{1}{4\sqrt{n}}\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}^n_{n^{-2}k}\right)\right)\mathbb{E}_\eta\left(\frac{1}{n}e^{-\frac{1}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k}+\eta_{1/n})}\right)\\
&+\frac{1}{n}1_{Y^n_{k/n^2}=0}-1_{Y^n_{k/n^2}=K_1}\frac{1}{n}e^{-\frac{1}{\sqrt{n}}\sum_{l'=1}^{nK_1-1}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k})}\\
= \;& \frac{1}{2n}\left(\mathbb{E}_\eta\left(e^{-\frac{1}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k}+\eta_{1/n})}\right)-e^{-\frac{1}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}-1}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k})}\right)\\
&+\frac{1}{4n^{3/2}}\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}^n_{n^{-2}k}\right)\left(e^{-\frac{1}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}-1}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k})}\right.\\
&\qquad\qquad\left.+\mathbb{E}_\eta\left(e^{-\frac{1}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k}+\eta_{1/n})}\right)\right)\\
&+\frac{1}{n}\cdot 1_{Y^n_{k/n^2}=0}-\frac{1}{n}\cdot 1_{Y^n_{k/n^2}=K_1}e^{-\frac{1}{\sqrt{n}}\sum_{l'=1}^{nK_1-1}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k})}\\
= \;& \frac{1}{2n}\left(\mathbb{E}_\eta\left(e^{-B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k}+\eta_{1/n})}\right)-e^{-B_{Y^n_{k/n^2}-1/n}(\hat{\mathbb{W}}^n_{n^{-2}k})}\right)\\
&+\frac{1}{4n^{3/2}}\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}_{n^{-2}k}\right)\left(e^{-B_{Y^n_{k/n^2}-1/n}(\hat{\mathbb{W}}^n_{n^{-2}k})}+\mathbb{E}_\eta\left(e^{-B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k}+\eta_{1/n})}\right)\right)\\
&+\frac{1}{n}\cdot 1_{Y^n_{k/n^2}=0}-\frac{1}{n}\cdot 1_{Y^n_{(k-1)/n^2}=K_1-1,Y^n_{k/n^2}=K_1}e^{-B_{K_1-1/n}(\hat{\mathbb{W}}^n_{n^{-2}k})}\\
= \;& I_{1,n,k}+I_{2,n,k}+I_{3,n,k}-I_{4,k,n}\,,
\end{aligned}
$$

where we also used the definition of B. We begin with an estimate of $\mathbb{E}_\eta\left(e^{-B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k}+\eta_{1/n})}\right)$. By Itô's formula we get

$$
\begin{aligned}
&\mathbb{E}_\eta\left(e^{-B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k}+\eta_{1/n})}\right)=e^{-B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k})}\\
&\quad+\mathbb{E}_\eta\left(\int_0^{1/n}e^{-B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k}+\eta_s)}\left(-\frac{1}{2}\Delta_x B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k}+x)|_{x=\eta_s}\right.\right.\\
&\quad\left.\left.+\frac{1}{2}\sum_{i=1}^d\left(\frac{\partial}{\partial x_i}B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k}+x)|_{x=\eta_s}\right)^2\right)ds\right).
\end{aligned}
$$

The first term at the right side above can be further decomposed as

$$
\begin{aligned}
&e^{-B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k})}\\
= \;& e^{-B_{Y^n_{k/n^2}-1/n}(\hat{\mathbb{W}}^n_{n^{-2}k})}\left(1-\frac{1}{n^{1/2}}\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}_{n^{-2}k}\right)+\frac{1}{2n}\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}_{n^{-2}k}\right)^2\right.\\
&\left.+O(n^{-3/2})\Theta\left(\left|\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}_{n^{-2}k}\right)\right|^3\right)\right)
\end{aligned}
$$

where $\Theta(x)$ is some point in $[-x,x]$. We get

$$\begin{aligned}
I_{1,n,k} &= \frac{1}{2n}\left(e^{-B_{Y^n_{k/n^2-1/n}}(\hat{\mathbb{W}}^n_{n^{-2}k})}\left(-\frac{1}{n^{1/2}}\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}_{n^{-2}k}\right)+\frac{1}{2n}\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}_{n^{-2}k}\right)^2\right.\right.\\
&\quad \left. +O(n^{-3/2})\Theta\left(\left|\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}_{n^{-2}k}\right)\right|^3\right)\right)\\
&\quad +\mathbb{E}_\eta\left(\int_0^{1/n} e^{-B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k}+\eta_s)}\left(-\frac{1}{2}\Delta_x B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k}+x)|_{x=\eta_s}\right.\right.\\
&\quad \left.\left.\left. +\frac{1}{2}\sum_{i=1}^d\left(\frac{\partial}{\partial x_i}B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k}+x)|_{x=\eta_s}\right)^2\right)\,ds\right)\right).
\end{aligned}$$

To handle $I_{2,n,k}$, denote

$$\begin{aligned}
R_{n,k}(s) &= e^{-B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k}+\eta_s)}\left(-\frac{1}{2}\Delta_x B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k}+x)|_{x=\eta_s}\right.\\
&\quad \left.+\frac{1}{2}\sum_{i=1}^d\left(\frac{\partial}{\partial x_i}B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k}+x)|_{x=\eta_s}\right)^2\right).
\end{aligned}$$

Then,

$$\begin{aligned}
I_{2,n,k} &= \frac{1}{4n^{3/2}}\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}_{n^{-2}k}\right)\left(e^{-B_{Y^n_{k/n^2-1/n}}(\hat{\mathbb{W}}^n_{n^{-2}k})}\left(2-\frac{1}{n^{1/2}}\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}_{n^{-2}k}\right)\right.\right.\\
&\quad \left.\left.+O(n^{-1})\Theta\left(\left|\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}_{n^{-2}k}\right)\right|^2\right)\right)+\mathbb{E}_\eta\left(\int_0^{1/n}R_{n,k}(s)\,ds\right)\right).
\end{aligned}$$

All together we get

$$\begin{aligned}
I_{1,n,k}+I_{2,n,k} &= \frac{1}{2n}\mathbb{E}_\eta\left(\int_0^{1/n}R_{n,k}(s)\,ds\right)+O(n^{-5/2})\Theta\left(\left|\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}_{n^{-2}k}\right)\right|^3\right)\\
&\quad +O(n^{-3/2})\Theta\left(\left|\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}_{n^{-2}k}\right)\right|\right)\mathbb{E}_\eta\left(\int_0^{1/n}R_{n,k}(s)\,ds\right).
\end{aligned}$$

From this it follows that for any $t>0$

$$\begin{aligned}
\sum_{k=1}^{\lfloor n^2t\rfloor}(I_{1,n,k}+I_{2,n,k}) &= \frac{1}{2n^2}\sum_{k=1}^{\lfloor n^2t\rfloor}n\mathbb{E}_\eta\left(\int_0^{1/n}R_{n,k}(s)\,ds\right)\\
&\quad +O(n^{-5/2})\sum_{k=1}^{\lfloor n^2t\rfloor}\Theta\left(\left|\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}_{n^{-2}k}\right)\right|^3\right)\\
&\quad +O(n^{-5/2})\sum_{k=1}^{\lfloor n^2t\rfloor}\Theta\left(\left|\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}_{n^{-2}k}\right)\right|\right)n\mathbb{E}_\eta\left(\int_0^{1/n}R_{n,k}(s)\,ds\right)\\
&\quad \to\int_0^t e^{-B_{Y_s}(\hat{\mathbb{W}}_s)}\left(-\frac{1}{2}\Delta_x B_{Y_s}(\hat{\mathbb{W}}_s)+\frac{1}{2}\sum_{i=1}^d\left(\frac{\partial}{\partial x_i}B_{Y_s}(\hat{\mathbb{W}}_s)\right)^2\right)\,ds,
\end{aligned}$$

where the second and third terms on the right side of the first equality converge to 0 and the first term converges to the first term on the right side of (5.1).

Now we will treat $I_{3,n,k}$ and $I_{4,n,k}$. By definition of the approximate local time $\ell^{n,n^{-1}m}$ we get

$$\sum_{k=0}^{\lfloor n^2 t\rfloor}(I_{3,n,k}+I_{4,n,k}) \;=\; \ell_t^{n,0}-\int_0^t \mathbb{E}_\eta\left(e^{-B_{K_1-\frac{1}{n}}(\hat{\mathbb{W}}^n_{s+1/n^2})}\right)\ell^{n,K_1-1/n}(ds).$$

Then pass to the limit, use the uniform on compacts convergence of ℓ^n and $\mathbb{W}^n$ to ℓ and $\mathbb{W}$, and the continuity of B to get that

$$\sum_{k=0}^{\lfloor n^2 t\rfloor}(I_{3,n,k}+I_{4,n,k}) \;\to\; \ell_t^0-\int_0^t e^{-B_{K_1}(\hat{\mathbb{W}}_s)}\ell^{K_1}(ds),$$

as $n\to\infty$. Thus, we obtain the second and the third terms in (5.1). ∎

Define the bracket process for the martingale M^n:

$$\langle M^n_\cdot\rangle_m \;\equiv\; \sum_{k=0}^{m-1}\mathbb{E}\left(\left(M^n_{k+1}-M^n_k\right)^2|\mathcal{F}_k\right),\quad m=1,2,\ldots. \tag{5.2}$$

Then we have

Lemma 5.2

$$\langle M^n_\cdot\rangle_{\lfloor n^2 t\rfloor}\to\int_0^t e^{-2B_{Y_s}(\hat{\mathbb{W}}_s)}ds,\quad \text{as } n\to\infty.$$

Proof: It is easy to check that for any $m\geq 1$,

$$\langle M^n_\cdot\rangle_m \;=\; \sum_{k=0}^{m-1}\mathbb{E}\left(V_{k+1}^2|\mathcal{F}_k\right)-\sum_{k=0}^{m-1}\left(\mathbb{E}\left(V_{k+1}|\mathcal{F}_k\right)\right)^2.$$

By Lemma 5.1 we know that as $n\to\infty$

$$\begin{aligned}\sum_{k=0}^{\lfloor n^2 t\rfloor}\mathbb{E}\left(V_{k+1}|\mathcal{F}_k\right) \;\to\;& \int_0^t e^{-B_{Y_s}(\hat{\mathbb{W}}_s)}\left\{-\frac{1}{2}\Delta B_{Y_s}(\hat{\mathbb{W}}_s)+\frac{1}{2}\sum_{i=1}^d\left(\frac{\partial}{\partial x_i}B_{Y_s}(\hat{W}_s)\right)^2\right\}ds\\ &+\ell_t^0-\int_0^t e^{-B_{K_1}(\hat{\mathbb{W}}_s)}\,\ell^{K_1}(ds)\end{aligned}$$

which is a process of bounded variation. From this it is easy to deduce that

$$\sum_{k=0}^{\lfloor n^2 t\rfloor}\left(\mathbb{E}\left(V_{k+1}|\mathcal{F}_k\right)\right)^2 \;\to\; 0,$$

as $n \to \infty$. Hence it is enough to consider the limiting behavior of

$$\sum_{k=0}^{\lfloor n^2 t\rfloor} \mathbb{E}\left(V_{k+1}^2|\mathcal{F}_k\right).$$

By repeating the argument in the proof of Lemma 5.1 we get

$$\begin{aligned}
&\mathbb{E}(V_{k+1}^2|\mathcal{F}_k)\\
&= \mathbb{P}\left(Y^n_{(k+1)/n^2} < Y^n_{k/n^2}|\mathcal{F}_k\right)\\
&\quad \times \mathbb{E}\left(\frac{1}{n^2}e^{-\frac{2}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}-1}\xi_{l'}\left(\mathbb{W}^n_{n^{-2}k}\left(Y^n_{k/n^2}\right)\right)}\middle| Y^n_{(k+1)/n^2} < Y^n_{k/n^2},\ \mathcal{F}_k\right)\\
&\quad +\mathbb{P}\left(Y^n_{(k+1)/n^2} > Y^n_{k/n^2}|\mathcal{F}_k\right)\\
&\quad \times \mathbb{E}\left(\frac{1}{n^2}e^{-\frac{2}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}}\xi_{l'}\left(\mathbb{W}^n_{n^{-2}(k+1)}\left(Y^n_{k/n^2}+1/n\right)\right)}\middle| Y^n_{(k+1)/n^2} > Y^n_{k/n^2},\ \mathcal{F}_k\right)\\
&= \left(\frac{1}{2}-\frac{1}{4\sqrt{n}}\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}^n_{n^{-2}k}\right)\right)\frac{1}{n^2}e^{-\frac{2}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}-1}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k})}\\
&\quad +\left(\frac{1}{2}+\frac{1}{4\sqrt{n}}\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}^n_{n^{-2}k}\right)\right)\mathbb{E}_\eta\left(\frac{1}{n^2}e^{-\frac{2}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k}+\eta_{1/n})}\right)\\
&\quad +\frac{1}{n^2}1_{Y^n_{k/n^2}=0}+1_{Y^n_{k/n^2}=K_1}\frac{1}{n^2}e^{-\frac{2}{\sqrt{n}}\sum_{l'=1}^{nK_1-1}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k})}\\
&= \frac{1}{2n^2}\left(\mathbb{E}_\eta\left(e^{-\frac{2}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k}+\eta_{1/n})}\right)+e^{-\frac{2}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}-1}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k})}\right)\\
&\quad +\frac{1}{4n^{5/2}}\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}^n_{n^{-2}k}\right)\left(-e^{-\frac{2}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}-1}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k})}\right.\\
&\qquad \left.+\mathbb{E}_\eta\left(e^{-\frac{2}{\sqrt{n}}\sum_{l'=1}^{nY^n_{k/n^2}}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k}+\eta_{1/n})}\right)\right)\\
&\quad +\frac{1}{n^2}\cdot 1_{Y^n_{k/n^2}=0}+\frac{1}{n^2}\cdot 1_{Y^n_{k/n^2}=K_1}e^{-\frac{2}{\sqrt{n}}\sum_{l'=1}^{nK_1-1}\xi_{l'}(\hat{\mathbb{W}}^n_{n^{-2}k})}\\
&= \frac{1}{2n^2}\left(\mathbb{E}_\eta\left(e^{-2B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k}+\eta_{1/n})}\right)+e^{-2B_{Y^n_{k/n^2}-1/n}(\hat{\mathbb{W}}^n_{n^{-2}k})}\right)\\
&\quad +\frac{1}{4n^{5/2}}\xi_{Y^n_{k/n^2}}\left(\hat{\mathbb{W}}_{n^{-2}k}\right)\left(-e^{-2B_{Y^n_{k/n^2}-1/n}(\hat{\mathbb{W}}^n_{n^{-2}k})}+\mathbb{E}_\eta\left(e^{-2B_{Y^n_{k/n^2}}(\hat{\mathbb{W}}^n_{n^{-2}k}+\eta_{1/n})}\right)\right)\\
&\quad +\frac{1}{n^2}\cdot 1_{Y^n_{k/n^2}=0}+\frac{1}{n^2}\cdot 1_{Y^n_{k/n^2}=K_1}e^{-2B_{K_1-1/n}(\hat{\mathbb{W}}^n_{n^{-2}k})}\\
&= I_{1,n,k}+I_{2,n,k}+I_{3,n,k}+I_{4,n,k},
\end{aligned}$$

Using the bounds from the proof of Lemma 5.1 it is easy to see that

$$\sum_{k=1}^{\lfloor n^2 t\rfloor}(I_{2,n,k}+I_{3,n,k}+I_{4,n,k}) \to 0,$$

as $n \to \infty$. As for $I_{1,n,k}$, again using the convergence of $(\mathbf{W}^n, Y^n)$ and the continuity of B, it is easy to see that

$$\sum_{k=1}^{\lfloor n^2 t \rfloor} I_{1,n,k} \to \int_0^t e^{-2B_{Y_s}(\hat{\mathbb{W}}_s)} ds, \quad \text{as } n \to \infty,$$

and we are done. ∎

Corollary 5.3 *As $n \to \infty$, M^n converges to a continuous martingale M such that*

$$\langle M.\rangle_t \quad = \quad \int_0^t e^{-2B_{Y_s}(\hat{\mathbb{W}}_s)} ds, \quad t \geq 0. \tag{5.3}$$

Proof: The continuity of M is immediate from the continuity of the limiting process Y and Lemma 5.1. The rest is immediate from Lemma 5.2. ∎

Corollary 5.4 *There exists a Brownian motion β such that*

$$M_t \quad = \quad \int_0^t e^{-B_{Y_s}(\hat{\mathbb{W}}_s)} d\beta_s, \quad t \geq 0. \tag{5.4}$$

Proof: Immediate from the previous corollary.

Proof of Theorem 1.3: Immediate from Lemma 5.1, Corollary 5.3 and Corollary 5.4. ∎

Finally, we describe the snake process when g is constant. The description for the general case, more specifically, the uniqueness of the solution for the martingale problem (1.21) remains a challenging *open* problem.

When g is constant, say $g = 1$, we have that $B_t(x) = B_t$ is a Brownian motion with constant drift ν. It follows from the martingale problem (1.21) that

$$\int_0^{Y_t} e^{B_r} dr = \ell_t^0 - e^{-B_{K_1}} \ell_t^{K_1} + \int_0^t e^{B_{Y_s}} d\beta_s.$$

Therefore, Y_t is the Brox diffusion reflected at 0 and K_1 (see the Appendix for a description when $\nu = 0$).

Next, we consider the conditional (given the lifetime process) path process. Let $w = (\mathbf{w}, \zeta_w)$ be an element in $\mathcal{W}$. Fix $a \in [0, \zeta_w]$ and $b \geq a$. Similar to LeGall ([9], p54), we define $R_{a,b}(w, dw')$ as the unique probability measure on $\mathcal{W}$ such that

(i) $\zeta_{w'} = b, \;\; R_{a,b}(w, dw')$ a.s.

(ii) $w'(t) = w(t)$ for all $t \le a$, $R_{a,b}(w, dw')$ a.s.
(iii) Under $R_{a,b}(w, dw')$, $(\mathbf{w}'(a+t) : t \in [0, b-a])$ is a Brownian motion.

Denote the time set $Q_n = \{n^{-2}k : k = 0, 1, 2, \cdots\}$. From the construction of the discrete snake, it follows that $\mathbb{W}^n_s, \ \ s \in Q_n$ is a conditional (given Y^n) Markov chain with transition probability

$$R_{m^n(s,s'),Y^n(s')}(w, dw'), \qquad s < s' \in Q_n,$$

where $m^n(s, s') = \inf\{Y^n(r) : r \in [s, s'] \cap Q_n\}$.

Taking $n \to \infty$, we see that the limit $\{\mathbb{W}_s, \ s \ge 0\}$ is a conditional (given Y) Markov process with transition probability

$$R_{m(s,s'),Y(s')}(w, dw'), \qquad s < s',$$

where $m(s, s') = \inf\{Y(r) : r \in [s, s']\}$. Namely, it has the same conditional law as LeGall's Brownian snake.

6 Appendix: Convergence to a reflected Brox diffusion

We provide in this appendix a short, direct proof of Corollary 1.5 that bypasses the study of the branching process, relying instead on an embedding of a random walk in random environment (RWRE) into a diffusion in random environment, in the spirit of [13]. For backround on Brownian motion in random environments we refer to [2], [12], [15] and to the nice overview in [13]. Background for RWRE can be found in [16].

Recall that a Brownian motion in random environment (BMRE) is a process X_t given by

$$dX_t = d\beta_t - \frac{1}{2}V'(X_t)dt, \tag{6.1}$$

where β_t is a Brownian motion and V is called the random potential. When V is itself a Brownian motion independent of β, this (formal) process is the Brox diffusion [2].

We need to consider reflecting BMRE's. Let h be the periodic function with period $2K_1$ and $h(x) = |x|$ for $|x| \le K_1$. Let V be a Brownian motion on $x \in [0, K_1]$ and set $\hat{V}(x) = V(h(x))$ for $x \in \mathbb{R}$. Set formally

$$dZ_t = d\beta_t - \frac{1}{2}\hat{V}'(Z_t)dt. \tag{6.2}$$

(In case V is not smooth, a precise meaning is given to (6.2) by the procedure described in [13, Section 2]). Let $Y_t = h(Z_t)$. A *formal* application of the Itô-Tanaka formula yields

$$\begin{aligned} dY_t &= h'(Z_t)dZ_t + d\ell_t^{Y,0} - d\ell_t^{Y,K_1} \qquad (6.3)\\ &= h'(Z_t)d\beta_t - h'(Z_t)\frac{1}{2}\hat{V}'(Z_t)dt + d\ell_t^{Y,0} - d\ell_t^{Y,K_1} \\ &= d\tilde{\beta}_t - \frac{1}{2}V'(Y_t)dt + d\ell_t^{Y,0} - d\ell_t^{Y,K_1}, \end{aligned}$$

where $\tilde{\beta}$ is a Brownian motion. To justify (6.3), one argues as follows. First, an application of Ito's formula for Dirichlet processes, see e.g. [6], gives that for any g which is twice differentiable, and with $Y_t^g = g(Z_t)$,

$$dY_t^g = g'(Z_t)dZ_t + \frac{1}{2}g''(Z_t)dt\,. \tag{6.4}$$

Now note that, by definition of the local time as the occupation time density, the local times of Z and Y at levels 0 and K_1 are equal up to multiplicative constant 2. Therefore a standard approximation of h by smooth functions g, together with (6.4), yields (6.3), provided that the local time $\ell_t^{Z,x}$ of $Z_\cdot$ is jointly continuous in t and x, the latter at $x = 0$ and $x = K_1$. However, $\ell_t^{Z,x}$ is a continuous transformation of the local time of the Brownian motion β_t (see e.g. Equation (10) in [1] for an explicit formula which holds for any environment—not necessarily for the two sided white noise), and thus is jointly continuous in its arguments. This yields (6.3). Therefore, $Y_\cdot$ is a reflecting (at 0 and K_1) Brox diffusion.

6.1 Embedding

In this subsection, we introduce an environment and represent Y^n as a RWRE, which we then proceed (after scaling of the environment) to embed in a diffusion in random environment.

Let the environment be given by a family $\{\xi^n(i),\ i \in \mathbb{Z}_+\}$ of independent random variables with mean 0 and variance 1. We further assume that $|\xi^n(i)| \le \sqrt{n}$. Define the potential $V^n(\cdot)$ on $\mathbb{R}_+$ by

$$V^n(x) = \sum_{i=1}^{[x]} \log \frac{\frac{1}{2} - \frac{1}{4\sqrt{n}}\xi^n(i)}{\frac{1}{2} + \frac{1}{4\sqrt{n}}\xi^n(i)},$$

and set $\hat{V}^n(x) = V^n(nh(x/n))$ and let $\hat{Z}^n$ be the BMRE with potential $\hat{V}^n$. Set $Z^n(t) = n^{-1}\hat{Z}^n(n^2 t)$. Define the stopping times $\sigma_0^n = 0$ and

$$\sigma_{m+1}^n = \inf\{t > \sigma_m^n :\ \ |Z^n(t) - Z^n(\sigma_m^n)| = 1/n\}\,.$$

By Schumacher's theorem (cf. Schumacher [12] and Shi [13]), we have

Lemma 6.1 *Let $\tilde{Z}_m^n = nZ^n(\sigma_m^n)$, $\ m = 0, 1, 2, \cdots$. Then $\tilde{Z}^n$ is a RWRE with*

$$\mathbb{P}^\xi\left(\tilde{Z}_{m+1}^n = i \pm 1 \middle| \tilde{Z}_m^n = i\right) = \frac{1}{2} \pm \frac{1}{4\sqrt{n}}\xi^n(nh(i/n)),$$

where $\mathbb{P}^\xi$ is the probability measure conditioned on the environment ξ.

The next proposition is crucial for the proof of Corollary 1.5.

Proposition 6.2 *The sequence of processes $\left\{\frac{1}{n}\tilde{Z}_{\lfloor tn^2 \rfloor}^n, t \ge 0\right\}_{n\ge 1}$ converges weakly in $D_{\mathbb{R}}[0,\infty)$ to the process Z which satisfies (6.2).*

Remark 6.3 *Note that* $\tilde{Y}^n \equiv h(\tilde{Z}^n)$ *is a sequence of reflecting (at* 0 *and* nK_1*) RWRE such that*

$$\mathbb{P}^{\xi}\left(\tilde{Y}^n_{m+1} = i \pm 1 \middle| \tilde{Y}^n_m = i\right) = \frac{1}{2} \pm \frac{1}{4\sqrt{n}}\xi^n(i), \quad i = 1, \ldots, nK_1 - 1,$$

and hence by the continuity of the function h *and the discussion in the beginning of the appendix, in order to prove Corollary 1.5 it is sufficient to prove Proposition 6.2.*

The rest of the appendix is devoted to the proof of Proposition 6.2.

The following is a straight-forward consequence of Section 3 of [13].

Lemma 6.4 Z^n *is the Brownian motion in random environment with potential* $\hat{V}^n(nx)$*.*

Proof: Let

$$\hat{A}^n_x = \int_0^x e^{\hat{V}^n(y)} dy.$$

As $\hat{Z}^n$ is the BMRE with potential $\hat{V}^n$, it is well-known (see (2.3) in [13]) that $\hat{A}^n_{\hat{Z}^n(t)}$ is a local martingale with quadratic variation $\hat{\Theta}^n(t)$ such that

$$(\hat{\Theta}^n)^{-1}(t) = \int_0^t e^{-2\hat{V}^n(\hat{A}^n_{\hat{Z}^n(u)})} du.$$

We now rescale. Let

$$A^n_x = \int_0^x e^{\hat{V}^n(ny)} dy.$$

Then

$$\hat{A}^n_{\hat{Z}^n(t)} = \int_0^{nZ^n(n^{-2}t)} e^{\hat{V}^n(z)} dz = n \int_0^{Z^n(n^{-2}t)} e^{\hat{V}^n(ny)} dy = nA^n_{Z^n(n^{-2}t)}.$$

Thus $A^n_{Z^n(t)}$ is a local martingale with quadratic variation process $\Theta^n(t) = n^2\hat{\Theta}^n(n^2 t)$. Thus,

$$(\Theta^n)^{-1}(t) = n^{-2} \int_0^{n^{-2}t} e^{-2\hat{V}^n(nA^n_{Z^n(n^{-2}u)})} du = \int_0^t e^{-2\hat{V}^n(nA^n_{Z^n(u)})} du.$$

Therefore (see again (2.3) and (2.5) in [13]), Z^n is the BMRE with potential $\hat{V}^n(nx)$. ∎

6.2 Scaling limit

As was proved in the previous subsection (see Lemma 6.1), the scaled RWRE is related to BMRE by

$$\frac{1}{n}\tilde{Z}^n_{[n^2t]} = Z^n(\sigma^n_{[n^2t]}). \tag{6.5}$$

In this section, we first prove that

$$\sigma^n_{[n^2t]} \to t, \quad \text{as } n \to \infty, \tag{6.6}$$

by the strong law of large numbers. Then, we prove that the scaled potential for Z^n converges to $\hat{V}$, and hence Z^n converges to a BMRE with potential $\hat{V}$. This by (6.5) and (6.6) will provide the proof of Proposition 6.2.

Lemma 6.5 *As $n \to \infty$, we have*

$$\sigma^n_{[n^2t]} \to t, \qquad a.s.$$

uniformly on compact sets.

Proof: By Proposition 3.2 in [13] (or a direct computation involving a time change), we see that $\theta_i = n^2(\sigma^n_i - \sigma^n_{i-1})$, $i = 1, 2, \cdots$, are i.i.d. with the same distribution as

$$\theta = \inf\{t > 0 : \ |W(t)| = 1\},$$

where W is a standard Brownian motion. Note that $\mathbb{E}\theta = 1$. By the strong law of large numbers, we get that

$$\sigma^n_{[n^2t]} = t\frac{1}{n^2t}\sum_{i=1}^{[n^2t]} \theta_i \to t$$

uniformly on compacts. ∎

For the next lemma, recall that Z is the processes that satisfies (6.2).

Lemma 6.6 *As $n \to \infty$, $Z^n \Longrightarrow Z$ weakly in $C_{\mathbb{R}}[0, \infty)$.*

Proof: First we consider the weak convergence of $V^n(nx)$. Note that

$$V^n(nx) = \sum_{i=1}^{[nx]} \frac{1}{\sqrt{n}}\xi^n(i) + o(1) \equiv M^n_x + o(1).$$

Regarding x as the time-parameter, $\{M^n_x, \ x > 0\}$ is a martingale with predictable quadratic variation process

$$\langle M^n \rangle_x = \sum_{i=1}^{[nx]} \mathbb{E}\left(\frac{1}{\sqrt{n}}\xi^n(i)\right)^2 \to x$$

uniformly on the compacts. Thus, by Theorem 4.13 ([7], P358), M^n converges weakly in $D_{\mathbb{R}}[0,\infty)$ to a Brownian motion $V(x), x \geq 0$. By switching to another probability space if necessary, we may and will assume that all weak convergences hold almost surely. Then we get, $V^n(n\cdot) \to V$, a.s.. Note that by the continuity of h, we immediately get that

$$\hat{V}^n(x) \to \hat{V}(x) = V(h(x)), \quad a.s..,$$

uniformly on the compacts of $\mathbb{R}_+$. Note that (see (2.6) in [13]),

$$Z^n(t) = (A^n)^{-1}(W^n((T^n)^{-1}(t))),$$

where

$$A^n_x = \int_0^x e^{\hat{V}^n(ny)} dy,$$

$$T^n(t) \quad = \quad \int_0^t e^{-2\hat{V}^n\left(n(A^n)^{-1}_{W^n(u)}\right)} du,$$

and W^n is a Brownian motion. Since W^n trivially converges weakly to the Brownian motion W, we assume as before that the convergence holds a.s.. Then we have

$$A^n_x \to \int_0^x e^{\hat{V}(y)} dy = A_x, \qquad \text{as } n \to \infty,$$

and

$$T^n(t) \quad \to \quad \int_0^t e^{-2\hat{V}\left(A^{-1}_{W(u)}\right)} du = T(t), \qquad \text{as } n \to \infty.$$

Note that all the convergence above are a.s. and uniform on compacts. We see that

$$Z^n(t) \to A^{-1}_{W(T^{-1}(t))} \equiv Z(t). \tag{6.7}$$

By stochastic calculus as in Section 2 of [13], it follows that (6.7) defines a BMRE $Z(t)$ with potential $\hat{V}$. ∎

Now Proposition 6.2 follows from Lemmas 6.5, 6.6, and (6.5). Then as we have mentioned already in Remark 6.3, Corollary 1.5 follows immediately from Proposition 6.2.

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