Preface to a collection of papers on Computational Complexity and Local Algorithms

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This volume contains a collection of studies in the areas of complexity theory and local algorithms. These studies were conducted during the last seven years. Although these works were posted in the public domain[1] and several of them have been cited in the literature, none of them was formally published before. Nevertheless, all works included in this volume were read by at least one keen reader (other than myself) and comments of these readers have benefited the revisions of these works. In fact, all works were revised, some more significantly than others, in preparation to their publication in this volume.

Indeed, this volume is quite unusual, but not without precedence. In fact, I published volumes of similar nature in 2011 and 2020, under the titles *Studies in Complexity Theory and Cryptography* (LNCS, Vol. 6650) and *Computational Complexity and Property Testing* (LNCS, Vol. 12050), respectively.

The decision to publish such volumes is justified in the prefaces of the previous volumes. In a nutshell, I think that the standard publication mechanism, consisting of conferences and journals, has become dysfunctional. The source of trouble is over-preoccupation with competition, and neglect of the original goals of providing accessibility and supporting dissemination. In contrast, I am an old-timer who still believes in these goals. I believe that science is a communal project not a competition. I want the works included in this volume to be read out of interest in their contents, not as means towards ranking them.

A common theme in almost all works included in this volume is *the interplay between randomness and computation*. This interplay is pivotal to some parts of complexity theory and is essential for local (a.k.a. sublinear)

[1] Most works were posted on *ECCC*, and all were posted on my web-site.

algorithms. The works included in this volume address a variety of topics in the areas of complexity theory and local algorithms.

Within complexity theory the topics include approximation algorithms, counting problems, enumeration problems, explicit construction of expander graphs, fine grained complexity, interactive proof systems, PPT-search and pseudodeterminism, space complexity, and worst-case to average-case reductions.

Within local algorithms the focus is mostly on property testing and on locally testable and decodable codes. In particular, many of the works are part of an agenda that seeks to advance the study of testing graph properties in the bounded-degree graph model; a handful of the papers do so explicitly and three others were initially motivated by this agenda but seem to be of independent interest. Other topics in property testing include testing group properties and testing properties of affine subspaces.

PPT-search and Pseudodeterminism. While most of complexity theory focuses on decision problems, search problems are at least as important for practice. In many cases, the study of search problem can be reduced to the study of decision problems, but in the context of probabilistic polynomial-time (PPT) computation an explicit study is called for. Two issues at hand are how to define PPT-search problems and when can a PPT algorithm almost always return one of a small number of possible solutions. The following papers address these issues.

- On defining PPT-search problems.
- Multi-pseudodeterministic algorithms.

It is fair to say that both works are conceptual and explorative in nature. They present definitions that extend existing ones and shed light on them.

Counting and enumeration problems. When considering search problems, one may be interested in the number of solutions. Hence, the problems of counting, approximate counting, and enumerating solutions arise naturally. The following papers address these types of problems.

- On counting t-cliques Mod 2.
- On coarse and fine approximate counting of t-cliques.
- On the complexity of enumerating ordered sets.

The first two works present quantitative improvements over what was known before, but their main contribution is in presenting alternative proofs and ideas, which are arguably more appealing. The third work is explorative in nature.

Space complexity. While much of complexity theory focuses on time complexity, space complexity is also of natural interest. In that context, the Tree Evaluation problem was proposed as a candidate for a super-logarithmic lower bound. This conjecture has been shaken recently, as illustrated in the following papers.

- On the Cook-Mertz Tree Evaluation procedure.
- Solving Tree Evaluation in $o(\log n \cdot \log \log n)$ space.

The first work is mainly an exposition of a known result using an alternative abstraction, which leads to a small improvement that is presented in the second work. These works are an exception in the current volume, since randomness plays no role in them.

Surveys in complexity theory. The first survey revisit a well-known topic within complexity theory, whereas the second survey provides an exposition of a surprising recent result.

- On parallel repetitions of interactive proof systems.
- On locally-characterized expander graphs (a survey).

Here, again, the focus is on alternative abstractions. In both cases, the surveys do not provide a full proof of the stated results, but do provide a self-contained presentation of some key aspects.

Locally testable and decodable codes. The study of local algorithm for various coding theoretic tasks was introduced a few decades ago. The following papers present relatively recent results in this research direction.

- On the Locally Testable Code of Dinur et al. (2021).
- On the lower bound on the length of relaxed Locally Decodable Codes.
- On the relaxed LDC of BGHSV: A survey that corrects the record.

The three works are expositional in nature, but the perspectives they suggest are different from those that are suggested in the original works. While the first work falls very short of providing a proof of the main result, the other two works provide full proofs of the corresponding main results.

New topics in local computation. The notions of the effective support size (of distributions) and robust/local self-ordering (of graphs) are relatively new. Both are related to testing graph properties, but are of independent interest. The following papers study these notions.

- On the complexity of estimating the Effective Support Size.
- Robust Self-Ordering versus Local Self-Ordering.

In both cases, a concrete motivation (from the study of testing graph properties in the bounded-degree graph model) led to a systematic study of the corresponding topic. The first work was improved by a subsequent work, which builds on its basic observations. The second work raises several interesting open problems.

Testing bounded-degree graphs. The bounded-degree graph model is the second most popular model within the context of testing graph properties. The following papers study a variety of natural testing problems in this model.

- On Testing Hamiltonicity in the Bounded Degree Graph Model.
- Testing Isomorphism in the Bounded-Degree Graph Model.
- On Testing Isomorphism to a fixed graph in the Bounded-Degree Graph Model.
- On Testing Asymmetry in the Bounded Degree Graph Model.
- On the query complexity of testing local graph properties in the Bounded-Degree Graph Model.
- Testing in the bounded-degree graph model with degree bound two.

These works represents a concious and systematic attempt to enrich the study of property testing in the bounded-degree graph model. The first and last works present ultimate results on their subject-matter, but the other works make progress and leave much to be explored.

Other topics in property testing. The first paper presents a basic result regarding property testing at large. In contrast, the other two papers consider specific testing problems.

- On properties that are non-trivial to test.
- One-sided error testing of Monomials and Affine Subspaces.
- On testing group properties.

The first two works present satisfactory results on their subject-matter, whereas the third work begs for continuation.