Absence of a wetting transition for a pinned harmonic crystal in dimensions three and larger

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Abstract

We consider a free lattice field (a harmonic crystal) with a hard wall condition and a weak pinning to the wall. We prove that in a weak sense the pinning always dominates the entropic repulsion of the hard wall condition when the dimension is a least three. This contrasts with the situation in dimension one, where there is a so called wetting transition, as has been observed by Michael Fisher. The existence of a wetting transition in the delicate two dimensional case was recently proved by Caputo and Velenik.

1 Introduction

The so called harmonic crystal is a Gaussian random field on a d-dimensional hypercubic lattice whose covariance operator is given by the inverse of the discrete Laplacian, i.e. the standard lattice Greens function. To be precise, let A be a finite subset of $\mathbb{Z}^d$. We denote by P_A the probability law on $\mathbb{R}^A$ defined as follows:

$$P_A(dx_A) = \frac{1}{Z_A} \exp(-\frac{1}{8d} \sum_{\substack{i,j \in A \cup \partial A \\ |i-j|=1}} (x_i - x_j)^2) dx_A \tag{1.1}$$

where $x_A = (x_i)_{i \in A}$ $dx_A = \prod_{i \in A} dx_i$, ∂A is the outer boundary of A: $\partial A = \{j \in \mathbb{Z}^d \backslash A : \exists i \in A \text{ with } |i-j| = 1\}$, and $x_i \equiv 0$ for $i \in \partial A$. Z_A is the normalizing constant

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$$Z_A = \int_{\mathbb{R}^A} \exp(-\frac{1}{8d} \sum_{\substack{i,j \in A \cup \partial A \\ |i-j|=1}} (x_i - x_j)^2) dx_A \tag{1.2}$$

Let $V_N = [-N, N]^d \cap \mathbb{Z}^d$. We usually write P_N and Z_N instead of P_{V_N} and Z_{V_N}.

We modify this measure now, by introducing a pinning to the "hard wall" $x \equiv 0$.

$$\hat{P}_{N,J}(dx_{V_N}) = \frac{1}{\hat{Z}_{N,J}} \exp(-\frac{1}{8d} \sum_{\substack{i,j \in V_N \cup \partial V_N \\ |i-j|=1}} (x_i - x_j)^2) \prod_{j \in V_N} (dx_j + e^J \delta_o(dx_j)) \tag{1.3}$$

where $\delta_o(dx)$ is the Dirac measure at 0 and $J \in \mathbb{R}$ is a parameter regulating the strength of the pinning. A slightly different model is obtained by defining

$$\hat{P}_{N,a,b}(dx_{V_N}) = \frac{1}{\hat{Z}_{N,a,b}} \exp(-\frac{1}{8d} \sum_{\substack{i,j \in V_N \cup \partial V_N \\ |i-j|=1}} (x_i - x_j)^2 - \sum_{i \in V_N} \psi(x_i)) \prod_{j \in V_N} dx_j \tag{1.4}$$

where ψ is a symmetric function $\mathbb{R} \to \mathbb{R}$ of finite support, having a small dip near 0, for instance $\psi(x) = -b1_{[-a,a]}(x)$, $a, b > 0$, see ([6],[11]). The results we derive here apply to both models. We discuss in details the delta-pinned case defined by (1.3), and will give the modification needed to handled the other case (1.4) in Section 3. The effect of this pinning force to the "wall" $\{x : x_i \equiv 0\}$ is quite marked. For all pinning parameters and for any dimension, the field becomes localized in a very strong sense, meaning that

$$\sup_N \sup_{i \in V_N} \hat{E}_{N,J}(X_i^2) < \infty \tag{1.5}$$

(X_i are the coordinate mappings), and there exists $\delta_J > 0$ with

$$\sup_N \sup_{\substack{i,j \in V_N \\ |i-j| \geq k}} \hat{E}_{N,J}(X_i X_j) \leq \exp(-\delta_J k) \tag{1.6}$$

for k large enough. This had been proved for the model (1.4) in dimensions larger or equal to 3 in [6]. In the delicate two dimensional case, (1.5) had been first been proved in [11], and then the positivity of the mass, i.e. (1.6) has been proved in [1], [10], and finally under rather general conditions in [13]. A discussion of a discrete one dimensional version of this problem can be found in [5] and [15].

The main aim of the present paper however is to discuss what happens in the presence of a so called "hard wall" condition. This simply means that the field is conditioned to stay positive. Let, for any set $A \subset V_N$,

$$\Omega_A^+ = \{x \in \mathbb{R}^{V_N} : x_i \geq 0, \forall i \in A\}, \ \Omega_N^+ := \Omega_{V_N}^+ ,$$

and

$$\hat{P}_{N,J}^+ = \hat{P}_{N,J}(\,.\,|\Omega_N^+)\,.$$

We recall that under the hard wall condition but without pinning, i.e. when $J = -\infty$, or $b = 0$, the field is repelled at height $\sqrt{\log N}$ (for $d \geq 3$), $\log N$ (for $d = 2$), and $\sqrt{N}$ (for $d = 1$). That is,

$$\hat{E}_{N,-\infty}^+(X_0) \sim \begin{cases} \sqrt{\log N}, & d \geq 3 \\ \log N, & d = 2 \\ \sqrt{N}, & d = 1\,, \end{cases}$$

cf. [3], [8]. A very interesting observation first made by Michael Fisher [12] in a slightly different model is that for $d = 1$, there is a transition from localization to delocalization if the parameter J varies: If J is large, then (1.5) and (1.6) hold true, but if J is small, then the field delocalizes, i.e.

$$\hat{E}_{N,J}^+(X_0) \sim \sqrt{N}.$$

Such a transition is called a wetting transition. Fisher had considered a random walk case, but the results are the same in our Gaussian model. For the convenience of the reader, we will include a discussion of the simple one dimensional case in Sect. 3.

This one dimensional case raises the question if a similar wetting transition occurs also in higher dimensions. Unfortunately, we are not able to discuss that fully here. What we can and do show here is that for dimensions at least three, there is always localization, at least in a somewhat weaker sense than expressed in (1.5) and (1.6). To formulate our results, let ξ_N be the number of zeros of the field:

$$\xi_N = \sum_{i \in V_N} 1_{X_i = 0}.$$

Then we have

Theorem 1 *Assume $d \geq 3$, and let $J \in \mathbb{R}$ be arbitrary. Then there exist $\varepsilon_J, \eta_J > 0$ such that*

$$\hat{P}_{N,J}^+(\xi_N \geq \varepsilon_J |V_N|) \geq 1 - \exp(-\eta_J N^d) \tag{1.7}$$

provided N is large enough.

It appears overwhelmingly plausible that a statement like (1.7) should imply (1.5) and (1.6), but this seems to be a quite delicate question which we had not been able to settle. For large enough J such statements have actually been proved (in a slightly different setting) by Lemberger [16] using cluster expansion techniques. Probably his methods would carry over to our situation, but they appear to be powerless for proving the result also for small J.

Define the two partition functions

$$\hat{Z}_N^+ = \int_{\Omega_N^+} \exp(-\frac{1}{8d} \sum_{\substack{i,j \in V_N \cup \partial V_N \\ |i-j|=1}} (x_i - x_j)^2) \prod_{j \in V_N} dx_j,$$

and

$$\hat{Z}_{N,J}^+ = \int_{\Omega_N^+} \exp(-\frac{1}{8d} \sum_{\substack{i,j \in V_N \cup \partial V_N \\ |i-j|=1}} (x_i - x_j)^2) \prod_{j \in V_N} (dx_j + e^J \delta_0(dx_j))$$

then Theorem 1 is actually an easy consequence of part b) of the following result

Theorem 2 *a) For any J, and any dimension*

$$\delta_J^+ = \lim_{N \to \infty} \frac{1}{N^d} \log \frac{\hat{Z}_{N,J}^+}{\hat{Z}_N^+},$$

exists and $J : \mathbb{R} \cup \{-\infty\} \longrightarrow \delta_J^+$ is a nonnegative convex function with $\delta_{-\infty}^+ = 0$.

b) For $d \geq 3$, $\delta_J^+ > 0$ holds true for any J.

c) For $d = 1$, there exists $J_0 \in \mathbb{R}$ such that $\delta_J^+ = 0$ for $J \leq J_0$, and $\delta_J^+ > 0$ for $J > J_0$.

The existence of the limit in a) is easy and has been proved for $d = 2$ in [4]. The argument there (for this issue) does not depend on the dimension and carries over essentially verbatim, so we will not prove it here. Let

$$\delta_{N,J}^+ = \frac{1}{N^d} \log \frac{\hat{Z}_{N,J}^+}{\hat{Z}_N^+}$$

then

$$\frac{\partial}{\partial J} \delta_{N,J}^+ = \frac{1}{N^d} \hat{E}_{N,J}^+(\xi_N)$$

is the expected density of zeros of the field and

$$\frac{\partial^2}{\partial J^2} \delta_{N,J}^+ = \mathrm{Var}_{N,J}^+(N^{-d/2} \xi_N) \geq 0$$

is the (rescaled) variance. In particular this implies the convexity of $\delta_\cdot^+$. Also $\delta_J^+ > 0$ implies a positive density of "pinned" configurations, that is the effect of J is felt at the thermodynamical level.

As remarked above, the part c) is essentially due to M. Fisher. We include the simple proof here in Section 3. In dimension 1 it is actually very easy to prove the stronger statements (1.5) and (1.6), but we leave that to the reader.

Part b) is the main result of this paper and will be proved in Section 2. Our method gives no information in the two dimensional case. After this paper was written, Caputo and Velenik [7] succeeded, by adapting a construction due to Chalker, to prove the existence of a wetting transition in two dimensions.

We end this section by showing rigorously how the positivity of δ_J^+ implies the statement of Theorem 1. First note that

$$\frac{1}{N^d}\log\frac{\hat{Z}^+_{N,J}}{\hat{Z}_{N,J}} = \frac{1}{N^d}\log\frac{\hat{Z}^+_{N,J}}{Z_N} - \frac{1}{N^d}\log\frac{\hat{Z}^+_{N}}{Z_N}$$

where $\frac{\hat{Z}^+_N}{Z_N} = P_N(\Omega_N^+)$ satisfies

$$\lim_{N\to\infty} -\frac{1}{N^{d-1}}\log P_N(\Omega_N^+) < \infty,$$

cf. [8]. On the other hand, expanding the product in (1.3) we see that

$$\frac{\hat{Z}^+_{N,J}}{Z_N} = \sum_{A\subset V_N} e^{J|A^{\complement}|}\frac{Z_A}{Z_N}P_A(\Omega_A^+)$$

so that an alternative definition of δ_J^+ is given by

$$\delta_J^+ = \lim_{N\to\infty}\frac{1}{N^d}\log\frac{Z^+_{N,J}}{Z_N} = \lim_{N\to\infty}\frac{1}{N^d}\log\sum_{A\subset V_N} e^{J|A^{\complement}|}\frac{Z_A}{Z_N}P_A(\Omega_A^+).$$

Next, let $\varepsilon > 0$. By expanding again the product in (1.3), we get

$$\hat{P}_{N,J}(\xi_N < \varepsilon|V_N|, \Omega_N^+) = \sum_{\substack{A\subset V_N\\ |A|>(1-\varepsilon)|V_N|}} e^{J|A^{\complement}|}\frac{Z_A}{\hat{Z}_{N,J}}P_A(\Omega_A^+)$$

$$\leq \frac{Z_N}{\hat{Z}_{N,J}}\sum_{\substack{A\subset V_N\\ |A|>(1-\varepsilon)|V_N|}} e^{J|A^{\complement}|}\frac{Z_A}{Z_N}.$$

On the other hand

$$\hat{P}_{N,J}(\Omega_N^+) = \frac{Z_N}{\hat{Z}_{N,J}}\sum_{A\subset V_N} e^{J|A^{\complement}|}\frac{Z_A}{Z_N}P_A(\Omega_A^+).$$

If $\delta_J^+ > 0$, (1.7) follows therefore easily, once we have proved

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N^d}\log\sum_{\substack{A\subset V_N\\ |A|>(1-\varepsilon)|V_N|}} e^{J|A^{\complement}|}\frac{Z_A}{Z_N}=0. \tag{1.8}$$

Using the fact proved in [4, Lemma 2.3.1 (a)] (note that the argument given there extends to all d),

$$\exp(-c(|V_N|-|A|))\leq\frac{Z_A}{Z_N}\leq 1, \tag{1.9}$$

for some constant $c>0$, (1.8) follows from the estimate

$$\sharp\left\{A\subset V_N:|A^{\complement}|<\varepsilon|V_N|\right\}\leq\frac{|V_N|^{\varepsilon|V_N|}}{(\varepsilon|V_N|)!}\sim\frac{\left(\frac{1}{\varepsilon}\right)^{\varepsilon|V_N|}e^{\varepsilon|V_N|}}{\sqrt{2\pi\varepsilon|V_N|}},$$

by Stirling's formula.

2 Proof of Theorem 2 b)

The strategy of the proof is to construct enough "pinning" configurations for which a lower bound on the probability of the hard wall conditioning can be found. The pinning configurations we construct are rather regular, and form a small perturbation of a regular sub-grid of V_N of step Δ. The desired lower bound then follows by a change of measure argument, which, as in [3] and [9], needs first a variance reduction step in order to be tight. In this change of measure the transience of simple random walk in dimension $d\geq 3$ plays a crucial role, for it allows to push the Gaussian field high enough even in the immediate vicinity of a pinned point without too large a penalty.

Turning to our construction, fix $\Delta>0$, independent of N, and let $l_N^{\Delta}=\{z_i\}_{i=1}^{|l_N^{\Delta}|}$ denote a finite collection of points $z_i\in V_N$, such that for each $y\in V_N\cap\Delta\mathbb{Z}^d$, there is *exactly* one $z\in l_N^{\Delta}$ such that $|z-y|<\Delta/10$. Note that the number of different possible configurations l_N^{Δ} is bounded below by $(\frac{\Delta}{5})^{d(\frac{2N+1}{\Delta})^d}=\exp((\frac{2N+1}{\Delta})^d(d\log\Delta+c_0))$. Let $A_{l_N^{\Delta}}=V_N\setminus l_N^{\Delta}$. Our main technical estimate is the following:

Proposition 3 *Assume $d\geq 3$, and let $t\geq 0$. Then there exists a constant $c_1=c_1(t)>0$ such that, for all Δ integer large enough,*

$$\liminf_{N\to\infty}\inf_{\{l_N^{\Delta}\}}\frac{1}{(2N+1)^d}\log P_{A_{l_N^{\Delta}}}(X_i\geq t,i\in A_{l_N^{\Delta}})\geq-\frac{d\log\Delta}{\Delta^d}+c_1\frac{\log\log\Delta}{\Delta^d}.$$

By choosing Δ large enough (depending on J) and $t=0$, and using (1.9), it is evident that this estimate proves $\delta_J^+>0$ for all $J\in\mathbb{R}$, i.e. part b) of Theorem 2 follows.

Proof of Proposition 3. Fix $\alpha > d$ independent of N, such that Δ^α is an integer, and fix a particular configuration l_N^Δ. All our constants below will be independent of the particular configuration l_N^Δ.

For $k \in \mathbb{Z}^d$, let $\bar{V}_{\alpha,\Delta}^{(k)} = V_{\Delta^\alpha} + k\Delta^\alpha$, and let $V_{\alpha,\Delta}^{(k)} = \{i \in \bar{V}_{\alpha,\Delta}^{(k)} : \mathrm{dist}(i, (\bar{V}_{\alpha,\Delta}^{(k)})^\complement) \geq 1\}$, and $\partial V_{\alpha,\Delta}^{(k)} = \bar{V}_{\alpha,\Delta}^{(k)} \setminus V_{\alpha,\Delta}^{(k)}$,

$$\mathcal{V}_{\alpha,\Delta} = \{\, k : V_{\alpha,\Delta}^{(k)} \subset V_N, \mathrm{dist}(V_{\alpha,\Delta}^{(k)}, \partial V_N) \geq \Delta^\alpha \,\}.$$

The strategy of the proof is to cover V_N with disjoints boxes $\{\bar{V}_{\alpha,\Delta}^{(k)}\}$ and estimate from below the probability of the event $\mathcal{E}_\alpha := \{X_i \geq t, i \in A_{\ell_N^\Delta}\}$ using FKG on each box $V_{\alpha,\Delta}^{(k)}$. In doing so, we will be able to use entropy inequalities on these boxes, which were chosen such that on the one hand, the probability of $\mathcal{E}_\alpha$, restricted to a single box, after the change of measure is close to 1 (this forces the box not to be too large), while on the other hand the loss due to imposing zero boundary conditions on the boxes is negligible.

Note that $|\mathcal{V}_{\alpha,\Delta}| = \lfloor \frac{(2N+1)}{\Delta^\alpha} \rfloor^d (1 + o(1))$, where throughout this proof $o(1)$, $O(1)$ etc. are taken with respect to $N \to \infty$.

Throughout, we let $\{X_\cdot\}$ denote the free Gaussian field, of covariance $\sigma(z, z')$, and let $\{X_\cdot^0\}$ denote the free Gaussian field on $\mathbb{Z}^d$ *pinned* at the points $l_N^{\Delta,\alpha} := l_N^\Delta \cup V_N^c \cup_k \partial V_{\alpha,\Delta}^{(k)}$. That is, $\{X_\cdot^0\}$ is a zero mean Gaussian field with $X_z^0 = 0$ for $z \in l_N^{\Delta,\alpha}$, whose covariance $\sigma_0(a,b)$ for $a, b \in V_N \setminus l_N^{\Delta,\alpha}$ equals that of $X_\cdot$ conditioned on $\sigma(X_z : z \in l_N^{\Delta,\alpha})$. By the usual random walk representation, c.f. [17], [2], we have that

$$\sigma_0(a,b) = \mathbb{E}_a\left(\sum_{n=0}^{\tau} 1_{\{w_n^a = b\}}\right) \tag{2.1}$$

where w_n^a is a simple random walk on $\mathbb{Z}^d$ starting from a, and $\tau = \min\{n : w_n \in l_N^{\Delta,\alpha}\}$. Here and in the sequel, we denote probabilities related to w_n^a by $\mathbb{P}_a(\cdot), \mathbb{E}_a(\cdot)$, etc. Because of (2.1), $\sigma_0(a,b) \geq 0$, and hence, due to the FKG property,

$$\begin{aligned}
\mathcal{P} \;&:=\; P_{A_{l_N^\Delta}}(X_i \geq t, i \in A_{l_N^\Delta}) \qquad\qquad (2.2)\\
&\geq \prod_{k \in \mathcal{V}_{\alpha,\Delta}} P(X_z^0 \geq t, z \in V_{\alpha,\Delta}^{(k)} \setminus l_N^\Delta) \cdot \prod_{z \in A_{l_N^\Delta} \setminus \bigcup_{k \notin \mathcal{V}_{\alpha,\Delta}} V_{\alpha,\Delta}^{(k)}} P(X_z \geq t)\\
&\quad \cdot \prod_{z \in \bigcup_k \partial V_{\alpha,\Delta}^{(k)}} P(X_z \geq t).
\end{aligned}$$

Because $E(X_z) = 0$ and $\mathrm{Var}(X_z) \geq 1$ for any $z \in A_{l_N^\Delta}$, we have that $P(X_z \geq t) \geq c_0$, for any such z. Hence,

$$\mathcal{P} \geq c_0^{4d\Delta^\alpha (2N+1)^{d-1}} \prod_{k \in \mathcal{V}_{\alpha,\Delta}} P(X_z^0 \geq t, z \in V_{\alpha,\Delta}^{(k)} \setminus l_N^\Delta). \tag{2.3}$$

Next, fix a box $V^{(k)}_{\alpha,\Delta}$, denoted hereafter as V_α. On each box V_α we estimate the probability of the repulsion following the approach of [9, Section 4]: we decompose $X^0_\cdot = Y_\cdot + Z_\cdot$, where $Y_\cdot$, $Z_\cdot$ are independent, zero mean Gaussian fields, with $Y_a = Z_a = 0$ on $a \in l^\Delta_N \cup V^c_\alpha$, such that $Z_\cdot$ has exponenentially decaying correlations while $Y_\cdot$ exhibits long range dependence but is "small". More precisely, for $a, b \in V_\alpha \setminus l^\Delta_N$,

$$\begin{aligned}\bar\sigma_{0,\varepsilon}(a,b) := E(Y_a Y_b) &= \sigma_0(a,b) - (\sigma_0^{-1} + \varepsilon^2)^{-1}(a,b),\\ E(Z_a Z_b) &= (\sigma_0^{-1} + \varepsilon^2)^{-1}(a,b).\end{aligned}$$

Here, with $\bar L^2$ denoting the space of functions $f \in L^2(\mathbb{Z}^d)$ with $f|_{l^\Delta_N \cup V^c_\alpha} = 0$, σ_0^{-1} is the operator on $\bar L^2$ determined by $\sigma_0^{-1} f = g$ if $f = \sigma_0 g$, and an explicit expression for σ_0^{-1} is $\sigma_0^{-1} = I - Q_0$, where Q_0 is the transition matrix of simple random walk killed when hitting $l^\Delta_N \cup V^c_\alpha$. $\varepsilon = \varepsilon(\Delta)$ is taken as $\varepsilon = (\log\Delta)^{-\gamma}$ for some γ large enough (whose precise value will become clearer in the course of the proof).

The idea of [9] is to lift the field Y at a certain height. In our case, we have to take special care around the obstacle ℓ^Δ_N (this is where transience will be crucial). More precisely, fix $1 > \beta > 0$ (the precise value of β will also become clearer in the course of the proof), and let $R = \Delta^\beta$, where β is chosen such that Δ^β is an integer.

For each $x_i \in V_\alpha \cap l^\Delta_N$, define $b_i = \{z \in \mathbb{Z}^d : \mathrm{dist}(x_i, z) \le R\}$, and let $b = \bigcup_{i: x_i \in V_\alpha \cap l^\Delta_N} b_i$.

We let x denote a fixed constant (eventually, we take $x = 2d\sigma - c_g \frac{\log\log\Delta}{\log\Delta}$, where $c_g = c_g(\alpha,\beta,\gamma) > 0$. The logarithmic correction term can be best understood as coming from the factor $1/x$ in (2.5) below). Let $v(\cdot) : \mathbb{Z}^d \to R$ denote the σ_0^{-1}-harmonic function, solution of the problem

$$\begin{cases}\sigma_0^{-1} v = 0 & \text{on } b \setminus l^\Delta_N,\\ v = 0 & \text{on } (l^\Delta_N \cap \partial V_\alpha)\\ v = 1 & \text{on } V_\alpha \setminus b.\end{cases}$$

This harmonic function is used below to perform a Gaussian change of measure, while controling the associated relative entropy. The following lemma plays a crucial role in our proof:

Lemma 4 *There exists $\varepsilon_1 > 0$, independent of Δ, α, β, γ, such that*

$$\min_{z \in b \setminus l^\Delta_N} v(z) \ge \varepsilon_1.$$

Proof. By the transience of the simple random walk $w^z_\cdot$ ∎

Returning to the proof of the Proposition, we compute

$$
\begin{aligned}
&P(X^0_z \geq t, z\in V_\alpha\setminus l_N^\Delta)\\
&\quad = P(Y_z+Z_z\geq t, z\in V_\alpha\setminus l_N^\Delta)\\
&\quad \geq P(Y_z+Z_z\geq t, Y_z\geq v(z)\sqrt{x\log\Delta}, z\in V_\alpha\setminus l_N^\Delta)\\
&\quad \geq P(Z_z\geq t-v(z)\sqrt{x\log\Delta}, Y_z\geq v(z)\sqrt{x\log\Delta}, z\in V_\alpha\setminus l_N^\Delta)\\
&\quad = P(Z_z\geq t-v(z)\sqrt{x\log\Delta}, z\in V_\alpha\setminus l_N^\Delta)P(Y_z\geq v(z)\sqrt{x\log\Delta}, z\in V_\alpha\setminus l_N^\Delta)
\end{aligned}
\tag{2.4}
$$

due to the independence of the fields $Z_\cdot$ and $Y_\cdot$. Note that the covariance of $Z_\cdot$ corresponds to the Green function of simple random walk, killed at rate ε^2. Hence, the entries of the covariance are positive, and by the FKG property, using that $\mathrm{Var}(Z_z)\leq\sigma_0(z,z)\leq\sigma(z,z)=\sigma$, and denoting $\Phi(x)=\int_x^\infty e^{-\theta^2/2}d\theta/\sqrt{2\pi}$,

$$
\begin{aligned}
&P(Z_z > t-v(z)\sqrt{x\log\Delta}, z\in V_\alpha\setminus l_N^\Delta)\\
&\quad\geq \prod_{z\in V_\alpha\setminus l_N^\Delta} P(Z_z>t-v(z)\sqrt{x\log\Delta})\\
&\quad\geq \prod_{z\in V_\alpha\setminus l_N^\Delta}\left(1-\Phi((v(z)\sqrt{x\log\Delta}-t)/\sqrt{\mathrm{Var}(Z_z)})\right)\\
&\quad\geq \left(1-\frac{e^{-x\log\Delta/2\sigma}}{c_2\sqrt{x\log\Delta}}\right)^{|V_\alpha|}\left(1-\frac{e^{-\varepsilon_1^2 x\log\Delta/2\sigma}}{c_2\sqrt{x\log\Delta}}\right)^{|b|},
\end{aligned}
$$

where we used Lemma 4 and the inequality, valid for all $x>1$,

$$
\Phi(x)\leq\frac{e^{-x^2/2}}{c_3 x}\,. \tag{2.5}
$$

With $|V_\alpha|=(2\Delta+1)^{\alpha d}$ and $|b|=c_4\Delta^{(\alpha-1)d}\Delta^{\beta d}$, we conclude that, for all Δ large enough,

$$
P(Z_z>t-v(z)\sqrt{x\log\Delta}, z\in V_\alpha\setminus l_N^\Delta)\geq\exp\left(-c_5\frac{\Delta^{-x/2\sigma}\Delta^{\alpha d}}{\sqrt{\log\Delta}}\right) \tag{2.6}
$$

as soon as $\beta<\varepsilon_1$ (recall that $x<2d\sigma$!).

We next turn to the evaluation of the second term in (2.4). Let $\hat{Y}_z = Y_z+v(z)\sqrt{x'\log\Delta}$, where $\sqrt{x'}=\sqrt{x}+(\log\Delta)^{-2}$ and let $\hat{P}$ denote the law of $\{\hat{Y}_z\}_{z\in\mathbb{Z}^d}$. Clearly,

$$
H(\hat{P}\,|\,P)=\frac{x'\log\Delta}{2}\langle v,\bar{\sigma}_{0,\varepsilon}^{-1}v\rangle_{\mathbb{Z}^d}
$$

where $H(P\,|\,Q)$ denotes the relative entropy of P with respect to Q. The following lemma is crucial in the evaluation of (2.6).

Lemma 5 *There exists a $\chi=\chi(\alpha,\beta,\gamma)>0$ such that*

$$
\langle v,\bar{\sigma}_{0,\varepsilon}^{-1}v\rangle_{\mathbb{Z}^d}\leq(\sigma^{-1}+\Delta^{-\chi})\frac{(2\Delta^\alpha+1)^d}{\Delta^d}.
$$

Proof of Lemma 5. We write

$$\langle v, \bar\sigma_{0,\varepsilon}^{-1} v\rangle_{\mathbb{Z}^d} = \langle v, (\bar\sigma_{0,\varepsilon}^{-1} - \sigma_0^{-1})v\rangle_{\mathbb{Z}^d} + \langle v, \sigma_0^{-1} v\rangle_{\mathbb{Z}^d} := I + II. \tag{2.7}$$

Note that

$$\langle v, (\bar\sigma_{0,\varepsilon}^{-1} - \sigma_0^{-1})v\rangle_{\mathbb{Z}^d} = \left\langle \sigma_0^{-1} v, \frac{\sigma_0^{-1}}{\varepsilon^2} v \right\rangle = \frac{1}{\varepsilon^2}\langle \sigma_0^{-1} v, \sigma_0^{-1} v\rangle_{\mathbb{Z}^d}. \tag{2.8}$$

We next claim that

$$\sum_{z\in\partial b_i} \sigma_0^{-1} v(z) \le \frac{1}{\sigma} + \frac{c_6}{R^{d-2}} \tag{2.9}$$

and that

$$\sum_{z\in\partial b_i} (\sigma_0^{-1} v(z))^2 \le \frac{c_6}{R^{d-2}}. \tag{2.10}$$

Indeed, with $\tau = \min\{\, n \ge 1 : w_n^z \in (\partial b_i \cup x_i)\,\}$ and $\theta \in \mathbb{Z}^d$ such that $\theta = z - x_i$, it holds that

$$\sigma_0^{-1} v(z) = \mathbb{P}_z(w_\tau^z = x_i) = \mathbb{P}_\theta(w_{\bar\tau}^\theta = 0)$$

where $\bar\tau = \min\{\, n \ge 1 : |w_n^{\cdot}| = 0 \text{ or } |w_n^{\cdot}| = R\,\}$ (note that the definition of $\bar\tau$ is the same for any starting point of the random walk $w_n^{\cdot}$). But, introducing $\sigma_R(x,y) = \mathbb{E}_x[\sum_{n=0}^{\tau_R} 1_{w_n^x = y}]$ where $\tau_R = \inf\{n \ge 0 : |w_n^0| = R\}$, we have in view of [14] (1.38)

$$\begin{aligned}
\sum_{z\in\partial b_i} \sigma_0^{-1} v(z) &= \sum_{\theta:|\theta|=R} \mathbb{P}_\theta(w_{\bar\tau}^\theta = 0) \\
&= \sum_{\theta:|\theta|=R} \sum_n \mathbb{P}_\theta(\bar\tau = n, w_{\bar\tau}^\theta = 0) \\
&= \sum_{\theta:|\theta|=R} \sum_n \mathbb{P}_0(w_{\bar\tau}^0 = \theta, \bar\tau = n) \\
&= \sum_n \mathbb{P}_0(|w_{\bar\tau}^0| = R, \bar\tau = n) \\
&= \mathbb{P}_0(|w_{\bar\tau}^0| = R) = \frac{1}{\sigma_R(0,0)} \le \frac{1}{\sigma} + \frac{c_6}{R^{d-2}},
\end{aligned}$$

for some constant c_6, where in last inequality we have used $\sigma_R(0,0) \ge \sigma(0,0) - c_7 R^{2-d}$, cf. [14] Prop. 1.5.9. To see (2.10), note that

$$\sum_{z\in\partial b_i} (\sigma_0^{-1} v(z))^2 \le \max_{z\in\partial b_i} \sigma_0^{-1} v(z) \sum_{z\in\partial b_i} \sigma_0^{-1} v(z).$$

But,

$$\sigma_0^{-1} v(z) \le \mathbb{P}_z(w_{\cdot}^z \text{ hits } x_i) = \frac{\sigma(0, z - x_i)}{\sigma(0,0)} \le \frac{c_6}{R^{d-2}},$$

and (2.10) follows (increasing c_6 if necessary, and using (2.9)).

Similarly, for $z \in \partial V_\alpha$, we have that

$$\sigma_0^{-1} v(z) \leq 1 - \left(\frac{1}{2d}\right)^2.$$

Hence, since the contribution to (2.8) comes only from $z \in \partial b \cup \partial V_\alpha$, we have that

$$I \leq \frac{1}{\varepsilon^2}\left[\frac{(2\Delta^\alpha+1)}{\Delta}\right]^d \cdot \left(\frac{c_6}{R^{d-2}}\right) + \frac{1}{\varepsilon^2}\left[1-\left(\frac{1}{2d}\right)^2\right]^2 2d(2\Delta^\alpha+1)^{d-1}$$

while

$$II \leq \left[\frac{(2\Delta^\alpha+1)}{\Delta}\right]^d \cdot \left(\frac{1}{\sigma} + \frac{c_6}{R^{d-2}}\right) + \left[1-\left(\frac{1}{2d}\right)^2\right]^2 2d(2\Delta^\alpha+1)^{d-1}.$$

Lemma 5 follows as soon as $\alpha > d$. ■

We remark that while (2.10) is not sharp the estimate in Lemma 5 is sharp as the reverse inequality holds true to the leading order (see [9] for a similar remark).

Equipped with Lemma 5, let us complete the proof of the Proposition. Note first that

$$\begin{aligned}
&\hat{P}(\hat{Y}_z > v(z)\sqrt{x \log \Delta}, z \in V_\alpha \setminus l_N^\Delta) \\
&\quad \geq 1 - \sum_{z \in V_\alpha \setminus l_N^\Delta} \hat{P}(\hat{Y}(z) \leq v(z)\sqrt{x \log \Delta}) \\
&\quad \geq 1 - \sum_{z \in V_\alpha \setminus l_N^\Delta} P(Y(z) \leq v(z)(\sqrt{x} - \sqrt{x'})\sqrt{\log \Delta}) \\
&\quad \geq 1 - \sum_{z \in V_\alpha \setminus l_N^\Delta} c_{71} \exp\left(-\frac{\varepsilon_1}{2\,\mathrm{Var}(Y(z))(\log \Delta)}\right).
\end{aligned}$$

Note that $\mathrm{Var}(Y_i) = \bar{\sigma}_{0,\varepsilon}(i,i) \leq c_{72}\varepsilon = c_{72}(\log \Delta)^{-\gamma}$, and choosing γ large enough, we have that

$$\hat{P}(\hat{Y}_z > v(z)\sqrt{x \log \Delta}, z \in V_\alpha \setminus l_N^\Delta) \geq 1 - \frac{1}{\Delta}. \tag{2.11}$$

We continue by an application of a specific entropy inequality (cf. e.g. [3, p. 421])

$$\log \frac{P(Y_z > v(z)\sqrt{x \log \Delta}, z \in V_\alpha \setminus l_N^\Delta)}{\hat{P}(\hat{Y}_z > v(z)\sqrt{x \log \Delta}, z \in V_\alpha \setminus l_N^\Delta)} \geq -\frac{H(\hat{P} \mid P) + e^{-1}}{\hat{P}(\hat{Y}_z > v(z)\sqrt{x \log \Delta}, z \in V_\alpha \setminus l_N^\Delta)}.$$

Using (2.11) and Lemma 5, we see that

$$\begin{aligned}
&P(Y_z > v(z)\sqrt{x \log \Delta}, z \in V_\alpha \setminus l_N^\Delta) \\
&\quad \geq c_8 \exp\left(-\frac{x'}{2}\left(1 + \frac{1}{\Delta}\right) \log \Delta (\sigma^{-1} + \Delta^{-\chi}) \frac{(2\Delta^\alpha + 1)^d}{\Delta^d}\right).
\end{aligned}$$

Using now (2.3), (2.4), (2.6) and the above, we obtain that

$$
\begin{aligned}
\mathcal{P} \geq \exp(-O(N^{d-1})) & \left[\exp\left(-c_5 \frac{\Delta^{-d}\Delta^{\alpha d}}{\sqrt{\log\Delta}}(\log\Delta)^{c_g/2\sigma}\right)\right]^{\left(\frac{N}{\Delta^\alpha}\right)^d} \\
\times & \left[c_8 \exp\left(-d\left(1+\frac{1}{\Delta}\right)(\log\Delta)\left(1+\frac{\Delta^{-\chi}}{\sigma}\right)\frac{(2\Delta^\alpha+1)^d}{\Delta^d}\right)\right. \\
& \times \exp\left(\frac{c_g}{2\sigma}(\log\log\Delta)\frac{(2\Delta^\alpha+1)^d}{\Delta^d}\right) \\
& \left.\times \exp\left(c_{10}\frac{(2\Delta^\alpha+1)^d}{\Delta^d}(\Delta^{-\chi}+\Delta^{-1})\log\Delta\right)\right]^{\left(\frac{2N+1}{2\Delta^\alpha+1}\right)^d}
\end{aligned}
$$

The claim follows at once, by choosing $c_g/2\sigma < 1/2$. ■

3 The square potential case

In this section we briefly show how the argument of the previous section should be adapted to the so called square potential case. More precisely, for $b, a > 0$ consider the square potential

$$\psi(x) = -b1_{[-a,a]}(x), \qquad x \in \mathbb{R}.$$

Next, denote by $\hat{P}_{N,b,a}$ and $\hat{P}^+_{N,b,a}$, the corresponding measures and let

$$\xi_N = \sum_{i \in V_N} 1_{|X_i| \leq a}$$

be the number of sites with values in the interval $[-a, a]$.

Theorem 6 *Assume $d \geq 3$, and let $b, a > 0$ be arbitrary. Then there exist $\varepsilon_{b,a}, \eta_{b,a} > 0$ such that,*

$$\hat{P}^+_{N,b,a}(\xi_N \geq \varepsilon_{b,a} N^d) \geq 1 - \exp(-\eta_{b,a} N^d) \tag{3.1}$$

provided N is large enough.

Actually, we will see that $\varepsilon_{b,a}$ and $\eta_{b,a}$ depend only on the strength $S(a,b) = \big((2a) \wedge 1\big)(e^b - 1)$. Thus, if we let $a \searrow 0$ and $b \uparrow \infty$ such that $J = \log S(a,b) \in \mathbb{R}$ is kept fixed, then, with respect to the weak convergence of measures,

$$\hat{P}^+_{N,b,a} \Longrightarrow \hat{P}^+_{N,J},$$

so that in some sense we could view Theorem 1 as a Corollary of Theorem 6.

Proof. Let us show that we can find $\varepsilon > 0$ such that

$$\limsup_{N \to \infty} \frac{1}{N^d} \log \hat{P}_{N,b,a}(\xi_N < \varepsilon N^d \,|\, \Omega^+_N) < 0 \tag{3.2}$$

The main idea in proving Theorem 6 is to write

$$\exp(\sum_{i\in V_N} b1_{|x_i|\le a}) = \prod_{i\in V_N}\big(e^b 1_{|x_i|\le a} + 1_{|x_i|>a}\big) = \prod_{i\in V_N}\big((e^b-1)1_{|x_i|\le a}+1\big)$$

and therefore

$$\hat P_{N,b,a}(\cdot) = \frac{Z_N}{\hat Z_{N,b,a}} \sum_{A\subset V_N} (e^b-1)^{|A^{\complement}|} P_N(\Omega_{A^{\complement}}(a)) P_N\big(\,\cdot\,\big|\Omega_{A^{\complement}}(a)\big)$$

where $\Omega_{A^{\complement}}(a)=\{x\in\mathbb{R}^{V_N}: |x_i|\le a, i\in A^{\complement}\}$. Note that $P_N\big(\,\cdot\,\big|\Omega_{A^{\complement}}(a)\big)$ being a conditioning on $\Omega_{A^{\complement}}(a)=\cap_{i\in A^{\complement}}\Omega_{\{i\}}(a)$ satisfies FKG, cf. §. 6 of [10], and therefore

$$\begin{aligned}\hat P_{N,b,a}(\Omega_N^+) &= \frac{Z_N}{\hat Z_{N,b,a}} \sum_{A\subset V_N}\big(e^b-1\big)^{|A^{\complement}|} P_N(\Omega_{A^{\complement}}(a)) P_N(\Omega_N^+\,|\Omega_{A^{\complement}}(a)) \\ &\ge \frac{Z_N}{\hat Z_{N,b,a}} \sum_{A\subset V_N}\big(e^b-1\big)^{|A^{\complement}|} P_N(\Omega_{A^{\complement}}(a)) P_N(\Omega_A^+\,|\Omega_{A^{\complement}}(a)) P_N(\Omega_{A^{\complement}}^+\,|\Omega_{A^{\complement}}(a))\end{aligned}$$

Using again the FKG property of $P_N\big(\,\cdot\,\big|\Omega_{A^{\complement}}(a)\big)$, we know that

$$P_N(\Omega_{A^{\complement}}^+\,|\Omega_{A^{\complement}}(a)) \ge \prod_{i\in A^{\complement}} P_N(X_i\ge 0\,|\Omega_{A^{\complement}}(a)) = \exp\big(-\log 2|A^{\complement}|\big).$$

An application of Lemma 6.2 of [10] shows that

$$P_N(\Omega_A^+\,|\Omega_{A^{\complement}}(a)) \ge P_N(X_i\ge a, i\in A\,\big|X_j=0, j\in A^{\complement}) = P_{A^{\complement}}(X_i\ge a, i\in A).$$

Next, note that for $d\ge 3$, the variance of X_i remains bounded, thus in view of Lemmas 6.4 and 6.5 of [10], there exists a constant $c>0$ such that

$$P_N(|X_i|\le a, i\in A^{\complement}) \ge \big(c((2a)\wedge 1)\big)^{|A^{\complement}|}. \tag{3.3}$$

Putting things together, we see that

$$\hat P_{N,b,a}(\Omega_N^+) \ge \frac{Z_N}{\hat Z_{N,b,a}} \sum_{A\subset V_N} e^{J'|A^{\complement}|} P_{A^{\complement}}(X_i\ge a, i\in A)$$

where $J'=\log c+\log((2a)\wedge 1)-\log 2+\log(e^b-1)=\log S(a,b)+\log c-\log 2\in\mathbb{R}$.

On the other hand, note that

$$P_N(\Omega_{A^{\complement}}(a)) \le \big((2a)\wedge 1\big)^{|A^{\complement}|}$$

This follows simply from the estimate, valid for $i\in V_N\setminus A$,

$$\sup_{X_j, j\ne i} P_N(|X_i|\le a\,\big|\,X_j, j\ne i) \le (2a)\wedge 1$$

since $P_N(X_i \in \cdot \,|X_j, j \neq i)$ is the normal distribution with variance 1 and mean $\frac{1}{2d}\sum_{j:|i-j|=1} X_j$. Thus

$$
\begin{aligned}
&\hat{P}_{N,b,a}(\xi_N < \varepsilon N^d; \Omega_N^+) \\
&= \frac{Z_N}{\hat{Z}_{N,b,a}} \sum_{A \subset V_N, |A^{\complement}| < \varepsilon N^d} e^{b|A^{\complement}|} P_N(\Omega_{A^{\complement}}(a)) P_N(\{|X_j| > a, j \in A\} \cap \Omega_N^+ \,|\, \Omega_{A^{\complement}}(a)) \\
&\leq \frac{Z_N}{\hat{Z}_{N,b,a}} ((2a) \wedge 1)^{\epsilon N^d} e^{\varepsilon b N^d} \sharp \left\{ A \subset V_N : A^{\complement} < \varepsilon N^d \right\}.
\end{aligned}
$$

That is, with $e^J = ((2a) \wedge 1) e^b$,

$$
\hat{P}_{N,b,a}(\xi_N < \varepsilon N^d \,|\, \Omega_N^+) \leq \frac{e^{\epsilon J N^d} \sharp \left\{ A \subset V_N : A^{\complement} < \varepsilon N^d \right\}}{\sum_{A \subset V_N} e^{J' |A^{\complement}|} P_{A^{\complement}}(X_i \geq a, i \in A)}
$$

and the result follows from Stirling formula and Proposition 3. ■

4 The one dimensional case

In this section we prove part c) of Theorem 2. If $|A^{\complement}| = k$, $A^{\complement} = \{x_1, x_2, \ldots, x_k\}$, $-N \leq x_1 < x_2 < \cdots < x_k \leq N$, and $l_1 = x_1 + N + 1$, $l_2 = x_2 - x_1, \ldots, l_k = x_k - x_{k-1}$, $l_{k+1} = N - x_k + 1$, then

$$
Z_A = \prod_{j=1}^{k+1} z_{l_j}, P_A(\Omega_A^+) = \prod_{j=1}^{k+1} p_{l_j}^+,
$$

where

$$
z_l = \begin{cases} 1 & l = 1 \\ Z_{[1,l-1]}, & l > 1, \end{cases}
$$

and

$$
p_l^+ = \begin{cases} 1 & l = 1 \\ P_{[1,l-1]}(\Omega_{[1,l-1]}^+), & l > 1. \end{cases}
$$

Evidently

$$
z_l = \frac{(2\sqrt{\pi})^{l-1}}{\sqrt{l}}.
$$

We will also use the fact that

$$
p_l^+ = 1/l. \tag{4.1}
$$

We postpone the proof of this. Collecting these facts, we see that

$$\begin{aligned}\varrho_N &:= \sum_{A\subset V_N} e^{J|A^0|}\frac{Z_A}{Z_N}P_A(\Omega_A^+)\\ &= \sum_{k=0}^{2N+1} e^{Jk}(2\sqrt{\pi})^{-(k+1)} \sum_{\substack{l_1,\cdots,l_{k+1}\\ \Sigma l_j=2N+2}} \frac{\sqrt{2N+2}}{\sqrt{l_1\ldots l_{k+1}}}\prod_{j=1}^{k+1}p^+(l_j).\end{aligned}$$

If $t>0$, we conclude that

$$\sum_{N=0}^{\infty}\frac{\varrho_N}{\sqrt{2N+2}}t^{2N+2} = e^{-J}\sum_{k=1}^{\infty}\left(\frac{e^J}{2\sqrt{\pi}}\right)^k\left(\sum_{l=1}^{\infty}t^l\frac{p^+(l)}{\sqrt{l}}\right)^k.$$

For $t=1$ and $J<\log(2\sqrt{\pi})+\log(\sum_{l=1}^{\infty}\frac{p^+(l)}{\sqrt{l}})$ this is convergent, and it follows that $\delta_J^+=0$. On the other hand, if $J>\log(2\sqrt{\pi})+\log(\sum_{l=1}^{\infty}\frac{p^+(l)}{\sqrt{l}})$, then $\sum_{N=0}^{\infty}\frac{\varrho_N}{\sqrt{2N+2}}t^{2N+2}$ is divergent for some $t<1$ sufficiently close to 1, as follows from the continuity of $\Sigma t^l p^+(l)/\sqrt{l}$ on $[0,1]$. From that it is evident that $\delta_J^+>0$. We therefore see that the critical value J_0 is

$$J_0=\log(2\sqrt{\pi})+\log\left(\sum_{l=1}^{\infty}\frac{p^+(l)}{\sqrt{l}}\right)=\log(2\sqrt{\pi})+\log\sum_{l=1}^{\infty}l^{-3/2}.$$

We finally complete the proof of (4.1). Let $\xi_1,\ldots,\xi_l$ be a random vector, whose joint distribution is the law of i.i.d. Gaussian random variables of zero mean and unit variance, conditioned on $\sum_{i=1}^{l}\xi_i=0$. We use the fact that this distribution is invariant under permutations, and in particular under cyclic permutations, and that almost surely, $\sum_{i\in I}\xi_i\neq 0$ for any strict subset I of $\{1,\ldots,l\}$. Let

$$Z=\min_{\alpha,\beta\in 1,\ldots,l}\sum_{i=\alpha}^{(\alpha+\beta)\mathrm{mod}\,l}\xi_i\,.$$

Then, the minimum in the definition of Z is achieved at exactly one pair (α,β). Let $\xi_i^o=\xi_{(i+\alpha+\beta)\mathrm{mod}\,l}$, and define $y_j^o=\sum_{i=1}^{j}\xi_i^o=\sum_{i=1}^{j}\xi_{(i+\alpha+\beta)\mathrm{mod}\,l}$. Then $y_j^o>0$ for $j\neq l$ and $y_l^o=0$, a.s., while for any other cyclic shift, i.e. for any sequence $y_j^\gamma=\sum_{i=1}^{j}\xi_{(i+\gamma)\mathrm{mod}\,l}$ such that $\gamma\neq(\alpha+\beta)\mathrm{mod}\,l$, there exists some j with $y_j^\gamma<0$. It thus follows that, with θ denoting a random variable distributed uniformly on $\{1,\ldots,l\}$,

$$\begin{aligned}P(\sum_{i=1}^{t}\xi_i\geq 0, t=1,\ldots,l) &= E(P(\sum_{i=1}^{t}\xi_{(i+\theta)\mathrm{mod}\,l}\geq 0, t=1,\ldots,l))\\ &= P((\alpha+\beta)=l)=1/l\,.\end{aligned}$$

(4.1) follows at once.

Remark 7 *It is actually easy to see that for $J < J_0$ (1.6) and (1.5) are satisfied, and that for $J > J_0$ the rescaled field*

$$Y_N(t) = \frac{1}{\sqrt{N}} X_{[Nt]}, \quad -1 \leq t \leq 1,$$

converges to a Brownian excursion on $[-1, 1]$. We leave that to the reader.

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