Non-adaptive vs Adaptive Queries in the Dense Graph Testing Model

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Abstract

We study the relation between the query complexity of adaptive and non-adaptive testers in the dense graph model. It has been known for a couple of decades that the query complexity of non-adaptive testers is at most quadratic in the query complexity of adaptive testers. We show that this general result is essentially tight; that is, there exist graph properties for which any non-adaptive tester must have query complexity that is almost quadratic in the query complexity of the best general (i.e., adaptive) tester.

More generally, for every $q : \mathbb{N} \to \mathbb{N}$ such that $q(n) \leq \sqrt{n}$ and constant $c \in [1, 2]$, we show a graph property that is testable in $\Theta(q(n))$ queries, but its non-adaptive query complexity is $\Theta(q(n)^c)$, omitting $\mathrm{poly}(\log n)$ factors and ignoring the effect of the proximity parameter ϵ. Furthermore, the upper bounds hold for one-sided error testers, and are at most quadratic in $1/\epsilon$.

These results are obtained through the use of general reductions that transport properties of ordered structured (like bit strings) to those of unordered structures (like unlabeled graphs). The main features of these reductions are query-efficiency and preservation of distance to the properties. This method was initiated in our prior work (*ECCC*, TR20-149), and we significantly extend it here.

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Contents

1 Introduction

The fundamental relation between adaptive and non-adaptive oracle machines has been studied in a variety of models. In particular, this relation has been studied also in the context of property testing in various settings. Specifically, in the setting of testing the satisfiability of linear constraints, it was shown that adaptivity offers absolutely no gain [2]. A similar result holds for testing monotonicity of sequences of positive integers [4]. In contrast, an exponential gap between the adaptive and non-adaptive complexities exists in the context of testing other properties of functions [4]. Lastly, we mention that an even more dramatic gap exists in the setting of testing graph properties in the bounded-degree model [14] (see also [5, Thm. 9.2]).

We follow [12, 9] in studying the relation between adaptive and non-adaptive oracle machines in the context of testing graph properties in the dense graph model. This is definitely a natural model, and the study is quite refined because it is known that, in this model, the gap between the query complexities of adaptive and non-adaptive machines is at most quadratic [1, 10]. Our results answer several natural open problems regarding possible relations between these complexities (cf. [5, Sec. 8.5.4]): Essentially, we show that any relation that is not ruled out by the quadratic upper bound is actually possible (i.e., occurs for some graph properties).

Our results are outlined in Section 1.2, following a brief review of the model (provided in Section 1.1). In Section 1.3 we describe some of the ideas used towards proving these results. We hint that the notion of robustly self-ordered graphs and local self-ordering procedures for them, introduced and studied in our prior work [11], play a central role in our proofs.

1.1 Property testing in the dense graph model

Property testing refers to algorithms of sublinear query complexity for *approximate decision*; that is, given oracle access to an object, these algorithms (called testers) distinguish objects that have a predetermined property from objects that are far from the property. Different models of property testing arise from different query access and different distance measures.

In the last couple of decades, the area of property testing has attracted significant attention (see, e.g., [5]). Much of this attention was devoted to testing graph properties in a variety of models including the dense graph model, introduced in [7] and surveyed in [5, Chap. 8]. In this model graphs are represented by their adjacency predicate and distances are measured as the ratio of the number of differing adjacencies to the maximal number of vertex-pairs.

Specifically, a graph $G = ([n], E)$ is represented by the adjacency predicate $g : [n] \times [n] \to \{0, 1\}$ such that $g(u, v) = 1$ if and only if $\{u, v\} \in E$, and oracle access to a graph means oracle access to its adjacency predicate (equiv., adjacency matrix). The distance between the graphs $G = ([n], E)$ and $G' = ([n], E')$ is defined as the fraction of entries (in the adjacency matrix) on which the two graphs disagree. Lastly, recall that a graph property is a set of graphs that is closed under isomorphism; that is, if G is isomorphic to G' and Π is a graph property, then $G \in \Pi$ if and only if $G' \in \Pi$. (In other words, graph properties are actually properties of unlabeled graphs.)

Definition 1.1 (testing graph properties in the dense graph model): *A* tester *for a graph property* Π *is a probabilistic oracle machine that, on input parameters* n *and* ϵ*, and oracle access to an* n*-vertex graph* $G = ([n], E)$ *outputs a binary verdict that satisfies the following two conditions.*

1. *If* $G \in \Pi$*, then the tester accepts with probability at least* $2/3$*.*

2. *If G is ϵ-far from Π, then the tester accepts with probability at most $1/3$, where G is ϵ-*far *from Π if for every n-vertex graph $G' = ([n], E') \in \Pi$ the adjacency matrices of G and G' disagree on more than $\epsilon \cdot n^2$ entries.*

We say that the tester is non-adaptive *if it determines all its queries based on its explicit input parameters (n and ϵ) and its internal coin tosses, independently of answers provided to prior queries. Otherwise, we say that the tester is* adaptive.

The query complexity of a tester for Π is a function (of the parameters n and ϵ) that represents the number of queries made by the tester on the worst-case n-vertex graph, when given the proximity parameter $\epsilon > 0$. The dependency of the complexity on n is the primary concern, and one often views ϵ as an arbitrary small constant. (Definitely $\epsilon \geq n^{-2}$, although one often envisions $\epsilon > n^{-\Omega(1)}$ and even $\epsilon > 1/\log n$.) In light of these dispositions, when stating that the query complexity is $\Omega(q(n))$, we mean that this bound holds for all sufficiently small $\epsilon > 0$; that is, there exists a constant $\epsilon_0 > 0$ such that distinguishing between n-vertex graphs in Π and n-vertex graphs that are ϵ_0-far from Π requires $\Omega(q(n))$ queries.

1.2 Our results

The focus of this paper is on the relation between the query complexity of adaptive and non-adaptive testers in the dense graph model. It has been known for a couple of decades that the query complexity of non-adaptive testers is at most quadratic in the query complexity of adaptive testers [1, 10], and the question of whether this relation is tight has been opened since.

The first indication that non-adaptive testers may need more queries than adaptive ones was provided in [12] in the context of promise problems. In that context (of promise problems), an almost quadratic separation was proved in [9, Thm. 5.7].[1] In the standard context (i.e., no promise), separations were proved in [9], with the largest one being roughly a power of $3/2$; that is, a graph property was presented for which the general query complexity is $\widetilde{O}(q)$ but non-adaptive testers require $\Omega(q^{3/2})$ queries. Our main result is an almost quadratic separation in the standard context; that is, we prove the following –

Theorem 1.2 (an almost quadratic gap between non-adaptive and adaptive query complexities): *There exists a graph property of n-vertex graphs that is testable by an adaptive oracle machine that make $O(\epsilon^{-1} \cdot \sqrt{n} \cdot \log n)$ queries but testing it non-adaptively requires $\Omega(n)$ queries.*

Theorem 1.2 asserts that the gap between the query complexity of non-adaptive testers and adaptive ones may be quadratic. Answering open problems raised in [9, Sec. 1.3] (see also [5, Sec. 8.5.4]), we prove that the gap can take the form of any function that is at most (almost) quadratic; that is, we prove the following –

Theorem 1.3 (generalizing Theorem 1.2 to smaller gaps between non-adaptive and adaptive query complexities): *For every function $g : \mathbb{N} \to \mathbb{N}$ such that $g(n) \leq \sqrt{n}$, there exists a graph property Π that satisfies the following two conditions:*

1. *There exists a general* (i.e., adaptive) *tester of Π that makes $O(\epsilon^{-1} \cdot \sqrt{n} \cdot \log n)$ queries, and any such tester must make $\Omega(\sqrt{n})$ queries.*

[1] Both the aforementioned results hold also in a model in which the graph property may depend on the proximity parameter ϵ.

2. *Any non-adaptive tester must make $\Omega(g(n) \cdot \sqrt{n})$ queries, and there exists such a tester that makes $O(\epsilon^{-2} \cdot g(n) \cdot \sqrt{n} \cdot \log n)$ queries.*

   Actually, the upper bound is $O(\epsilon^{-2} \cdot g(n) \cdot \sqrt{n} + \epsilon^{-1} \cdot \sqrt{n} \cdot \log n)$.

(Recall that n denotes the number of vertices in the tested graph.)

Note that in Theorem 1.2 there was no need to state the lower bound for adaptive testers and the upper bound for non-adaptive testers, since they roughly follows from the fact that the query complexity of non-adaptive testing is at most quadratic in the query complexity of adaptive testing [1, 10].

Theorem 1.3 refers to graph properties that are testable in query complexity $q(n)/\epsilon$ such that $q(n) = \widetilde{O}(n^{-0.5})$. It establishes a conjecture in [9, Sec. 1.3] (which referred to the case of $g(n) = q(n)^{1-\frac{2}{t}}$, for every constant $t \in \mathbb{N}$) as a special case. Theorem 1.3 also answers (negatively) an open problem in [9, Sec. 1.3] that asks whether the forgoing relations (i.e., g's of the form $g(n) = q(n)^{1-\frac{2}{t}}$ for some $t \in \mathbb{N}$) are the only possible ones. We further generalized the result to obtain analogous bounds for any lower level of query complexity; that is, we replace the special case of $q(n) = O(n^{-0.5} \cdot \log n)$ by $q(n) = O(f(n) \log n)$ for any $f(n) \leq n^{-0.5}$.

Theorem 1.4 (generalizing Theorem 1.3 to lower levels of query complexity): *For every functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $f(n) \leq \sqrt{n}$ and $g(m) \leq m$, there exists a graph property Π that satisfies the following two conditions:*

1. *There exists a general* (i.e., adaptive) *tester of Π that makes $O(\epsilon^{-1} \cdot f(n) \cdot \log n)$ queries, and any such tester must make $\Omega(f(n))$ queries.*

2. *Any non-adaptive tester must make $\Omega(g(f(n)) \cdot f(n))$ queries, and there exists such a tester that makes $O(\epsilon^{-2} \cdot g(f(n)) \cdot f(n) \cdot \log n)$ queries.*

   Actually, the upper bound is $O(\epsilon^{-2} \cdot g(f(n)) \cdot f(n) + \epsilon^{-1} \cdot f(n) \cdot \log n)$.

We mention that Part 1 establishes a hierarchy theorem for testing graph properties in the dense graph model. This theorem is slightly weaker than [8, Thm. 4] (i.e. it applies to any $f(n) \leq \sqrt{n}$ rather than to any $f(n) \leq n$ and has a $\log n$ factor gap between its bounds), but its proof seems simpler and is definitely totally different.

Lastly, we prove that essentially the same upper bounds can be obtained by one-sided error testers (i.e., testers that always accept graphs in Π).

Theorem 1.5 (a one-sided error version of Theorem 1.4):[2] *For every functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $f(n) \leq \sqrt{n}$ and $g(m) \leq m$, there exists a graph property Π that satisfies the following two conditions:*

1. *There exists a one-sided error tester of Π that makes $O(\epsilon^{-1} \cdot f(n) \cdot \log^3 n)$ queries, and any tester must make $\Omega(f(n))$ queries.*

2. *Any non-adaptive tester must make $\Omega(g(f(n)) \cdot f(n))$ queries, and there exists a one-sided error non-adaptive tester that makes $O(\epsilon^{-2} \cdot g(f(n)) \cdot f(n) \cdot \log^3 n)$ queries.*

We stress that the lower bounds hold also for two-sided error testers. Note that the upper bounds have extra $\mathrm{poly}(\log n)$ factors (compared to Theorem 1.4).

[2] Actually, the upper bounds hold provided that $\epsilon > n^{-0.49}$. Otherwise, the trivial tester that checks all entries works using $O(1/\epsilon^{4.1})$ queries.

1.3 Techniques

The notion of robustly self-ordered graphs and local self-ordering procedures for them, introduced and studied in our prior work [11], play a central role in our proofs. These notions and the relevant results are reviewed in the preliminaries (i.e., Section 2).[3]

Loosely speaking, a *robustly self-ordered graph* is as far as possible from having even approximate automorphisms; that is, for any t, any permutation of the vertices that displaces t out of the n vertices must "displace" $\Omega(t \cdot n)$ edges. In other words, a graph $G = ([n], E)$ is called robustly self-ordered if for every permutation $\pi : [n] \to [n]$ it holds that G and $\pi(G)$ differ on $\Omega(n) \cdot |\{v \in [n] : \pi(v) \neq v\}|$ vertex-pairs, where $\pi(G) = ([n], \{\{\pi(u), \pi(v)\} : \{u, v\} \in E\})$; that is, if π has t non-fixed-points, then the symmetric difference between the graphs is $\Omega(t \cdot n)$.

As for *local self-ordering procedure for G*, given oracle access to an arbitrary isomorphic copy of G, denoted G', and a vertex v in G', it is required to identify the name (or location) of v in G, while making few queries to G'. That is, a local self-ordering procedure for G is a randomized algorithm that, on input $v \in [n]$ and oracle access to $G' = \pi(G)$, makes $\text{poly}(\log n)$ queries to G' and outputs $\pi^{-1}(v)$ (with probability at least $1 - n^{-c}$, for any desired constant c). We stress that π is *a priori* uknown to this procedure, but indeed $\pi^{-1}(v)$ is partial information about π that is obtained by the procedure.

1.3.1 The basic ideas

Our first contribution is showing that local self-ordering procedure do exist for some robustly self-ordered graphs. Specifically, we use the fact that random graphs are robustly self-ordered [11, Prop. 7.1] and prove that they have a local self-ordering procedure (in a very strong sense).[4]

As we advocated in [11], working with robustly self-ordered graphs offers a way of embedding (naturally ordered) bit strings in *unlabeled* graphs such that relative distances are approximately preserved. As for local self-ordering procedures, they offer a way of sampling these embedded bits *along with their location*. Hence, the robust self-ordering allows to transport lower bounds regarding ordered objects to the domain of unlabeled graphs, whereas local self-ordering offer a way to transport upper bounds in the same direction.

The source of the quadratic gap. Let us first present a testing problem (on ordered structures – matrices) that is the source of our quadratic gap, while noting that robust self-ordering and local self-ordering will come into play when embedding this problem in an unordered structure (unlabeled graphs). The property consists of a pair of n-by-n matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ coupled with bijections (on their rows and columns), denoted $\pi_{\mathtt{r}}, \pi_{\mathtt{c}}, \phi_{\mathtt{r}}, \phi_{\mathtt{c}} : [n] \to [n]$, such that $a_{\pi_{\mathtt{r}}(i),\pi_{\mathtt{c}}(j)} = b_{\phi_{\mathtt{r}}(i),\phi_{\mathtt{c}}(j)}$. In this case, we may write $\pi(A) = \phi(B)$. We are given oracle access both to entries in the two matrices and to the four bijections; that is, we can ask for the $(i,j)^{\text{th}}$ entry of each of the matrices as well as for the value of each of the bijections at any point of our choice. The testing task is to determine whether such a sextuple satisfies the property or is far from satsfying it.[5]

Note that a non-adaptive tester must make $\Omega(n)$ queries to the entries of the matrices, because otherwise it is unlikely that it will make a query (i_1, j_1) to matrix A and a query (i_2, j_2) to matrix

[3] The following formulations are slightly different from those used in [11].

[4] This strong sense allows us to incur a lower overhead than what would follow from using a generic local self-ordering procedure. Specifically, our overhead is $O(\log n)$ rather than $\text{poly}(\log n)$.

[5] The definition of distance will be weighted, making each oracle have equal weight. Alternatively, each of the bijections is repeated n times.

B such that $(\pi_{\mathtt{r}}(i_1), \pi_{\mathtt{c}}(j_1)) = (\phi_{\mathtt{r}}(i_2), \phi_{\mathtt{c}}(j_2))$. Failing to make such a pair of queries does not allow for distinguishing the case that $\pi(A) = \phi(B)$ is random from the case that A and B are random and independent of one another, where in both cases all bijections are random. Here the Birthday Paradox is (negatively) applied to the set of entries.

In contrast, an adaptive tester may make $O(\sqrt{n})$ queries to each of the bijections and find (w.h.p.) i_1 and i_2 such that $\pi_{\mathtt{r}}(i_1) = \phi_{\mathtt{r}}(i_2)$ as well as j_1 and j_2 such that $\pi_{\mathtt{c}}(j_1) = \phi_{\mathtt{c}}(j_2)$. In this case, it has obtained two pairs (i_1, j_1) and (i_2, j_2) such that $(\pi_{\mathtt{r}}(i_1), \pi_{\mathtt{c}}(j_1)) = (\phi_{\mathtt{r}}(i_2), \phi_{\mathtt{c}}(j_2))$. Then, it can make the corresponding queries to A and B, respectively, which is something that a non-adaptive algorithm cannot do. Note that the answers are different with probability that equals the relative distance between the permuted matrices $\pi(A)$ and $\phi(B)$. Here the Birthday Paradox is (positively) applied to the set of rows (and columns), which are quadratically smaller.

Embedding the two-matrix problem in a graph property. Loosely speaking, we embed the two matrices in the connection between two robustly self-orefdered graphs that have local self-ordering procedures. In this embedding, the local self-ordering procedures play the role of the (inverses of the) foregoing bijections, which are not given to us explicitly in the setting of testing graph properties. This offer us a way to transport the foregoing adaptive tester into one that tests graphs in which the two matrices are embedded. Loosely speaking, we let the matrix determine a bipartite graph between two parts of a robustly self-ordered graph, and use the local self-ordering procedure to emulate the foregoing bijections. On the other hand, the robustness of the self-ordering of the graph guarantees that one must respect the structure of the matrix as embedded (rather than freely permute rows and columns in a way that is not consistent with the original bijections).

We stress that *local self-ordering procedures for robustly self-ordered graphs* play a central role in our testers, and proving their existence in the dense graph model is one of the contributions of this paper. We actually present two different procedures: One that carries a small failure probability on each input (see Theorem 2.6), and one that makes no failures on almost all inputs (see Claim 5.5).

1.3.2 Beyond basic ideas

As stated above, our proofs are based on embedding Boolean matrices in robustly self-ordered graphs, but the question is how exactly is this done. One way is proposed in our prior work [11]: We take two robustly self-ordered graphs that exhibit a *gap between their ranges of vertex degrees* and connect them by a bipartite graph that corresponds to the Boolean matrix. As argued implicitly in [11, Lem. 10.3], this yields a robustly self-ordered graph, no matter which matrix is embedded (equiv., bipartite graph is used).

Specifically, in [11, Const. 10.2], the two graphs had the same edge density, since they are taken from the same construction, but they have a different size (by a constant factor) and so their ranges of vertex degrees are sufficiently far apart.[6] We use this idea when proving Theorems 1.2–1.4, but it relies on the fact that when we sample vertices of the combined graph we are likely to get a proportional number of vertices in each of the original two graphs. Hence, this method is not adequate for one-sided error testing (i.e., proving Theorem 1.5).

The alternative, employed in our proof of Theorem 1.5, is using two robustly self-ordered graphs of vastly *different edge density* (e.g., one with edge-density 0.1 and one with edge density 0.9). Furthermore, each induced subgraph of logarithmic size in each of these graphs will have approximately

[6] Specifically, all vertex degrees in one graph are at most $1.1k$, all vertex degrees in another are at least $3.4k$. The first graph has $2k$ vertices and the second has $7k$ vertices, and we connect them by two k-by-k bipartite graphs.

the same edge density as the entire graph. Moreover, in order to maintain the degree separation also in the combined graph, we use only bipartite graphs of edge density 0.5 ± 0.1 (with induced subgraphs that maintain this density).[7]

The fact that each induced subgraph of logarithmic size in the combined graph provides a good approximation of the degrees of almost all vertices allows us to determine to which part almost each vertex belongs. Analogous requirements are imposed on the local self-ordering procedures that we use towards obtaining one-sided error testers: Each subset of logarithmic size allows for self-ordering almost all vertices of each graph that satisfies the tested property (i.e., with the exception of a logarithmic number of vertices). In both cases, random subsets are used to detect graphs that are far from the property, whereas no subset causes rejection of graphs in the property.

Needless to say, the local self-ordering procedures for these non-explicit graphs are non-explicit themselves, and their computational complexity is huge. What is small (i.e., logarithmic in the size of the graph) is the number of queries that they make. The same holds also for the local self-ordering procedure that we use for the two-sided error tester. Although we know of explicit constructions of robustly self-ordered graphs [11, Part II], we do not know of a local self-ordering procedures for them (not even in the query complexity sense that suffices here). Instead, in this case too, we use local self-ordering procedures that work on the non-explicit robustly self-ordered graphs.

1.4 Organization

In Section 2 we review the definitions of robustly self-ordered graphs and local self-ordering procedures for them, and prove their existence (Theorem 2.6). These notions and the corresponding results play a central role in establishing our property testing results (i.e., Theorems 1.2–1.5). The almost qudratic gap between non-adaptive and adaptive testers (Theorem 1.2) is proved in Section 3.

The proof of Theorem 1.2 also serves as basis for modifications that establish Theorems 1.3–1.5. The modifications used to establish Theorems 1.3 and 1.4 are presented in Section 4. While these modifications are relatively simple, establishing Theorem 1.5 requires several additional ideas (see, e.g., the proof of Claim 5.5, the design of Construction 5.7, and the proof of Proposition 5.9). In fact, the proof of Theorem 1.5, presented in Section 5, is the most technically involved part of this paper. An overview of this proof is provided at the beginning of Section 5.

The appendices contain proofs that have appeared elsewhere. They are included for sake of self-containment and because we shall refer to sub-claims that appear in them.

2 Preliminaries

The following definitions and results are mostly reproduced from [11], with a few modifications, which will be spelled out. One of these modifications in merely in notation, and others are substantial. Unless explicitly stated differently, by graphs we mean labeled (simple) graphs.

Self-ordered graphs. For a graph $G = (V, E)$, and a bijection $\phi : V \to V'$, we denote by $\phi(G)$ the graph $G' = (V', E')$ such that $E' = \{\{\phi(u), \phi(v)\} : \{u, v\} \in E\}$, and say that G' is isomorphic to

[7] We do not provide explicit constructions for these ingredients; for example, a bipartite graph with the foregoing parameters constitutes an optimal two-source extractor (for which explicit constructions are still unknown, cf. [15, 3]).

G. The set of automorphisms of the graph $G = (V, E)$, denoted $\mathtt{aut}(G)$, is the set of permutations that preserve the graph G; that is, $\pi \in \mathtt{aut}(G)$ if and only if $\pi(G) = G$. We say that a graph is asymmetric (equiv., self-ordered) if its set of automorphisms is a singleton, which consists of the trivial automorphism (i.e., the identity permutation). Following [11], we actually prefer the term *self-ordered*, because we take the perspective that is offered by the following equivalent definition.

Definition 2.1 (self-ordered (a.k.a asymmetric) graphs): *The graph $G = ([n], E)$ is* self-ordered *if for every graph $G' = (V', E')$ that is isomorphic to G there exists a unique bijection $\phi : V' \to [n]$ such that $\phi(G') = G$.*

In other words, given an isomorphic copy $G' = (V', E')$ of a fixed graph $G = ([n], E)$, there is a unique bijection $\phi : V' \to [n]$ that orders the vertices of G' such that the resulting graph (i.e., $\phi(G')$) is identical to G. Indeed, if $G' = G$, then this unique bijection is the identity permutation.

Robustly self-ordered graphs. In this work, we use a notion, introduced by us in [11] and called *robust self-ordering*, which is a quantitative version self-ordering. Unlike in [11, Part I], our focus in this work is on dense graphs with a high level of robust self-ordering. Hence, we say that a graph $G = ([n], E)$ is robustly self-ordered if, for every permutation $\pi : [n] \to [n]$, the size of the symmetric difference between G and $\pi(G)$ (i.e., $|E \triangle \{\{\pi(u), \pi(v)\} : \{u, v\} \in E\}|$) is $\Omega(n)$ times the number of non-fixed-points under π (i.e., $|\{i \in [n] : \pi(i) \neq i\}|$).[8]

Definition 2.2 (robustly self-ordered graphs): *A graph $G = ([n], E)$ is said to be γ-*robustly self-ordered *if for every permutation $\pi : [n] \to [n]$ it holds that*

$$|E \triangle \{\{\pi(u), \pi(v)\} : \{u, v\} \in E\}| \;\geq\; \gamma \cdot n \cdot |\{i \in [n] : \pi(i) \neq i\}|. \tag{1}$$

An infinite family of graphs $\{G_n = ([n], E_n)\}_{n \in \mathbb{N}}$ is called robustly self-ordered *if there exists a constant $\gamma > 0$ such that for every n the graph G_n is γ-robustly self-ordered.*

Note that $|E_n \triangle \{\{\pi(u), \pi(v)\} : \{u, v\} \in E_n\}| \leq 2n \cdot |\{i \in [n] : \pi(i) \neq i\}|$ always holds.

While our focus in [11, Part II] was on constructing robustly self-ordered graphs, here we are content with their mere existence. Actually, it will be beneficial for us to use the fact that, with high probability, a random graph is robustly self-ordered.

Theorem 2.3 (a random graph is robustly self-ordered [11, Prop. 7.1]): *A random n-vertex graph $G_n = ([n], E_n)$ is $\Omega(1)$-robustly self-ordered with probability $1 - \exp(-\Omega(n))$.*

For sake of self-containment, the proof is reproduced in Appendix A. We mention that the result is implicit in the proof of [13, Thm. 3.1].

[8] This notation is a "scaled up" version of the notation in [11]. Specifically, in [11, Def. 1.2] a graph was defined to be robustly self-ordered if, for every permutation $\pi : [n] \to [n]$, the size of the symmetric difference between G and $\pi(G)$ is $\Omega(1)$ times the number of non-fixed-points under π. Likewise, the parameter γ in [11, Def. 1.2] is the constant hidden in the $\Omega(1)$ notation (rather than in the $\Omega(n)$ notation).

Locally self-ordering a graph. By Definition 2.1 a graph $G = ([n], E)$ is called self-ordered if for every graph $G' = (V', E')$ that is isomorphic to G there exists a unique bijection $\phi : V' \to [n]$ such that $\phi(G') = G$. One reason for our preferring the term "self-ordered" over the classical term "asymmetric" is that we envision being given such an isomorphic copy $G' = (V', E')$ and asked to find its unique isomorphism to G, which may be viewed as ordering the vertices of G' according to (their name in) G. While the foregoing formulation is global in nature (i.e., one is given the entire graph and is asked to find the entire isomorphism), here we are interested in its *local* version: Given a vertex in G' (and oracle access to the adjacency predicate of G'), we wish to find the corresponding vertex in G while making $\mathrm{poly}(\log n)$ many queries to G'.

We stress that, unlike in [11, Sec. 4.4], we *do not* require this task to be performed in $\mathrm{poly}(\log n)$-time; we only bound the number of queries to G'. Another difference is that we allow randomized algorithms; these algorithms may fail (but not err) with small probability. We mention that, even under these relaxations, it is not clear whether graphs having local self-ordering procedures exist[9],

Definition 2.4 (locally self-ordering a self-ordered graph): *We say that a self-ordered graph $G = ([n], E)$ is* locally self-ordered *if there exists a randomized algorithm that, given a vertex v in any graph $G' = (V', E')$ that is isomorphic to G and oracle access to the adjacency predicate of G', makes $\mathrm{poly}(\log n)$ queries to G', fail with probability at most $1/2$, and otherwise output $\phi(v) \in [n]$ for the unique bijection $\phi : V' \to [n]$ such that $\phi(G') = G$.*

Indeed, the isomorphism ϕ orders the vertices of G' in accordance with the original (or target) graph G. We stress that the foregoing algorithm may depend arbitrarily on the graph G and may not be efficient. Effectively, it is only required that, given $v \in V'$, with probability at least $1/2$, the answers given to the queries to G' determine $\phi(v) \in [n]$, and otherwise we get a failure notice rather than a wrong answer. Clearly, the failure probability can be reduced to $n^{-\omega(1)}$ by repetitions, while maintaining a query complexity of $\mathrm{poly}(\log n)$, but since we care about the specific bounds we define a stronger notion that we can achieve at low cost (see Theorem 2.6). This notion, called *reliable locators*, is a subset S of the vertices of G such that the subgraph of G induced by S is self-ordered and unique in G, and every other vertex has a unique signature with respect to S (i.e., its sequence of adjacencies with S is unique).

Definition 2.5 (reliable locator of a self-ordered graph): *A set of vertices $S \subset [n]$ is called a* reliable locator *of a graph $G = ([n], E)$ if the following two conditions hold*

1. *The subgraph of G induced by S is self-ordered and is not isomorphic to any other induced subgraph of G.*
2. *For every $v \in [n] \setminus S$, the adjacencies of v with S uniquely determine v; that is, for every $u \neq v$ in $[n] \setminus S$ there exists $s \in S$ such that $\{u, s\} \in E$ if and only if $\{v, s\} \notin E$.*

We stress that it is *not* required that all induced subgraphs of G are not isomorphic to one another; it is only required that none of the other induced subgraphs is isomorphic to the reliable locator. Hence, the first condition allows for identifying reliable locators in any graph G' that is isomorphic to G, since S' is a reliable locator of G' if and only if the subgraph of G' induced by S' is isomorphic to the subgraph of G induced by any of its reliable locators. The first condition also allows for

[9] An indication to the non-triviality of this problem arises from the fact that most graphs (equiv., random graphs) do not have a deterministic local self-ordering procedure. This fact can be proved analogously to Footnote 11.

self-ordering the vertices of S' (i.e., these vertices can be identified via structure of the (unlabeled) induced subgraph). The second condition then allows us to locally self-order each other given vertex in G', where a vertex outside S' is identified via its adjacencies to the identified vertices of S'.

We shall prove a stronger statement for random graphs, asserting that, in a random graph (whp), at least 99% of the *logarithmically-sized* sets are reliable locators. Note that the fact that reliable locators exist will not be good enough for our applications, where we shall be given oracle access to an isomorphic copy of G and need to find a reliable locator for it (rather than for G). In contrast, the fact that random subsets are reliable locators of G will allow us to find reliable locators in any graph that is isomorphic to G.

Theorem 2.6 (locally self-ordering a random graph):[10] *With probability* $1-(1/\mathrm{poly}(n))$ *over the uniform distribution of* n*-vertex graphs* G_n*, all but* $1/\mathrm{poly}(n)$ *fraction of the* $O(\log n)$*-sized sets are reliable locators for* G_n*, where* poly *denotes any polynomial* (and the hidden constant in the O-notation is proportional to its degree).

We stress again that a reliable locator of G' that is isomorphic to G is uniquely identified by the unlabeled subgraph of G' induced by it, and that given this subgraph the set is recognized as a reliable locator by a thought experiment on G. That is, if $G'=(V',E')$ is isomorphic to G and $G=\phi(G')$ for some unknown to us bijection $\phi:V'\to[n]$, then given S' we can determine whether S' is a reliable locator of G' (equiv., $\phi(S')$ is a reliable locator of G) by inspecting the subgraph of G' induced by S' and checking whether this subgraph is isomorphic to the subgraph induced by some reliable locator of G. We stress that the latter check is a thought experiment that is based on the fact that we know G, and so we know all its reliable locators and the subgraphs that they induce.

Proof: The driving observation is that, for a random graph, with overwhelmingly high probability, disjoint subsets of logarithmic size induce subgraphs that are not isomorphic to one another. An analogous claim holds also for non-disjoint subsets, but the probability is less overwhelming in this case. In particular, it is not true that *each* such subset induces a subgraph that is not isomorphic to any other induced subgraph.[11] But it is true that almost all subsets have this feature.

Claim 2.6.1 (a fixed $O(\log n)$-subset in a random n-vertex graph satisfies Condition 1 of Definition 2.5): *For sufficiently large* $\ell=O(\log n)$ *it holds that for every fixed* ℓ*-subset* S *of* $[n]$*, with probability* $\exp(-\Omega(\ell))$ *over the choice of* G_n*, the subgraph of* G_n *induced by* S *is not isomorphic to the subgraph of* G_n *induced by any other* ℓ*-subset of* $[n]$*. Furthermore, with probability* $\exp(-\Omega(\ell))$ *over the choice of* G_n*, the subgraph of* G_n *induced by* S *is self-ordered.*

[10] We mention that the same holds for graphs that are generated by a $O(\log n)^2$-wise independent process. This is the case since the proof takes a (careful) union bound over events that refer to subgraphs of the random n-vertex graph that are induced by vertex-sets of size $O(\log n)$.

[11] Consider a collection $\mathcal{C}$ of $(\ell-1)$-subsets of $[n/2]$ such that the pairwise intersections are of size at most $\log_2 n<\ell/2$. Specifically, we may have $|\mathcal{C}|=\exp(\Omega(\log n)^2)$. Now, for each $S\in\mathcal{C}$ and $\{i,j\}\in\binom{\{(n/2)+1,...,n\}}{2}$, consider the 0-1 random variable $\zeta_{S,\{i,j\}}=\zeta_{S,\{i,j\}}(G)$ indicating whether the subgraphs of a random graph G that is induced by $S\cup\{i\}$ is identical to the subgraph of G induced by $S\cup\{j\}$. Note that $p=\mathrm{E}[\zeta_{S,\{i,j\}}]=2^{-(\ell-1)}\gg 1/|\mathcal{C}|$, and that $\zeta_{S_1,\{i_1,j_1\}}$ and $\zeta_{S_2,\{i_2,j_2\}}$ are independent if $\{i_1,j_1\}\neq\{i_2,j_2\}$. Furthermore, for different sets $S_1,S_2\in\mathcal{C}$, the co-variance of $\zeta_{S_1,\{i,j\}}$ and $\zeta_{S_2,\{i,j\}}$ is upper-bounded by $q=\Theta(2^{-|S_1\cup S_2|})=O(2^{-(2(\ell-1)-\log_2 n)})=O(n\cdot p^2)$. Hence, the variance of the sum of all $\zeta_{S,\{i,j\}}$'s is $O(|\mathcal{C}|\cdot n^2\cdot p)+O(|\mathcal{C}|^2\cdot n^2\cdot q)$, which equals $O(|\mathcal{C}|^2\cdot n^3\cdot p^2)$. Using Chebyshev's inequality, it follows that the probability that none of the events hold is $\frac{O(|\mathcal{C}|^2\cdot n^3\cdot p^2)}{(|\mathcal{C}|\cdot n^2\cdot p)^2}=O(n^{-1})$.

Note that the failure probability is not small enough to support a union bound over all ℓ-subsets of $[n]$. Claim 2.6.1 follows as a special case of [6, Clm. 8.2], which referred to the case that only the adjacencies of vertices in S are random. For sake of self-containment, the proof is reproduced in Appendix B.

Combining Claim 2.6.1 with the fact that, in a random graph G_n, with probability $\binom{n}{2} \cdot 2^{-\ell}$, the vertices in $[n] \setminus S$ have different neighborhood in S (i.e., satisfies Condition 2), we conclude that each ℓ-subset is a reliable locator of at least a $1 - \exp(-\Omega(\ell))$ fraction of the n-vertex graphs. The claim follows by an averaging argument, while recalling that we can pick $\ell = O(\log n)$ to be large enough so that $\exp(-\Omega(\ell)) = 1/\mathrm{poly}(n)$. ■

Corollary 2.7 (by Theorems 2.3 and 2.6): *There exists a family of robustly self-ordered graphs $\{G_n = ([n], E_n)\}_{n\in\mathbb{N}}$ such that G_n is an n-vertex graph for which all but at most a $1/\mathrm{poly}(n)$ fraction of the $O(\log n)$-sized sets are reliable locators. Furthermore, each vertex in G_n has degree $(0.5 \pm 0.01) \cdot n$.*

3 The Main Result

In this section we prove Theorem 1.2, while hinting that Theorems 1.3–1.5 will be proved (in Sections 4 and 5) by modifications to the proof presented here.

Theorem 3.1 (Theorem 1.2, restated): *There exists a graph property Π that is testable by adaptive algorithms that make $O(\epsilon^{-1} \cdot \sqrt{n} \cdot \log n)$ queries but testing it non-adaptively requires $\Omega(n)$ queries, where n denotes the number of vertices in the tested graph and ϵ denotes the proximity parameter.*

Proof: For every n, we set $k = n/9$ and $\ell = O(\log k)$, and use a family of robustly self-ordered graphs $\{G_k = ([k], E_k) : k \in \mathbb{N}\}$ such that for each G_k all but an $\exp(-\Omega(\ell))$ fraction of the ℓ-sized sets are reliable locators. Furthermore, each vertex in G_k has degree $(0.5 \pm 0.01) \cdot k$. Recall that such graphs are provided by Corollary 2.7. Using these graphs we consider the following construction that consists of a copy of G_{2k} and a copy of G_{7k} such that the first (resp., last) k vertices of G_{2k} and the first (resp., last) k vertices of G_{7k} are connected according to some k-by-k Boolean matrix. These two matrices will be identical in the case of graphs in the property, and will be far apart in the construction used for showing the lower bound on non-adaptive testers.

Construction 3.1.1 (the graph property Π): *Let $G_{2k} = ([2k], E_{2k})$ and $G_{7k} = ([7k], E_{7k})$ be as postulated in Corollary 2.7.*

- *For two k-by-k Boolean matrices $A = (a_{i,j})$ and $B = (b_{i,j})$, we define the graph $G_{A,B} = ([9k], E_{A,B})$ such that*

$$\begin{aligned} E_{A,B} \;=\; & E_{2k} \cup \{\{2k+i, 2k+j\} : \{i,j\} \in E_{7k}\} \\ & \cup \{\{i, 2k+j\} : i,j \in [k] \wedge a_{i,j} = 1\} \cup \{\{k+i, 8k+j\} : i,j \in [k] \wedge b_{i,j} = 1\}. \end{aligned} \tag{2}$$

  That is, $G_{A,B}$ consists of a copy of G_{2k} and a copy of G_{7k} that are connected by two bipartite graphs that are determined by A and B, respectively. The first bipartite graph connects $[k]$ to $\{2k+1, ..., 3k\}$ and the second bipartite graph connects $\{k+1, ..., 2k\}$ to $\{8k+1, ..., 9k\}$. See Figure 1.

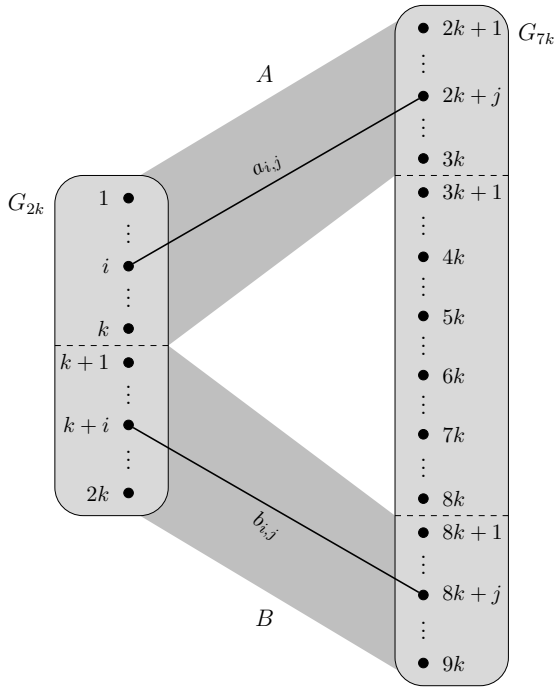

Figure 1: The graph $G_{A,B}$.

- *We define the property* $\Pi = \bigcup_{n\in\mathbb{N}} \Pi_n$ *such that* Π_n *is the set of all graphs that are isomorphic to some* n*-vertex graph* $G_{A,A}$*; that is,*

$$\Pi_{9k} = \left\{\pi(G_{A,A}) : A \in \{0,1\}^{k\times k} \wedge \pi \in \mathrm{Sym}_{9k}\right\} \tag{3}$$

  where Sym_{9k} *denote the set of all permutations over* $[9k]$.

Note that, given a graph of the form $\pi(G_{A,A})$, the vertices of G_{2k} are easily identifiable (as having degree at most $0.51 \cdot 2k + 2k < 3.1k$).[12]

Claim 3.1.2 (lower bound for non-adaptively testing Π): *Any non-adaptive tester for* Π_{9k} *has query complexity* $\Omega(k)$.

Proof: We first show that $\Omega(k)$ non-adaptive queries are required in order to distinguish the following two distributions.

1. The uniform distribution over Π_{9k}; that is, the distribution generated by picking uniformly $A \in \{0,1\}^{k\times k}$ and $\pi \in \mathrm{Sym}_{9k}$, and outputting $\pi(G_{A,A})$.
2. The distribution generated by picking uniformly and independently $A, B \in \{0,1\}^{k\times k}$ and $\pi \in \mathrm{Sym}_{9k}$, and outputting $\pi(G_{A,B})$.

[12] In contrast, the vertices of G_{7k} have degree at least $0.49 \cdot 7k > 3.4k$.

The key point here is that a non-adaptive machine determines its queries beforehand, and so it suffices to analyze the probability that a fixed sequence of q queries distinguishes the two distributions. Furthermore, the two q-long sequences of answers provided by these two distributions to a fixed q-long sequence of queries are distributed identically, unless the sequence of queries contains two queries (u_1, v_1) and (u_2, v_2) that refer to corresponding entries in the matrices A and B; that is, unless $(\pi^{-1}(u_1), \pi^{-1}(v_1) - 2k) \in [k]^2$ equals $(\pi^{-1}(u_2) - k, \pi^{-1}(v_2) - 8k)$.[13] However, the probability that this event occurs is upper-bounded by $\binom{q}{2} \cdot \frac{2k^2}{(9k)^2} \cdot \frac{1}{(9k)^2-1} < q^2/6000k^2$.

On the other hand, distinguishing these two distributions is essential for testing Π, since the first distribution is supported by Π, whereas (as shown next) graphs selected according to the second distribution are $\Omega(1)$-far from Π, with overwhelmingly high probability. To prove the latter claim, consider a random graph $\pi(G_{A,B})$ generated according to the second distribution and an arbitrary graph $\phi(G_{C,C}) \in \Pi$. Then, with probability at least $1 - \exp(-\Omega(k^2))$, the matrices A and B disagree on at least $k^2/3$ of their entries. Fixing such A and B (as well as π), we consider the following cases, where $\gamma \in (0, 1]$ is a constant such that all G_k's are γ-robustly self-ordered (see Definition 2.2).

***Case 1*:** $d \stackrel{\text{def}}{=} |\{i \in [9k] : \pi(i) \in [2k] \wedge \phi(i) \notin [2k]\}| \geq \gamma \cdot k/9$.

Using the different in degrees between the vertices in G_{2k} and G_{7k}, which is at least $0.3k$, it follows that the symmetric difference between $\pi(G_{A,B})$ and $\phi(G_{C,C})$ is at least $d \cdot 0.3k > 0.03\gamma \cdot k^2$.

***Case 2*:** $d < \gamma \cdot k/9$ and $d' \stackrel{\text{def}}{=} |\{i \in [9k] : \pi(i) \neq \phi(i)\}| \geq k/4$.

Using the γ-robust self-ordering of the graphs G_{2k} and G_{7k}, it holds that the symmetric difference between $\pi(G_{A,B})$ and $\phi(G_{C,C})$ is at least $(\gamma \cdot (d' - d) - d) \cdot 2k$, which is at least $(\gamma \cdot d' - 2d) \cdot 2k = \Omega(k^2)$.

(Letting $D_1 = \{i \in [9k] : \pi(i) \in [2k] \wedge \phi(i) \notin [2k]\}$ and $D_2 = \{i \in [9k] : \pi(i) \notin [2k] \wedge \phi(i) \in [2k]\}$, we only count on the contribution of the vertices in either $\{i \in [9k] \setminus D_1 : \pi(i) \in [2k] \setminus \{\phi(i)\}\}$ or $\{i \in [9k] \setminus D_2 : \pi(i) \in \{2k+1, ..., 9k\} \setminus \{\phi(i)\}\}$, and subtract for each contribution d units per their potential neighbours in either D_1 or D_2.)

***Case 3*:** $d' < k/4$.

In this case, we consider the set $D = \{i \in [9k] : \pi(i) \neq \phi(i)\}$, which is smaller than $k/4$, and the set

$$\Delta = \{(i,j) \in [k]^2 : a_{i,j} \neq b_{i,j} \ \& \ i, k+i \notin D \ \& \ 2k+j, 8k+j \notin D\}.$$

Recalling that $|\{(i,j) \in [k]^2 : a_{i,j} \neq b_{i,j}\}| \geq k^2/3$, it follows that $|\Delta| \geq \frac{k^2}{3} - |D| \cdot k = \Omega(k^2)$. Lastly note that each entry in Δ contributes one unit to the symmetric difference between $\pi(G_{A,B})$ and $\phi(G_{C,C})$, since the corresponding adjacencies in $\pi(G_{A,B})$ are different whereas these adjacencies are equal in $\phi(G_{C,C})$. We stress that $(i,j) \in \Delta$ contributes to the symmetric difference because this entry appears in the same location in both graphs (i.e., $\pi(i) = \phi(i)$, $\pi(k+i) = \phi(k+i)$, $\pi(2k+j) = \phi(2k+j)$, and $\pi(8k+j) = \phi(8k+j)$), whereas $a_{i,j} \neq b_{i,j}$ implies that either $a_{i,j} \neq c_{i,j}$ or $a_{i,j} \neq c_{i,j}$.

[13] Indeed, assuming that $\pi^{-1}(u_1), \pi^{-1}(u_2) - k \in [k]$ and $\pi^{-1}(v_1) - 2k, \pi^{-1}(v_2) - 8k \in [k]$, these queries are answered by the values $a_{\pi^{-1}(u_1),\pi^{-1}(v_1)-2k}$ and $b_{\pi^{-1}(u_2-k),\pi^{-1}(v_2)-8k}$, and these values are informative (towards distinguishing the two distributions) if and only if $(\pi^{-1}(u_1), \pi^{-1}(v_1) - 2k) = (\pi^{-1}(u_2 - k), \pi^{-1}(v_2) - 8k)$.

Hence, we have shown that in all cases the symmetric difference between $\pi(G_{A,B})$ and $\phi(G_{C,C})$ is $\Omega(k^2)$, whereas these are $O(k)$-vertex graphs. ■

Claim 3.1.3 (upper bound for adaptively testing Π): *There exists an adaptive tester for Π_{9k} of query complexity $O(\epsilon^{-1} \cdot \sqrt{k} \cdot \log k)$.*

Proof: The basic idea is for the tester to first identify the copies of G_{2k} and G_{7k} in the tested graph G', and then to index vertices in G' according to their location in G_{2k} and G_{7k} via the local self-ordering procedure (specifically via reliable locators). If these procedures fail, then we can definitely reject. More generally, we test whether G' is close to some graph of the form $\pi(G_{A,B})$ for some matrix pair (A, B) and a bijection $\pi : [9k] \to [9k]$. Assuming that this is the case, we generate samples of $O(\sqrt{k}/\epsilon)$ vertices in the two parts of G', and expect to obtain $\Omega(1/\epsilon)$ pairs of vertices (of G') that correspond to vertex-pairs $(i, k+i)$ in G_{2k} and $\Omega(1/\epsilon)$ pairs of vertices (of G') that correspond to vertex-pairs $(j, 6k+j)$ in G_{7k}. These pairs allow us to compare $\Omega(1/\epsilon)^2$ corresponding entries in the two matrices, since the $(i,j)^{\text{th}}$ entry in A (resp., in B) is represented by the adjacency of the i^{th} vertex of G_{2k} and the j^{th} vertex of G_{7k} (resp., the $k+i^{\text{th}}$ vertex of G_{2k} and the $6k+j^{\text{th}}$ vertex of G_{7k}). Each of these entries is uniformly distributed in $[k] \times [k]$, and $\Omega(1/\epsilon)$ of them are independently distributed (and so we shall use them).

The crucial point is that we apply the Birthday paradox to the k rows (resp., columns) of these matrices rather than to their k^2 entries. Adaptivity is used in order to query the relevant entries (rather than querying all entries in the generalized rectangle spanned by the sampled rows and columns). We stress that the collisions (of rows and of columns, which are indexed by vertices in G_{2k} and G_{7k}, respectively) can be identified only after self-ordering the sampled vertices, and identification is performed using the local self-ordering procedure.

The actual tester. Recalling that $\ell = O(\log n)$ and following the foregoing outline, our tester first selects a sample that will be used to identifies the G_{2k} and G_{7k} parts of G' and to index the vertices in these parts (see Steps 1 and 2). Then, an additional sample is chosen and it is used for testing that G' has the form $\pi(G_{A,B})$ for some matrices A and B and a permuation π (see Steps 3–5). This additional sample is large enough to cause collisions among rows (resp., among columns) of the two matrices, which in turn allow for testing the equality of A and B (in Step 6). Specifically, the tester proceeds as follows, on input $\epsilon > 0$ and oracle access to $G' = ([9k], E')$.

1. *Selecting a sample for degree estimation*: We select uniformly at random a set of 9ℓ vertices, denoted S, and inspect the subgraph of G' induced by S. If it contains less than 1.9ℓ or more than 2.1ℓ vertices of degree at most 3.1ℓ, then we halt and reject. (We may also reject if any of the other vertices has degree less than 3.4ℓ, but this is unuseful because this check is performed in Step 3 based on a larger sample.)

   In the sequel, we say that a vertex has low degree if it contains at most 3.1ℓ neighbors in S; otherwise, we say that it has high degree.

   (Recall that a graph in Π_{9k} has $2k$ vertices of degree at most $0.51 \cdot 2k + 2k < 3.1k$ and $7k$ vertices of degree at least $0.49 \cdot 7k > 3.4k$.)

2. *Finding reliable locators in the initial sample*: Let S_1 be a random ℓ-subset of the low degree vertices selected in Step 1, and S_2 be a random ℓ-subset of the high degree vertices. (Recall that S contains more than ℓ low (resp., high) degree vertices.) If the subgraph of G' induced

by S_1 (resp., by S_2) is not isomorphic to the subgraph of G_{2k} (resp., of G_{7k}) that is induced by some reliable locator of G_{2k} (resp., of G_{7k}), then we halt and reject.

Otherwise, we let π_1 (resp., π_2) denote the unique isomorphism between the subgraph of G' induced by S_1 (resp., S_2) and the subgraph of G_{2k} (resp., of G_{7k}) induced by the (unique) reliable locator that we identified for G_{2k} (resp., for G_{7k}).

(Note that no queries are made in this step: The subgraphs of G_{2k} and G_{7k} that are induced by the various ℓ-subsets are hard-wired in the tester.)

3. *Sampling the two parts of the graph and locating the sampled vertices*: We select uniformly at random a set of $O(\sqrt{k}/\epsilon)$ vertices, denoted R, and let R_1 and R_2 denote the low and high degree vertices, where these degrees are approximated according to S. That is, for each vertex $r \in R$, we check its neighborhood in S, by making $|S|$ queries.

   If R_2 contains a vertex of degree less than 3.4ℓ, then we reject.

   Using the local self-ordering procedure (based on the reliable locator S_1 (resp., S_2)), we determine the location in G_{2k} (resp., G_{7k}) of each vertex in R_1 (resp., R_2). Specifically, we determine that vertex v in R_1 corresponds to vertex i in G_{2k} if the sequence of adjacencies of v with S_1 (in G') match the adjacencies of i with $\pi_1(S_1)$ (in G_{2k}); that is, if for every $s \in S_1$ it holds that $\{v, s\} \in E'$ if and only if $\{i, \pi_1(s)\} \in E_{2k}$. Ditto for $v \in R_2$ (using S_2 and π_2).

   If any of these invocation fails, then we reject. Ditto if two different vertices were assigned the same location. Otherwise, we define $\pi_1 : R_1 \to [2k]$ and $\pi_2 : R_2 \to [7k]$ (or rather extend them from S_i to $S_i \cup R_i$) accordingly; that is, $\pi_1(v)$ (resp., $\pi_2(v)$) is the location of $v \in R_1$ (resp., $v \in R_2$) in G_{2k} (resp., G_{7k}).

4. *Using the sample to test isomorphism of each part of G' to the corresponding part in a generic $G_{A,B}$*: We test that the subgraph of G' induced by R_1 (resp., R_2) equals the subgraph of G_{2k} induced by $\pi_1(R_1)$ (resp., the subgraph of G_{7k} induced by $\pi_2(R_2)$). This test is performed by checking $O(1/\epsilon)$ vertex-pairs at random; that is, for every selected vertex-pair $(u, v) \in R_1 \times R_1$, we check whether its adjacency in G' fits the adjacency of $(\pi_1(u), \pi_1(v))$ in G_{2k} (and ditto for R_2 and G_{7k} (via π_2)).

5. *Using the sample to complete a test of isomorphism of G' to a generic $G_{A,B}$*: We test that there are no edges between $R_{1,1} \stackrel{\text{def}}{=} \{u \in R_1 : \pi_1(u) \in [k]\}$ and $R_{2,2} \stackrel{\text{def}}{=} \{v \in R_2 : \pi_2(v) \in \{k+1, ..., 7k\}\}$, and ditto between $R_{1,2} \stackrel{\text{def}}{=} \{u \in R_1 : \pi_1(u) \in \{k+1, ..., 2k\}\}$ and $R_{2,1} \stackrel{\text{def}}{=} \{v \in R_2 : \pi_2(v) \in [6k]\}$. This test is performed by checking $O(1/\epsilon)$ vertex-pairs at random; that is, for every selected vertex-pair $(u, v) \in (R_{1,1} \times R_{2,2}) \cup (R_{1,2} \times R_{2,1})$, we reject if u is connected to v in G'.

6. *Testing that $A = B$*: We call a pair $(i, j) \in [k]^2$ a matrix-collision if there exists $u_1, v_1 \in R_1$ and $u_2, v_2 \in R_2$ such that $\pi_1(u_1) = i = \pi_1(v_1) - k$ and $\pi_2(u_2) = j = \pi_2(v_2) - 6k$. In such a case, we call i a row-collision, and call j a column-collision.

   (Note that, if $G' \in \Pi$, then we are likely to see $\Theta(1/\epsilon)$ row-collision (resp., column-collision). Furthermore, in this case, every matrix-collision (i, j) satisfies $\{\pi_1^{-1}(i), \pi_2^{-1}(j)\} \in E'$ if and only if $\{\pi_1^{-1}(k+i), \pi_2^{-1}(6k+j)\}$, since these edges correspond to the $(i, j)^{\text{th}}$ entry in the matrix A such that G' is isomorphic to $G_{A,A}$.)

Letting I (resp., J) denote the set of row-collisions (resp., column-collision), we select a random set of $\min(|I|, |J|, \Theta(1/\epsilon))$ disjoint pairs $P \subseteq I \times J$ (i.e., distinct $(i_1, j_1), (i_2, j_2) \in P$ satisfy both $i_1 \neq i_2$ and $j_1 \neq j_2$). For each $(i, j) \in P$, we query G' on the pairs $(\pi_1^{-1}(i), \pi_2^{-1}(j))$ and $(\pi_1^{-1}(k+i), \pi_2^{-1}(6k+j))$, and reject if the answers are different.

If we did not reject so far, then we accept.

We stress that in Step 6 we only make $O(1/\epsilon)$ queries; these queries are determined adaptively based on the information gathered in Step 3 (i.e., the values $(\pi_1(v) : v \in R_1)$ and $(\pi_2(v) : v \in R_2)$ that were determined there). Hence, we only query $O(1/\epsilon)$ pairs out of all $|R_1| \cdot |R_2| = O(k/\epsilon^2)$ pairs.

Analysis of the foregoing algorithm. The query complexity of the algorithm is $O(\epsilon^{-1} \cdot \sqrt{k} \cdot \log k)$, where Step 3 dominates the number of queries made. Specifically, for each vertex in R, we invoked the local self-ordering procedure, which results in quering its neighborhood in S; that is, we queried all pairs in $R \times S$.

Let us first verify that graphs in Π are accepted with probability at least $1 - \exp(-\Omega(\ell))$, where the unlikely rejection events are solely due to Steps 1–3 (i.e., either a wrong approximation of a vertex's degree (in either Step 1 or Step 3) or failure to sample a reliable locator in Step 2).[14] In contrast, if all vertices of $G' \in \Pi$ are correctly categorized (based on their approximated degrees per their neighborhood in S) and a reliable locator was found, then π_1 (resp., π_2) equals the unique isomorphism between the subgraph of G' induced by the low degree vertices and G_{2k} (resp., the subgraph of G' induced by the high degree vertices and G_{7k}). In this case, all subsequent checks will be successful, and our tester will always accept.

On the other hand, suppose that $G' = ([9k], E')$ is ϵ-far from Π. As a warm-up, consider the case that $G' = \pi(G_{A,B})$, for some matrix pair (A, B) and a bijection $\pi : [9k] \to [9k]$. In this case, B must be 81ϵ-far from A, since the adjacencies determined by B constitute a $1/81$ fraction of all vertex-pairs. It follows that if we reach Step 6, then we reject with high probability, since we are likely to inspect $\Omega(1/\epsilon)$ disjoint matrix-collisions, which are uniformly and independently distributed in $[k]^2$. Intuitively, if G' is close enough to some $\pi(G_{A,B})$, then the same argument holds, and otherwise Steps 1–5 reject with high probability. The actual analysis follows.

Let V_1 denote the set of vertices of G' that have degree at most $3.2k$, and $V_2 \subseteq [9k] \setminus V_1$ denote the set of vertices of G' that have degree at least $3.3k$. We may assume that the degrees of all vertices are well approximated by their neighborhoods in the sample S (selected in Step 1), since this holds with probability at least $1 - \exp(-\Omega(\ell))$. It follows that all vertices in $V_1 \cup V_2$ are correctly categorized as low and high degree vertices. We may also assume that $|V_1 \cup V_2| \geq 9k - \epsilon' k$, since otherwise Step 3 rejects with high probability (because vertices of degree in $(3.2k, 3.3k)$ are most likely to be placed in R_2 and cause rejection per their too low degree).

Assuming that Steps 1 and 2 were completed successfully (i.e., without rejection), we define a function $\phi_1 : V_1 \to [2k] \cup \{\perp\}$ such that $\phi_1(v)$ denotes the answer of the local self-ordering (based on S_1) to the input v, which may be a failure symbol, denoted $\perp$, where failure may occur because the subgraph of G' induced by V_1 is not necessarily isomorphic to G_{2k}. Similarly, we define $\phi_2 : V_2 \to [7k] \cup \{\perp\}$. Note that ϕ_i agrees with π_i on R_i, where π_i is defined exactly in this manner in Step 3.

[14] Note that a wrong approximation of a vertex degree may lead to placing it in the wrong R_i, which may lead to rejection at later steps. Failure to find a reliable locator in Step 2 may also be due to S containing too few low (resp., high) degree vertices, which is also highly unlikely. Specifically, a degree approximation error occurs with probability at most $n \cdot \exp(-\Omega(\ell)) = \exp(-\Omega(\ell))$, whereas failure to sample a reliable locator occurs with probability at most $\exp(-\Omega(\ell))$.

Letting $\epsilon' = \Omega(\epsilon)$, we may assume that ϕ_i does not evaluate to $\perp$ on more than $\epsilon' k$ points, since otherwise such a point is sampled (w.h.p.) by Step 3, leading it to reject. Likewise, ϕ_i does not have more than $\epsilon' k$ points that have an image with several pre-images under ϕ_i, where here a Birthday argument proves the claim (since the collision probability is at least $1/\epsilon' k$). In particular, it follows that $|V_1| \leq (2+2\epsilon') \cdot k$ (resp. $|V_2| \leq (7+2\epsilon') \cdot k$), because otherwise ϕ_1 (resp., ϕ_2) would have had more than $\epsilon' k$ points that have an image with several pre-images under it.

Denote by $V_i' \subseteq V_i$ the set of vertices that were not discarded above; that is, $V_i' = \{v \in V_i : \phi_i(v) \neq \perp \,\&\, |\phi_i^{-1}(\phi_i(v))| = 1\}$. Using $|V_1| \geq 9k - |V_2| - \epsilon' k \geq (2-3\epsilon') \cdot k$ (resp. $|V_2| \geq (7-3\epsilon') \cdot k$), we note that $|V_1'| \geq (2-5\epsilon') \cdot k$ and $|V_2'| \geq (7-5\epsilon') \cdot k$ must hold, and define ϕ_i' as the restriction of ϕ_i to V_i'. Lastly, let $\phi' : V_1' \cup V_2' \to [9k]$ such that $\phi'(v) = \phi_i'(v)$ if $v \in V_1'$ and $\phi'(v) = 2k + \phi_i'(v)$ otherwise, and let ϕ be an arbitrary extension of ϕ' to a permutation over $[9k]$.

Relying on Step 4 (and assuming that it does not reject w.h.p), we infer that the subgraph of $\phi(G')$ induced by $[2k]$ (resp., by $\{2k+1, ..., 9k\}$) is $6\epsilon'$-close to G_{2k} (resp., to G_{7k} (when relabeling its vertices by subtracting $2k$ to each label)). Relying on Step 5 (and assuming that it does not reject w.h.p), we infer that $\phi(G')$ is $7\epsilon'$-close to some $G_{A,B}$ for some matrices A and B. Recalling that G' is ϵ-far from Π (and using $\epsilon = 8\epsilon'$), we infer that A and B must disagree on more than $\epsilon' \cdot 81k^2$ entries, which means that they are $81\epsilon'$-far apart. We claim that in this case Step 6 rejects with high probability.

We stress that so far we have not conditioned the sets R_1 and R_2; the foregoing inferences that involve Steps 3–5 were thought experiments.[15] Recalling that $|V_1| = (2k \pm 0.3k)$, we observe that with high probability (over the choice of R_1 and R_2), Step 6 defines a set of at least $1/\epsilon$ disjoint matrix-collision pairs, and that these pairs are independently and uniformly distributed in $[k] \times [k]$. Such a pair (i,j) causes rejection if the following conditions hold:

1. A and B disagree on (i,j) (i.e., $a_{i,j} \neq b_{i,j}$);
2. $\phi_1^{-1}(i)$ and $\phi_1^{-1}(k+i)$ are in V_1', which implies $\pi_1(\phi_1^{-1}(i)) = i$ and $\pi_1(\phi_1^{-1}(k+i)) = k+i$;
3. $\phi_2^{-1}(j), \phi_2^{-1}(6k+j) \in V_2'$, which implies $\pi_2(\phi_2^{-1}(j)) = j$ and $\pi_2(\phi_2^{-1}(6k+j)) = 6k+j$.

The probability that this event holds is lower-bounded by $81\epsilon' - 5\epsilon' - 5\epsilon' > 8\epsilon$, and the claim follows. ■

Combining Claims 3.1.2 and 3.1.3, the theorem follows. ■

Digest. Note that the proof of Claim 3.1.2 only uses the hypothesis that the graphs $\{G_k\}_{k \in \mathbb{N}}$ are robustly self-ordered, whereas the proof of Claim 3.1.3 only uses the hypothesis that these graphs can be locally self-ordered (via a reliable locator of logarithmic size). The robust self-ordering is used to infer that if $\pi(G_{A,B})$ is far from $\pi'(G_{A',B'})$, then either π is far from π' or (A,B) is far from (A',B'). The local self-ordering procedure is used in order to associate vertices of $\pi(G_{A,B})$ with rows or columns of either A or B.

We note that, in contrast to the situation in the bounded-degree graph model (cf. [11, Thm. 4.7]), the local self-ordering procedures that we used are not computationally efficient. This is the reason that our testers are also not computationally efficient. The logarithmic gap between the square

[15] We argued that if certain conditions do not hold, then one of these steps would reject with high probability (when using random R_1 and R_2). However, we continued assuming that the conditions do hold, and did not condition the distribution of R_1 and R_2.

root of the lower bound on the query complexity of non-adaptive testers and the query complexity of our adaptive tester is due to the query complexity of our local self-ordering procedure, which we indeed tried to minimize. Lastly, we comment that "local reversed self-ordering" (cf. [11, Def. 4.8]) requires very high query complexity in the dense graph model (unlike in the bounded-degree graph model [11, Thm. 4.9]).

4 Generalizations

In this section we prove Theorems 1.3 and 1.4. The proofs are obtained by gradual modifications of the proof of Theorem 3.1, presented in Section 3. While these modifications are relatively simple, establishing Theorem 1.5 requires several additional ideas, which will be presented in Section 5.

4.1 Smaller complexity gaps

Theorem 3.1 asserts that the gap between the query complexity of non-adaptive testers and adaptive ones may be almost quadratic. Answering open problems raised in [9, Sec. 1.3] (see also [5, Sec. 8.5.4]), we next prove that the gap can take the form of any function that is at most (almost) quadratic. This exhibits a richer range of gaps than conjectured in [9].

Theorem 4.1 (Theorem 1.3, restated): *For every function $g : \mathbb{N} \to \mathbb{N}$ such that $g(n) \leq \sqrt{n}$, there exists a graph property Π that satisfies the following two conditions:*

1. *The general* (adaptive) *query complexity of testing Π lies between $\Omega(\sqrt{n})$ and $O(\epsilon^{-1} \cdot \sqrt{n} \cdot \log n)$.*
2. *The non-adaptive query complexity of testing Π lies between $\Omega(g(n) \cdot \sqrt{n})$ and $O(\epsilon^{-2} \cdot g(n) \cdot \sqrt{n} + \epsilon^{-1} \cdot \sqrt{n} \cdot \log n)$.*

Note that in Theorem 3.1 there was no need to state the non-adaptive upper bound and the adaptive lower bound, since they roughly follows from the fact that the non-adaptive complexity is at most quadratic in the general complexity [1, 10]; specifically, for $g(n) = \sqrt{n}$, Theorem 3.1 implies an upper bound of $O(O(\epsilon^{-1} \cdot \sqrt{n} \cdot \log n)^2) = O(\epsilon^{-2} \cdot n \cdot \log^2 n)$ on the query complexity of non-adaptive testers, and a lower bound of $\Omega(\Omega(n)^{1/2}) = \Omega(\sqrt{n})$ on the query complexity of adaptive testers.

Proof Sketch: We use the same property as in the proof of Theorem 3.1, except that we utilize matrices in which each column is repeated $\Theta(n/g(n)^2)$ times (in consecutive columns). Intuitively, this redundancy effectively shrinks the matrix size from n^2 (in Theorem 3.1) to $n \cdot g(n)^2$, thus reducing the complexity of non-adaptive tester from $O(\sqrt{n^2}) = O(n)$ to $O(\sqrt{g(n) \cdot n}) = O(g(n) \cdot \sqrt{n})$. On the other hand, a lower bound that is a square root of the effictive matrix-size still holds. Turning to the complexity of general (adaptive) testing, we observe that it remains a square root of the number of rows, which did not change (not even effectively). Details follow.

We first outline the modification to Construction 3.1.1. Setting $k = n/9$ (as before), we let $t = k/g(n)^2$ (be the number of repetitions), and modify the property Π as follows.

- For two k-by-k Boolean matrices $A = (a_{i,j})$ and $B = (b_{i,j})$, we define the graph $G_{A,B} = ([9k], E_{A,B})$ exactly as in Eq. (2).

- We say that a matrix $A = (a_{i,j})$ is t-column-redundant if $a_{i,j} = a_{i,j'}$ for every $i, j, j' \in [k]$ such that $\lceil j/t \rceil = \lceil j'/t \rceil$.

  We define the property $\Pi = \bigcup_{n \in \mathbb{N}} \Pi_n$ such that Π_n is the set of all graphs that are isomorphic to some n-vertex graph $G_{A,A}$ for some t-column-redundant matrix A. That is, the only difference between Π_n and its definition in Construction 3.1.1 is that A is t-column-redundant; each of its columns is repeated t times, in consecutive columns.

The proofs of Claims 3.1.2 and 3.1.3 remain almost intact under the relevant modifications. In particular, the *lower bound for non-adaptive testers* is still a square root of the actual information contents of the matrices, which in this case equals $\sqrt{k \cdot (k/t)}$, which in turn equals $\Omega(\sqrt{n \cdot g(n)^2}) = \Omega(g(n) \cdot \sqrt{n})$.

Specifically, we consider the problem of distinguishing the uniform distribution on Π_n and the distribution generated by picking two random and independent t-column-redundant matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ and outputting $\pi(G_{A,B})$ for a random $\pi \in \mathrm{Sym}_n$. In this case, the probability that two specific queries correspond to entries (i, j) in A and (i, j') in B such that $\lceil j/t \rceil = \lceil j'/t \rceil$ is $\frac{1}{k} \cdot \frac{t}{k}$, which means that distinguishing these distributions requires $\Omega(\sqrt{k^2/t})$ non-adaptive queries. On the other hand, with overwhelmingly high probability, two random and independent t-column-redundant matrices disagree on more than $k^2/3$ of their entries.

The *adaptive tester* is as in the proof of Claim 3.1.3, except that we slightly modify Step 6; specifically, the pair $(i, j) \in [k]^2$ is considered a matrix-collision if there exists $u_1, v_1 \in R_1$ and $u_2, v_2 \in R_2$ such that $\pi_1(u_1) = i = \pi_1(v_1) - k$ and $\lceil \pi_2(u_2)/t \rceil = \lceil j/t \rceil = \lceil (\pi_2(v_2) - 6k)/t \rceil$. (In this case, as in the proof of Claim 3.1.3, for each selected matrix-collision (i, j), we query G' on the pairs $(\pi_1^{-1}(i), \pi_2^{-1}(j))$ and $(\pi_1^{-1}(k + i), \pi_2^{-1}(6k + j))$, and reject if the answers are different.)

The query complexity of this modified algorithm is $O(\epsilon^{-1} \cdot \sqrt{k} \cdot \log k)$ as before, and the analysis of Step 6 is extended to show that if $\phi(G')$ is far from Π but close to some $G_{A,B}$, then, for any t-column-redundant matrix C it holds that either A or B is far from C. To see that, in this case, the modified Step 6 rejects (w.h.p) consider $\alpha, \beta \in \{0,1\}^t$ such that for every $\gamma \in \{0^t, 1^t\}$ either α or β disagrees with γ on at least d entries. Then, the probability that a random entry in α disagrees with a random entry in β is at least d/t.[16]

We now turn to the two remaining claimed bounds. The *upper bound on non-adaptive testers* follows by considering a non-adaptive version of the foregoing adaptive tester, with a crucial variation in Step 6. First note that Steps 1–5 use adaptivity only in their identification of the S_i's and R_i's. We can avoid this adaptivity by querying the corresponding samples; specifically, denoting the sample taken in Step 1 (resp., Step 3) by S (resp., R), we query $S \times R$ (rather than querying $\bigcup_{i \in \{1,2\}}(S_i \times R_i)$), and use the identification of the S_i's and R_i's only in the interpretation of the answers. Note that $|S| \cdot |R| = O(\ell \cdot \sqrt{k}/\epsilon) = O(\epsilon^{-1}\sqrt{n} \log n)$. (In particular, we benefit from the fact that the local self-ordering procedure, which relies on a reliable locator, is non-adaptive.)[17]

As for Step 6, the key observation is that we can select a random subset R' of $O(g(n)/\epsilon)$ vertices of R, and make non-adaptive queries to all pairs in $R \times R'$, which means making $O(g(n) \cdot \sqrt{n}/\epsilon^2)$

[16] Denoting the fraction of 1-entries in α (resp., β) by p (resp., q), observe that the probability that random entries disagree equals $p(1-q) + (1-p)q = p + q - 2pq$. Assuming, w.l.o.g., that $p + q \leq 1$, we have $p + q - 2pq \geq (p + q)/2$. On the other hand, the relative distance of α from a constant string is at least $\min(p, 1-p)$, and ditto for β. Hence, $\min(p, 1-p, q, 1-q) \geq d/t$, and $(p + q)/2 \geq d/t$ follows.

[17] This is not so crucial given that the query complexity of this procedure is polylogarithmic, and such a bound would have been preserved under the transformation from adaptive to non-adaptive algorithms. Still, this saves us a polylogarithmic factor.

queries. The point is that the number of non-repeated columns is $k/t = g(n)^2$, and R' is likely to yield $\Theta(1/\epsilon)$ column-collisions, where j is a column-collision if there exists $u_2, v_2 \in R_2$ such that $\lceil \pi_2(u_2)/t \rceil = \lceil j/t \rceil = \lceil (\pi_2(v_2) - 6k)/t \rceil$. Again, the identification of the S_i's and R_i's as well as of the matrix-collisions takes place only in the interpretation of the answers.

Lastly, we turn the *lower bound on adaptive testers*. This bound follows merely by considering collisions among samples of the k rows of the matrix. Specifically, we used the same distributions as in the proof of the lower bound for non-adaptive testers (i.e., the modified Claim 3.1.2), and observe that $\Omega(\sqrt{k})$ adaptive queries are needed in order to find vertices of G' that are mapped (by π) to location i and $i+k$, for some $i \in [k]$. ■

4.2 Lower levels of complexity

Using the idea of redundancy, we can get gaps at lower levels of complexity. Specifically, starting from Theorem 4.1, we replace $\sqrt{n}$ by $f(n) \leq \sqrt{n}$ (and replace $g(n)$ by $g(f(n))$).[18]

Theorem 4.2 (Theorem 1.4, restated): *For every functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $f(n) \leq \sqrt{n}$ and $g(m) \leq m$, there exists a graph property Π that satisfies the following two conditions:*

1. *The general* (adaptive) *query complexity of testing Π lies between $\Omega(f(n))$ and $O(\epsilon^{-1} \cdot f(n) \cdot \log n)$.*
2. *The non-adaptive query complexity of testing Π lies between $\Omega(g(f(n)) \cdot f(n))$ and $O(\epsilon^{-2} \cdot g(f(n)) \cdot f(n) + \epsilon^{-1} \cdot f(n) \cdot \log n)$.*

Proof Sketch: As in the proof of Theorem 4.1, we use the same property as in the proof of Theorem 3.1, except that we utilize matrices in which each row is repeated $\Theta(n/f(n)^2)$ times and each column is repeated $\Theta(n/(g(f(n)) \cdot f(n))^2)$ times. Specifically, we set $k = n/9$, $m = f(n)$, $t' = k/m^2$ and $t = k/g(m)^2 = t' \cdot m^2/g(m)^2 \geq t'$, and modify the definition of $\Pi = \bigcup_{n \in \mathbb{N}} \Pi_n$ such that Π_n is the set of all graphs that are isomorphic to some n-vertex graph $G_{A,A}$ such that $A = (a_{i,j})$ satisfies $a_{i,j} = a_{i,j'}$ for every $i, i', j, j' \in [k]$ such that $\lceil i/t' \rceil = \lceil i'/t' \rceil$ and $\lceil j/t \rceil = \lceil j'/t \rceil$. We call such matrices (t', t)-redundant

The proofs of all four bounds remain almost intact under the relevant modifications. Starting with the *lower bound for non-adaptive testers*, we observe that it is still a square root of the actual information contents of the matrices, which in this case equals $\sqrt{(k/t') \cdot (k/t)}$, which in turn equals $\Omega(\sqrt{f(n)^2 \cdot g(f(n))^2}) = \Omega(g(f(n)) \cdot f(n))$. Specifically, we consider the problem of distinguishing the uniform distribution on Π_n and the distribution generated by picking two random and independent (t', t)-redundant matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ and outputting $\pi(G_{A,B})$ for a random $\pi \in \mathrm{Sym}_n$. In this case the probability that two specific queries correspond to entries (i, j) in A and (i', j') in B such that $\lceil i/t' \rceil = \lceil i'/t' \rceil$ and $\lceil j/t \rceil = \lceil j'/t \rceil$ is $\frac{t'}{k} \cdot \frac{t}{k}$, which means that distinguishing these distributions requires $\Omega(\sqrt{k^2/t't})$ non-adaptive queries. On the other hand, with high probability, two random and independent (t', t)-redundant matrices disagree on more than $k^2/3$ of their entries.[19]

The *adaptive tester* is again a modification of the tester presented in the proof of Claim 3.1.3, where the modification is confined to taking a smaller sample in Step 3, and adapting Step 6

[18] Indeed, it would have been more consistent to replace $g(n)$ by $g(f(n)^2)$, but doing so would have made the current statement slightly more complex.

[19] This holds provided that $t' = o(n)$, and otherwise the claims hold vacuously.

to the current redundancy; specifically, in Step 3 we take a sample of $O(f(n)/\epsilon)$ vertices (i.e., $|R| = O(f(n)/\epsilon)$), and in Step 6 we consider the pair $(i,j) \in [k]^2$ to be a matrix-collision if there exists $u_1, v_1 \in R_1$ and $u_2, v_2 \in R_2$ such that $\lceil \pi_1(u_1)/t' \rceil = \lceil i/t' \rceil = \lceil (\pi_1(v_1) - k)/t' \rceil$ and $\lceil \pi_2(u_2)/t \rceil = \lceil j/t \rceil = \lceil (\pi_2(v_2) - 6k)/t \rceil$. The query complexity of this modified algorithm is $O(\epsilon^{-1} \cdot f(n) \cdot \log n)$, and the analysis of Step 6 is extended to show that if $\phi(G')$ is far from Π but close to some $G_{A,B}$, then, for any (t',t)-redundant matrix C it holds that either A or B is far from C.[20]

The *upper bound on non-adaptive testers* follows by considering a non-adaptive version of the foregoing adaptive tester. The observations and modifications regarding this matter that were made in the proof of Theorem 4.1 apply here (without any change). Specifically, recall that for Step 6 we can select a random subset R' of $O(g(f(n))/\epsilon)$ vertices of R, and make non-adaptive queries to all pairs in $R \times R'$, which means making $O(g(f(n)) \cdot f(n)/\epsilon^2)$ queries. The point is that the number of non-repeated rows (resp., columns) is $k/t' = f(n)^2$ (resp., $k/t = g(f(n))^2$), and R (resp., R') is likely to yield $\Theta(1/\epsilon)$ row-collisions (resp., column-collisions), where i is a row-collision if there exists $u_1, v_1 \in R_1$ such that $\lceil \pi_1(u_1)/t' \rceil = \lceil i/t' \rceil = \lceil (\pi_1(v_1) - k)/t' \rceil$ and j is a column-collision if there exists $u_2, v_2 \in R_2$ such that $\lceil \pi_2(u_2)/t \rceil = \lceil j/t \rceil = \lceil (\pi_2(v_2) - 6k)/t \rceil$.

Lastly, we turn the *lower bound on adaptive testers*. This bound follows merely by considering collisions (in the revised sense) among samples of the k rows of the matrix. Specifically, we used the same distributions as in the proof of the lower bound for non-adaptive testers, and observe that $\Omega(\sqrt{k/t'}) = \Omega(f(n))$ adaptive queries are needed in order to find vertices of G' that are mapped (by π) to location i and $i' + k$, for some $i, i' \in [k]$ such that $\lceil i/t' \rceil = \lceil i'/t' \rceil$. ■

5 Accommodating one-sided error testers

Recall that a tester for a graph property Π is said to have one-sided error probability if it accepts every graph in Π with probability 1; otherwise, we say that it has two-sided error probability.

We show that a (significant) variant of the graph property defined in the proof of Theorem 4.2 can be tested with one-sided error probability within almost the same query complexity. In particular, although we shall define some new notions and establish some new results before getting to the lower bounds, the proof of the lower bounds will use simple modifications of the prior proofs. The new notions and results will be mostly used in the description and analysis of the testers, which are quite complex. (Of course, some of these modifications will appear in the graph property itself, and the proof of the lower bound will need to deal with them, but doing so will not be difficult.) In any case, our goal is to prove the following –

Theorem 5.1 (Theorem 1.5, restated): *For every functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $f(n) \leq \sqrt{n}$ and $g(m) \leq m$, there exists a graph property Π that satisfies the following two conditions:*

1. *The general* (adaptive) *query complexity of testing Π lies between $\Omega(f(n))$ and $O(\epsilon^{-1} \cdot f(n) \cdot \log^3 n)$, where the upper bound holds for a one-sided error tester and the lower bound holds for two-sided error testers.*

2. *The non-adaptive query complexity of testing Π lies between $\Omega(g(f(n)) \cdot f(n))$ and $O(\epsilon^{-2} \cdot g(f(n)) \cdot f(n) \cdot \log^2 n + \epsilon^{-1} \cdot f(n) \cdot \log^3 n)$, where the lower bound holds for two-sided error*

[20] To see that, in this case, the modified Step 6 rejects (w.h.p) consider $\alpha, \beta \in \{0,1\}^{t' \times t}$ such that for every $\gamma \in \{0^{t' \times t}, 1^{t' \times t}\}$ either α or β disagrees with γ on at least d entries. Then, as detailed in Footnote 16, the probability that a random entry in α equals a random entry in β is at least $d/t't$.

testers and the upper bound holds for a one-sided error tester.

Actually, the upper bounds hold provided that $\epsilon > n^{-0.49}$.

Otherwise (i.e., if $\epsilon \leq n^{-0.49}$), the trivial tester that checks all entries works using $O(1/\epsilon^{4.1})$ queries.

Overview of the proof of Theorem 5.1. The issues that we shall address are rooted in the proof of Theorem 3.1, but we deal with them in a way that also accommodates the modification made to derive Theorem 4.2. Hence, we refer to the proof of Theorem 4.2 rather than to that of Theorem 3.1.

As stated above, we shall use a variant of graph property Π that was used in the proof of Theorem 4.2, and so the lower bounds remain valid (although we shall have to show this too). Furthermore, we will need to modify the testers (which are based on the tester presented in the proof of Claim 3.1.3), which evidently have two-sided error probability. This error probability (on graphs in Π) has three sources:

1. A wrong categorization of vertices (w.r.t the S_i's and R_i's) caused by a wrong approximation of their degrees (which takes place in Steps 1 and 3, respectively).

2. Failure to sample (in Step 1) a proportional number of vertices on each side of the graph, where we refer to the copies of G_{2k} and G_{7k} used in the construction of graphs in Π_{9k}.

3. Failure to find a reliable locator (for each side) in Step 2.

These errors seem unavoidable. For starters, when taking a small sample of the graph vertices, it may happen with (small) positive probability that the sample does not provide a correct estimate for the degrees of some vertices in the graph (e.g., all sampled vertices may happen to be neighbors of some vertex). Likewise, it may happen (although rarely) that all vertices sampled in Step 1 hit one side of the graph. And, similarly, it may happen that a small sample (as taken in Step 2) does not contain a reliable locator, because not all polylogarithmic-sized subsets of the vertex-set are reliable locators.

Faced with these problems, we take two steps. First, we move away from notions that require correct performance with respect to all the graph's vertices, but only guarantee it with high probability. Instead, we devise notions that allow for few exceptional vertices, but guarantee such performance with probability 1 (when the graph is in the property). For example, rather than seeking a sample that estimates correctly the degrees of all vertices in a graph, we seek samples that (for graphs in the property) always estimate correctly the degrees of all but logarithmically many vertices. (Needless to say, these samples will provide good estimations, with high probability, for any vertex in any graph.) An analogue relaxation is employed to the notion of a reliable locator. Furthermore, we show that both these relaxations can be achieved.

Second, given that we cannot rely on Step 1 to hit the two sides of the graph in proportion to their size, we use a different way of obtaining two robustly self-ordered graphs with significantly different vertex degrees. Specifically, we use random graphs with different edge density. Note that we can detect the case that a sample is extremely biased towards vertices of one of the two sides (or graphs), and in this case we proceed without rejecting (rather than reject based on statistic evidence). In this case, graphs that are far from the property will be rejected by the subsequent steps, which will produce (with high probability) an absolute witness for violation of the property.

Organization of the proof. In Section 5.1, we develop the tools that will be used in the actual proof. These include a relaxed notion of reliable locators, called *pseudo-locators*, that allows for location failures on very few vertices of the graph (Definition 5.4), and a proof that in a random graph all logarithmically-sized subsets are pseudo-locators (Claim 5.5). The actual proof is presented in Section 5.2. Its core is the descriptions of the modified graph property (Construction 5.7) and of the modified adaptive tester (presented in the proof of Proposition 5.9).

5.1 Developing tools for the proof

As noted above, in previous sections, distinguishing the vertices in the two sides of the graph (i.e., in copies of G_{2k} and G_{7k}) was based on the fact that these vertices have different degrees, but this difference is detected based on the assumption that the sample taken in Step 1 hits each side in proportion to its size. Although this happens with high probability, it does not happen with probability 1, and so we cannot rely on this reasoning for one-sided error testers. The solution is to use different edge densities in the two sides, which means that we need an alternative to Construction 3.1.1. Specifically, for a small constant $p > 0$ (e.g., $p = 0.1$), we replace the graphs G_{2k} and G_{7k} used there, with two $2k$-vertex graphs, one of edge density p, and the other of edge density $1-p$. Both graphs will be robustly self-ordered and their edge density will be preserved by any induced subgraph of size $\Omega(\log k)$. Hence, we first prove the following –

Claim 5.2 (robust self-ordering of $\mathcal{G}(n,p)$ and vertex degrees in it): *For any constant $p \in (0,1)$, let $\mathcal{G}(n,p)$ be the Erdos–Renyi random n-vertex graph in which each pair of vertices is connected with probability p independently of all other vertex-pairs. Then:*

1. *With probability $1 - \exp(-\Omega(n))$, the graph is robustly self-ordered.*
2. *For sufficiently large $\ell = O(\log n)$, with probability $1 - \exp(-\Omega(\ell^2))$, the following two conditions hold:*
   (a) *Every set of ℓ vertices induces a subgraph of edge density $(1 \pm 0.01) \cdot p$.*
   (b) *For each set of ℓ vertices S there are at least $n - 2\ell$ vertices $v \in [n] \setminus S$ such that the number of neighbors that v has in S is $(1 \pm 0.01) \cdot p \cdot \ell$.*

We mention that Part 1 is implicit in the proof of [13, Thm. 3.1]. We highlight the fact that *all* ℓ-subsets provide a good approximation of the degrees of *almost all* vertices (Part 2b). Indeed, for every vertex there exists ℓ-subsets that fail to provide a good approximation to its degree, but no ℓ-subset fails for more than ℓ vertices (outside it).[21]

Proof Sketch: Part 2a follows by the fact that the probability that any fixed ℓ-subset induces a subgraph with deviating edge density is $\exp(-\Omega(\min(p, 1-p) \cdot \ell^2))$, which is $\exp(-\Omega(\ell^2))$ since p is constant.[22] Hence, the effect of p is hidden by the Ω-notation, and the same phenomenon will happen in the later parts.

[21] The choice of ℓ as the number of exceptional vertices is immaterial; we could have chosen $c\ell$ for any small or larger positive constant c. (Making c smaller can be supported by making $\ell = O(\log n)$ larger, whereas our applications can tolerate $O(\ell)$ exceptional vertices with no real effect.) Analogous comments apply to our relaxation of the notion of a reliable locator (see Definition 5.4 and Claim 5.5).

[22] Using a union bound over all $\binom{n}{\ell}$ subsets, the claim follows.

Part 2b follows by the fact that, for each ℓ-set S and each $v \in [n] \setminus S$, the probability that the number of neighbors that v has in S is not $(1 \pm 0.01) \cdot p \cdot \ell$ is $\exp(-\Omega(\ell))$. Applying a union bound, it follows that the probability that there exists an ℓ-set for which there are $\ell + 1$ deviating vertices is at most

$$\binom{n}{\ell} \cdot \binom{n}{\ell+1} \cdot \exp(-\Omega(\ell))^{\ell+1},$$

which is $\exp(-\Omega(\ell^2))$. (We highlight the fact that allowing ℓ exceptional vertices, which means that violation requires $\ell + 1$ vertices rather than 1, raises the probability to the power of $\ell + 1$, which in turn enables the application of a union bound over all ℓ-sets of $[n]$.)

Part 1 follows by a simple adaptation of the proof of Theorem 2.3, which is presented in Appendix A. Specifically, the probability that two different vertex-pairs disagree on whether or not they form an edge is $q = 2p(1-p)$ rather than $1/2$, and Eq. (8) is modified accordingly (i.e., $n \cdot |T|/20$ is replaced by $n \cdot |T| \cdot q/10$). ■

Connecting graphs sampled from $\mathcal{G}(n, 0.1)$ and $\mathcal{G}(n, 0.9)$. For any constant $p < 1/2$, part 2b of Claim 5.2 offers a way of distinguishing vertices in (a graph sampled from) $\mathcal{G}(n, p)$ from vertices in (a graph sampled from) $\mathcal{G}(n, 1-p)$, but this way may be frustrated if the two graphs are connected arbitrarily (or almost so) as in Construction 3.1.1. We solve this problem by connecting these two graphs by a bipartite graph in which the edge density is $0.5 \pm p$, for say $p = 0.1$, where the slackness offers enough room to encode information (of the embedded matrices). Furthermore, we need every ℓ-by-ℓ bipartite subgraph to have the same density up to a factor of 1 ± 0.01. Actually, we use the following claim, which is formulated in terms of matrices rather than in terms of bipartite graphs, since this allows to capture the slackness (reflected by $*$-entries) in a more elegant manner.

Claim 5.3 (all logarithmically sized submatrices are balanced in a random p-slacked matrix): *For any constant $p \in [0, 0.5)$, let $\mathcal{M}(n, p)$ be the distribution on n-by-n matrices over $\{0, 1, *\}$ such that each entry is 0* (resp., 1) *with probability $q = (1-p)/2$, independently of all other entries, and equals $*$ otherwise. Then, for sufficiently large $\ell = O(\log n)$, with probability $1 - \exp(-\Omega(\ell^2))$, every ℓ-by-ℓ submatrix has $(1 \pm 0.01) \cdot q \cdot \ell^2$ entries of value 0, and ditto for value 1. Furthermore, each row* (resp., column) *in the matrix has $(1 \pm 0.01) \cdot q \cdot n$ entries of value 0, and ditto for value 1.*

The proof of Claim 5.3 is by a straightforward counting argument (as used in the proof of Part 2a of Claim 5.2).[23] We shall use an arbitrary matrix M that satisfies the condition of Claim 5.3.

Intuitively, for some small constant $p > 0$ (e.g., $p = 0.1$), the graphs in our revised property Π will consist of a pair of graphs, denoted $G_{2k}^{(p)}$ and $G_{2k}^{(1-p)}$, drawn from $\mathcal{G}(2k, p)$ and $\mathcal{G}(2k, 1-p)$ respectively, that are connected by biparite graphs that are determined by two matrices in which the $*$-entries of M are replaced by Boolean values. But before presenting this construction, we introduce another feature that we wish the graphs $G_{2k}^{(\cdot)}$'s to possess.

The foregoing modifications to Construction 3.1.1 are aimed at removing the sources of errors of Types 1 & 2 discussed in the overview of the proof. We are now going to deal with errors of Type 3 (i.e., failure to sample a reliable locator).

[23] Indeed, for $p = 0$, Claim 5.3 yields a non-explicit two-source extractor (cf. [15]) with almost optimal parameters.

Relaxing the notion of a locator. Recall that the remaining source of error (on graphs in Π) in our original tester is the failure to sample a reliable locator (for each side) in Step 2. The problem is that not all ℓ-subsets are reliable locators (although almost all of them are). We resolve this problem by defining a relaxed notion of a locator, and showing that, with high probability, in both $\mathcal{G}(2k,p)$ and $\mathcal{G}(2k,1-p)$, all ℓ-subset satisfy this relaxed definition. The nature of the relaxation is allowing the locator to correctly identify *all but very few* of the vertices (as opposed to *all* in the original definition). When using the following definition, the threshold parameter τ will be set to $p \cdot (1-p)$.

Definition 5.4 (pseudo-locators): *A set of vertices $S \subset [n]$ is a* pseudo-locator (for threshold τ) *of a graph $G = ([n], E)$ if the following two conditions hold*

1. *The subgraph of G induced by S is not isomorphic to any subgraph of G that is induced by any $|S|$-set that misses more than $\tau \cdot |S|$ vertices of S* (i.e., any $|S|$-set S' such that $|S' \cap S| < (1-\tau) \cdot |S|$).

2. *There exists a set $U \subseteq [n] \setminus S$ of $n - 2 \cdot |S|$ vertices such that, for every $v \in U$, the adjacencies of v with any subset that induces the same unlabeled subgraph as S uniquely determine v; that is, for every $u \neq v$ in U and every S' such that the subgraph of G_n induced by S' is isomorphic via ϕ to the subgraph of G_n induced by S, there exists $s \in S$ such that $\{u,s\} \in E$ if and only if $\{v, \phi(s)\} \notin E$.*

   In other words: For every $u \neq v$ in U and every S', let H (resp., H') *denote the subgraph of G_n induced by S* (resp., S')*, and suppose that $\phi(H) = H'$. Then, there exists $s \in S$ such that $\{u,s\} \in E$ if and only if $\{v, \phi(s)\} \notin E$.* (U stands for *unexceptional*.)

(In light of Condition 1, it suffices to consider in Condition 2 sets S' such that $|S' \cap S| \geq (1-\tau) \cdot |S|$).

Definition 5.4 relaxes both conditions of Definition 2.5: In Condition 1 of Definition 5.4 we requires that only $|S|$-subsets that intersect S in less than $(1-\tau) \cdot |S|$ vertices induce a subgraph that is not isomorphic to the one induced by S; that is, we discard $|S|$-subsets that have a larger intersection with S and do not require that the subgraph induced by S is asymmetric. In light of the fact that Condition 1 was relaxed, the revised Condition 2 refers not only to the adjacencies with S but also to the adjacencies with any subset that induces the same unlabeled subgraph as S. On the other hand, in the revised Condition 2 we allow for $|S|$ exceptional vertices (even in case $S' = S$, which is the only case considered in Definition 2.5). Using both relaxations, we shall prove that, in a random graph, all logarithmically-sized subsets are pseudor-locators (Claim 5.5). But before proving this claim, let us spell out how a pseudo-locator will be used.

Suppose that $G' = (V', E')$ is isomorphic to $G = ([n], E)$; that is, $G = \phi(G')$ for some unknown to us bijection $\phi : V' \to [n]$. Now, if S' is a pseudo-locator of G' (equiv., $\phi(S')$ is a pseudo-locator of G), then, given $v \in V' \setminus S'$ we can identify $\phi(v)$ by inspecting the subgraph of G' induced by $S' \cup \{v\}$. Specifically, we look in G for an $|S'|$-subset $S \subset [n]$ and a vertex $i \in [n] \setminus S$ such that the subgraph of G induced by $S \cup \{i\}$ is isomorphic to the subgraph of G' induced by $S' \cup \{v\}$ and this isomorphism maps i to v. (In other words, we seek a bijection $\pi : S \cup \{i\} \to S' \cup \{v\}$ such that $\pi(i) = v$ and for every $j, j' \in S \cup \{i\}$ it holds that $\{j,j'\} \in E$ if and only if $\{\pi(j), \pi(j')\} \in E'$.) If G' is indeed isomorphic to G and v is not one of the ℓ exceptional vertices, then there exists a single i that satisfies this condition (although S need not be unique). If G' is not isomorphic to G, then anything may happen, but if the number of i's satisfying the foregoing condition is not one, then

we shall announce failure. We stress that, while we actually query G' for the subgraph induced by $S' \cup \{v\}$, looking for suitable S and i in G is a thought experiment (since G is fixed and known to us).

Claim 5.5 (pseudo-locators in a random graph): *For any constant $p \in (0,1)$ and sufficiently large $\ell = O(\log n)$, with probability $1 - \exp(-\Omega(\ell^2))$ over the choice of $\mathcal{G}(n,p)$, all ℓ-subsets of the vertex-set are pseudo-locators for threshold $p \cdot (1-p)$.*

Proof: Fixing an arbitrary ℓ-subset, S, we shall upper-bound by $\exp(-\Omega(\ell^2))$ the probability that S is not a pseudo-locator of $\mathcal{G}(n,p)$ (for threshold $p \cdot (1-p)$). This probability is low enough so to support a union bound over all ℓ-subsets; specifically, $\binom{n}{\ell} \cdot \exp(-\Omega(\ell^2)) = \exp(-\Omega(\ell^2))$, for sufficiently large $\ell = O(\log n)$.

For $p = 1/2$, Condition 1 of being a pseudo-locator (i.e., sufficiently different ℓ-subsets induce non-isomorphic subgraphs) is implicit in the proof of Claim 2.6.1, which is presented in Appendix B. Specifically, the probability that the subgraphs of a random graph induced by S and S' are isomorphic was upper-bounded in Eq. (9)&(10) by

$$\frac{\ell!}{|S \cap S'|!} \cdot 2^{-\Omega((\ell - |S \cap S'|) \cdot \ell)}. \tag{4}$$

Using $|S \cap S'| < 3\ell/4$, the foregoing is upper-bounded by $\exp(-\Omega(\ell^2))$, which allows us to apply a union bound (over all relevant ℓ-subsets S''s) and get an upper bound of $\binom{n}{\ell} \cdot \exp(-\Omega(\ell^2)) = \exp(-\Omega(\ell^2))$. As for a general $p \in (0,1)$, it is handled by observing that all that changes is the probability of disagreement, which is $q = 2p(1-p)$ rather than $1/2$, and the threshold (which is $p(1-p)$ rather than $1/4$). Specifically, as in the proof of Part 2 of Claim 5.2, the probability that two different vertex-pairs disagree is $q = 2p(1-p)$ rather than $1/2$, and Eq. (4) is modified accordingly (cf. Eq. (5) below), while noting that the derived bound is

$$\frac{\ell!}{|S \cap S'|!} \cdot q^{\Omega((\ell - |S \cap S'|) \cdot \ell)} \;<\; 2^{\ell} \ell^{0.5q\ell} \cdot q^{\Omega(q \cdot \ell^2)} \;=\; \exp(-\Omega(\ell^2)),$$

where we use $|S \cap S'| < (1 - 0.5q) \cdot \ell$ and $q = \Omega(1)$.

We seize the opportunity to state a related fact that will be used for establishing Condition 2. We claim that, for every ℓ-subset S' and a bijection $\pi : S \to S'$, the probability that the subgraph of $\mathcal{G}(n,p)$ induced by S is isomorphic via π to the subgraph induced by $\pi(S)$ (i.e., $\{w, w'\}$ is an edge in the first subgraph iff $\{\pi(w), \pi(w')\}$ is an edge in the second subgraph) is upper-bounded by

$$\min\left(q^{|\mathtt{FP}(\pi)| \cdot (\ell - |\mathtt{FP}(\pi)|)/3}, q^{\binom{(\ell - |\mathtt{FP}(\pi)|)/3}{2}}\right) \tag{5}$$

where $\mathtt{FP}(\pi) \stackrel{\text{def}}{=} \{v \in S : \pi(v) = v\}$. This fact is actually established *en route* to establishing Condition 1 as outlined above. Note that if $|\mathtt{FP}(\pi)| > 0.1q \cdot \ell$, then Eq. (5) is upper-bounded by $q^{\Omega(q \cdot \ell^2)} = \exp(-\Omega(\ell^2))$.

Before proceeding to Condition 2 (of being pseudo-locator), we highlight the fact that the $\log_2 n$ unit lower bound on the symmetric difference between the subsets S and S', which yields a probability upper bound of $\exp(-\Omega(\ell^2))$, enables the application of a union bound on all ℓ-subsets. Analogously, as we shall see, allowing for ℓ exceptions in localization, which means that failure

requires more that ℓ mistakes (rather than one), implies that the probability that an ℓ-subset is not a pseudo-locator vanishes exponentially with ℓ^2 (i.e., another ℓ factor on top of what we get for one failure). Again, such an upper bound enables the application of a union bound over all ℓ-subsets.

Turning to Condition 2 of Definition 5.4, we stress that we shall show that S can be used to locate almost all vertices in $[n] \setminus S$, although the vertices of S itself may not be all uniquely located. Still, using Eq. (5) it follows that most vertices in S are uniquely located, and we shall show that almost all vertices in $[n] \setminus S$ have significantly different adjacencies in S. Hence, the uncertainty regarding the location of few vertices in S is compensated by the large difference between the adjacencies in S of almost all vertices in $[n] \setminus S$. Details follow, where we start with a description of a process that identifies the exceptional vertices in $[n] \setminus S$.

We consider an iterative process that starts with $U = [n] \setminus S$ and searches for a pair of vertices in U such that their adjacencies to the set S disagree on at most $0.9 \cdot q \cdot \ell$ entries; that is, we see $w \neq w'$ in U such that $|\{s \in S : \{w, s\} \in E \Leftrightarrow \{w', s\} \notin E\}| \leq 0.9 \cdot q \cdot \ell$. If such a pair does not exist, then we halt and output U, which is considered a success. Otherwise, we omit both vertices in the pair from U, and proceed to the next iteration. (We may halt without output after $\ell/2$ unsuccessful iterations).

Vertices in U are correctly located by S. We show that if the process outputs U (after at most $\ell/2$ iterations), then U constitutes a set as required in Condition 2. Recall that here we consider arbitrary $u \neq v$ in U and any bijection $\phi : S \cup \{u\} \to S' \cup \{v\}$ (for any S') such that $\phi(u) = v$ and the subgraphs induced by S and by $S' = \phi(S)$ are isomorphic via ϕ. Using Condition 1, it suffices to consider sets S' such that $|S' \cap S| \geq (1 - 0.5q) \cdot \ell$, since the other sets induce subgraphs that are not isomorphic to the one induced by S. Furthermore, using Eq. (5) and a union bound, we may assume that $\phi(s) = s$ for at least $(1 - 0.6q) \cdot \ell$ of the vertices $s \in S \cap S'$, because the probability that this does not hold for some S and ϕ is $\exp(-\Omega(\ell^2))$ (over the choice of $\mathcal{G}(n,p)$).[24] The punchline is that, by construction of U, *the neighborhoods of different vertices u and v in U disagree on more than $0.9 \cdot q \cdot \ell$ entries* (i.e., $|\{s \in S : \{u, s\} \in E \Leftrightarrow \{v, s\} \notin E\}| > 0.9q\ell$), and (using $|S' \cap S| \geq (1 - 0.5q) \cdot \ell$ and the lower bound on the number of fixed-points of ϕ) this implies that *there exists $s \in S \cap S'$ such that $\{u, s\} \in E$ if and only if $\{v, \phi(s)\} \notin E$*, where we also used $\phi(s) = s$.

Hence, we have shown that, with probability at least $1 - \exp(-\Omega(\ell^2))$ over the choice of $\mathcal{G}(n,p)$, if the iterative process outputs U, then, for every $u \neq v \in U$ and any $\phi : S \cup \{u\} \to S' \cup \{v\}$, the hypothesis that $\phi(u) = v$ and the subgraphs induced by S and by $S' = \phi(S)$ are isomorphic via ϕ (i.e., $\{w, w'\}$ is an edge in the first subgraph iff $\{\phi(w), \phi(w')\}$ is an edge in the second subgraph) implies that there exists $s \in S$ such that $\{u, s\} \in E$ if and only if $\{v, \phi(s)\} \notin E$.

The process is successful with overwhelmingly high probability. It remains to prove that, with probability at least $1 - \exp(-\Omega(\ell^2))$ over $\mathcal{G}(n,p)$, the foregoing process is successful (i.e., does halt with output in at most $\ell/2$ iterations). This is the case because the probability that the first $\ell/2$ iterations find pairs of the type sought (equiv., that $\ell/2$ such disjoint pairs exist), in $\mathcal{G}(n,p)$, is upper-bounded by

$$\binom{n}{2}^{\ell/2} \cdot \exp(-\Omega(q \cdot \ell))^{\ell/2} = \exp(-\Omega(\ell^2)), \tag{6}$$

[24] Specifically, if at least $0.1 \cdot q \cdot \ell$ of the vertices in $S \cap S'$ are not fixed-points of ϕ, then the subgraphs of $\mathcal{G}(n,p)$ induced by S and $\phi(S)$ are equal (i.e., for every $w, w' \in S$ it holds that $\{w, w'\}$ is an edge iff $\{\phi(w), \phi(w')\}$ is an edge) with probability at most $q^{\binom{0.1q\ell/3}{2}}$. Using a union-bound over all ℓ-sets S' and $\phi : S \to S'$, we get a probability bound of $n^\ell \cdot \exp(-\Omega(\ell^2)) = \exp(-\Omega(\ell^2))$.

where each fixed pair of vertices disagrees on at most $0.9 \cdot q\ell$ of their neighbors with probability at most $\exp(-\Omega(q \cdot \ell))$, since they differs on neighboring any other vertex with probability q. Again, the bound in Eq. (6) is small enough for allowing the application of a union bound over all $\binom{n}{\ell}$ sets (i.e., the S's). ■

Corollary 5.6 (by Claims 5.2 and 5.5): *For every $p \in (0,1)$ and sufficiently large $\ell = O(\log n)$, there exists a family of robustly self-ordered graphs $\{G_n^{(p)} = ([n], E_n^{(p)})\}_{n \in \mathbb{N}}$ such that any ℓ-subset $S \subset [n]$ of its vertices satisfies the following four conditions.*

1. *The subgraph of $G_n^{(p)}$ induced by S has $(1 \pm 0.01) \cdot p \cdot \binom{\ell}{2}$ edges.*
2. *For at least $n - 2\ell$ vertices $v \in [n] \setminus S$, the number of neighbors that v has in S is $(1 \pm 0.01) \cdot p\ell$.*
3. *The set S is a pseudo-locator for threshold $p \cdot (1-p)$ of $G_n^{(p)}$.*

Furthermore, each vertex in $G_n^{(p)}$ has degree $(1 \pm 0.01) \cdot p \cdot n$.

Combining Corollary 5.6 with Claim 5.3, we are finally ready to present the (revised) graph property Π, and proceed to the actual proof.

5.2 The actual proof

We shall use a graph property that is a variant of the property used in Construction 3.1.1, as adapted in Section 4.2. Here we start with the graphs $G_{2k}^{(\delta)}$ and $G_{2k}^{(1-\delta)}$ provided by Corollary 5.6, and connect them using four different bipartite graphs that are derived from a fixed k-by-k matrix M (provided by Claim 5.3) and two arbitrary k-by-k matrices A and B. The latter matrices will be identical in the case of graphs in the property, but will be random and independent of one another when proving the lower bounds. Unlike in Construction 3.1.1, only some of the entries of A and B (i.e., those that correspond to $*$-entries of M) will be encoded in the construction, but these entries constitute a constant fraction of all entries and so the proofs of the lower bounds can handle this change. Recall that this complication is introduced so to maintain a gap between the degrees of the vertices that belong to the different sides of the constructed graph.

In accordance with the foregoing, we present the following (revised) graph property Π, which is a variant of the property used in Construction 3.1.1, as adapted in Section 4.2 for any functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $f(n) \leq \sqrt{n}$ and $g(m) \leq m$. Indeed, the case of $f(n) = \sqrt{n}$ and $g(m) = m$ corresponds to Construction 3.1.1 itself, where no redundancy is used, and it suffices for establishing a quadratic gap (between adaptive one-sided error testers and non-adaptive two-sided error testers).[25]

Construction 5.7 (the revised graph property Π): *For any $\delta \in (0, 0.5)$ and sufficiently large $\ell = O(\log k)$, let $G_{2k}^{(\delta)} = ([2k], E_{2k}^{(\delta)})$ and $G_{2k}^{(1-\delta)} = ([2k], E_{2k}^{(1-\delta)})$ be as postulated in Corollary 5.6, and let M be a k-by-k matrix that satisfies the conditions of Claim 5.3.*

[25] In such a case, the second item may be skipped, and the third item may be simplified by allowing any k-by-k matrix A.

- *For two k-by-k Boolean matrices $A = (a_{i,j})$ and $B = (b_{i,j})$, we define the graph $G_{A,B} = ([4k], E_{A,B})$ such that*

$$\begin{aligned} E_{A,B} \;=\; & E^{(\delta)}_{2k} \cup \{\{2k+i, 2k+j\} : \{i,j\} \in E^{(1-\delta)}_{2k}\} \\ & \cup \{\{i, 2k+j\} : i,j \in [k] \;\wedge\; m_{i,j} \odot a_{i,j} = 1\} \\ & \cup \{\{k+i, 3k+j\} : i,j \in [k] \;\wedge\; m_{i,j} \odot b_{i,j} = 1\} \\ & \cup \{\{i, 3k+j\}, \{k+i, 2k+j\} : i,j \in [k] \;\wedge\; m_{i,j} = 1\} \end{aligned} \tag{7}$$

  where $m_{i,j} \odot \sigma = m_{i,j}$ if $m_{i,j} \in \{0,1\}$ and $m_{i,j} \odot \sigma = \sigma$ otherwise (i.e., if $m_{i,j} = *$).

  That is, $G_{A,B}$ consists of a copy of $G^{(\delta)}_{2k}$ and a copy of $G^{(1-\delta)}_{2k}$ that are connected by four bipartite graphs that is determined by the matrices $M \odot A$, $M \odot B$ and M, respectively, where $\odot$ is defined such that Boolean entries of M dominate and entries of the other matrix are used only for entries in which M is undetermined (as indicated by $*$). *In other words, only entries of A and B that correspond to $*$-entries of M are actually encoded in this construction.*[26]

  The first bipartite graph is determined by $M \odot A$ and connects $[k]$ to $\{2k+1, ..., 3k\}$, the second bipartite graph is determined by $M \odot B$ and connects $\{k+1, ..., 2k\}$ to $\{3k+1, ..., 4k\}$, whereas the other two bipartite graphs are determined by M and connect $[k]$ to $\{3k+1, ..., 4k\}$ and $\{k+1, ..., 2k\}$ to $\{2k+1, ..., 3k\}$.[27] *The first two bipartite graphs, which will be called* informative, *encode entries of A and B that correspond to $*$-entries of M.*

- *For $t', t \in [k]$, we say that a k-by-k Boolean matrix $A = (a_{i,j})$ is (t',t)-*redundant *if $a_{i,j} = a_{i,j'}$ for every $i, i', j, j' \in [k]$ such that $\lceil i/t' \rceil = \lceil i'/t' \rceil$ and $\lceil j/t \rceil = \lceil j'/t \rceil$.*

- *For $f, g : \mathbb{N} \to \mathbb{N}$ such that $f(n) \leq \sqrt{n}$ and $g(m) \leq m$, we let $k = n/4$, $m = f(n)$, $t' = k/m^2$ and $t = k/g(m)^2$, and define the property $\Pi^{(f,g)} = \Pi = \bigcup_{n \in \mathbb{N}} \Pi_n$ such that Π_n is the set of all graphs that are isomorphic to some n-vertex graph $G_{A,A}$ such that A is (t',t)-redundant.*

Note that, given a graph of the form $\pi(G_{A,A})$, the vertices in the copy of $G^{(\delta)}_{2k}$ are easily identifiable as having degree (in $\pi(G_{A,A})$) at most $1.01 \cdot \delta \cdot 2k + (1.01 \cdot (1-\delta)/2 + 1.01 \cdot \delta) \cdot 2k = 1.01 \cdot (1+3\delta) \cdot k < 1.4k$, where we use $\delta \leq 0.1$. (In contrast, the vertices in the copy of $G^{(1-\delta)}_{2k}$ have degree (in the $4n$-vertex graph $\pi(G_{A,A})$) at least $0.99 \cdot (1-\delta) \cdot 2k + (0.99 \cdot (1-\delta)/2) \cdot 2k = 0.99 \cdot 3 \cdot (1-\delta) \cdot k > 2.6k$.)

Claim 5.8 (lower bound for non-adaptively testing Π): *Any non-adaptive tester for Π_n has query complexity $\Omega(g(f(n)) \cdot f(n))$.*

Proof Sketch: Following the proof of Claim 3.1.2 (modulo the adaptations presented in Sections 4.1 and 4.2), we observe that the lower bound is still a square root of the actual information contents of the matrices, which yields a bound of $\Omega(\sqrt{(k/t') \cdot (k/t)}) = \Omega(f(n) \cdot g(f(n)))$.

[26] As noted above, these entries constitute a constant fraction of all entries, and this will suffice for our purpose: Specifically, for uniformly and independently distributed A and B, with very high probability, the pair $(M \odot A, M \odot B)$ is far from any pair of the form $(M \odot C, M \odot C)$.

[27] The latter two bipartite graphs, which are determined solely by M, are used in order to simplify the identification of vertices based on their adjacencies in a small set T. Specifically, in this case, $|T \cap [2k]|/|T|$ determines the expected number of neighbours that $v \in [2k]$ (resp., $v \in \{2k+1, ..., 4k\}$) has in T upto $\pm 2\delta \cdot |T|$. Not using these two bipartite graphs would have required either knowledge of $|T \cap \{(i-1)k+1, ..., ik\}|/|T|$ for $i = 0, 1, 2$ or using far more crude approximations.

Specifically, we consider the problem of distinguishing the uniform distribution on Π_n and the distribution generated by picking two random and independent (t',t)-redundant matrices $A=(a_{i,j})$ and $B=(b_{i,j})$ and outputting $\pi(G_{A,B})$ for a random $\pi\in\mathrm{Sym}_n$. In this case, the probability that two specific queries correspond to entries (i,j) in $M\odot A$ and (i',j') in $M\odot B$ such that $\lceil i/t'\rceil=\lceil i'/t'\rceil$ and $\lceil j/t\rceil=\lceil j'/t\rceil$ is $\frac{t'}{k}\cdot\frac{t}{k}$, which means that distinguishing these distributions requires $\Omega(\sqrt{k^2/t})$ non-adaptive queries. (This ignore the fact that distinguishing requires that $m_{i,j}=*$, which only acts in our favor.) On the other hand, with high probability, for two random and independent (t',t)-redundant matrices A and B, the matrices $M\odot A$ and $M\odot B$ disagree on more than $\delta\cdot k^2/3$ of their entries. This is because the entries in A and B that correspond to the $*$-entries of M are random and sufficiently independent (i.e., entries in different t'-by-t submatrices are independent)[28] and their number is at least $0.99\cdot\delta\cdot k^2$. Note that a small δ merely means that the lower bound holds for a smaller value of the proximity parameter. ■

Proposition 5.9 (upper bound for adaptively testing Π): *There exists an adaptive tester for Π_n of query complexity $O(\epsilon^{-1}\cdot f(n)\cdot\log^3 n)$ if $\epsilon>n^{-0.49}$ and $O(1/\epsilon^{4.1})$ otherwise.*

The alternative tester (for $\epsilon\leq n^{-0.49}$) is trivial: In this case we can afford exploring the entire graph, since $n^2\leq\epsilon^{-4.1}$.[29]

Proof: Our starting point is the tester used in the proof of Claim 3.1.2 (modulo the adaptations presented in Sections 4.1 and 4.2), but we need to significantly modify it so to avoid the (small) error probability on graphs in Π. As outlined above, this is done by avoiding any rejection decision that is based on statistical evidence. The main modifications are the following (whereas the actual description of the tester will be provided later).

- *A modification of Step 1*: We select uniformly at random a sample of size $\ell'=O(\ell)$, denoted T, to be used in all degree estimations (but not as a pool that contains the two locators, S_1 and S_2).[30] In particular, we shall use the edge density in the subgraph induced by T in order to estimate how many vertices of T reside in the copy of $G^{(\delta)}_{2k}$, and determine the thresholds that distinguishes low and high degree vertices accordingly.

  We stress that, in contrast to the original Step 1, we perform no checks in this modified step (and, needless to say, never reject in it).

  Actually, we may (and do) pick T arbitrarily (rather than at random).

- *Branching and modification of Step 2*: Since T may provide a wrong degree estimation for few vertices, the identification of locators in the original Step 2 may be wrong, and lead to subsequent errors (in locating vertices), which in turn may lead to rejecting graphs in Π. We overcome the problem by branching to many executions, which use different samples for the locators, such that in almost all branches the locators are correctly identified (based on correct estimation of the vertices' degrees). We shall reject only if most branches reject (or rather if most sub-branches (introduced later) reject).

[28] In the case of $f(n)=\sqrt{n}$ and $g(m)=m$ (equiv., $t'=t=1$), the entries are totally independent.

[29] Indeed, 0.49 stands for any constant smaller than 0.5, and $4.1>2/0.49$.

[30] Recall that in the original tester, the random sample S was used both for degree estimation and as a pool for the selection of reliable locators (in Step 2).

Specifically, we branch to ℓ' executions, using the same degree approximation sample, T, in all of them. In the p^{th} branch we select, almost independently of other branches, a sample $S^{(p)}$, identify its low and high degree vertices (according to T), and select random ℓ-sets, $S_1^{(p)}$ and $S_2^{(p)}$, as in Step 2. These sets will be used as pseudo-locators (for threshold $\delta \cdot (1-\delta)$). If $S^{(p)}$ contains less that ℓ vertices of one type, we reset $S_i^{(p)}$ to be empty, but do not reject. We do reject, *in this branch*, if $S^{(p)}$ contains at least ℓ vertices of type i, but the selected $S_i^{(p)}$ is not a pseudo-locator, where from this point onwards whenever we mention a pseudo-locator we means a pseudo-locator for threshold $\delta \cdot (1-\delta)$.

We make sure that the samples $S^{(p)}$ taken in the different branches are dispersed in the sense that any set of ℓ vertices is intersected by $O(\ell)$ samples. This guarantees that the (at most) ℓ vertices that were wrongly categorised based on their degree (as approximated by T) only affect the execution of $O(\ell) < \ell'/100$ of the branches.

- *Secondary branching and adaptation of subsequent steps*: Since a pseudo-locator may wrongly locate few vertices, the outcomes provided by the original Step 3 may be wrong, and lead to errors in the tests performed in subsequent steps, which in turn may lead to rejecting graphs in Π. We overcome the problem by branching to many executions, which use different samples for the subsequent steps, such that in almost all sub-branches all vertices are correctly located.

  Specifically, in the p^{th} foregoing branch, we branch again to ℓ' executions, all using the same pseudo-locators $S_1^{(p)}$ and $S_2^{(p)}$. In the q^{th} sub-branch we select, almost independently of other sub-branches, a sample $R^{(p,q)}$, identify its low and high degree vertices, denoted $R_1^{(p,q)}$ and $R_2^{(p,q)}$, and locally self-order $R_i^{(p,q)}$ using $S_i^{(p)}$. The later local self-ordering is performed analogously to Step 3, although the $S_i^{(p)}$'s may only be pseudo-locators. (If $S_i^{(p)}$ is empty, we define $R_i^{(p,q)}$ as empty.)

  We stress that if one of the $S_i^{(p)}$ is empty, then we invoke the locally self-order only for the other $R_{i'}^{(p,q)}$; in this case, if no failures or collision occurs, then we halt and accept. In any case, if some failure or collision occurs when performing local self-ordering, then we halt and reject *in this sub-branch*. (Recall that failure means that the procedure returns a failure symbol on some vertex in $R_{i'}^{(p,q)}$, and collision means that two different vertices in $R_{i'}^{(p,q)}$ were assigned the same location.)

  We make sure that the samples $R^{(p,q)}$ taken in the different sub-branches (referring to $S^{(p)}$) are dispersed in the very sense defined above. This guarantees that the (at most) ℓ vertices that were wrongly located by $S_i^{(p)}$ only affect the execution of $O(\ell) < \ell'/100$ of the sub-branches.

  The remaining steps are performed almost as in the original testers, where rejection in them will mean rejection in the corresponding sub-branch. The only significant modification is that Step 5 is replaced by a testing of the fixed part of the bipartite graphs (i.e., the part determined by the Boolean entries in the matrix M).

  At the end, we reject if a majority of the sub-branches rejected; otherwise, we accept.

We first observe that the modified procedure never rejects a graph in Π. Intuitively, this holds because we either eliminated the sources of rejection (i.e., in Step 1) or confined them to few branches and sub-branches. Specifically, the *disperseness* condition guarantees that the ℓ exceptional vertices

of $G^{(\delta)}$ whose degree is badly approximated affect the execution of $O(\ell)$ branches. Ditto for the ℓ exceptional vertices of $G^{(1-\delta)}$. Analogously, the ℓ exceptional vertices of $G^{(\delta)}$ that are incorrectly located by $S_1^{(p)}$ affect the execution of $O(\ell)$ sub-branches of the p^{th} branch. Hence, in total, only $O(\ell)\cdot\ell' + (\ell' - O(\ell))\cdot O(\ell) = O(\ell\cdot\ell')$ of the sub-branches are affected, whereas a vast majority (e.g., more than 99%) of the other sub-branches vote for acceptance (and ℓ' is picked to dominate $O(\ell)$). We shall also have to show that the resulting algorithm still rejects graphs that are far from Π. In fact, we shall show that almost all sub-branches do so. But, first, let us spell out the algorithm.

The algorithm (claimed to be a one-sided error tester of Π). On input $\epsilon > n^{-0.49}$ and oracle access to $G' = ([4k], E')$, we proceed as follows.

1. *Selecting a sample for degree estimation*: We pick an arbitrary set of $\ell' = O(\ell)$, denoted T, and inspect the subgraph of G' induced by T. Let d denote the average degree of vertices in this subgraph, and let $\widetilde{\rho} = 1 - (d/\ell')$ be a rough approximation of the fraction of T that resides in the copy of $G_{2k}^{(\delta)}$ (assuming $G' \in \Pi$).

   Indeed, if $G' \in \Pi$ and ρ denotes the (actual) fraction of vertices of T that resides in the copy of $G_{2k}^{(\delta)}$, then $d = (1\pm0.01)\cdot((1-\rho)\pm\delta)\cdot\ell' \pm 2\ell$. Furthermore, in this case, almost all vertices that reside in $G_{2k}^{(\delta)}$ have $(1\pm0.01)\cdot(0.5 - 0.5\rho + (\rho\pm0.5)\cdot\delta)\cdot\ell'$ neighbors in T, whereas the other vertices have $(1\pm0.01)\cdot(1 - 0.5\rho - (1\pm0.5)\cdot\delta)\cdot\ell'$ neighbors in T.

   In light of the foregoing, we shall rule that a vertex has low degree if it has less than $(0.75 - 0.5\cdot\widetilde{\rho})\cdot\ell'$ neighbors in T.

2. *Finding reliable locators for each branch*: As stated above, we branch to ℓ' executions, using the same degree approximation sample, T, in all of them. In the p^{th} branch we select, almost independently of other branches (see below), a sample $S^{(p)}$ of ℓ' vertices (in $[4k]\setminus T$), and identify its low and high degree vertices (according to T).

   If $S^{(p)}$ contains less that ℓ vertices of low (resp., high) degree, then we set $S_1^{(p)}$ (resp., $S_2^{(p)}$) to be empty, but do not reject; otherwise (i.e., $S^{(p)}$ contains at least ℓ vertices of low (resp., high) degree), we select $S_1^{(p)}$ (resp., $S_2^{(p)}$) uniformly at random among all ℓ-subsets of $S^{(p)}$ that consists of vertices of low (resp., high) degree.

   If the subgraph of G' induced by $S_1^{(p)} \neq \emptyset$ (resp., by $S_2^{(p)} \neq \emptyset$) is not isomorphic to the subgraph of $G_{2k}^{(\delta)}$ (resp., of $G_{2k}^{(1-\delta)}$) that is induced by some pseudo-locator of $G_{2k}^{(\delta)}$ (resp., of $G_{2k}^{(1-\delta)}$), then we halt and *reject*.

   Recall that we make sure that the samples $S^{(p)}$ taken in the different branches are dispersed in the sense that any set of ℓ vertices is intersected by $O(\ell)$ samples. As argued below (see paragraph on *bounding the deviation caused by the disperseness condition*), this can be done using $S^{(p)}$'s that are distributed *almost independently* of one another.

3. *Sampling the two sides of the graph in each sub-branch and locating the sampled vertices*: In the p^{th} foregoing branch, we branch again to ℓ' executions, all using the same pseudo-locators $S_1^{(p)}$ and $S_2^{(p)}$. In the q^{th} sub-branch we select, almost independently of other sub-branches, a sample $R^{(p,q)}$ of $O(f(n)/\epsilon)$ vertices (in $[4k]\setminus(T\cup S^{(p)})$), and identify its low and high degree vertices (according to T). Denoting the corresponding sets by $R_1^{(p,q)}$ and $R_2^{(p,q)}$, we locally self-order the vertices of $R_i^{(p,q)}$ using $S_i^{(p)}$. If $S_i^{(p)}$ is empty, then we reset $R_i^{(p,q)}$ to be empty.

We locally self-order the vertices of $R_1^{(p,q)}$ using $S_1^{(p)}$ by querying G' on all pairs $R_1^{(p,q)} \times S_1^{(p)}$ and looking at our hard-wired copy of $G_{2k}^{(\delta)}$. Specifically, letting $S_1^{(p)} = \{s_1, ..., s_\ell\}$ and $s_0 = v \in R_1^{(p,q)} \times S_1^{(p)}$ be the vertex that we wish to locate, we look for an ℓ-subset $W = \{w_1, ..., w_\ell\}$ and a vertex $w_0 \in [2k]$ such that $\{s_j, s_{j'}\} \in E'$ if and only if $\{w_j, w_{j'}\} \in E^{(\delta)}$ for every $j, j' \in \{0, 1, ..., \ell\}$. If the foregoing condition identifies a unique w_0, then we assign v to location w_0; otherwise, we consider the local self-ordering as failing on v. Ditto for $v \in R_2^{(p,q)}$ (using $S_2^{(p)}$).

If any of these invocation fails, then we *vote for rejection* in the corresponding sub-branch and suspend its execution. Ditto if two different vertices were assigned the same location. Otherwise, we define a bijection $\pi_i^{(p,q)} : R_i^{(p,q)} \to [2k]$ according to the local self-ordering procedure; that is, $\pi_i^{(p,q)}(v)$ is the location of $v \in R_i^{(p,q)}$ in $G_{2k}^{(c_i)}$, where $c_1 = \delta$ and $c_2 = 1 - \delta$.

Recall that we make sure that the samples $S^{(p)}$ taken in the different sub-branches are dispersed in the same sense as in Step 2.

4. *Testing isomorphism of each side of G' to the corresponding side in a generic $G_{A,B}$*: For every p and q, we test that the subgraph of G' induced by $R_1^{(p,q)}$ equals the subgraph of $G_{2k}^{(\delta)}$ induced by $\pi_1^{(p,q)}(R_1^{(p,q)})$, and ditto for $R_2^{(p,q)}$ and $G_{2k}^{(\delta)}$. As in the original tester, this test is performed by checking $O(1/\epsilon)$ vertex-pairs at random. Again, a failed check leads to *voting for rejection* in the corresponding sub-branch (and suspension of its execution).

5. *Completing a test of isomorphism of G' to a generic $G_{A,B}$*: For every p and q, we test that the subgraph of G' induced by $R_1^{(p,q)} \times R_2^{(p,q)}$ fits the Boolean entries of the matrix M. Specifically, we pick $O(1/\epsilon)$ random vertex-pairs (u, v) in $R_1^{(p,q)} \times R_2^{(p,q)}$ and check whether the adjacency of u and v fits the $(\pi_1^{(p,q)}(u), \pi_2^{(p,q)}(v))^{\text{th}}$ entry in the matrix M; that is, letting $(i, j) \stackrel{\text{def}}{=} (\pi_1^{(p,q)}(u), \pi_2^{(p,q)}(v))$, we *vote for rejection* in the current sub-branch if the relevant condition among the following four conditions holds:

   (a) For $i, j \in [k]$: if either $m_{i,j} = 0$ and $\{u, v\} \in E'$ or $m_{i,j} = 1$ and $\{u, v\} \notin E'$.

   (b) For $i, j \in \{k + 1, ..., 2k\}$: if either $m_{i-k,j-k} = 0$ and $\{u, v\} \in E'$ or $m_{i-k,j-k} = 1$ and $\{u, v\} \notin E'$.

   (c) For $i \in [k]$ and $j \in \{k + 1, ..., 2k\}$: if either $m_{i,j-k} \neq 1$ and $\{u, v\} \in E'$ or $m_{i,j-k} = 1$ and $\{u, v\} \notin E'$.

   (d) For $i \in \{k + 1, ..., 2k\}$ and $j \in [k]$: if either $m_{i-k,j} \neq 1$ and $\{u, v\} \in E'$ or $m_{i-k,j} = 1$ and $\{u, v\} \notin E'$.

   The first (resp., second) condition refers to edges in the first (resp., second) informative bipartite graph of a generic $G_{A,B}$, where a bipartite graph is called informative if it depends on the matrix A (resp., B). The third (resp., fourth) condition refers to edges in the first (resp., second) non-informative bipartite graph. Note that in the non-informative case $m_{i,j} = *$ mandates a non-edge, whereas in the informative case determining connectivity depends on $a_{i,j}$ (resp., $b_{i,j}$), and this will be checked by Step 6.

6. *Testing that $A = B$*: For every p and q, we call a pair $(i, j) \in [k]^2$ a matrix-collision if there exists $u_1, v_1 \in R_1^{(p,q)}$ and $u_2, v_2 \in R_2^{(p,q)}$ such that $\lceil \pi_1^{(p,q)}(u_1)/t' \rceil = \lceil i/t' \rceil = \lceil (\pi_1^{(p,q)}(v_1) - k)/t' \rceil$

and $\lceil \pi_2^{(p,q)}(u_2)/t \rceil = \lceil j/t \rceil = \lceil (\pi_2^{(p,q)}(v_2) - k)/t \rceil$. In such a case, we call i a row-collision, and call j a column-collision.

Letting I (resp., J) denote the set of row-collisions (resp., column-collision), we select a random set of $\min(|I|, |J|, \Theta(1/\epsilon))$ disjoint pairs $P \subseteq I \times J$ (i.e., distinct $(i_1, j_1), (i_2, j_2) \in P$ satisfy both $i_1 \neq i_2$ and $j_1 \neq j_2$). For each $(i, j) \in P$, we query G' on the corresponding pairs (u_1, u_2) and (v_1, v_2), where $\lceil \pi_1^{(p,q)}(u_1)/t' \rceil = \lceil i/t' \rceil = \lceil (\pi_1^{(p,q)}(v_1) - k)/t' \rceil$ and $\lceil \pi_2^{(p,q)}(u_2)/t \rceil = \lceil j/t \rceil = \lceil (\pi_2^{(p,q)}(v_2) - k)/t \rceil$, and *vote for rejection* in this sub-branch if *all* the following conditions hold.

(a) The first pair (i.e., (u_1, u_2)) corresponds to a non-fixed entry of M;
that is, $m_{\pi_1^{(p,q)}(u_1), \pi_2^{(p,q)}(u_2)} = *$.

(b) The second pair (i.e., (v_1, v_2)) corresponds to a non-fixed entry of M;
that is, $m_{\pi_1^{(p,q)}(v_1)-k, \pi_2^{(p,q)}(v_2)-k} = *$.

(c) The answers to these two queries disagree;
that is, $\{u_1, u_2\} \in E'$ if and only if $\{v_1, v_2\} \notin E'$.

(Note that in case one of the first conditions does not hold, we can check consistency of the answer with the relevant Boolean entry of M. However, this is unnecessary in light of Step 5.)

If the majority of all sub-branches vote for rejection, then we reject. Otherwise we accept.

The query complexity of the algorithm. Step 3 makes $\sum_{i \in [2], p, q \in [\ell']} |R_i^{(p,q)} \times S_i^{(p)}| = O(\epsilon^{-1} \cdot f(n) \cdot \log^3 n)$ queries, and dominates the total number of queries made by the foregoing algorithm. The extra $O(\log^2 n)$ factor (in comparison to the Theorem 4.2) is due to the branching.

Analysis of the execution on graphs in Π. The key point in proving that graphs in Π are accepted with probability 1 is showing that the categorization of vertices to low and high degree based on T is always correct (i.e., it is correct for all but at most 2ℓ vertices outside T, no matter which set T is chosen in Step 1). This calls for an analysis of the categorization rule defined in Step 1, which we undertake next.

Let $G' = \pi(G_{A,A})$ and $T_1 = T \cap \pi([2k])$. Then, the sum of the number of neighbors that vertices in T_1 have in T is

$$\begin{aligned} & |T_1| \cdot \left((1 \pm 0.01) \cdot \delta \cdot (|T_1| - 1) + ((1 \pm 0.01) \cdot \frac{1 \pm \delta}{2} \cdot |T_2| \right) \pm 2\ell \cdot |T| \\ \approx \;\; & (1 \pm 0.01) \cdot \left(\rho \cdot \delta + (1 - \rho) \cdot \frac{1 \pm \delta}{2} \right) \cdot |T| \cdot |T_1| \pm 2\ell \cdot |T|, \end{aligned}$$

where $\rho \stackrel{\text{def}}{=} |T_1|/|T|$. Recalling that $|T| = \ell'$, this means that the average degree (in the subgraph induced by T) of vertices in T_1 is approximately $(1 \pm 0.01) \cdot ((0.5 - 0.5\rho) + (\rho \pm 0.5(1-\rho)) \cdot \delta) \cdot \ell' \pm (2\ell/\rho)$. Similarly, for vertices in $T_2 = T \setminus T_1$ the average degree is approximately $(1 \pm 0.01) \cdot ((1 - 0.5\rho) - (1 - \rho \pm 0.5\rho) \cdot \delta) \cdot \ell' \pm (2\ell/(1-\rho))$. Hence, using $\ell' > 400\ell$, the average degree of vertices in the subgraph induced by T is $(1 \pm 0.02) \cdot (1 - \rho + (2\rho - 1 \pm \rho(1-\rho))\delta) \cdot |T|$, which means that $\widetilde{\rho} \approx \rho$ (where $\widetilde{\rho}$ is the estimate of ρ that was determined in Step 1). The foregoing calculations presume that $|T_1|, |T_2| \geq \ell$, but the claims are trivial if this condition does not hold (because in that case

$\min(\rho, 1-\rho) \ll \delta$). Hence, a threshold of $(0.75 - 0.5 \cdot \widetilde{\rho}) \cdot \ell'$ for the number of neighbors in T correctly distinguishes all but ℓ of the vertices in $\pi([2k]) \setminus T$ from those in $([4k] \setminus \pi([2k])) \setminus T$.

Using the hypothesis that the $S^{(p)}$'s are dispersed in the sense that the 2ℓ exceptional vertices appear in at most $O(\ell)$ of the ℓ' branches, it follows that in the other branches the vertices of $S^{(p)}$'s and all corresponding $R^{(p,q)}$ are categorized correctly. In these branches, all but at most 2ℓ of the vertices in $[n]$ are correctly located, which means that in at least $\ell' - O(\ell)$ sub-branches all vertices in $R^{(p,q)}$ are correctly located. (Recall that whenever the vertices of $S^{(p)}$ are categorized correctly, we obtain corresponding pseudo-locators.) Setting the hidden constant in the O-notation to 300 (see below), it follows that at least $\ell' \cdot (\ell' - 600\ell)$ of the sub-branches will not reject, which implies that the tester accepts (provided that $\ell' > 1200\ell$).

Bounding the deviation caused by the disperseness condition. The foregoing analysis relies on the disperseness of the collection of $S^{(p)}$'s and $R^{(p,q)}$'s. We show that almost independent sampling of the $S^{(p)}$'s and $R^{(p,q)}$'s satisfies this condition while not affecting the distribution of the sequence of sub-branches too much. This is indeed the case per our choice of the constant 300 and the hypothesis that $\epsilon > n^{-0.49}$ (and $f(n) \leq n^{0.5}$). Specifically, the probability that, when selecting ℓ' subsets of size $O(f(n)/\epsilon) = O(n^{0.99})$ in $[n] \setminus T$ (resp., in $[n] \setminus (T \cup S^{(p)})$), there exists ℓ vertices that occur in 150ℓ sets is upper-bounded by

$$\binom{n}{\ell} \cdot \binom{\ell'}{150\ell} \cdot \left(\frac{\ell \cdot O(n^{0.99})}{n - \ell - \ell'}\right)^{150\ell} = n^{-(0.5-o(1))\ell}$$

where $\frac{\ell \cdot O(n^{0.99})}{n-\ell-\ell'}$ represents the probability of one of the ℓ vertices appearing in a random $O(n^{0.99})$-subset.

Analysis of the execution on graphs that are far from Π. We now turn to the case that G' is ϵ-far from Π, and show that each sub-branch rejects with high probability (while recalling that these executions are almost independent). As a warm-up, consider the case that $G' = \pi(G_{A,B})$, for some pair (A, B) of (t', t)-redundant matrices and a bijection $\pi : [4k] \to [4k]$. In this case, B must be $15\delta^{-1} \cdot \epsilon$-far from A, since the adjacencies determined by A and B constitute a $(1 \pm 0.01) \cdot \delta/16$ fraction of all vertex-pairs. It follows that Step 6 rejects with high probability, since it is likely to inspect $\Omega(1/\epsilon)$ disjoint matrix-collisions, which are uniformly and independently distributed in $[k]^2$.[31] Intuitively, if G' is close enough to some $\pi(G_{A,B})$, then the same argument holds, and otherwise Steps 2–5 reject with high probability. The actual analysis follows.

Let V_1 denote the set of vertices of G' that are categorized as being of low degree (according to T), and $V_2 = [4k] \setminus V_1$. We consider a single sub-branch, and simplify notations accordingly; that is, we omit the superscripts (p) and (p, q). The following argument repeats some of the arguments made in the proof of Claim 3.1.3, but these repetitions are called for since some of the objects (i.e., T and the V_i's) are different here.

Assuming that Steps 1–2 were completed successfully (i.e., without rejection), we define a function $\phi_1 : V_1 \to [2k] \cup \{\perp\}$ such that $\phi_1(v)$ denotes the answer of the local self-ordering (based on S_1) to the input v, which may be a failure symbol, denoted $\perp$, where failure may happen because the subgraph of G' induced by V_1 is not necessarily isomorphic to $G^{(\delta)}_{2k}$. Similarly, we define $\phi_2 : V_2 \to [2k] \cup \{\perp\}$ based on S_2. Note that ϕ_i agrees with π_i on R_i.

Letting $\epsilon' = \Omega(\epsilon)$, we may assume that ϕ_i does not evaluate to $\perp$ on more than $\epsilon' k$ points, since otherwise such a point is sampled (w.h.p.) by Step 3, leading it to reject. Likewise, ϕ_i

[31] Indeed, if $G' = \pi(G_{A,B})$, then Step 6 is always reached.

does not have more than $\epsilon' k$ points that have an image with several pre-images under ϕ_i, where here a Birthday argument proves the claim (since the collision probability is at least $1/\epsilon' k$). In particular, it follows that $|V_i| \leq (2+2\epsilon') \cdot k$, because otherwise ϕ_i would have had more than $\epsilon' k$ points that have an image with several pre-images under it. (The latter assertion also uses the randomness of S, which guarantees that each V_i is sampled proportionally by S, which implies that if $|V_i| > (2+2\epsilon') \cdot k$ then (w.h.p.) S_i is not empty.) Denote by $V'_i \subseteq V_i$ the set of vertices that were not discarded above; that is, $V'_i = \{v \in V_i : \phi_i(v) \neq \perp \,\&\, |\phi_i^{-1}(\phi_i(v))| = 1\}$. Using $|V_i| \geq (2-2\epsilon') \cdot k$, we note that $|V'_i| \geq (2-4\epsilon') \cdot k$ must hold, and define ϕ'_i as the restriction of ϕ_i to V'_i. Lastly, let $\phi' : V'_1 \cup V'_2 \to [4k]$ such that $\phi'(v) = \phi'_i(v)$ if $v \in V'_1$ and $\phi'(v) = 2k + \phi'_i(v)$ otherwise, and let ϕ be an arbitrary extension of ϕ' to a permutation over $[4k]$.

Relying on Step 4 (and assuming that it does not reject w.h.p), we infer that the subgraph of $\phi(G')$ induced by $[2k]$ (resp., by $\{2k+1, ..., 4k\}$) is $5\epsilon'$-close to $G^{(\delta)}_{2k}$ (resp., to $G^{(1-\delta)}_{2k}$, when relabeling its vertices by subtracting $2k$ to each label). Relying on Step 5 (and assuming that it does not reject w.h.p), we infer that $\phi(G')$ is $6\epsilon'$-close to some $G_{A,B}$ for some matrices A and B (which are not necessarily (t',t)-redundant). Recalling that G' is ϵ-far from Π (and using $\epsilon' = \epsilon/7$), we infer that A and B must disagree on more than $\epsilon' \cdot 16 \cdot k^2$ entries that are not fixed in M. We claim that in this case Step 6 rejects with high probability.

Recalling that $|V_i| = (2k \pm 0.2k)$, we observe that with high probability (over the choice of R_1 and R_2), Step 6 defines a set of at least $10/\epsilon$ disjoint matrix-collision pairs, and that these pairs are independently and uniformly distributed in $[k] \times [k]$. Such a pair (i,j) causes rejection if there exists $u_1, v_1 \in R_1$ and $u_2, v_2 \in R_2$ such that the following conditions hold:

1. The corresponding queries, (u_1, u_2) and (v_1, v_2), are not fixed in M; that is, $m_{\pi_1(u_1),\pi_2(u_2)} = m_{\pi_1(v_1),\pi_2(v_2)} = *$.
2. These entries that are supposed to be equal in A and B; that is, $\lceil \pi_1(u_1)/t' \rceil = \lceil i/t' \rceil = \lceil (\pi_1(v_1) - k)/t' \rceil$ and $\lceil \pi_2(u_2)/t \rceil = \lceil j/t \rceil = \lceil (\pi_2(v_2) - k)/t \rceil$.
3. Yet, the foregoing entries are not identical in A and B; that is, $a_{\pi_1(u_1),\pi_2(u_2)} \neq b_{\pi_1(v_1),\pi_2(v_2)}$.

The probability that this event holds (i.e., all three conditions hold) is lower-bounded by $15\epsilon' - 4\epsilon' - 4\epsilon' = \epsilon$, and the claim follows. ■

Claim 5.10 (upper bound for non-adaptively testing Π): *There exists a non-adaptive tester for Π_n of query complexity $O(\epsilon^{-2} \cdot g(f(n)) \cdot f(n) \cdot \log^2 n + \epsilon^{-1} \cdot f(n) \cdot \log^3 n)$, if $\epsilon > n^{-0.49}$ and $O(1/\epsilon^{4.1})$ otherwise.*

Proof Sketch: We use a non-adaptive version of the adaptive tester presented in the proof of Proposition 5.9. The observations and modifications regarding this matter that were made in the proof of Theorem 4.1 apply here (without any change). Specifically, for Step 6, we select a random subset R' of $O(g(f(n))/\epsilon)$ vertices of each relevant $R = R^{(p,q)}$, and make non-adaptive queries to all pairs in $R \times R'$, which means making $O(g(f(n)) \cdot f(n)/\epsilon^2)$ queries.[32] (This quantity has to be multiplied by the number of sub-branches, whereas the $O(\epsilon^{-1} \cdot f(n) \cdot \log^3 n)$ term arises from Step 3.) ■

[32] See also the relevant part of the proof of Theorem 4.2.

Claim 5.11 (lower bound for adaptively testing Π): *Any non-adaptive tester for Π_n has query complexity $\Omega(f(n))$.*

Proof Sketch: This bound follows merely by considering matrix-collisions among samples of the k rows of the matrix. Specifically, we used the same distributions as in the proof of the lower bound for non-adaptive testers, and observe that $\Omega(\sqrt{k/t'}) = \Omega(f(n))$ adaptive queries are needed in order to find vertices of G' that are mapped (by π) to location i and $i' + k$, for some $i, i' \in [k]$ such that $\lceil i/t' \rceil = \lceil i'/t' \rceil$. ■

Conclusion. Having established all four bounds, Theorem 5.1 follows.

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Appendix A: Proof of Theorem 2.3

For each (non-trivial) permutation $\mu : [n] \to [n]$, letting $T \stackrel{\text{def}}{=} \{i \in [n] : \mu(i) \neq i\}$ denote its (non-empty) set of non-fixed-points, we show that, with probability $1 - \exp(-\Omega(n \cdot |T|))$, the size of the symmetric different between a random n-vertex graph $G_n = ([n], E_n)$ and $\mu(G_n)$ is $\Omega(n \cdot |T|)$.

For every $u, v \in [n]$ such that $u < v$, let $\chi_{u,v} = \chi^{\mu}_{u,v}(G_n)$ represent the event that *the pair* $(\mu(u), \mu(v))$ *contributes to the symmetric difference between* G_n *and* $\mu(G_n)$; that is, $\chi_{u,v} = 1$ if exactly one of the edges $\{\mu(u), \mu(v)\}$ and $\{u, v\}$ is in G_n, since $\{u, v\}$ is an edge of G_n if and only if $\{\mu(u), \mu(v)\}$ is an edge of $\mu(G_n)$. We shall prove that

$$\Pr_{G_n}\left[\sum_{u<v\in[n]} \chi^{\mu}_{u,v}(G_n) < \frac{n \cdot |T|}{20}\right] = \exp(-\Omega(n \cdot |T|)). \tag{8}$$

We prove Eq. (8) by using a $\lceil |T|/3 \rceil$-subset $I \subseteq T$ such that $I \cap \mu(I) = \emptyset$. Let $T' = T \setminus (I \cup \mu^{-1}(I))$, which implies $T' \cap I = \emptyset$ and $\mu(T') \cap I = \emptyset$. Let $J = ([n] \setminus T) \cup T'$, and note that $|J| = n - |T| + (|T| - 2 \cdot \lceil |T|/3 \rceil) \geq n - (2|T|/3) - 2 \geq (n/3) - 2$. Observe that, for every $(u, v) \in J \times I$, it holds that $u \neq v$ and $\Pr[\chi_{u,v} = 1] = 1/2$, where the equality is due to $\{u, v\} \neq \{\mu(u), \mu(v)\}$, which holds since $(u, v) \in J \times I$ but $\mu(u), \mu(v) \in [n] \setminus I$. Furthermore, the events the correspond to the pairs in $J \times I$ are independent, because the sets $\{\{u, v\} : (u, v) \in J \times I\}$ and $\{\{\mu(u), \mu(v)\} : (u, v) \in J \times I\}$ are disjoint; that is, $(u, v) \in J \times I$ implies $(\mu(u), \mu(v)) \in ([n] \setminus I) \times ([n] \setminus I)$. Hence (using $n \leq 3(|J| + 2)$ and $|T| \leq 3|I|$ (as well as $3(|J| + 2) \cdot 3|I| < 9.9 \cdot |J| \cdot |I|$)), the l.h.s. of Eq. (8) is upper-bounded by

$$\begin{aligned}\Pr_{G_n}\left[\sum_{(u,v)\in J\times I} \chi^{\mu}_{u,v}(G_n) < \frac{3(|J|+2) \cdot 3|I|}{20}\right] &\leq \Pr_{G_n}\left[\sum_{(u,v)\in J\times I} \chi^{\mu}_{u,v}(G_n) < \frac{0.99 \cdot |J| \cdot |I|}{2}\right] \\ &= \exp(-\Omega(|J| \cdot |I|))\end{aligned}$$

which is $\exp(-\Omega(n \cdot |T|))$. Having established Eq. (8), the claim follows by a union bound (over all non-trivial permutations $\mu : [n] \to [n]$); specifically, denoting the set of non-trivial permutations by P_n, we upper-bound the probability that G_n is not 0.05-robust by

$$\begin{aligned}&\sum_{\mu\in P_n} \Pr_{G_n}[\mu \text{ violates the condition in Eq. (8)}] \\ &\leq \sum_{t\in[n]} \binom{n}{t} \cdot (t!) \cdot \exp(-\Omega(n \cdot t)) \\ &< n \cdot \max_{t\in[n]}\{n^t \cdot \exp(-\Omega(n \cdot t))\} \\ &= \exp(-\Omega(n))\end{aligned}$$

where t represents the size of the set of non-fixed-points (w.r.t μ). ■

Appendix B: Proof of Claim 2.6.1

We start with the main claim. Recall that G_n denotes a uniformly distributed n-vertex graph, S is a fixed ℓ-subset of $[n]$, and our goal is to upper-bound the probability that the subgraph of G_n induced by S is isomorphic to the subgraph induced by some other subset. Hence, we let $S' \neq S$ be an arbitrary ℓ-subset of $[n]$, and upper-bound the probability that the subgraphs of G_n induced by S and S' are isomorphic.

The case of $S' \cap S = \emptyset$ is easy, because in this case the we may fix the subgraph of G_n induced by S', whereas a random ℓ-vertex graph (i.e., the subgraph of G_n induced by S) is isomorphic to this fixed graph with probability at most $(\ell!) \cdot 2^{-\binom{\ell}{2}} \ll \binom{n}{\ell}^{-1}$, where the inequality uses a sufficiently large $\ell = O(\log n)$. Hence, we can afford to take a union bound over all ℓ-subsets that are disjoint of S. However, for sets that are not disjoint of S, the foregoing probability bound does not hold, and a more careful analysis is called for. Nevertheless, the foregoing analysis does provide a good warm-up towards the rest.

Turning to the general case, for each ℓ-set $S' \subset [n]$ such that $S' \neq S$, we shall upper-bound the probability that the subgraphs of G_n induced by S and S' are isomorphic as a function of $|S \cap S'|$. For every bijection $\pi : S \to S'$, let $\mathtt{FP}(\pi) \stackrel{\text{def}}{=} \{v \in S : \pi(v) = v\}$ denote the set of fixed-points of π, and note that $|\mathtt{FP}(\pi)| \leq \ell - 1$ (since $S \neq S'$). Now, let $G = G_n$ denote the random n-vertex graph and G_R denote the subgraph of G induced by R. Then, we claim that the probability that there exists a bijection $\pi : S \to S'$ such that $\pi(G_S) = G_{S'}$ is upper-bounded by

$$\sum_{\pi: S \stackrel{1\text{-}1}{\to} S'} \min\left(2^{-|\mathtt{FP}(\pi)| \cdot (\ell - |\mathtt{FP}(\pi)|)/3}, 2^{-\binom{(\ell - |\mathtt{FP}(\pi)|)/3}{2}}\right) \tag{9}$$

$$\leq \sum_{f \in \{0,\ldots,|S \cap S'|\}} \frac{\ell!}{f!} \cdot 2^{-\max(6 \cdot f \cdot (\ell - f), (\ell - f) \cdot (\ell - f - 1))/18}$$

$$< \frac{\ell!}{|S \cap S'|!} \cdot 2^{-\Omega((\ell - |S \cap S'|) \cdot \ell)} \tag{10}$$

where f represents the size of $\mathtt{FP}(\pi)$.

Justifying Eq. (9). To justify the upper bound claimed in Eq. (9), consider an arbitrary bijection $\pi : S \to S'$, and identify a set $I \subseteq S \setminus \mathtt{FP}(\pi)$ such that $\pi(I) \cap I = \emptyset$ and $|I| \geq (\ell - |\mathtt{FP}(\pi)|)/3$. Define random variables $\chi_{u,v}(\cdot)$ for $\{u, v\} \in \binom{[n]}{2}$ such that $\chi_{u,v}(G) = 1$ if $\{u, v\}$ is an edge in G and $\chi_{u,v}(G) = 0$ otherwise, and observe that $\pi(G_S) = G_{S'}$ if and only if $\chi_{\pi(u),\pi(v)}(\pi(G)) = \chi_{\pi(u),\pi(v)}(G)$ for every $\{u, v\} \in \binom{S}{2}$. Noting that $\chi_{\pi(u),\pi(v)}(\pi(G)) = \chi_{u,v}(G)$, the first bound in Eq. (9) is justified by

$$\begin{aligned}
&\Pr_G\left[\forall (u,v) \in \binom{S}{2} : \quad \chi_{\pi(u),\pi(v)}(\pi(G)) = \chi_{\pi(u),\pi(v)}(G)\right] \\
&\leq \Pr_G\left[\forall (u,v) \in \mathtt{FP}(\pi) \times I : \quad \chi_{u,v}(G) = \chi_{\pi(u),\pi(v)}(G)\right] \\
&= \prod_{(u,v) \in \mathtt{FP}(\pi) \times I} \Pr_G\left[\chi_{u,v}(G) = \chi_{u,\pi(v)}(G)\right]
\end{aligned}$$

$$
\begin{aligned}
&= 2^{-|\mathtt{FP}(\pi)|\cdot|I|} \\
&\leq 2^{-|\mathtt{FP}(\pi)|\cdot(\ell-|\mathtt{FP}(\pi)|)/3}
\end{aligned}
$$

where the equalities are due to the disjointness of the sets $\mathtt{FP}(\pi) \times I$ and $\mathtt{FP}(\pi) \times \pi(I)$ (to the fact that $\pi(u) = u$ for every $u \in \mathtt{FP}(\pi)$), and to the fact that the different $\chi_{u,v}(G)$'s are independent and uniformly distributed in $\{0,1\}$. Similarly, we justify the second bound in Eq. (9) by

$$
\begin{aligned}
&\Pr_G\left[\forall\{u,v\}\in\binom{S}{2}: \quad \chi_{\pi(u),\pi(v)}(\pi(G)) = \chi_{\pi(u),\pi(v)}(G)\right] \\
&\leq \Pr_G\left[\forall\{u,v\}\in\binom{I}{2}: \quad \chi_{u,v}(G) = \chi_{\pi(u),\pi(v)}(G)\right] \\
&= \prod_{\{u,v\}\in\binom{I}{2}} \Pr_G\left[\chi_{u,v}(G) = \chi_{\pi(u),\pi(v)}(G)\right] \\
&= 2^{-\binom{|I|}{2}} \\
&\leq 2^{-\binom{(\ell-|\mathtt{FP}(\pi)|)/3}{2}}
\end{aligned}
$$

where the equalities are due to the disjointness of the sets $\binom{I}{2}$ and $\binom{\pi(I)}{2}$, and to the fact that the different $\chi_{u,v}(G)$'s are independent and uniformly distributed in $\{0,1\}$.

Finishing the proof of the main claim. Combining Eq. (9)&(10) with a union bound over all ℓ-subsets $S' \subset [n]$ that are different from S, we upper-bound the probability that the subgraphs of G induced by S and by some other ℓ-set are isomorphic by

$$
\sum_{S'\in\binom{[n]}{\ell}\setminus\{S\}} \frac{\ell!}{|S\cap S'|!}\cdot 2^{-\Omega((\ell-|S\cap S'|)\cdot\ell)} = \sum_{i\in\{0,\ldots,\ell-1\}} \binom{\ell}{i}\cdot\binom{n-i}{\ell-i}\cdot\frac{\ell!}{i!}\cdot 2^{-\Omega((\ell-i)\cdot\ell)} \tag{11}
$$

where the index i represents the size of the intersection with S. Using a sufficiently large $\ell = O(\log n)$, we have

$$
\begin{aligned}
\sum_{i\in\{0,\ldots,\ell-1\}} \binom{\ell}{i}\cdot\binom{n-i}{\ell-i}\cdot\frac{\ell!}{i!}\cdot 2^{-\Omega((\ell-i)\cdot\ell)} &= \sum_{i\in\{0,\ldots,\ell-1\}} \binom{\ell}{i}^2\cdot\binom{n-i}{\ell-i}\cdot\frac{(n-i)!}{(n-\ell)!}\cdot 2^{-\Omega((\ell-i)\cdot\ell)} \\
&< \sum_{i\in\{0,\ldots,\ell-1\}} n^{\ell-i}\cdot\binom{\ell}{i}^2\cdot 2^{-\Omega((\ell-i)\cdot\ell)} \\
&< \ell\cdot\max_{i\in\{0,\ldots,\ell-1\}}\left\{n^{\ell-i}\cdot\binom{\ell}{i}^2\cdot 2^{-\Omega((\ell-i)\cdot\ell)}\right\} \\
&= \ell\cdot\left(n\cdot\ell^2\cdot 2^{-\Omega(\ell)}\right)
\end{aligned}
$$

and the main claim follows.

Proving the furthermore claim. The furthermore claim follows by observing that the probability that the subgraph of G induced by S is self-ordered is upper-bounded by an expression analogous to Eq. (9), where $S' = S$ and the difference is that the identity permutation is excluded from the sum. Hence, $|\mathtt{FP}(\pi)| < \ell$ still holds, and so do the justifications given to Eq. (9). ■