LLN's and CLT's for

RANDOM WALKS in MIXING

RANDOM ENVIRONMENTS

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(Based on joint work with F. Comets)

Model RWRE in $\mathbb{Z}^d$

Environment: S is $2d$-simplex , $\Omega = S^{\mathbb{Z}^d}$,

$\omega = \{\omega(z,e)\}_{z\in\mathbb{Z}^d}$, $|e|_1 = 1$.

Walk is conditional on ω (quenched), Markov:

$P_\omega(X_{n+1} = x + e | X_n = x) = \omega(x,e)$

Questions: LLN? CLT?

Simplest model: P, law on Ω, ergodic (i.i.d.)

Walk is **ballistic** if $\frac{X_n}{n} \underset{n\to\infty}{\longrightarrow} v \neq 0$, $\mathbb{P}$-a.s.

$$(\mathbb{P} = P \times P_\omega \text{ "annealed"})$$

LDP: not today; CGZ '00 d=1; Zerner '98; Varadhan '02; Rassoul-Agha '03

Embarrassing open problem: 0-1 law:

For each direction ℓ, set $A_\ell^\pm := \{X_n \cdot \ell \to \pm\infty\}$.

Prove that $\mathbb{P}(A_\ell^+) \in \{0,1\}$.

Known:

$\mathbb{P}(A_\ell^+ \cup A_\ell^-) \in \{0,1\}$ (Kalikow '81).

0-1 true (P i.i.d.) for $d = 2$ (Zerner-Merkl '02).

Not true for certain ergodic environments:

Zerner-Merkl ('02), $d = 2$, but example is *not uniformly elliptic*.

$d \geq 3$, Bramson-Z.-Zerner ('03): counter example for a uniformly elliptic ergodic environment (polynomial decay of correlations).

Remark

(Zerner '02, P i.i.d.):

If 0-1 law holds then LLN holds (with possibly 0 speed).

Similar result for certain ergodic environments by F. Rassoul-Agha (different, LD-based proof).

LLN, CLT in $\mathbb{Z}^d$:

For $d = 1$, define $\tau_1 = \inf\{n : X_n = 1\}$. Then, using that path cannot go around 1 if $X_n \to \infty$, get recursions that allow to compute $E_P(\tau_1)$, $E_P(\tau_1^2)$, etc. Ergodic arguments then give LLN and CLT, or stable limit laws.

(Solomon, '75; Kozlov-Kesten-Spitzer '75; Sinai '82; Kozlov '85; Alili '99; Bremont '02; Mayer-Wolf, Roitershtein, Z. '03).

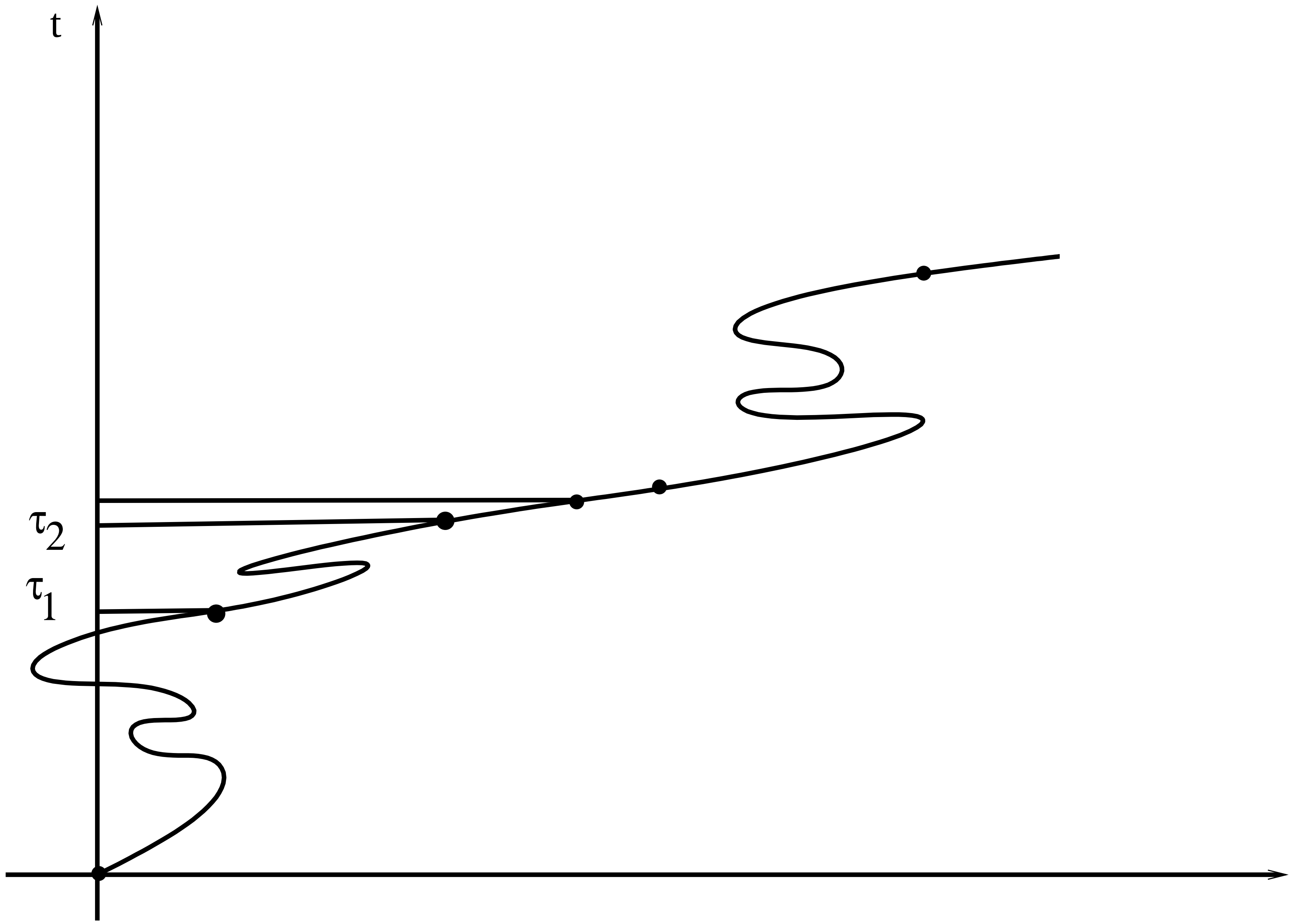
For $d > 1$:
Cannot compute $E_P(\tau_1)$ (even for $\bar{\tau}_1 = \inf\{n : X_n \cdot e_1 = 1\}$)

Alternative: introduce regeneration times.

For RW on random trees: Kesten-Grimmett ($\sim$ '84), Lyons-Peres-Pemantle ('95), Piau ('98), Dembo-Gantert-Peres-Z. ('01).

For RWRE in $\mathbb{Z}^d$: introduced and exploited by Sznitman-Zerner ('99).

Plot $X_n \cdot e_1$:



Fact: If P is i.i.d.,

$$\Big(\eta_i := (\tau_{i+1} - \tau_i), \Big\{X_{\tau_i+n} - X_{\tau_i}\Big\}_{n=1,\cdots,\eta_i},$$

$$\{\omega(z,\cdot)\}_{X_{\tau_i}\cdot e_1 \le z\cdot e_1 < X_{\tau_{i+1}}\cdot e_1}\Big)$$

for $i \geq 1$ are i.i.d. under $\mathbb{P} = P \times P_\omega$ (annealed) (and, with $\tau_0 = 0$, η_0 has, under conditioning that walk does not backtrack, the same law as η_1).

Theorem (Sznitman-Zerner '99)

a) If $E_{\mathbb{P}}(\tau_2 - \tau_1) < \infty$ then walk is ballistic.

b) $E_{\mathbb{P}}(\tau_2 - \tau_1) < \infty$ if Kalikow's condition holds.

Kalikow's condition: $U \subset Z^d, 0 \in U$

$$\widehat{P}_U(x, x+e) = \frac{\mathbb{E}\left(\sum_{n=1}^{\tau_{U^c}} \mathbf{1}_{\{X_n = x\}} \omega(x,e)\right)}{\mathbb{E}\left(\sum_{n=1}^{\tau_{U^c}} \mathbf{1}_{\{X_n = x\}}\right)}$$

τ_{U^c} = exit time from U; $\widehat{P}_U$ and P possess same exit law from U.

Kalikow's drift: $\hat{d}_U(x) = \sum_{|e|_1 = 1} e \widehat{P}_U(x, x+e)$

Kalikow's condition: $\inf_{U, x \in U} \hat{d}_U(x) \cdot e_1 > 0$

(There are easy to check sufficient conditions).

Improvements Sznitman ('00,'01,'02), P- i.i.d.:

- CLT + tail estimates under Kalikow's condition.
- Milder conditions (T,T'), leading also to CLT + tail estimates.
- Criteria for T, T' conditions.

Based on analysis (via renormalization arguments) of tails of $\tau_2 - \tau_1$.

$(d \geq 2)$

No known examples of sub-diffusive behavior!

No known **criteria** for transience-recurrence.

Conjectures (P i.i.d.):

1) (T') characterizes ballistic behavior

2) No sub-diffusive behavior for $d \geq 2$.

For P not i.i.d., regeneration times less useful, even in the uniformly elliptic case:

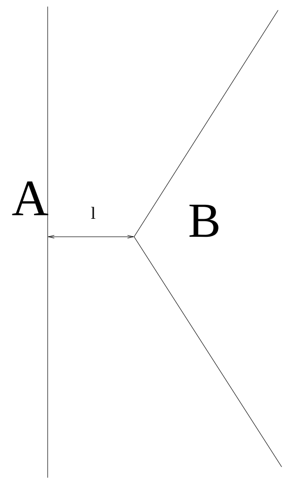
$$\{\tau_{i+1} - \tau_i\} \text{ not even stationary!}$$

Cone of vertex x, direction ℓ and angle $\cos^{-1}(\zeta)$,

$$C(x,\ell,\zeta) = \{y \in \mathbb{R}^d\,;\ (y-x)\cdot\ell \geq \ \zeta|y-x||\ell|\}\,.$$

Mixing Assumption Environment P is uniformly elliptic, stationary and ergodic, and satisfies mixing condition on ℓ-cones: for all positive ζ small enough there exists a function $\phi(r) \underset{r\to\infty}{\to} 0$ such that for any two events A, B with $P(A) >$ 0, $A \in \sigma\{\omega_z; z\cdot\ell \leq 0\}$ and $B \in \sigma\{\omega_z; z \in C(r\ell,\ell,\zeta)\}$ it holds that

$$\left|\frac{P(A\cap B)}{P(A)} - P(B)\right| \leq \phi(r|\ell|)\,.$$



Remark: for $d \geq 2$, taking a hyperplane instead of a cone (i.e. $\zeta = 0$) *implies* that $\phi(R) = 0$ for some R (Bradley). Hence, such condition is too strong.

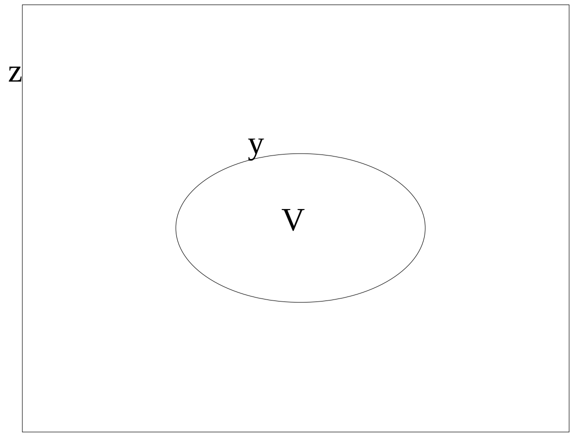
Mixing assumption (with exponentially decaying ϕ) hold e.g. for Gibbs fields satisfying the Dobrushing-Shlosman weak mixing assumption (e.g., Ising at uniqueness regime):

$\exists$ constants $\gamma > 0$, $C < \infty$ such that for all finite subsets $V \subset \Lambda \subset \mathbb{Z}^d$,

$$\sup\left\{\|\pi_\Lambda(\cdot \mid \omega) - \pi_\Lambda(\cdot \mid \omega')\|_V ; \quad \omega, \omega' \in S^{\Lambda^c}\right\}$$

$$\leq C \sum_{y \in V,\, z \in \partial\Lambda^k} \exp(-\gamma |z - y|) \ ,$$

with $\|\mu - \nu\|_V = \sup\{\mu(A) - \nu(A); A \subset \sigma((\omega_x)_{x \in V})\}$.



Modified Kalikow assumption: $\hat{d}_U(x)$ is Kalikow's drift conditioned on environment outside of U. Then, assume there exists a $\delta(\ell) > 0$ deterministic such that

$$\inf_{U, x \in U} \hat{d}_U(x) \cdot \ell \geq \delta(\ell), \ P - a.s..$$

(The infimum is taken over all connected finite subsets of $\mathbb{Z}^d$ containing 0.)

Modifications of regeneration times Suppose walk is uniformly elliptic with ellipticity constant 2κ. Couple walk with independent coins: if

$\omega(x, x+e) \geq 2\kappa$, generate RWRE by ε_n i.i.d.,

$Q_\varepsilon(\varepsilon_1 = e) = \kappa$, $Q_\varepsilon(\varepsilon_1 = 0) = 1 - 2d\kappa$,

$$\tilde{P}_{\omega,\varepsilon}\Big(X_{n+1} = z + e \big| X_n = z\Big) = \mathbf{1}_{\varepsilon_{n+1}=e} + \frac{\mathbf{1}_{\varepsilon_{n+1}=0}}{1-2d\kappa}\left(\omega(z,e) - \kappa\right)$$

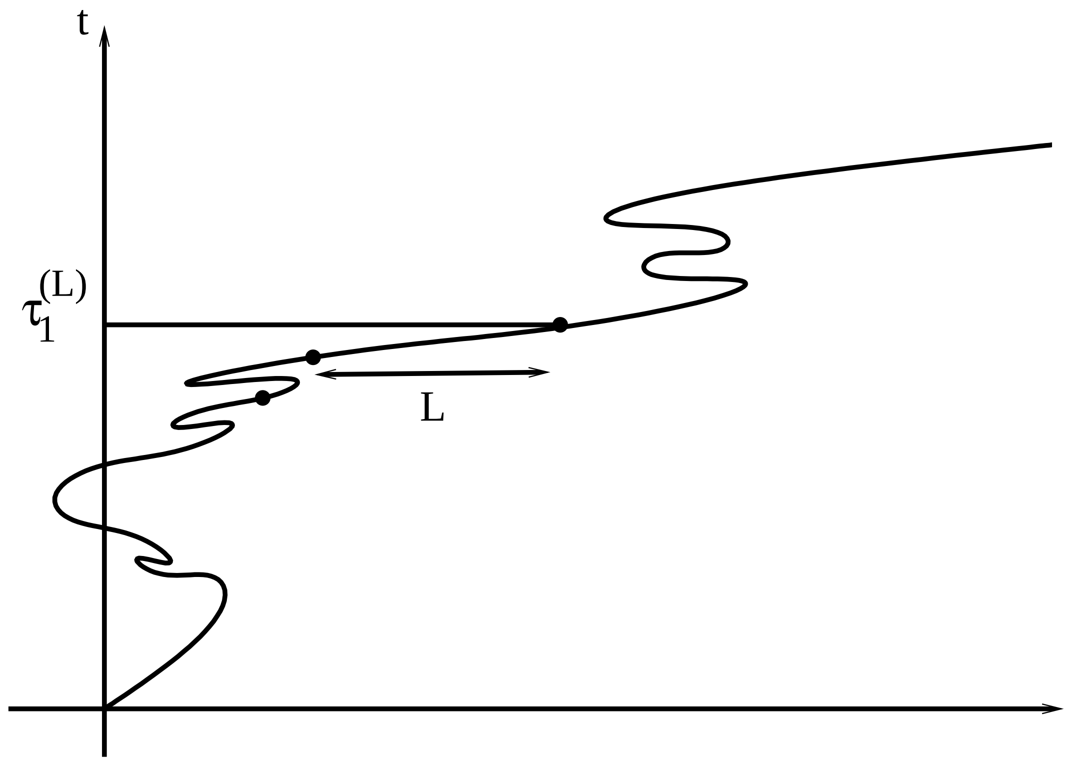
Average over the $\{\varepsilon_n\}$ gives quenched RWRE.

Call a point an *L-reg point* if reg point

(with respect to cones, not hyperplanes!)

$+$ used ε coins L times just before, in dir. ℓ

$+$ fresh point before those.



$\{\tau_{i+1}^{(L)} - \tau_i^{(L)}\}$ not iid, but have better and better (in L) mixing properties.

To get LLN, need extra uniform integrability of $\tau_1^{(L)}$: Assume $\exists \alpha > 1, M = M(L)$ such that, with $1/\alpha' = 1 - 1/\alpha$,

$$\phi'(L)^{1/\alpha'} M(L)^{1/\alpha} \underset{L \to \infty}{\longrightarrow} 0$$

and with $D' = \inf\{n : X_n \cdot \ell < 0\}$,

$$P\Big(\mathbb{E}^o((\kappa^L \tau_1^{(L)})^\alpha \mid D' = \infty, \mathcal{F}_0^L) > M \Big) = 0\,,$$

where $\mathcal{F}_0^L = \sigma(\omega(y,\cdot) : \ell \cdot y < -L)$.

Theorem (Comets-Z. '02) Assume above. Then $X_n/n \to_{n\to\infty} v$, $\mathbb{P}$-a.s., with v deterministic and $v \cdot \ell > 0$.

Assumptions are satisfied for a class of Ising-like models with large enough Kalikow coefficient (checking integrability conditions involves a multi-scale argument).

In case of non-nestling walks, that is RWRE with $d(0) \cdot \ell > \delta > 0$, P-a.s., where $d(0)$ is drift at origin, any mixing rate on cones suffices.

Proof is based on coupling the $\tau_{i+1}^{(L)} - \tau_i^{(L)}$ sequence to a sequence of i.i.d. variables, using LLN (+integrability) for the i.i.d. sequence, and use $L \to \infty$ to control error.

Different approach via environment viewed from particle and regeneration to control law: F. Rassoul-Agha ('02).

What about CLT?

Intuitive idea: take $L \to \infty$. However, errors in coupling pile up, so even if assume second moment finite, all one gets is:

Theorem (Comets and Z., '03): Under above assumption, if (annealed) fluctuations are of scale n (diffusive), then there is a deterministic sequence v_n with $v_n/n \to v$ and a deterministic sequence of non-degenerate covariance matrices R_n such that $(X_n - v_n)R_n^{-1/2}$ converges in distribution to a standard centered Gaussian.

To improve result to a full CLT, impose stronger mixing:

Say that P is strong mixing if it is Gibbsian and $\exists \gamma > 0$, $C < \infty$ such that for all finite subsets $V \subset \Lambda \subset \mathbb{Z}^d$,

$$\sup\{\|\pi_\Lambda(\cdot \mid \omega) - \pi_\Lambda(\cdot \mid \omega')\|_V; \quad \omega, \omega' \in S^{\Lambda^c},$$

$$\omega_x = \omega'_x \text{ except at } x = y\}\}$$

$$\leq C \sum_{z \in \partial V^k} \exp(-\gamma |z - y|) \,,$$

(Assumption holds e.g. for Ising, high temp.)

Theorem (Comets and Zeitouni '03)
Assume Gibbsian strong mixing + second moment uniform integrability of regeneration times (for L large but fixed). Then, $\exists R, v$ such that $[X_{[nt]} - nvt]/\sqrt{n}$ converges in distribution (under the annealed law $\mathbb{P}^o$) to a standard Brownian motion of covariance R.

Idea of proof: define "truncated cone environments and paths"

$$\mathcal{T} = \{(M, S^{C(0,\ell,\zeta,M)}, \mathcal{P}_M)\}_{M=y\cdot\ell>0,\text{some } y\in\mathbb{Z}^d},$$

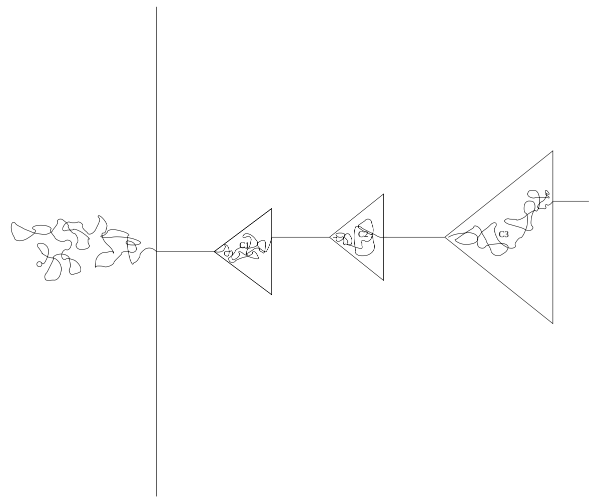
$$\mathcal{P}_M = \{\underline{x} = (x_1, \dots, x_k) \in C(0,\ell,\zeta,M)^{\mathbb{N}_*} :$$

$$x_0 = 0, |x_{i+1} - x_i| = 1\},$$

$$\mathcal{W} = \overline{\mathcal{T}}^{\mathbb{N}} \qquad (\mathcal{T}\text{-words})$$

(adjoining to $\mathcal{T}$ a "stop" symbol to create $\overline{\mathcal{T}}$; M corresponds to regeneration distance, i.e. length of cone).

Elements of $\mathcal{W}$ consist of the portion of cones between regeneration locations, as well as the RWRE path and the ε-coins between regeneration times.



RWRE defines a transition kernel from $\mathcal{W}$ to $\mathcal{W}$ by concatenating the new cone-path to the word.

By use of ε coins, if L is large enough then transitions are dominated below by a fixed measure. Hence, using general results from the theory of chains with infinite dependence, get LLN and CLT statements.

Book of Iosifescu and Grigorescu '90.