Control and Optimal Control

Assignment 8, due February 25, 2012

Consider the linear quadratic optimal control system

$$\begin{aligned} \text{minimize} \quad & \int_{t_0}^{t_1} (x^T Q(t) x + u^T R(t) u) dt \\ \text{subject to} \quad & \frac{dx}{dt} = A(t)x + B(t)u \\ & x(t_0) = x_0 \\ & x(t_1) = x_1 \end{aligned}$$

with $(A(t), B(t))$ controllable on the interval, $Q(t)$ and $R(t)$ positive definite for every t, and no constraints on the control.

Define the extended attainable set of the system as the set of all pairs (x, r) such that x is the attainable set and r is bigger or equal to the infimal cost of stirring x_0 to x.

1. Prove that the extended attainable set is convex and closed.
2. Prove that there is a unique optimal solution to the optimal control problem.
3. Show that the optimal solution is obtained in a feedback form as

   $$u(t) = \frac{1}{2} R(t)^{-1} B(t)^T p^T(t)$$

   where $p(t)$ is a solution of

   $$\frac{dp}{dt} = -pA(t) + 2x^T(t)Q(t)$$

   (hint: Use the Pontryagin maximum principle to formulate a necessary condition for the optimal $u(t)$. Then use a "completion to squares" technique to derive the explicit form for the optimal control. Then use the convexity of question 1 to deduce that the condition is sufficient.
4. Solve the optimal control problem of steering a point $x_0 = (\zeta_0, \eta_0)$ to the origin, while minimizing the energy $\int_0^{2\pi} u(t)^2 dt$, where the equation is the harmonic oscillator system $\frac{d\zeta}{dt} = \eta$, $\frac{d\eta}{dt} = -\zeta + u$, with no restrictions on the control.