Uniform Decay and Equicontinuity for Normalized, Parameter Dependent, Ito Integrals

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Abstract

Let $\{M_t(\theta), t \geq 0\}_{\theta \in I\!R^d}$ be a collection of continuous, continuous-time martingales such that for all $t > 0$, the associated increasing processes satisfy $< M(\theta) >_t \to \infty$ as $\|\theta\| \to \infty$. We show that if $< M(\theta) >_t$ grows with $\|\theta\|$ sufficiently fast, then $M_t(\theta)/ < M(\theta) >_t \to 0$ as $\|\theta\| \to \infty$ uniformly in $t \in [t_0, \infty), t_0 > 0$. An equicontinuity property for normalized, parameter dependent stochastic integrals follows. These results serve in the study of the maximum likelihood estimation problem, over unbounded sets, for diffusion processes.

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1 Introduction

This paper is motivated by a problem which arises in the study of parameter estimation in continuous-time stochastic processes with unbounded parameter sets.

Consider the family of processes $\left\{x_t^\theta, \quad t \geq 0\right\}_{\theta \in \mathbb{R}^d}$ which satisfy the SDEs,

$$dx_t^\theta = m(\theta, x_t^\theta, t)\, dt + dw_t \ , \qquad x_0^\theta = 0 \tag{1.1}$$

where $\{w_t, \quad t \geq 0\}$ is standard Brownian motion and the drift function m satisfies some regularity conditions which ensure that the measures induced by $\left\{x^\theta\right\}_{\theta \in \mathbb{R}^d}$ on $C[0,T]$ are mutually equivalent $\forall\, T < \infty$. Assume that there exists a $\theta^* \in \mathbb{R}^d$ s.t. $x = x^{\theta *}$ is the observed process. In this case, the well known log-likelihood is of the form [1, Vol. II]

$$L_t(\theta) = M_t(\theta) - \frac{1}{2} < M(\theta) >_t \tag{1.2}$$

where the martingale $M(\theta)$ and its increasing process $< M(\theta) >$ are,

$$M_t(\theta) = \int_0^t [m(\theta, x_s, s) - m(\theta^*, x_s, s)] dw_s \ , \ < M(\theta) >_t = \int_0^t [m(\theta, x_s, s) - m(\theta^*, x_s, s)]^2 ds.$$

In the analysis of the behavior of the maximum likelihood estimator (MLE), one faces the problem of explosion. Frequently used approaches are either to restrict estimation to a bounded domain (in which θ^* is assumed to lie) see e.g. [3-4], or to show the existence of a consistent root to the estimation equation $\nabla L_t(\theta) = 0$ [5], which is essentially a local-type method. On the other hand, in the global optimization approach the true MLE fits the data best and is in general different from the "projected" estimator (at least on finite time horizons). In this case, the explosion of $\hat{\theta}_t (=$ MLE at time $t)$ can be ruled out when large θ's are highly penalized. More precisely, if for some (deterministic) $t_0 > 0$,

$$\lim_{\|\theta\| \to \infty} \sup_{t \geq t_0} L_t(\theta) = -\infty \qquad \text{a.s.} \tag{1.3}$$

then, it can be deduced that $\sup_{t \geq t_0} \|\hat{\theta}_t\| < \infty$ a.s. This relies on the fact that by definition, $L_t(\hat{\theta}_t) \geq L_t(\theta) \ \forall\, \theta \in \mathbb{R}^d$ and in particular, $L_t(\hat{\theta}_t) \geq L_t(\theta^*) = 0$.

Remark: In fact, stability of $\hat{\theta}$ could be ensured if (1.3) is relaxed to $\lim_{\|\theta\| \to \infty} \sup_{t \geq t_0} L_t(\theta) < 0$ a.s. However, weak conditions under which this holds (and (1.3) does not) are very hard to find.

Now, by definition (1.2) it is seen that (1.3) holds under the following two conditions:

$$\lim_{\|\theta\|\to\infty} < M(\theta) >_{t_0} = \infty \qquad \text{a.s.} \quad , \qquad t_0 > 0 \tag{1.4}$$

$$\lim_{\|\theta\|\to\infty} \sup_{t\geq t_0} |M_t(\theta)| / < M(\theta) >_t = 0 \qquad \text{a.s.} \quad , \qquad t_0 > 0 \tag{1.5}$$

The proof of (1.5) (under some strengthening of (1.4)) is the main concern of this paper.

We note that contrary to the classical LLN for continuous-time martingales (i.e. $M_t / < M >_t \to 0$ as $t \to \infty$ a.e. on $\{< M >_t \uparrow \infty\}$ [2]), (1.5) (under (1.4)) is far from obvious. The main difficulty lies in the need for a sufficiently fast growth rate of the LHS in (1.4). Consider (as a simplified counterexample) a collection $\{U_n(t), t \geq 0\}_{n\geq 1}$ of mutually independent, continuous-time martingales s.t. $\lim_{n\to\infty} < U_n >_{t_0} = \infty$ a.s. $(t_0 > 0)$. If

$$\lim_{n\to\infty} (\log n)^{-1} < U_n >_{t_0} = 0 \quad \text{a.s.} \quad , \tag{1.6}$$

then, using random time change, i.e. representing the martingales $\{U_n\}$ as time changed Brownian motions, it can rather easily be shown that $P(\sup_{t\geq t_0} |U_n(t)| / < U_n >_t > \varepsilon \text{ i.o.}) = 1$ $(\forall\, \varepsilon > 0)$, which implies that (1.5) fails to hold for $\{U_n\}$.

Remark: The case under study in this paper differs from the example above by the fact that the martingales $\{M(\theta)\}_{\theta\in\mathbb{R}^d}$ are mutually dependent (contrary to the independence of $\{U_n\}_n$).

While this example could suggest that (1.4) with a growth rate of $\log\|\theta\|$ will do, our result relies on a slightly higher rate of $\|\theta\|^\gamma$ (any $\gamma > 0$). It is important to remark that if $\gamma > 1$, the main result (i.e. $M / < M > \to 0$ uniformly in $[t_0, \infty]$ as $\|\theta\| \to \infty$) could be proved with relative ease (under some strengthening of condition C below, see [3, assumption A_6-ii]). However, in light of the previous discussion which suggests that a logarithmic rate suffices, this is a somewhat weak result. Moreover, while one may expect that with M (or more precisely its integrand) satisfying some regularity conditions (see A,B,C below), it holds that $\|\theta\|^{-\beta} M_t(\theta) / < M(\theta) >_t \to 0$ as $\|\theta\| \to \infty$ (uniformly in $[t_0, \infty)$) $\forall \beta > 0$ (take for example M independent of θ), a "cheap" proof based on Kolmogorov's continuity criterion works only for the case $\beta > 1$. Our method of proof reveals that indeed $\beta > 0$ suffices, see corollary 3.4.

The paper is organized as follows: The basic setup is introduced in the next section. Section 3 is devoted to the main result which is based on two technical lemmas whose proofs are given in the appendix. An equicontinuity theorem for normalized, parameter dependent stochastic integrals is presented in the last section. This theorem, which is implied by lemma 3.3 below, serves in the analysis of the asymptotic behavior of the MLE in diffusion processes, see e.g. [7,8] and a related application in [3].

Notations: For any cube $A \in I\!\!R^n$, $I\!\!B(A)$ denotes the Borel σ-algebra on A with $I\!\!B^n = I\!\!B(I\!\!R^n)$. $\|\cdot\|$ stands always for the Euclidean norm . Θ denotes an arbitrary countable, dense subset of $I\!\!R^d$. Finally, for any $a \in I\!\!R^d, b > 0$, let $B(a,b)$ denote the open ball in $I\!\!R^d$ of center a and radius b.

2 Problem Formulation

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $(w_t, \mathcal{F}_t)$ a standard Brownian motion, and let $f : I\!\!R^d \times \Omega \times I\!\!R_+ \to I\!\!R$ satisfy A,B,C below:

A $\{\theta, \omega, s \in [0,t] \mid f(\theta,\omega,s) \in A\} \in I\!\!B^d \otimes \mathcal{F}_t \otimes I\!\!B[0,t] \quad \forall\, A \in I\!\!B^1, \quad t < \infty$

B $\exists \quad \eta > \varepsilon \geq 0, \quad h(\omega,t) \in \mathcal{F}_t \otimes I\!\!B[0,t]$ s.t.
(i) $|f(\theta,\omega,t) - f(\theta',\omega,t)|^2 \leq h^2(\omega,t)(\|\theta-\theta'\|^{1+\varepsilon} \vee \|\theta-\theta'\|^{1+\eta}) \quad \forall \theta,\theta' \in I\!\!R^d,\ t \in I\!\!R_+, P-$ a.s.
(ii) $f^2(0,\omega,t) \leq h^2(\omega,t) \quad \forall\, t \in I\!\!R_+, P-$ a.s.

C $\exists$ integer $\ell \geq 1$ s.t.

$$\sup_{t>s\geq 0, t\geq 1} \frac{1}{t^{2m(\ell-1)}(t-s)} \int_s^t Eh^{2m}(\omega,r)dr < \infty \quad \forall\, m \geq 1$$

Remarks:

1. A is a progressive measurability-type condition.
2. Condition B-i implies that f is Hölder continuous with parameter larger than $1/2$, whereas B-ii is implied by B-i if there exists a bounded "zero term" θ_0 (may be random or t-dependent) s.t. $f(\theta_0,\omega,t) \equiv 0$ (e.g. in MLE application where $f(\theta,\omega,t) = m(\theta,\omega,t) - m(\theta_0,\omega,t)$).
3. Condition C may be interpreted as an $\ell - 1$-degree polynomial growth (in t) of h.

The collection of continuous time martingales under study is denoted by $\{F(\theta,t)\ , \quad \mathcal{F}_t, \quad t \geq 0\}_{\theta \in I\!\!R^d}$ where for each (θ,t), $F(\theta,t) = \int_0^t f(\theta,\omega,s)\, dw_s$ is an Ito stochastic integral, whose corresponding increasing process is $< F(\theta,\cdot) >_t = \int_0^t f^2(\theta,\omega,s)\, ds$.

Conditions A-C imply that F and $< F >$ are (θ, t) jointly continuous in mean square and hence in probability. This leads to the existence of measurable separable versions. Throughout, we consider only those jointly separable versions (denoted also by F and $< F >$).

Lemma 2.1 *Assume conditions A,B,C hold. Then F and its increasing process $< F >$ have jointly continuous paths over $I\!R^d \times I\!R_+$, a.s.*

Proof: Fix $K \subset\subset I\!R^d$, $T < \infty$ then $\forall\, \theta,\ \theta' \in K, \quad \|\theta - \theta'\| \leq 1, \quad m \geq 1$ we have, for some $C = C(m)$,

$$
\begin{aligned}
& E \sup_{0 \leq t \leq T} |F(\theta, t) - F(\theta', t)|^{2m} = E \sup_{0 \leq t \leq T} |\int_0^t [f(\theta, \omega, s) - f(\theta', \omega, s)] dw_s|^{2m} \\
& \leq CE(\int_0^T [f(\theta, \omega, s) - f(\theta', \omega, s)]^2 ds)^m \leq \|\theta - \theta'\|^{m(1+\varepsilon)} CE(\int_0^T h^2(\omega, s) ds)^m \\
& \leq \|\theta - \theta'\|^{m(1+\varepsilon)} CT^{m-1} \int_0^T Eh^{2m}(\omega, s) ds
\end{aligned}
$$

The last three inequalities follow, respectively, from the Burkholder-Gundy inequality, B-i and Hölder's inequality. Choose next $m = d + 1$ to conclude that

$$
E \sup_{0 \leq t \leq T} |F(\theta, t) - F(\theta', t)|^{2(d+1)} \leq \tilde{C} \|\theta - \theta'\|^{d+\mu}, \tilde{C} = \tilde{C}(d, T), \quad \text{some} \quad \mu > 0
$$

The separability, together with Kolmogorov's continuity theorem for random fields on $I\!R^d$ (see e.g. [2, thm. I-2.1]) imply that F has θ-continuous samples, uniformly in $t \in [0, T]$ a.s. that is,

$$
\lim_{\Delta \downarrow 0} \sup_{\substack{\theta, \theta' \in K \\ \|\theta - \theta'\| \leq \Delta}} \sup_{0 \leq t \leq T} |F(\theta, t) - F(\theta', t)| = 0 \qquad \text{a.s.} \tag{2.1}
$$

We next show that F is t-continuous, uniformly in $\theta \in K$. Towards this end, choose $\varepsilon > 0$ and define

$$
\eta(\theta, \epsilon) = \sup\{\delta > 0 \mid \sup_{0 \leq t \leq T} |F(\theta, t) - F(\theta', t)| < \varepsilon \text{ whenever } \theta' \in K, \|\theta - \theta'\| < \delta\}.
$$

Due to (2.1), $\eta(\theta, \epsilon) > 0$ a.s. . Clearly, $\bigcup_{\theta \in K} B(\theta, \eta(\theta, \epsilon))$ is a (random) open cover of K. Since Θ is dense in K, it follows that also $\bigcup_{\theta \in K \bigcap \Theta} B(\theta, \eta(\theta, 2\epsilon))$ is a random open cover of K. Hence, there exists a $k = k(\omega) < \infty$ a.s. and $\{\theta_i\}_{i=1}^k \in K \bigcap \Theta$ s.t. $K \subset \bigcup_{i=1}^k B(\theta_i, \eta(\theta_i, 2\epsilon))$ a.s. . Thus,

$$
\lim_{\delta \downarrow 0} \sup_{\substack{t, t' \in [0, T] \\ |t - t'| \leq \delta}} \sup_{\theta \in K} |F(\theta, t) - F(\theta, t')| <
$$

$$< 2\varepsilon + \lim_{\delta \downarrow 0} \sup_{\substack{t,t' \in [0,T] \\ |t-t'| \leq \delta}} \max_{1 \leq i \leq k} |F(\theta_i, t) - F(\theta_i, t')| \quad = \quad 2\varepsilon \quad \text{a.s.} \tag{2.2}$$

where the last equality is due to the (a.s.) sample path continuity of $\{F(\theta_i, \cdot)\}_{\theta_i \in \Theta}$, which holds because Θ is countable. Since (2.2) holds for all $\varepsilon > 0$, the a.s. joint continuity of F over $K \times [0,T]$ follows by using (2.1). The extension to $I\!R \times I\!R^+$ is obtained via monotone convergence by taking $K_n = [-n,n]^d$, $T_n = n$ and letting $n \to \infty$.

The corresponding statement for $< F >$ relies on the Hölder continuity B-i, the a.s. boundness of the Lebesgue integrals $\int_0^t h^2(\omega, s)\, ds$ and $\int_0^t f^2(\theta, \omega, s)\, ds$ (by C and B respectively) and the (a.s.) t-continuous samples of those integrals (where the properties of the latter are satisfied apriori outside a θ-dependent null set). The details which are technical and straightforward are omitted. □

3 The Main Result

The following condition is needed for the statement of our main result.

D There exists a $\gamma > 0$ s.t. $\forall\, t_0 > 0$

$$\lim_{\|\theta\| \to \infty} \inf_{t \geq t_0} \frac{< F(\theta, \cdot) >_t}{(t\|\theta\|)^\gamma} = \infty \quad \text{a.s.}$$

Theorem 3.1 *Let conditions A-D hold. Then*

$$\lim_{\|\theta\| \to \infty} \sup_{t \geq t_0} |F(\theta, t)| / < F(\theta, \cdot) >_t = 0 \quad a.s. \quad \forall\, t_0 > 0 \tag{3.1}$$

The proof of theorem (3.1) is based on lemmas 3.2-3.3 below, whose proofs are defered to the appendix. We first introduce some definitions.

For a fixed θ, define the $\mathcal{F}_t$-stopping times $\tau^\theta(s)$

$$\tau^\theta(s) = \inf\ \{t > 0 \mid < F(\theta, \cdot) >_t > s\} \tag{3.2}$$

where $\tau^\theta(s) = \infty$ if $< F(\theta, \cdot) >_\infty \leq s$.

Let $t < \infty$ be fixed, let $\nu \in (0, 1/2]$ and define the "truncated" random field $Z_t : I\!R^d \times I\!R_+ \to I\!R$

$$Z_t(\theta, s) = F(\theta, \tau^\theta(s) \wedge t)/\phi(s), \qquad Z_t(\theta, 0) = 0 \tag{3.3}$$

where

$$\phi(s) = s^{(1-\nu)/2}, \quad s \in [0,1); \quad \phi(s) = s^{(1+\nu)/2}, \quad s \in [1,\infty) \tag{3.4}$$

Lemma 3.2 *Assume A,B,C hold. Then Z possesses a modification (denoted also by Z) which satisfies*

$$\sup_{t\geq t_0} \sup_{\theta\in K} \sup_{s>0} t^{-\ell\nu} |Z_t(\theta,s)| < \infty \quad a.s. \quad \forall\, K \subset\subset I\!R^d, \quad t_0 > 0 \tag{3.5}$$

where ℓ is the integer in hypothesis C.

Lemma 3.3 *Under conditions A,B,C it holds that,*

$$\sup_{t\geq t_0} \sup_{\theta\in K} t^{-\ell\nu} \frac{|F(\theta,t)|}{\phi(< F(\theta,\cdot) >_t)} < \infty \quad a.s. \quad \forall\, K \subset\subset I\!R^d, \quad t_0 > 0 \tag{3.6}$$

Before we turn to the proof of the theorem, we outline the role of (3.6) in the proof of theorem 3.1 and the need of the auxiliary lemma 3.2 to by-pass some technical problems on the way to establish (3.6).

The main step in proving (3.1) is to bound the LHS by a product of two terms. The first, which tends to zero (as $\|\theta\| \to \infty$) under the growth condition D and the second, which is bounded in the form of (3.6). To illustrate this idea, consider the case $\theta \in I\!R^1$. Let $\tilde{F}(\beta,\cdot) = r(\beta)F(1/\beta,\cdot)$, $|\beta| \in (0,1]$ and $\tilde{F}(0,\cdot) = 0$ where $r : [-1,1] \to [0,1]$ is such that the corresponding integrand $\tilde{f}$ satisfies conditions A,B,C. Then, for all large N's

$$\sup_{|\theta|\geq N} \sup_{t\geq t_0} |F(\theta,t)| \Big/ < F(\theta,\cdot) >_t \leq \sup_{|\beta|\leq 1/N} \sup_{t\geq t_0} t^{\nu\ell}(\beta) \frac{\phi(< \tilde{F}(\beta,\cdot) >_t)}{< \tilde{F}(\beta,\cdot) >} \sup_{|\beta|\leq 1} \sup_{t\geq t_0} t^{-\nu\ell} \frac{|\tilde{F}(\beta,t)|}{\phi(< \tilde{F}(\beta,\cdot) >_t)}$$

Now, under D (and with a careful choice of $r(\cdot)$), by letting $N \to \infty$ the first term in the product on the RHS$\to 0$ while the second is bounded due to lemma 3.3. The transformation $F \to \tilde{F}$ is needed because $|\theta| \to \infty$ while (3.6) holds on compact parameter sets.

The a.s. boundness in (3.6) could be obtained (loosely speaking) by showing the (θ,t) (a.s.) joint continuity of $F/\phi(< F >)$ via Kolmogorov's continuity criterion. This requires to compute the α-moments (some $\alpha > 0$) of differences of $F/\phi(< F >)$ for fixed (θ,t), (θ',t') values, to obtain a bound which is of order $\|(\theta,t) - (\theta',t')\|^{d+\beta}$ (some $\beta > 0$). However, due to the random denominator such a task is very hard to perform (it may be impossible: For example, splitting terms by Hölder inequalities does not work). The idea is therefore to use time change to transform $F/\phi(< F >)$ to Z for which Kolmogorov's condition could be established (Z has a deterministic denominator), and then to use (3.5) and a separability argument to bound the LHS of (3.6).

Proof of Theorem 3.1

Let $K = [-1,1]^d$. Define $\varphi : K \setminus G \to I\!R^d$ and $r : K \setminus G \to (0,1]$ as follows

$$\varphi(\beta) = \{\varphi_i(\beta)\}_{i=1}^d \quad , \quad \varphi_i(\beta) = \begin{cases} 1/\beta_i - 1 \,, & \beta_i \in (0,1] \\ 1/\beta_i + 1 \,, & \beta_i \in [-1,0) \end{cases}$$

$$r(\beta) = \min_{1\le i\le d} |\beta_i|^{1+\eta} \qquad\qquad (\eta \quad \text{as in assumption} \quad B)$$

where $G = \{x = (x_1, \ldots, x_d) \in K \mid \ x_i = 0 \text{ for some } i\}$. With these definitions, construct the martingales $\tilde{F}$ and their associated increasing processes $< \tilde{F} >$:

$$\tilde{f}(\beta,\omega,t) = \begin{cases} r(\beta) f(\varphi(\beta),\omega,t) & , \ \min_{1\le i\le d} |\beta_i| > 0 \\ 0 & , \text{ otherwise} \end{cases}$$

$$\tilde{F}(\beta,t) = \int_0^t \tilde{f}(\beta,\omega,s)dw_s \ , \ < \tilde{F}(\beta,\cdot) >_t = \int_0^t \tilde{f}^2(\beta,\omega,s)ds$$

We check below that $\tilde{f}$ satisfies conditions A–C on K. Hence by applying lemma 3.3 we can conclude that

$$\sup_{t\ge t_0} \sup_{\beta\in K} t^{-\nu\ell} \frac{|\tilde{F}(\beta,t)|}{\phi(< \tilde{F}(\beta,\cdot) >_t)} = C(\omega) < \infty \quad \text{a.s.} \quad \forall\, \nu, \quad t_0 > 0 \tag{3.7}$$

Recall that by definition, for any $\theta \in I\!R^d$, $\beta = \varphi^{-1}(\theta) \in K \setminus G$ and

$$F(\theta,t) = F(\varphi(\beta),t) = \frac{\tilde{F}(\beta,t)}{r(\beta)}, < F(\theta) >_t = \frac{< \tilde{F}(\beta)_t >}{r^2(\beta)}.$$

Substituting in (3.1) we have

$$\begin{aligned}
&\limsup_{\|\theta\|\to\infty} \sup_{t\ge t_0} |F(\theta,t)| \Big/ < F(\theta,\cdot) >_t = \limsup_{0 < \min_i |\beta_i| \to 0} \sup_{t\ge t_0} r(\beta) \frac{|\tilde{F}(\beta,t)|}{< \tilde{F}(\beta,\cdot) >_t} \le \\
&\le \sup_{\beta\in K} \sup_{t\ge t_0} t^{-\nu\ell} \frac{|\tilde{F}(\beta,t)|}{\phi(< \tilde{F}(\beta,\cdot) >_t)} \limsup_{0 < \min_i |\beta_i| \to 0} \sup_{t\ge t_0} t^{\nu\ell} r(\beta) \frac{\phi(< \tilde{F}(\beta,\cdot) >_t)}{< \tilde{F}(\beta,\cdot) >_t} \\
&\le C(\omega) \limsup_{0 < \min_i |\beta_i| \to 0} \sup_{t\ge t_0} t^{\nu\ell} r(\beta) \frac{\phi(< \tilde{F}(\beta,\cdot) >_t)}{< \tilde{F}(\beta,\cdot) >_t}
\end{aligned} \tag{3.8}$$

Thus, the theorem is proved if we show that

$$\lim_{\|\theta\|_\infty\to\infty} \inf_{t\ge t_0} t^{-\nu\ell} \frac{1}{\|\theta\|_\infty^{1+\eta}} < F(\theta,\cdot) >_t / \phi\left(\frac{< F(\theta,\cdot) >_t}{\|\theta\|_\infty^{2(1+\eta)}}\right) = \infty \quad \text{a.s.} \tag{3.9}$$

where $\|\theta\|_\infty = \max_{1\le i\le d} |\theta_i|$ and

$$r(\beta) = r(\varphi^{-1}(\theta)) = \min_{1\le i\le d}\left(\frac{1}{|\theta_i|+1}\right)^{1+\eta} = (\|\theta\|_\infty + 1)^{-(1+\eta)} \approx \|\theta\|_\infty^{-(1+\eta)}$$

(for large $\|\theta\|_\infty$).

Equation (3.9) easily follows from assumption D and the definition of ϕ,

$$\frac{1}{t^{\ell\nu}}\frac{< F(\theta,\cdot) >_t}{\|\theta\|_\infty^{1+\eta}\,\phi(< F(\theta,\cdot) >_t / \|\theta\|_\infty^{2(1+\eta)})} = \begin{cases} \frac{1}{t^{\ell\nu}}\frac{<F(\theta,\cdot)>_t^{(1+\nu)/2}}{\|\theta\|_\infty^{(1+\eta)\nu}} & ;\ < F(\theta,\cdot) >_t / \|\theta\|_\infty^{2(1+\eta)} \le 1 \\ \frac{1}{t^{\ell\nu}}\|\theta\|_\infty^{(1+\eta)\nu} < F(\theta,\cdot) >_t^{(1-\nu)/2} & ;\ \text{otherwise} \end{cases}$$

$$= \begin{cases} \left(\frac{1}{t^{2\ell\nu/(1+\nu)}}\frac{<F(\theta,\cdot)>_t}{\|\theta\|_\infty^{2(1+\eta)\nu/(1+\nu)}}\right)^{(1+\nu)/2} & ;\ < F(\theta,\cdot) >_t \le \|\theta\|_\infty^{2(1+\eta)} \\ \left(\frac{1}{t^{2\ell\nu/(1-\nu)}}\|\theta\|_\infty^{2(1+\eta)\nu/(1-\nu)} < F(\theta,\cdot) >_t\right)^{(1-\nu)/2} & ;\ \text{otherwise} \end{cases}$$

Now note that $\|\theta\|_\infty \le \|\theta\|$ and take $0 < \nu \le 1/2 \wedge \gamma/4\ell(1+\eta)$ (which implies that $2(1+\eta)\nu/(1+\nu) \le 2\ell\nu(1+\eta) \le \gamma$ and $2\ell\nu/(1-\nu) \le \gamma$) to conclude that (under D) (3.9) holds.

It thus remains only to check that $\tilde{f}$ satisfies A-C. Since A is trivial, we concentrate on the latter two conditions.

Fix $\alpha, \beta \in K$ with $0 < \min|\beta_i| \le \min|\alpha_i| \le 1$, $(\|\alpha-\beta\|^2 \le 4d)$. Then

$$\begin{aligned} &|\tilde{f}(\alpha,\omega,t) - \tilde{f}(\beta,\omega,t)|^2 = |r(\alpha)f(\varphi(\alpha),\omega,t) - r(\beta)f(\varphi(\beta),\omega,t)|^2 \le \\ &\le 3r^2(\beta)|f(\varphi(\alpha),\omega,t) - f(\varphi(\beta),\omega,t)|^2 + 3(r(\alpha)-r(\beta))^2|f(\varphi(\alpha),\omega,t) - f(0,\omega,t)|^2 \\ &+3(r(\alpha)-r(\beta))^2 f^2(0,\omega,t) \end{aligned} \tag{3.10}$$

The evaluation of the three terms on the RHS of (3.10) is based on assumption B and the definitions of φ and r. Note that for α_i, β_i with opposite signs, say $\alpha_i > 0$, $\beta_i < 0$ it holds that

$$0 \le \varphi_i(\alpha) - \varphi_i(\beta) = \frac{1}{\alpha_i} - \frac{1}{\beta_i} - 2 \le \frac{1}{\alpha_i} - \frac{1}{\beta_i}$$

For the case of identical signs (for which $\varphi_i(\alpha) - \varphi_i(\beta) = \frac{1}{\alpha_i} - \frac{1}{\beta_i}$) and the fact that $|\varphi_i(\alpha)| \le |\frac{1}{\alpha_i}|$ (which implies that $\|\varphi(\alpha)\| \le \|1/\alpha\|$, $1/\alpha \triangleq \{1/\alpha_i\}_{i=1}^d$), we have

$$\begin{aligned} \text{(i)}\quad & r^2(\beta)|f(\varphi(\alpha),\omega,t) - f(\varphi(\beta),\omega,t)|^2 \le r^2(\beta)\|\varphi(\alpha)-\varphi(\beta)\|^{1+\eta}h^2(\omega,t) \\ & \le \min_{1\le i\le d}|\beta_i|^{2(1+\eta)}\left[\sum_{j=1}^d\left(\frac{\alpha_j-\beta_j}{\alpha_j\beta_j}\right)^2\right]^{(1+\eta)/2} h^2(\omega,t) = \end{aligned}$$

$$\left[\sum_{j=1}^{d}\frac{\min\limits_{1\le i\le d}|\beta_i|^4}{(\alpha_j\beta_j)^2}(\alpha_j-\beta_j)^2\right]^{(1+\eta)/2} h^2(\omega,t)$$
$$\le\left[\sum_{j=1}^{d}(\alpha_j-\beta_j)^2\right]^{(1+\eta)/2} h^2(\omega,t)=\|\alpha-\beta\|^{1+\eta}h^2(\omega,t)$$
$$\le\left(\|\alpha-\beta\|^{1+\varepsilon}\vee\|\alpha-\beta\|^{1+\eta}\right)h^2(\omega,t)$$

(ii) $$(r(\alpha)-r(\beta))^2|f(\varphi(\alpha),\omega,t)-f(0,\omega,t)|^2\le(r(\alpha)-r(\beta))^2\|\varphi(\alpha)\|^{1+\eta}h^2(\omega,t)$$
$$\le\left(\min_{1\le i\le d}|\alpha_i|^{1+\eta}-\min_{1\le i\le d}|\beta_i|^{1+\eta}\right)^2\left\|\frac{1}{\alpha}\right\|^{1+\eta}h^2(\omega,t)=$$
$$=(1+\eta)^2u^{2\eta}\left(\min_i|\alpha_i|-\min_i|\beta_i|\right)^2\left\|\frac{1}{\alpha}\right\|^{1+\eta}h^2(\omega,t),\ \text{some } u\in[\min_i|\beta_i|,\min_i|\alpha|]$$
$$=(1+\eta)^2\left|\min_i|\alpha_i|-\min_i|\beta_i|\right|^{1+\varepsilon}\left[(\min_i|\alpha_i|-\min_i|\beta_i|)\|\tfrac{1}{\alpha}\|\right]^{1-\varepsilon}\left[u\|\tfrac{1}{\alpha}\|\right]^{2\eta}\|\tfrac{1}{\alpha}\|^{\varepsilon-\eta}h^2(\omega,t)$$
$$\le(1+\eta)^2\max_{1\le i\le d}|\alpha_i-\beta_i|^{1+\varepsilon}\left[\sum_{j=1}^{d}\min_i\alpha_i^2/\alpha_j^2\right]^{(1-\varepsilon)/2}\left[\sum_{j=1}^{d}\left(\frac{u}{\alpha_j}\right)^2\right]^{\eta}h^2(\omega,t)$$
$$\le(1+\eta)^2d^{(1+\eta)/2}\max_{1\le i\le d}|\alpha_i-\beta_i|^{1+\varepsilon}h^2(\omega,t)\le(1+\eta)^2d^{(1+\eta)/2}\left(\|\alpha-\beta\|^{1+\varepsilon}\vee\|\alpha-\beta\|^{1+\eta}\right)h^2(\omega,t)$$

In a similar way we obtain (using B-ii)

(iii) $$(r(\alpha)-r(\beta))^2f(0,\omega,t)\le(1+\eta)^2\left(\|\alpha-\beta\|^{1+\varepsilon}\vee\|\alpha-\beta\|^{1+\eta}\right)h^2(\omega,t)$$

In the case where $\min\limits_{1\le i\le d}|\beta_i|=0,\ \tilde f(\beta,\omega,t)=0$ and

$$|\tilde f(\alpha,\omega,t)-\tilde f(\beta,\omega,t)|^2=\tilde f^2(\alpha,\omega,t)=r^2(\alpha)f^2(\varphi(\alpha),\omega,t)\le$$
$$\le 2r^2(\alpha)\left[\|\varphi(\alpha)\|^{1+\eta}+1\right]h^2(\omega,t)\le 2\min_{1\le i\le d}|\alpha_i|^{2(1+\eta)}[\|1/\alpha\|^{1+\eta}+1]h^2(\omega,t)$$
$$\le 2\min_{1\le i\le d}|\alpha_i|^{1+\eta}[d^{(1+\eta)/2}+1]h^2(\omega,t)\le 2\left[\sum_{j=1}^{d}(\alpha_j-\beta_j)^2\right]^{(1+\eta)/2}[d^{(1+\eta)/2}+1]h^2(\omega,t)$$
$$=2\left(\|\alpha-\beta\|^{1+\varepsilon}\vee\|\alpha-\beta\|^{1+\eta}\right)[d^{(1+\eta)/2}+1]h^2(\omega,t)$$

Combining this with (i-iii) and defining $\tilde h=ch,\ c^2=3(1+\eta)^2d^{(1+\eta)/2}$ results in

$$|\tilde f(\alpha,\omega,t)-\tilde f(\beta,\omega,t)|^2\le\left(\|\alpha-\beta\|^{1+\varepsilon}\vee\|\alpha-\beta\|^{1+\eta}\right)\tilde h^2(\omega,t)\tag{3.11}$$

which stands for condition B. Since $\tilde h=ch$, it follows that condition C holds for $\tilde h$. □

Corollary 3.4: *Let conditions A,B,C hold. Assume that there exists some* $\delta>0$ s.t.

$$\liminf_{\|\theta\|\to\infty} \inf_{t\geq t_0} t^{-\delta} < F(\theta,\cdot) >_t > 0 \quad \text{a.s.} \quad t_0 \geq 0 \tag{3.12}$$

Then, for every $\gamma > 0$ it holds that

$$\lim_{\|\theta\|\to\infty} \sup_{t\geq t_0} \frac{1}{\|\theta\|^\gamma} |F(\theta,t)| / < F(\theta,\cdot) >_t = 0 \quad \text{a.s.} \tag{3.13}$$

Proof: The proof follows the lines of the proof of theorem 3.1 with the redefinition of the function r as $r(\beta) = \min_{1\leq i\leq d} |\beta_i|^{1+p}$, $p = \eta + 2\gamma$ and the choice of $0 < \nu \leq 1/2 \wedge ((\gamma\wedge\delta)/4(1+\eta))$.

□

4 An Equicontinuity Theorem

The purpose of this section is to present an equicontinuity theorem which plays a central role in the study of asymptotic properties of the MLE in continuous, parameter dependent semimartingales, see e.g. theorem 3.1 in Borkar & Bagchi [3]. This theorem is a consequence of lemma 3.3 and is more general than the one presented in [3].

Theorem 4.1 *Assume f satisfies hypotheses A,B,C and let $\{A_t\}_{t\geq 0}$ $0 < A_t \uparrow \infty$ a.s. be a continuous, $\mathcal{F}_t$-adapted process. Let h be as in B,C and assume that*

$$\limsup_{t\to\infty} A_t^{-1} \int_0^t h^2(\omega,s)\,ds < \infty \quad a.s. \tag{4.1}$$

Moreover, assume that there exists a $\gamma > 0$ s.t.

$$\limsup_{t\to\infty} A_t^{-1} t^\gamma < \infty \quad a.s. \tag{4.2}$$

Then, for all compact $K \subset\subset I\!R^d$, and all $t_0 > 0$, the function set $\left\{A_t^{-1} F(\cdot,t)\right\}_{t\geq t_0}$ is equicontinuous, uniformly on $K, P-$ a.s.

Proof: Define the martingale $Y_t(\theta,\theta') = F(\theta,t) - F(\theta',t)$, $t \geq 0$ and its increasing process $< Y(\theta,\theta') >_t = \int_0^t [f(\theta,\omega,s) - f(\theta',\omega.s)]^2 ds$. Fix $K \subset\subset I\!R^d$, $t_0 > 0$. Choose $\nu \leq 1/2 \wedge \gamma/4\ell(1+\eta)$ and define ϕ as in (3.4). Then $\forall\, \Delta > 0$,

$$\sup_{\substack{\theta,\theta'\in K\\ \|\theta-\theta'\|\leq\Delta}} \sup_{t\geq t_0} A_t^{-1} |F(\theta,t) - F(\theta',t)| = \sup_{\substack{\theta,\theta'\in K\\ \|\theta-\theta'\|\leq\Delta}} \sup_{t\geq t_0} A_t^{-1} |Y_t(\theta,\theta')| \leq \tag{4.3}$$

$$\leq \sup_{\theta,\theta'\in K} \sup_{t\geq t_0} t^{-\ell\nu} \frac{|Y_t(\theta,\theta')|}{\phi(< Y(\theta,\theta') >_t)} \sup_{t\geq t_0} A_t^{-(1-\nu)/2} t^{\ell\nu} \sup_{\substack{\theta,\theta'\in K\\ \|\theta-\theta'\|\leq\Delta}} \sup_{t\geq t_0} A_t^{-(1+\nu)/2} \phi(< Y(\theta,\theta') >_t)$$

It is easy to see that the integrand $f(\theta,\omega,s) - f(\theta',\omega,s)$ (as a map from $I\!R^{2d} \times \Omega \times I\!R_+$ to $I\!R$) satisfies conditions A-C (with $\tilde{h} = 2h$) hence by lemma 3.3 it holds that

$$\sup_{\theta,\theta' \in K} \sup_{t \geq t_0} t^{-\ell\nu} \frac{|Y_t(\theta,\theta')|}{\phi(< Y(\theta,\theta') >_t)} < \infty \quad \text{a.s.} \quad \forall\, K \subset\subset I\!R^d,\ \nu,\ t_0 > 0 \tag{4.4}$$

Moreover, due to (4.2) and the choice of ν

$$\limsup_{t\to\infty} A_t^{-1} t^{2\ell\nu/(1-\nu)} \leq \limsup_{t\to\infty} A_t^{-1} t^{\gamma} < \infty \quad \text{a.s.} \tag{4.5}$$

which by the continuity of A imply that

$$\sup_{t \geq t_0} A_t^{-(1-\nu)/2} t^{\ell\nu} < \infty \quad \text{a.s.} \tag{4.6}$$

It remains to show that the last term on the RHS of (4.3) $\to 0$ as $\Delta \to 0$. To this end, note that by the definition of ϕ

$$A_t^{-(1+\nu)/2} \phi(< Y(\theta,\theta') >_t) = \begin{cases} (< Y(\theta,\theta') >_t / A_t)^{(1+\nu)/2}, & < Y(\theta,\theta') >_t \geq 1 \\ A_t^{-\nu} (< Y(\theta,\theta') >_t / A_t)^{(1-\nu)/2}, & < Y(\theta,\theta') >_t < 1 \end{cases}$$

which obviously leads to

$$\sup_{\substack{\theta,\theta' \in K \\ \|\theta-\theta'\| \leq \Delta}} \sup_{t \geq t_0} A_t^{-(1+\nu)/2} \phi(< Y(\theta,\theta') >_t) \leq \tag{4.7}$$

$$\leq (A_{t_0}^{-\nu} \wedge 1) \sup_{\substack{\theta,\theta' \in K \\ \|\theta-\theta'\| \leq \Delta}} \sup_{t \geq t_0} \max\{[(A_t^{-1} < Y(\theta,\theta') >_t)^{(1+\nu)/2},\ (A_t^{-1} < Y(\theta,\theta') >_t)^{(1-\nu)/2}]\}$$

Because $A_{t_0}^{-\nu} < \infty$ a.s. and since $g(x) = x^{\mu}$, $\mu > 0$ is increasing, it suffices to show that the term $\sup\sup A^{-1} < Y >$ on the RHS decays to zero as $\Delta \to 0$. This is easily obtained relying on B-i and (4.1):

$$\sup_{\substack{\theta,\theta' \in K \\ \|\theta-\theta'\| \leq \Delta}} \sup_{t \geq t_0} A_t^{-1} < Y(\theta,\theta') >_t = \sup_{\substack{\theta,\theta' \in K \\ \|\theta-\theta'\| \leq \Delta}} \sup_{t \geq t_0} A_t^{-1} \int_0^t [f(\theta,\omega,s) - f(\theta',\omega,s)]^2\, ds$$

$$\leq \sup_{\substack{\theta,\theta' \in K \\ \|\theta-\theta'\| \leq \Delta}} \sup_{t \geq t_0} \|\theta - \theta'\|^{1+\varepsilon} A_t^{-1} \int_0^t h^2(\omega,r)\, dr \leq$$

$$\leq \Delta^{1+\varepsilon} \sup_{t \geq t_0} A_t^{-1} \int_0^t h^2(\omega,r)\, dr \to 0 \quad \text{a.s.} \qquad \text{as } \Delta \to 0. \tag{4.8}$$

which follows from (4.1) together with C and the fact that A is continuous and increasing. Substituting (4.4), (4.6) and (4.8) into (4.3) completes the proof. □

Appendix

Proof of Lemma 3.2

Outline:

In order to make the proof more trackable we begin by proving that under A-C, for each T, Z_T possesses a modification (denoted also by Z_T) which satisfies

$$\sup_{\theta\in K}\sup_{s>0}|Z_T(\theta,s)|<\infty \quad \text{a.s.} \quad \forall\, K\subset\subset I\!R^d, \quad T<\infty \tag{A.1}$$

Towards this end, we first show that Z_T possesses a modification with a.s. jointly continuous paths over $K\times[0,1]$ $(K\subset\subset I\!R)$. This is done via Kolmogorov's continuity criterion. Then, by "time reversal", we prove that the same assertion holds for $\tilde Z_T$, $\tilde Z_T(\theta,s)=Z_T(\theta,1/s)$, $\tilde Z_T(\theta,0)=0$. These two statements easily lead to (A.1).

Building on the estimates obtained in the first part, the last step of the proof is conducted in a very similar way with V and $\tilde V$ where $V(\theta,s,r)=r^{\nu\ell}Z_{1/r}(\theta,s)$; $\tilde V(\theta,s,r)=r^{\nu\ell}\tilde Z_{1/r}(\theta,s)$, $(\theta,s,r)\in K\times[0,1]^2$ (where $V(\theta,s,0)=V(\theta,0,r)=0$). The extension to $K\times[0,1]\times[0,T]$, $\forall T<\infty$ does not differ from the case treated here.

Remark: The proof of (A.1) is actually obtained under a weaker condition than C. Namely, it suffices to assume that for all $m\ge 1$, $\int_0^t E\,h^{2m}(\omega,s)ds<\infty$.

First step:

Let m be a positive integer to be determined later. Fix $0<s\le t\le 1$, $\theta,\beta\in K$. Then,

$$E|Z_T(\theta,t)-Z_T(\beta,s)|^{2m}\le c_1\,E|Z_T(\theta,t)-Z_T(\theta,s)|^{2m}+c_1\,E|Z_T(\theta,s)-Z_T(\beta,s)|^{2m} \tag{A.2}$$

where $c_1=c_1(m)=2^{2m-1}$. Next,

$$\begin{aligned}E|Z_T(\theta,t)-Z_T(\theta,s)|^{2m}&=E\left|\frac{F(\theta,\tau^\theta(t)\wedge T)}{\phi(t)}-\frac{F(\theta,\tau^\theta(s)\wedge T)}{\phi(s)}\right|^{2m}\le\\ &c_1\phi(t)^{-2m}E|F(\theta,\tau^\theta(t)\wedge T)-F(\theta,\tau^\theta(s)\wedge T)|^{2m}+\\ &+c_1\left(\frac{\phi(t)-\phi(s)}{\phi(t)\phi(s)}\right)^{2m}E|F(\theta,\tau^\theta(s)\wedge T)|^{2m}\end{aligned} \tag{A.3}$$

Consider the first term on the RHS of (A.3). Since $\tau^\theta(\cdot)$ is a $\mathcal{F}_t$-stopping time, then there exists, by the Burkholder moment inequality [2, cor. IV-4.2], a universal constant $c_2=c_2(m)$ s.t.

$$E|F(\theta,\tau^\theta(t)\wedge T)-F(\theta,\tau^\theta(s)\wedge T)|^{2m}=E\left|\int_{\tau^\theta(s)\wedge T}^{\tau^\theta(t)\wedge T}f(\theta,\omega,r)\,dw_r\right|^{2m}\le \tag{A.4}$$

$$\leq c_2 E\left(\int_{\tau^\theta(s)\wedge T}^{\tau^\theta(t)\wedge T} f^2(\theta,\omega,r)\,dr\right)^m \leq c_2 E\left[\left(\int_{\tau^\theta(s)}^{\tau^\theta(t)} f^2(\theta,\omega,r)\,dr\right)^m \mathbf{1}_{\{<F(\theta,\cdot)>_\infty > s\}}\right] \leq c_2(t-s)^m$$

The last inequality follows from the definition of the stopping times τ^θ and the fact that

$$\int_0^{\tau^\theta(u)} f^2(\theta,\omega,r)\,dr = u\wedge < F(\theta,\cdot) >_\infty \leq u \quad \forall\, u \in I\!\!R_+ \tag{A.5}$$

In a similar way,

$$E|F(\theta,\tau^\theta(s)\wedge T)|^{2m} \leq c_2 s^m \tag{A.6}$$

By combining (A.3,A.4,A.6) we obtain

$$E|Z_T(\theta,t) - Z_T(\theta,s)|^{2m} \leq c_1 c_2 \phi(t)^{-2m}\left[(t-s)^m + \left(\frac{\phi(t)-\phi(s)}{\phi(s)}\right)^{2m} s^m\right], \quad 0 < s \leq t \tag{A.7}$$

By the definition of ϕ and the mean value theorem (for $g(x) = x^{1-\nu}$ and some $u \in [s,t]$),

$$\begin{aligned}\left(\frac{\phi(t)-\phi(s)}{\phi(s)}\right)^{2m} s^m &= s^{\nu m}\left(t^{(1-\nu)/2} - s^{(1-\nu)/2}\right)^{2m} = s^{\nu m}\left((1-\nu)(\sqrt{u})^{-\nu}(\sqrt{t}-\sqrt{s})\right)^{2m}, \\ &\leq s^{\nu m}\left(s^{-\nu/2}(\sqrt{t}-\sqrt{s})\right)^{2m} \leq (t-s)^m \end{aligned} \tag{A.8}$$

Substituting into (A.7) results in

$$E|Z_T(\theta,t) - Z_T(\theta,s)|^{2m} \leq \frac{2c_1c_2}{t^{(1-\nu)m}}(t-s)^m \leq 2\frac{c_1c_2}{(t-s)^{(1-\nu)m}}(t-s)^m = 2c_1c_2(t-s)^{\nu m} \tag{A.9}$$

For the case $s = 0, \quad 0 < t \leq 1$ we have by (A.6)

$$E\,|Z_T(\theta,t)|^{2m} = \phi(t)^{-2m} E\left|F(\theta,\tau^\theta(t)\wedge T)\right|^{2m} \leq c_2\phi(t)^{-2m}t^m = c_2 t^{\nu m}. \tag{A.10}$$

Combining the above with (A.9) yields

$$E|Z_T(\theta,t) - Z_T(\theta,s)|^{2m} \leq 2c_1c_2|t-s|^{\nu m} \quad \forall\, s,t \in [0,1], \quad \theta \in I\!\!R^d \tag{A.11}$$

We proceed with the evaluation of the second term on the RHS of (A.2). In order to simplify the notations, define the $\mathcal{F}_t$-stopping time $\sigma(s) = \tau^\theta(s)\wedge\tau^\beta(s)\wedge T$. Then,

$$E|Z_T(\theta,s) - Z_T(\beta,s)|^{2m} = \phi(s)^{-2m}E|F(\theta,\tau^\theta(s)\wedge T) - F(\beta,\tau^\beta(s)\wedge T)|^{2m} \leq$$

$$\leq c_1^2\phi(s)^{-2m}(E|F(\theta,\sigma(s)) - F(\beta,\sigma(s))|^{2m} + \tag{A.12}$$

$$+E|F(\theta,\tau^\theta(s)\wedge T) - F(\theta,\sigma(s))|^{2m} + E|F(\beta,\tau^\beta(s)\wedge T) - F(\beta,\sigma(s))|^{2m})$$

By the definition of F and σ together with Burkholder's inequality,

$$
\begin{aligned}
& E|F(\theta,\sigma(s)) - F(\beta,\sigma(s))|^{2m} = E\left|\int_0^{\sigma(s)} [f(\theta,\omega,r) - f(\beta,\omega,r)]\, dw_r\right|^{2m} \leq \\
& \leq c_2 E\left(\int_0^{\sigma(s)} [f(\theta,\omega,r) - f(\beta,\omega,r)]^2\, dr\right)^m \leq \\
& \leq c_2 E\left\{\left(\int_0^{\sigma(s)} [f(\theta,\omega,r) - f(\beta,\omega,r)]^2\, dr\right)^{mp} \left(2\int_0^{\sigma(s)} [f^2(\theta,\omega,r) + f^2(\beta,\omega,r)]\, dr\right)^{m(1-p)}\right\} \\
& \leq c_2 2^{2m(1-p)} s^{m(1-p)} E\left(\int_0^T [f(\theta,\omega,r) - f(\beta,\omega,r)]^2\, dr\right)^{mp}, \quad \forall\, p \in (0,1/2] \qquad \text{(A.13)}
\end{aligned}
$$

The last equality follows from the definition of σ and (A.5). Define the "distance"

$$
e_s(\theta,\beta) = \sup_{0\leq u\leq \sigma(s)} |< F(\theta,\cdot) >_u - < F(\beta,\cdot) >_u | \leq s
$$

and note that σ is bounded from below as follows,

$$
\begin{aligned}
\sigma(s) &= \inf\{t > 0\, |< F(\beta,\cdot) >_t > s\} \wedge \tau^\theta(s) \wedge T = \\
&= \inf\{t > 0\, |< F(\theta,\cdot) >_t > s + < F(\theta,\cdot) >_t - < F(\beta,\cdot) >_t\} \wedge \tau^\theta(s) \wedge T \\
&\geq \inf\{t > 0\, |< F(\theta,\cdot) >_t > s - \sup_{0\leq u\leq\sigma(s)} | < F(\theta,\cdot) >_u - < F(\beta,\cdot) >_u |\} \wedge T \\
&= \inf\{t > 0\, |< F(\theta,\cdot) >_t > s - e_s(\theta,\beta)\} \wedge T = \tau^\theta(s - e_s(\theta,\beta)) \wedge T \qquad \text{(A.14)}
\end{aligned}
$$

In a similar way,

$$
\sigma(s) \geq \tau^\beta(s - e_s(\theta,\beta)) \wedge T \qquad \text{(A.15)}
$$

The distance $e_s(\theta,\beta)$ satisfies,

$$
\begin{aligned}
e_s(\theta,\beta) &= \sup_{0\leq u\leq\sigma(s)} \left|\int_0^u [f^2(\theta,\omega,r) - f^2(\beta,\omega,r)]\, dr\right| \leq \\
&\leq \left(\int_0^{\sigma(s)} [f(\theta,\omega,r) - f(\beta,\omega,r)]^2\, dr \int_0^{\sigma(s)} [f(\theta,\omega,r) + f(\beta,\omega,r)]^2\, dr\right)^{1/2} \leq \\
&\leq \sqrt{2}\left(\int_0^{\sigma(s)} [f(\theta,\omega,r) - f(\beta,\omega,r)]^2\, dr \int_0^{\sigma(s)} [f^2(\theta,\omega,r) + f^2(\beta,\omega,r)]\, dr\right)^{1/2} \leq \\
&\leq 2s^{1/2}\left(\int_0^{\sigma(s)} [f(\theta,\omega,r) - f(\beta,\omega,r)]^2\, dr\right)^{1/2} \qquad \text{(A.16)}
\end{aligned}
$$

The last inequality is obtained exactly as the last inequality in (A.13). Note that, by the definition of τ, $< F(\theta,\cdot) >_{\tau^\theta(s)} - < F(\theta,\cdot) >_{\tau^\theta(s-e_s(\theta,\beta))} \leq e_s(\theta,\beta)$. Thus,

$$
\int_{\sigma(s)}^{\tau^\theta(s)\wedge T} f^2(\theta,\omega,r)dr \leq \int_{\tau^\theta(s-e_s(\theta,\beta))\wedge T}^{\tau^\theta(s)\wedge T} f^2(\theta,\omega,r)dr \leq e_s(\theta,\beta)
$$

Combining the above with (A.14-A.16) and the Burkholder inequality, it follows that

$$
\begin{aligned}
&E|F(\theta,\tau^{\theta}(s)\wedge T)-F(\theta,\sigma(s))|^{2m}=E\left|\int_{\sigma(s)}^{\tau^{\theta}(s)\wedge T} f(\theta,\omega,r)dw_r\right|^{2m}\leq\\
&\leq c_2E\left(\int_{\sigma(s)}^{\tau^{\theta}(s)\wedge T} f^2(\theta,\omega,r)dr\right)^{m}\leq c_2 2^m s^{m/2}E\left(\int_0^{\sigma(s)}[f(\theta,\omega,r)-f(\beta,\omega,r)]^2dr\right)^{m/2}\leq\\
&\leq c_2 2^m s^{m/2}E\left(\int_0^{\sigma(s)}[f(\theta,\omega,r)-f(\beta,\omega,r)]^2dr\right)^{mp}\left(2\int_0^{\sigma(s)}[f^2(\theta,\omega,r)+f^2(\beta,\omega,r)]dr\right)^{m(\frac{1}{2}-p)}\\
&\leq c_2 2^{2m(1-p)}s^{m(1-p)}E\left(\int_0^{\sigma(s)}[f(\theta,\omega,r)-f(\beta,\omega,r)]^2dr\right)^{mp}, \qquad p\in(0,1/2]. \qquad (A.17)
\end{aligned}
$$

Exactly in the same way

$$
E|F(\beta,\tau^{\beta}(s)\wedge T)-F(\beta,\sigma(s))|^{2m}\leq c_2 2^{2m(1-p)}s^{m(1-p)}E\left(\int_0^{\sigma(s)}[f(\theta,\omega,r)-f(\beta,\omega,r)]^2\,dr\right)^{mp} \qquad (A.18)
$$

Substituting (A.13, A.17-A.18) in (A.12) results in

$$
E|Z_T(\theta,s)-Z_T(\beta,s)|^{2m}\leq 3c_1^2c_2 2^{2m}\frac{s^{m(1-p)}}{\phi(s)^{2m}}E\left(\int_0^{\sigma(s)}[f(\theta,\omega,r)-f(\beta,\omega,r)]^2\,dr\right)^{mp} \qquad (A.19)
$$

By choosing $p=\nu/(1+\varepsilon)$ one has $\frac{s^{m(1-p)}}{\phi(s)^{2m}}\leq 1$ and therefore by B-i and (A.19)

$$
\begin{aligned}
&E|Z_T(\theta,s)-Z_T(\beta,s)|^{2m}\leq\|\theta-\beta\|^{m\nu}3c_1^2c_2 2^{2m}E\left(\int_0^T h^2(\omega,r)dr\right)^{m\nu/(1+\varepsilon)}\leq \qquad (A.20)\\
&\leq\|\theta-\beta\|^{m\nu}3c_1^2c_2 2^{2m}\left[E\left(\int_0^T h^2(\omega,r)dr\right)^{m}\right]^{\nu}\leq\|\theta-\beta\|^{m\nu}3c_1^2c_2 2^{2m}\left[T^{m-1}\int_0^T Eh^{2m}(\omega,r)dr\right]^{\nu}
\end{aligned}
$$

Combining (A.11, A.20) and (A.2) yields

$$
E|Z_T(\theta,t)-Z_T(\beta,s)|^{2m}\leq c_3(|t-s|\vee\|\beta-\theta\|)^{\nu m}\leq c_4(|t-s|\vee\max_{1\leq i\leq d}|\beta_i-\theta_i|)^{m\nu} \qquad (A.21)
$$

where

$$
c_4=d^{m\nu/2}c_3,\ c_3=c_3(m,T,\nu)=6c_1^3c_2 2^{2m}\left[T^{m-1}\int_0^T Eh^{2m}(\omega,r)dr\right]^{\nu} \qquad (A.22)
$$

Since m is arbitrary, taking $m\nu>d+1$ and using Kolmogorov's criterion [2, thm. I-2.1], one concludes that Z_T possesses a modification with a.s. jointly continuous paths over $K\times[0,1]$, hence (for this modification) it holds that

$$
\sup_{\theta\in K}\sup_{s\in[0,1]}|Z_T(\theta,s)|<\infty \quad \text{a.s.} \quad \forall\, K\subset\subset I\!R^d \qquad (A.23)
$$

To complete the proof of (A.1) it remains to obtain a corresponding statement for $s \in [1,\infty)$. To this end define

$$\tilde{Z}_T(\theta, s) = Z_T(\theta, 1/s) \quad , \qquad \tilde{Z}_T(\theta, \cdot) = 0 \tag{A.24}$$

As before, it suffices to show that $\tilde{Z}_T$ has a modification with a.s. jointly continuous paths over $K \times [0,1]$. This is done almost the same as for Z_T. We therefore omit most of the details. As in the case of Z_T (eq. (A.7)) with $0 < s < t \le 1$ one has,

$$\begin{aligned} E|\tilde{Z}_T(\theta,s) - \tilde{Z}_T(\theta,t)|^{2m} &= E|Z_T(\theta,1/s) - Z_T(\theta,1/t)|^{2m} \le \\ &\le c_1c_2\phi(1/s)^{-2m}\left[(1/s-1/t)^m + \left(\frac{\phi(1/s)-\phi(1/t)}{\phi(1/t)}\right)^{2m}(1/t)^m\right] \end{aligned} \tag{A.25}$$

First note that by the definition of ϕ and the fact that $1/s > 1/t \ge 1$,

$$\begin{aligned} \phi(1/s)^{-2m}(1/s-1/t)^m &= s^{m(1+\nu)}\left(\frac{t-s}{st}\right)^m = \frac{s^{m\nu}}{t^m}(t-s)^m = \\ &= \left(\frac{s}{t}\right)^{m\nu}\left(\frac{t-s}{t}\right)^{m(1-\nu)}(t-s)^{m\nu} \le (t-s)^{m\nu} \end{aligned} \tag{A.26}$$

Furthermore

$$\begin{aligned} \left(\frac{\phi(1/s)-\phi(1/t)}{\phi(1/s)\phi(1/t)}\right)^{2m}(1/t)^m &= (st)^{m(1+\nu)}\left[(1/s)^{(1+\nu)/2} - (1/t)^{(1+\mu)/2}\right]^{2m}(1/t)^m = \\ &= \frac{1}{t^m}\left[(\sqrt{t})^{1+\nu} - (\sqrt{s})^{1+\nu}\right]^{2m} = \frac{1}{t^m}(1+\nu)^{2m}u^{m\nu}(\sqrt{t}-\sqrt{s})^{2m}\ , \\ \text{some}\ \ & u \in [s,t] \le 2^{2m}\left(\frac{u}{t}\right)^{m\nu}\left(\frac{t-s}{t}\right)^{m(1-\nu)}(t-s)^{m\nu} \le \\ &\le 2^{2m}(t-s)^{m\nu} \end{aligned} \tag{A.27}$$

For the case $s = 0, \quad t \in (0,1]$ we have

$$\begin{aligned} E|\tilde{Z}_T(\theta,t)|^{2m} &= E|Z_T(\theta,1/t)|^{2m} = \phi(1/t)^{-2m}E|F(\theta,\tau^\theta(1/t)\wedge T)|^{2m} \le \\ &\le c_2\phi^{-2m}(1/t)(1/t)^m = c_2t^{m(1+\nu)}/t^m = c_2t^{\nu m} \end{aligned} \tag{A.28}$$

which together with (A.25-A.27) results in

$$E|\tilde{Z}_T(\theta,s) - \tilde{Z}_T(\theta,t)|^{2m} \le 2c_1c_2 2^{2m}|t-s|^{m\nu} \quad \forall\, s,t \in [0,1] \tag{A.29}$$

On the other hand, exactly as in the case of Z_T (eq. A.20)

$$\begin{aligned} E|\tilde{Z}_T(\theta,t) - \tilde{Z}_T(\beta,t)|^{2m} &= E|\tilde{Z}_T(\theta,1/t) - \tilde{Z}_T(\beta,1/t)|^{2m} \le \\ &\le \|\theta-\beta\|^{m\nu}3c_1^2c_2 2^{2m}\left[T^{m-1}\int_0^T Eh^{2m}(\omega,r)dr\right]^\nu \end{aligned} \tag{A.30}$$

By combining (A.29-30) and choosing $m > (d+1)/\nu$ it is seen that $\tilde{Z}_T$ satisfies (A.21). This implies that (A.23) is valid for a modification of $\tilde{Z}_T$ and therefore completes the proof of (A.1).

Second step:

Define $V : K \times I\!R_+ \times [0,1] \to I\!R$ by

$$V(\theta, s, r) = r^{\ell\nu} Z_{1/r}(\theta, s)\ ,\ V(\theta, s, 0) = V(\theta, 0, r) = 0 \tag{A.31}$$

Our first goal is to show that V has a modification which is a.s. jointly continuous on $K \times [0,1]^2$. This relies on the continuity proof of Z_T. Fix $0 < s < t \leq 1$, $0 \leq u < r \leq 1$, $\theta, \beta \in K$. Let m be some positive integer. Then, to apply Kolmogorov's criterion we evaluate

$$\begin{aligned} &E|V(\theta,t,r) - V(\beta,s,u)|^{2m} = E\left|r^{\ell\nu} Z_{1/r}(\theta,t) - u^{\ell\nu} Z_{1/u}(\beta,s)\right|^{2m} \leq \\ &a_1 r^{2\ell m\nu} E\left|Z_{1/r}(\theta,t) - Z_{1/r}(\beta,s)\right|^{2m} + a_1 (r^{\ell\nu} - u^{\ell\nu})^{2m} E\left|Z_{1/r}(\beta,s)\right|^{2m} + \\ &a_1 u^{2\ell m\nu} E\left|Z_{1/u}(\beta,s) - Z_{1/r}(\beta,s)\right|^{2m} \quad , \quad a_1 = a_1(m) \end{aligned} \tag{A.32}$$

Let $k = \ell - 1$ (ℓ as in condition C). An estimate of the first term on the RHS is immediately obtained from (A.21) and condition C, namely,

$$\begin{aligned} &r^{2\ell m\nu} E\left|Z_{1/r}(\theta,t) - Z_{1/r}(\beta,s)\right|^{2m} \leq \\ &\leq 6c_1^2 c_2 2^{2m} \left[\frac{r^{2\ell m}}{r^{m-1}} \int_0^{1/r} Eh^{2m}(\omega,\tau)d\tau\right]^{\nu} (|t-s| \vee \|\beta - \theta\|)^{m\nu} \leq \\ &\leq 6c_1^2 c_2 2^{2m} r^{m\nu} \left[r^{2km+1} \int_0^{1/r} Eh^{2m}(\omega,\tau)d\tau\right]^{\nu} (|t-s| \vee \|\beta-\theta\|)^{m\nu} \leq \\ &\leq a_2(|t-s| \vee \|\beta-\theta\|)^{m\nu}, a_2 = a_2(m,\nu) = 6c_1^2 c_2 2^{2m} \sup_{\substack{t>s\geq 0 \\ t\geq 1}} \frac{1}{t^{2km}(t-s)} \int_s^t Eh^{2m}(\omega,r)dr \end{aligned} \tag{A.33}$$

As for the second term on the RHS of (A.32) note that,

$$\begin{aligned} 0 \leq r^{k\nu} - u^{k\nu} &= \frac{r}{r^{1-\ell\nu}} - \frac{u}{r^{1-\ell\nu}} = \frac{u^{1-\ell\nu} r - r^{1-\ell\nu} u}{(ru)^{1-\ell\nu}} \\ &\leq \frac{1}{r^{1-\ell\nu}}(r-u) = \left(\frac{r-u}{r}\right)^{1-\nu/2} r^{\nu(\ell-1/2)}(r-u)^{\nu/2} \leq (r-u)^{\nu/2} \end{aligned} \tag{A.34}$$

Hence, using (A.10),

$$(r^{\ell\nu} - u^{\ell\nu})^{2m} E(Z_{1/r}(\beta,s)|^{2m} \leq c_2 s^{\nu m}(r-u)^{m\nu} \leq a_2(r-u)^{m\nu} \tag{A.35}$$

Finally,

$$u^{2\ell m\nu} E|Z_{1/u}(\beta,s) - Z_{1/r}(\beta,s)|^{2m} = u^{2\ell m\nu} \phi(s)^{-2m} e \left|\int_{\tau^\beta(s)\wedge 1/r}^{\tau^\beta(s)\wedge 1/u} f(\beta,\omega,\tau)dw_\tau\right|^{2m}$$

$$
\begin{aligned}
&\le c_2 \frac{s^{m(1-\nu)}}{\phi(s)^{2m}} u^{2\ell m\nu} \left(\int_{1/u}^{1/r} Ef^2(\beta,\omega,\tau)d\tau \right)^{m\nu} \le \\
&\le c_2 b^{m\nu} \left[u^{2\ell m} \left(\frac{1}{u} - \frac{1}{r} \right)^{m-1} \int_{1/u}^{1/r} Eh^{2m}(\omega,\tau)d\tau \right]^{\nu} \\
&= c_2 b^{m\nu} \left[\frac{u^{2mk}}{\left(\frac{1}{u}-\frac{1}{r}\right)} \int_{1/u}^{1/r} Eh^{2m}(\omega,\tau)d\tau \right]^{\nu} (r-u)^{m\nu} \le a_2 b^{m\nu}(r-u)^{m\nu} \qquad \text{(A.36)}
\end{aligned}
$$

The first inequality is obtained by following the basic steps of (A.13) (the case here is somewhat simpler). The second inequality relies on assumption B-ii (with $b = \sup\limits_{\beta\in K} \|\beta\|^{1+\varepsilon} + 1$) and the last one is based on condition C together with the definition of a_2 in (A.33). By combining (A.33-A.36) and choosing $m > \frac{d+2}{\nu}$ we obtain

$$
E|V(\theta,t,r) - V(\beta,s,u)|^{2m} \le a_3 \, \|\theta_1-\beta_1,\cdots\theta_d-\beta_d, t-s, r-u\|_\infty^{d+2+\mu\nu}, \text{ some } \mu>0 \quad \text{(A.37)}
$$

where $a_3 = 3d^{m\nu/2}a_2 b^{m\nu}$ (it can easily be seen that the case $u=0$ with the definition (A.31) fits into (A.37)). Thus, by Kolmogorov's continuity criterion, V possesses a modification, denoted $\hat{V}$, which is a.s. jointly continuous on $K\times[0,1]^2$. This leads to

$$
\sup_{\theta\in K} \sup_{s\in[0,1]} \sup_{T\ge 1} T^{-\ell\nu}|Z_T(\theta,s)| = \sup_{\theta\in K} \sup_{(s,r)\in[0,1]^2} |\hat{V}(\theta,s,r)| < \infty \quad \text{a.s.} \qquad \text{(A.38)}
$$

The corresponding statement for the sup over $s\in[1,\infty)$ is obtained as in the case of Z_T i.e. by defining

$$
\tilde{V}(\theta,s,r) = r^{\ell\nu}\tilde{Z}_{1/r}(\theta,s) = r^{\ell\nu}Z_{1/r}(\theta,1/s) \quad (= V(\theta,1/s,r))
$$

Since it suffices to examine only the first term on the RHS of (A.32) then, by following the lines from (A.24) to (A.30), we obtain (3.5) for $T_0 = 1$.

To complete the proof (for $T_0\in(0,1]$) note that by (A.1), a continuous modification of Z satisfies

$$
\sup_{T_0\le T\le 1} \sup_{\theta\in K} \sup_{s>0} T^{-\ell\nu}|Z_T(\theta,s)| \le T_0^{-\ell\nu} \sup_{\theta\in K} \sup_{s>0} |Z_1(\theta,s)| < \infty \quad \text{a.s.}
$$

$\square$

Proof of Lemma 3.3

For every fixed θ, $F(\theta,\cdot)$ is a continuous martingale, hence it is a τ^θ-continuous process [2, def. V-1.3]. That is, there exists a set $\mathcal{N}_\theta \subset \Omega$ with $P(\mathcal{N}_\theta)=0$ s.t. $\forall\,\omega\notin\mathcal{N}_\theta$, $\quad \tau^\theta(s)-$

$\tau^\theta(s^-) > 0 \Rightarrow F(\theta, r) = F(\theta, u)\ \forall\, r, u \in [\tau^\theta(s^-), \tau^\theta(s)]$, for all $s \geq 0$ (where by definition $< F(\theta,\cdot) >_{\tau^\theta(s)} \equiv < F(\theta,\cdot) >_{\tau^\theta(s^-)}$ ($= s$ for $\tau^\theta(s) < \infty$)). This implies that $\forall\, t_0,\ \nu > 0$

$$\begin{aligned}\sup_{t\geq t_0} t^{-\nu\ell} \frac{|F(\theta,t)|}{\phi(< F(\theta,\cdot) >_t)} &\leq \sup_{t\geq t_0} \sup_{0<r\leq t} t^{-\nu\ell} \frac{|F(\theta,r)|}{\phi(< F(\theta,\cdot) >_r)} = \\ &= \sup_{t\geq t_0} \sup_{0<\tau^\theta(s)\leq t} t^{-\nu\ell} \frac{|F(\theta,\tau^\theta(s))|}{\phi(< F(\theta,\cdot) >_{\tau^\theta(s)})} = \\ &= \sup_{t\geq t_0} \sup_{0<s\leq < F(\theta,\cdot)>_t} t^{-\nu\ell} \frac{|F(\theta,\tau^\theta(s)\wedge t)|}{\phi(s)} \leq \\ &\leq \sup_{t\geq t_0} \sup_{s>0} t^{-\nu\ell} |Z_t(\theta,s)| \quad \forall\, \omega \notin \mathcal{N}_\theta \end{aligned} \tag{A.39}$$

Since both $F(\theta,t)$ and $< F >$ are separable and ϕ is continuous, it is enough in order to prove the lemma to take the supremum over $\theta \in \Theta \bigcap K$. Let $V(\theta,s,u) = u^{\nu\ell} Z_{1/u}(\theta,s)$. Recall that for any fixed θ, $F(\theta,\tau^\theta(\cdot))$ and $F(\theta,\cdot)$ are continuous processes ($\forall\, \omega \notin \mathcal{N}_\theta$). It follows that, for each $\theta \in \Theta \bigcap K$, $V(\theta,s,u)$ is jointly continuous in $(s,u) \in (0,1] \times (0,T]$, for all finite fixed T. On the other hand, by the proof of Lemma 3.2 (starting with (A.31)), it follows that $V(\theta,s,u)$ possesses a jointly continuous version on $K \times [0,1]^2$, which we denote by $\hat V$. This version satisfies (A.38). With minor changes in the proof, this statement holds on $K \times [0,1] \times [0,T]$, $\forall\, T < \infty$. Next, for any $\theta \in \Theta \bigcap K$, $V(\theta,s,u) = \hat V(\theta,s,u)$, $\forall\,(s,u) \in (0,1] \times (0,1/t_0]$, outside an ω-null set $\mathcal{U}_\theta$ which may depend on θ but is independent of (s,u). Let $\mathcal{U} = \bigcup_{\theta\in\Theta} \mathcal{U}_\theta$. Then, $\forall\, \omega \notin \mathcal{N} \cup \mathcal{U}$

$$\sup_{\theta\in\Theta\bigcap K} \sup_{s\in(0,1]} \sup_{u\in(0,1/t_0]} |V(\theta,s,u)| \leq \sup_{\theta\in\Theta\bigcap K} \sup_{s\in[0,1]} \sup_{u\in[0,1/t_0]} |\hat V(\theta,s,u)| < \infty \tag{A.40}$$

where the last inequality follows from (A.38). Defining now $\tilde V(\theta,s,u) = u^{\nu\ell} Z_{1/u}(\theta,1/s)$ and repeating the same argument, it follows that

$$\sup_{\theta\in\Theta\bigcap K} \sup_{s\in(0,1]} \sup_{u\in(0,1/t_0]} |\tilde V(\theta,s,u)| < \infty \tag{A.41}$$

By definition

$$\sup_{t\geq t_0} \sup_{s>0} t^{-\nu\ell} |Z_t(\theta,s)| = \sup_{s\in(0,1]} \sup_{u\in(0,1/t_0]} \max\left\{ |V(\theta,s,u)|, |\tilde V(\theta,s,u)| \right\} \tag{A.42}$$

This, together with (A.39-A.41), completes the proof of the lemma. □

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