FROM CRYSTAL OPTICS TO DIRAC OPERATORS: A SPECTRAL THEORY OF FIRST-ORDER SYSTEMS

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HAPPY BIRTHDAY YAKAR

FIRST-ORDER SYSTEMS

$$L = \sum_{j=1}^{n} M_j(x) D_j + M_0(x), \; D_j = \frac{1}{i}\frac{\partial}{\partial x_j}, \quad x \in \mathbf{R}^n, \tag{1}$$

$M_0(x), \ldots, M_n(x)$ Hermitian $K \times K$ matrices , $x \in \mathbf{R}^n$, $n \geq 2$.

L symmetric (wrt $\mathcal{L}^2(\mathbf{R}^n; \mathbf{C}^K)$ scalar product) on $C_0^\infty(\mathbf{R}^n; \mathbf{C}^K)$.

EXAMPLES:

- Classical physics: acoustic, electromagnetic and elastic waves in inhomogeneous anisotropic media.

- Dirac equation of relativistic quantum electrodynamics .

Treatise of **Courant and Hilbert**: common features emphasized.

HIGH-ORDER SYSTEMS

$$L = \sum_{|\alpha| \le m} A_\alpha(x) D^\alpha, \ D^\alpha = \prod_{j=1}^{n} D_j^{\alpha_j}, \quad x \in \mathbf{R}^n, \tag{2}$$

$A_\alpha(x)$ Hermitian $p \times p$ matrices , $x \in \mathbf{R}^n$, $n \geq 2$.

L self-adjoint, positive in $\mathcal{L}^2(\mathbf{R}^n; \mathbf{C}^p)$. $m \geq n$.

GOAL: **Characterize the spectral family** E_λ.

CLASSICAL WORK: Y. KANNAI Thesis (Hebrew University, 1967):

- The kernel $E_\lambda(x, y)$ (''spectral function'') is smooth.
- Asymptotic behavior as $\lambda \to \infty$ (away from spectrum).

S. Agmon and Y. Kannai, On the asymptotic behavior of spectral functions and resolvent kernels of elliptic operators, Israel J. Math. (1967)

FIRST-ORDER CONSTANT COEFFICIENT SYSTEMS

$$L_0 = \sum_{j=1}^{n} M_j^0 D_j + M_0^0, \; D_j = \frac{1}{i}\frac{\partial}{\partial x_j}, \quad x \in \mathbf{R}^n. \tag{3}$$

$M_0^0, M_1^0, \ldots, M_n^0$ constant Hermitian $K \times K$ matrices (over $\mathbf{C}$). Addition of M_0^0 needed for massive Dirac operator .

Definition 1. $L_{0,hom} = \sum_{j=1}^{n} M_j^0 D_j$ *is* **strongly propagative** *if there exists a fixed* $0 \leq d < K$ *so symbol* $M(\xi) = \sum_{j=1}^{n} M_j^0 \xi_j$ *satisfies*

$$dim\, \mathit{ker}(M(\xi)) = d, \quad \forall \xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n \setminus \{0\}\,. \tag{4}$$

$\Rightarrow$ negative and positive eigenvalues of $M(\xi)$, $\xi \neq 0$, are

- Equal in number.
- Positive-homogeneous of degree 1.

$$Z = \{\xi \neq 0, \text{ the discriminant of } M(\xi) \text{ vanishes}\}.$$

- Z is a closed cone of Lebesgue measure zero .
- Every eigenvalue of $M(\xi)$ has constant multiplicity in $\mathbf{R}^n \setminus \{Z \cup \{0\}\}$.
- Distinct eigenvalues (real analytic)

$$\lambda_\rho(\xi) > \ldots > \lambda_1(\xi) > 0 > \lambda_{-1}(\xi) > \ldots > \lambda_{-\rho}(\xi), \ \xi \in \mathbf{R}^n \setminus \{Z \cup \{0\}\}$$

A more restricted class is of operators for which $Z = \emptyset$:

Definition 2. $L_{0,hom} = \sum_{j=1}^{n} M_j^0 D_j$ *is* **uniformly propagative** *if it is strongly propagative and, moreover, the eigenspace associated with every eigenvalue* $\lambda_k(\xi)$ *has a constant dimension , independent of* $\xi \in \mathbf{R}^n \setminus \{0\}$.

Definition 3. *The surfaces* $\left\{\lambda_j(\xi) = \pm 1\right\}$ *are called the* **slowness surfaces** *of the system.*

Courant-Hilbert:**normal surfaces**

The *nonhomogeneous system* ($M(\xi)$ is strongly propagative):

$$\widetilde{M}(\xi) = M(\xi) + M_0^0 = \sum_{j=1}^{n} M_j^0 \xi_j + M_0^0. \tag{5}$$

Proposition 4. *There exists a closed set $\tilde{Z} \subseteq \mathbf{R}^n$ of Lebesgue measure zero such that:*

For $\xi \in \mathbf{R}^n \setminus \tilde{Z}$ every eigenvalue of $\widetilde{M}(\xi)$ that is not an eigenvalue of M_0^0, has constant multiplicity .

$$\lambda_r(\xi) > \ldots > \lambda_1(\xi), \ \xi \in \mathbf{R}^n \setminus \tilde{Z},$$

with corresponding (orthogonal) projection operators

$$P_j(\xi), \ j = 1, \ldots, r.$$

SPECTRAL STRUCTURE OF NONHOMOGENEOUS SYSTEMS

$$L_0 = \sum_{j=1}^{n} M_j^0 D_j + M_0^0, \; D_j = \frac{1}{i}\frac{\partial}{\partial x_j}, \quad x \in \mathbf{R}^n.$$

$$\widetilde{M}(\xi) = M(\xi) + M_0^0 = \sum_{j=1}^{n} M_j^0 \xi_j + M_0^0.$$

$$E_\lambda = \text{spectral family of } L_0.$$

$$E_\lambda(\xi) = \text{symbol of } E_\lambda, \quad \xi \in \mathbf{R}^n.$$

$$U^\lambda = \left\{ \xi \in \mathbf{R}^n \setminus \tilde{Z}, \quad E_\lambda(\xi) \neq 0 \right\}.$$

Proposition 5. *There exist a finite set* $\Lambda \subseteq \mathbf{R}$ *and an integer-valued function* $k(\lambda) \geq 0,\ \lambda \in \mathbf{R} \setminus \Lambda$, *so that,*

1. $k(\lambda)$ *is constant in open intervals of* $\mathbf{R} \setminus \Lambda$.

2. *For* $k(\lambda) > 0$ *the sets*

$$\Gamma_j(\lambda) = \left\{\xi \in \mathbf{R}^n \setminus \widetilde{Z},\ \lambda_j(\xi) = \lambda\right\},\ j = 1, \ldots, k(\lambda),$$

are bounded smooth $(n-1)-$*manifolds, while*

$$\left\{\xi \in \mathbf{R}^n \setminus \widetilde{Z},\ \lambda_j(\xi) = \lambda\right\} = \emptyset,\ j > k(\lambda).$$

Proposition 6. *Let* $\lambda \in \mathbf{R} \setminus \Lambda$ *be such that* $k(\lambda) > 0$. *Then there exist nonempty open sets* $\left\{U_1^\lambda, \ldots, U_{k(\lambda)}^\lambda\right\}$:

1. $U^\lambda = \bigcup_{j=1}^{k(\lambda)} U_j^\lambda, \quad \partial U_j^\lambda \setminus \widetilde{Z} = \Gamma_j(\lambda).$

2. $U_j^\lambda = \left\{\xi \in \mathbf{R}^n \setminus \widetilde{Z},\ \lambda_j(\xi) < \lambda\right\},\ j \in \{1, \ldots, k(\lambda)\}.$

$$rank\, P_j(\xi) = constant \neq 0,\ \xi \in U_j^\lambda,$$

and , for every $\xi \in U^\lambda$,

$$E_\lambda(\xi) = \sum_{j=1}^{k(\lambda)} P_j(\xi). \tag{6}$$

EIGENFUNCTION EQUATION

$$L_0 v(x) = \lambda v(x), \quad x \in \mathbf{R}^n.$$

$$supp\, \widehat{v}(\xi) \subseteq \bigcup_j \left\{ \xi \in \mathbf{R}^n,\ \lambda_j(\xi) = \lambda \right\} = \overline{\bigcup_j \Gamma_j(\lambda)}$$

singularities in $\widetilde{Z}$.

EVOLUTION EQUATION

$$i\frac{\partial}{\partial t} u(x,t) = L_0 u(x,t), \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}.$$

$$u = \int_{\mathbf{R}} e^{it\lambda} dE_\lambda$$

$$u(x,t) = \int_{\mathbf{R}} \sum_{j=1}^{k(\lambda)} \int_{\Gamma_j(\lambda)} e^{i\lambda_j(\xi)t} e^{i\xi x} \widetilde{u_0}(\xi) \mathbf{dP_j}(\xi) d\lambda.$$

C. H. Wilcox, Asymptotic wave functions and energy distributions in strongly propagative anisotropic media, J. Math. Pures Appl. **57** (1978), 275-321.

(asymptotics for large time)

R. Weder, Analyticity of the scattering matrix for wave propagation in crystals, J. Math. Pures Appl. **64** (1985), 121-148.

(eigenfunction expansion using geometry of slowness surfaces).

O. Liess, Curvature properties of the slowness surface of the system of crystal acoustics for cubic crystals, Osaka J. Math. **45** (2008), 173-210.

(oscillatory integrals on slowness surfaces).

DIRAC OPERATOR

$$H_m = \boldsymbol{\alpha} \cdot D + m\beta, \; m \geq 0, \tag{7}$$

$$D = (D_1, D_2, D_3), \quad x \in \mathbf{R}^3. \tag{8}$$

$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is triplet of 4×4 Dirac matrices

$$\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3. \tag{9}$$

Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; , \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{10}$$

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \; I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{11}$$

DIRAC OPERATOR IS SQUARE ROOT:

$$(H_m)^2 = (-\Delta + m^2) \bigotimes I_4. \tag{12}$$

$$Dom(H_m) = \mathcal{H}^1(\mathbf{R}^3; \mathbf{C}^4). \tag{13}$$

symbol $H_m(\xi)$:

$$\widehat{H_m f}(\xi) = H_m(\xi)\widehat{f}(\xi), \quad f \in C_0^\infty(\mathbf{R}^3; \mathbf{C}^4),\ \xi \in \mathbf{R}^3. \tag{14}$$

$H_m(\xi)$ is a 4×4 Hermitian matrix

$$H_m(\xi) = \boldsymbol{\alpha} \cdot \xi + m\beta. \tag{15}$$

- Eigenvalues of $H_m(\xi)$: $\quad \lambda_\pm(\xi) = \pm\sqrt{|\xi|^2 + m^2}$,
  both of double multiplicity (except for $m = \xi = 0$).

- For any $\xi \in \mathbf{R}^3$ there exists a unitary matrix $U_m(\xi)$ such that

$$U_m(\xi)^* H_m(\xi) U_m(\xi) = \begin{pmatrix} \lambda_+(\xi)I_2 & 0 \\ 0 & \lambda_-(\xi)I_2 \end{pmatrix}. \tag{16}$$

$f \in \mathcal{L}^2(\mathbf{R}^3; \mathbf{C}^4)$ define:

$$\mathcal{G}f(\xi) = U_m(\xi)\widehat{f}(\xi),\ \xi \in \mathbf{R}^3. \tag{17}$$

$$\mathcal{G} : \mathcal{L}^2_x(\mathbf{R}^3; \mathbf{C}^4) \to \mathcal{L}^2_\xi(\mathbf{R}^3; \mathbf{C}^4) \quad \text{unitary}$$

Foldy-Wouthuysen-Tani transformation

$$\mathcal{G}H_m\mathcal{G}^{-1} = \begin{pmatrix} \lambda_+(\xi)I_2 & 0 \\ 0 & \lambda_-(\xi)I_2 \end{pmatrix}. \tag{18}$$

$E_m(\lambda)$ spectral family of H_m.

$f \in C_0^\infty(\mathbf{R}^3; \mathbf{C}^4),\ \lambda > 0$

$$(E_m(\lambda)f, f) = \left(\begin{pmatrix} \chi_{\lambda_+(\xi)\le\lambda} I_2 & 0 \\ 0 & I_2 \end{pmatrix} \mathcal{G}f, \mathcal{G}f \right). \tag{19}$$

$\chi_B(y) = 1$ (resp. $\chi_B(y) = 0$) if $y \in B$ (resp. $y \notin B$).

$$\mathcal{L}^{2,s} = \{f/\|f\|^2_{\mathcal{L}^{2,s}} := \int_{\mathbf{R}^n} (1+|x|^2)^s |f(x)|^2 dx < \infty\}, \quad s \in \mathbf{R}$$

$$\mathcal{H}^{m,s}(\mathbf{R}^n) := \{u(x) \quad /D^\alpha u \in \mathcal{L}^{2,s}, \quad |\alpha| \leq m, \|u\|^2_{m,s} = \sum_{|\alpha \leq m} \|D^\alpha u\|^2_{0,s}\}$$

$$\mathcal{H}^\sigma \equiv H^{\sigma,0} = \{\widehat{u} \quad /u \in \mathcal{L}^{2,\sigma}, \quad \|\widehat{u}\|_{\sigma,0} = \|u\|_{0,\sigma}\}, \quad \sigma \in \mathbf{R}.$$

$$\left\{\mathcal{H}^{-m,s}, \quad \|\cdot\|_{-m,s}\right\} \quad = \text{the dual space of } \mathcal{H}^{m,-s}.$$

$f \in \mathcal{H}^{-1,s}$ can be represented (not uniquely) as

$$(*) \quad f = f_0 + \sum_{k=1}^{n} i^{-1} \frac{\partial}{\partial x_k} f_k, \qquad f_k \in \mathcal{L}^{2,s}, \quad 0 \leq k \leq n.$$

$L_0 = \sum_{j=1}^{n} M_j^0 D_j + M_0^0, \quad \sigma(L_0) = \text{its spectrum.}$

THEOREM. *L_0 satisfies the LAP in $W = int\,\sigma(L_0)$. More precisely, let $s > 1$ and consider the resolvent $R_0(z) = (L_0 - z)^{-1}, \quad Im\,z \neq 0$, as a bounded operator from $\mathcal{L}^{2,s}(\mathbf{R}^n)$ to $\mathcal{L}^{2,-s}(\mathbf{R}^n)$.*

Then:

(a) $R_0(z)$ *is bounded wrt the $\mathcal{H}^{-1,s}(\mathbf{R}^n)$ norm. Density of $\mathcal{L}^{2,s}$ in $\mathcal{H}^{-1,s} \Rightarrow$*

$$R_0(z) \in B(\mathcal{H}^{-1,s}(\mathbf{R}^n), \mathcal{L}^{2,-s}(\mathbf{R}^n)).$$

$$(b) \qquad z \to R_0(z) \in B(\mathcal{H}^{-1,s}(\mathbf{R}^n), \mathcal{L}^{2,-s}(\mathbf{R}^n)), \qquad s > 1, \quad \pm Im\,z > 0, \quad (20)$$

can be extended continuously from
$\mathcal{C}^{\pm} = \{z / \pm Im\,z > 0\}$ to $\mathcal{C}^{\pm} \bigcup \{W \setminus \Lambda\}$ (in $B(\mathcal{H}^{-1,s}(\mathbf{R}^n), \mathcal{L}^{2,-s}(\mathbf{R}^n))$).

Notation:

$$R_0^{\pm}(\lambda) = \lim_{\varepsilon \to \pm 0} R_0(\lambda + i\varepsilon).$$

Spectrum of L_0 is absolutely continuous in open intervals $\subseteq \sigma(L_0)$.

BACKGROUND for LAP–Dirac and first-order systems:

- Massive Dirac
  Yamada (1973), Jensen-Kato (1979), Balslev-Helffer (1992)
- Homogeneous uniformly propagative
  Tamura (1981)
- Homogeneous strongly propagative
  Weder (1985)
- Nonhomogeneous systems
  ??

GENERAL FRAMEWORK FOR LAP

Abstract treatment of the "LIMITING ABSORPTION PRINCIPLE" (LAP) developed by MBA and A. Devinatz.

MBA, **Smooth spectral calculus** , In: M. Demuth, B.-W. Schulze and I. Witt (Eds.) "Partial Differential Equations and Spectral Theory", pp. 119-182 , Springer Basel, 2010.

Other abstract approaches:

1) (commutators): Mourre (1981)

2) (energy estimates): C. Gérard (2008).

H self-adjoint in Hilbert space $\mathcal{H}$,

$R(z) = (H - z)^{-1}, \quad Im\, z \neq 0. \qquad E(\lambda)$ spectral family.

$X \subseteq \mathcal{H} \subseteq X^*$ continuous, dense imbedding.

Definition 7. *H satisfies the*

LIMITING ABSORPTION PRINCIPLE *(LAP) at*

$\lambda \in \mathbf{R}$ *if* $\quad \exists \lim_{\varepsilon \to 0\pm} R(\lambda + i\varepsilon) = R^{\pm}(\lambda)$

in $\quad B(X, X^*) \quad$ *uniform topology.*

$\exists \lim_{\varepsilon \to 0\pm} R(\lambda + i\varepsilon) = R^{\pm}(\lambda)$

Definition 8. $U \subseteq \mathbf{R}$ *open,* $0 < \alpha \leq 1$.

H *is of* **type** $(\boldsymbol{X}, \boldsymbol{X}^*, \boldsymbol{\alpha}, \boldsymbol{U})$ *if* $\exists A(\lambda) \in B(X, X^*)$ *such that*

(i) $\frac{d}{d\lambda}(E(\lambda)x, y) = < A(\lambda)x, y >, \qquad x, y \in X, \quad \lambda \in U.$

(ii) $\|A(\lambda) - A(\mu)\|_{B(X,X^*)} \leq M_K |\lambda - \mu|^{\alpha}, \quad \lambda, \mu \in K \subseteq U$

K *compact.*

THEOREM. *If* H *is of type* $(\boldsymbol{X}, \boldsymbol{X}^*, \boldsymbol{\alpha}, \boldsymbol{U})$ *then*

it satisfies the LAP in U.

PROOF The Privalov-Korn theorem: $R(z) = \int \frac{dE(\lambda)}{\lambda - z} = \int \frac{A(\lambda) d\lambda}{\lambda - z}$.

REMARKS

1) $A(\lambda)$ is called **"density of states"** in physics literature.

2) If H satisfies the LAP , it does not mean $f(H)$ satisfies LAP, even for $f(H) = H^2$.

However, H of **type** $(\boldsymbol{X}, \boldsymbol{X}^*, \boldsymbol{\alpha}, \boldsymbol{U}) \Rightarrow f(H)$ of **type** $(\boldsymbol{X}, \boldsymbol{X}^*, \boldsymbol{\alpha}, \boldsymbol{f}(\boldsymbol{U}))$.

3) The abstract framework can be extended to $H + V$, where V is "short-range" in a suitable sense.

4) Still no good abstract theory for long-range perturbations.

$$\mathcal{L}^{2,s} = \{f / \|f\|^2_{\mathcal{L}^{2,s}} := \int_{\mathbf{R}^n} (1+|x|^2)^s |f(x)|^2 dx < \infty\}, \quad s \in \mathbf{R}$$

$$\mathcal{H}^s = \{\widehat{f} / \|\widehat{f}\|_{\mathcal{H}^s} := \|f\|_{L^{2,s}} < \infty\}.$$

The norm in $\mathcal{L}^2 = \mathcal{H}^0 = \mathcal{L}^{2,0}$ is denoted by $\| \ \|$.

Trace Lemma. *Let $\widehat{f} \in \mathcal{H}^s(\mathbf{R}^n), n \geq 3, \frac{1}{2} < s < \frac{3}{2}$. Then for any $r > 0$,*

$$\Big(\int_{|\xi|=r} |\widehat{f}(\xi)|^2 d\sigma_r \Big)^{1/2} \leq C \cdot Min\,(r^{s-\frac{1}{2}}, 1) \cdot \|\widehat{f}\|_{\mathcal{H}^s},$$

where $d\sigma_r$ is the Lebesgue surface measure on $|\xi| = r$ and the constant C depends only on s, n.

The spectral derivative $A(\lambda)$ is connected to the slowness surfaces

$$\Gamma_j(\lambda) = \left\{\xi \in \mathbf{R}^n \setminus \widetilde{Z},\ \lambda_j(\xi) = \lambda\right\},\ j = 1, \ldots, k(\lambda),$$

by the trace lemma.

Interpretation of $dP_j(\xi)$ in the oscillatory integral solution of $iu_t = L_0 u$

$$u(x,t) = \int_{\mathbf{R}} \sum_{j=1}^{k(\lambda)} \int_{\Gamma_j(\lambda)} e^{i\lambda_j(\xi)t} e^{i\xi x} \widetilde{u_0}(\xi) \mathbf{dP_j}(\xi) d\lambda.$$

Massive Dirac Operator

$$(E_m(\lambda)f,f)=\left(\begin{pmatrix}\chi_{\lambda_+(\xi)\le\lambda}I_2 & 0\\ 0 & I_2\end{pmatrix}\mathcal{G}f,\mathcal{G}f\right). \tag{21}$$

$$\frac{d}{d\lambda}(E_m(\lambda)f,f)=\frac{\lambda}{\sqrt{\lambda^2-m^2}}\int\limits_{|\xi|=\sqrt{\lambda^2-m^2}}|(\mathcal{G}f)_1(\xi)|^2 d\sigma_{\sqrt{\lambda^2-m^2}}, \tag{22}$$

where $(\mathcal{G}f)_1$ is a $2-$vector consisting of the first two components of $\mathcal{G}f$.

$$|(\mathcal{G}f)_1(\xi)| \leq |(\mathcal{G}f)(\xi)| = |\widehat{f}(\xi)|,$$

By Trace Lemma , for any $\lambda \in \mathbf{R}$,

$$\begin{aligned} &< A_m(\lambda)f, f >:= \frac{d}{d\lambda}(E_m(\lambda)f, f) \\ &\leq C \min\left(\frac{\lambda}{\sqrt{\lambda^2 - m^2}}, \lambda(\lambda^2 - m^2)^{s-1}\right)\|\widehat{f}\|^2_{s,0},\ s > \frac{1}{2}, \end{aligned} \tag{23}$$

$<,>= (\mathcal{L}^{2,-s}(\mathbf{R}^3, \mathbf{C}^4), \mathcal{L}^{2,s}(\mathbf{R}^3; \mathbf{C}^4))$ pairing.

$A_m(\lambda) = \frac{d}{d\lambda}(E_m(\lambda))$ uniformly bounded and uniformly Hölder continuous in the uniform operator topology of

$$B(\mathcal{L}^{2,s}(\mathbf{R}^3; \mathbf{C}^4), \mathcal{L}^{2,-s}(\mathbf{R}^3; \mathbf{C}^4)), \quad s > 1.$$

EXAMPLE: KLEIN-GORDON EQUATION

$$B = (-\Delta + m^2)^{1/2} = \mathcal{F}^{-1}(|\xi|^2 + m^2)^{1/2}\mathcal{F}, \quad m \geq 0.$$

$$(E(\lambda)f, g) = \int\limits_{\xi^2+m^2\leq\lambda^2} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi \quad , \quad f, g \in C_0^\infty(\mathbf{R}^n),$$

$$\frac{d}{d\lambda}(E(\lambda)f, g) = \frac{\lambda}{(\lambda^2 - m^2)^{1/2}} \int\limits_{\xi^2+m^2=\lambda^2} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\sigma_{\sqrt{\lambda^2-m^2}}.$$

$d\sigma_r$ is the Lebesgue surface measure on $|\xi| = r$.

$$\frac{d}{d\lambda}(E(\lambda)f,g) \equiv < A(\lambda)f,g > \,=\, \frac{\lambda}{(\lambda^2-m^2)^{1/2}} \int\limits_{\xi^2+m^2=\lambda^2} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\sigma_{\sqrt{\lambda^2-m^2}}.$$

Trace lemma $\Rightarrow \frac{d}{d\lambda}(E(\lambda)f,g)$ is continuous on $\mathcal{L}^{2,s} \times \mathcal{L}^{2,s}, s > \frac{1}{2}$, so it defines $A(\lambda) \in B(\mathcal{L}^{2,s}, \mathcal{L}^{2,-s}), \quad \lambda > m$ and with the same C, s,

$$| < A(\lambda)f,g > | \leq C^2 \cdot \lambda(\lambda^2-m^2)^{-\frac{1}{2}} \cdot \mathrm{Min}\left((\lambda^2-m^2)^{s-\frac{1}{2}}, 1\right) \cdot \|f\|_{\mathcal{L}^{2,s}}\|g\|_{\mathcal{L}^{2,s}}.$$

$\big(<,>$ is the pairing between $\quad L^{2,-s}, L^{2,s}\big)$.

Corollary. B **satisfies the LAP in** $B(\mathcal{L}^{2,s}, \mathcal{L}^{2,-s})$,
$\lambda > m, \quad s > 1/2$.

Remark. The Hölder continuity of $A(\lambda)$ follows by interpolation with large s, where it is Lipschitz by Sobolev imbedding.

THE CLASSICAL WAVE EQUATION

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \tag{24}$$

$$u(x,0) = u_0(x), \quad \partial_t u(x,0) = v_0(x). \quad x \in \mathbf{R}^n. \tag{25}$$

The Morawetz estimate :

$$\int_{\mathbf{R}} \int_{\mathbf{R}^n} |x|^{-3} |u(x,t)|^2 dx dt \leq C(\|\nabla u_0\|_0^2 + \|v_0\|_0^2), \quad n \geq 4, \tag{26}$$

For $n \geq 3$ we gave the estimate

$$\int_{\mathbf{R}} \int_{\mathbf{R}^n} |x|^{-2\alpha-1} |u(x,t)|^2 dx dt \leq C_\alpha(\||\nabla|^\alpha u_0\|_0^2 + \||\nabla|^{\alpha-1} v_0\|_0^2), \quad n \geq 3, \tag{27}$$

for every $\alpha \in (0,1)$.

Global weighted$-L^2$ spacetime estimates for Schrödinger operators well established.

Such estimates for Dirac operators obtained by

Boussaid (2006), D'Ancona-Fanelli (2008).

Illustrated here via Klein-Gordon.

APPLICATION TO KLEIN-GORDON EQUATION

$$(*) \quad u_{tt} - \Delta u + m^2 u = 0 \quad , \quad m \geq 0,$$

with Cauchy data,

$$u(x,0) = u_0(x) \quad , \quad u_t(x,0) = v_0(x), \quad x \in \mathbf{R}^n.$$

The solution $u(x,t)$ can be represented as,

$$u(x,t) = u_+(x,t) + u_-(x,t),$$

$$u_\pm(x,t) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbf{R}^n} e^{\pm it(|\xi|^2+m^2)^{1/2}} e^{i\xi x} \widehat{\varphi}_\pm(\xi) d\xi,$$

where

$$\widehat{\varphi_\pm}(\xi) = \frac{1}{2}[\widehat{u_0}(\xi) \mp i(|\xi|^2 + m^2)^{-\frac{1}{2}} \widehat{v_0}(\xi)],$$

$$(*) \quad u_{tt} - \Delta u + m^2 u = 0 \quad , \quad m \geq 0,$$

$$B = (-\Delta + m^2)^{1/2} = \mathcal{F}^{-1}(|\xi|^2 + m^2)^{1/2}\mathcal{F}, \quad m \geq 0.$$

THEOREM. *Let $m \geq 0, n \geq 3$, and let $u(x,t)$ be the solution to (*). There exists a constant $C > 0$, depending only on n, such that*

$$\int_{\mathbf{R}} \int_{\mathbf{R}^n} (1+|x|^2)^{-1} |u(x,t)|^2 dx dt \leq$$

$$(2\pi)^{1/2}(m^2+1)^{1/2} C[\|u_0\|^2_{\mathcal{L}^2(\mathbf{R}^n)} + \|B^{-1}v_0\|^2_{\mathcal{L}^2(\mathbf{R}^n)}].$$

Remark Note that the operator B^{-1} is bounded (in $\mathcal{L}^2(\mathbf{R}^n)$) only if $m > 0$.

Proof of Theorem $m \geq 0$

$$\widehat{v}(\xi,t) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbf{R}^n} v(x,t) e^{-i\xi x} dx,$$
$$\tilde{v}(x,t) = (2\pi)^{-\frac{1}{2}} \int\limits_{\mathbf{R}} v(x,t) e^{-it\tau} dt.$$

$(\cdot,\cdot), \langle\cdot,\cdot\rangle$ the scalar products in $\mathcal{L}^2(\mathbf{R}^n), \mathcal{L}^2(\mathbf{R}^{n+1})$, respectively.

$B = (-\Delta + m^2)^{1/2}$, a positive self-adjoint operator in $\mathcal{L}^2(\mathbf{R}^n)$ (in fact, $B \geq m$), with domain $D(B) = \mathcal{H}^1(\mathbf{R}^n)$.

$\{E(\lambda)\}$ its associated spectral family ($E(\lambda)$ is the projection operator on $(-\infty, \lambda)$).

$A(\lambda) = \frac{d}{d\lambda} E(\lambda)$ in the weak sense.

$$(i) \qquad \left|[A(\lambda)f,g]\right|^2 \le [A(\lambda)f,f]\cdot[A(\lambda)g,g],$$

$$(ii) \qquad \int_m^\infty [A(\lambda)f,f]d\lambda = \|f\|^2.$$

We may assume (by a standard density argument) that $\widehat{u_0}, \widehat{v_0} \in C_0^\infty(\mathbf{R}^n)$. To estimate $u_+(x,t) = e^{itB}\varphi_+$ use a duality argument. Take $w(x,t) \in C_0^\infty(\mathbf{R}^n)$. Then,

$$\langle u_+, w\rangle = \int_{-\infty}^\infty dt \int_m^\infty < e^{it\lambda}A(\lambda)\varphi_+, w(\cdot,t) > d\lambda$$

$$= \int_m^\infty < A(\lambda)\varphi_+, \int_{-\infty}^\infty e^{-it\lambda}w(\cdot,t)dt > d\lambda$$

$$= (2\pi)^{1/2}\int_m^\infty < A(\lambda)\varphi_+, \tilde{w}(\cdot,\lambda) > d\lambda.$$

Using the Cauchy-Schwartz inequality , and noting (i)-(ii)

$$\langle u_+, w\rangle = (2\pi)^{1/2} \int_m^\infty < A(\lambda)\varphi_+, \tilde{w}(\cdot,\lambda) >$$

$$(i) \qquad \left|[A(\lambda)f, g]\right|^2 \leq [A(\lambda)f, f] \cdot [A(\lambda)g, g],$$

$$(ii) \qquad \int_m^\infty [A(\lambda)f, f] d\lambda = \|f\|^2.$$

$$\left|\langle u_+, w\rangle\right| \leq (2\pi)^{1/2} \left(\int_m^\infty < A(\lambda)\varphi_+, \varphi_+ > d\lambda\right)^{1/2} \cdot \left(\int_m^\infty < A(\lambda)\tilde{w}(\cdot,\lambda), \tilde{w}(\cdot,\lambda) > d\lambda\right)^{1/2}$$

$$= (2\pi)^{1/2}\|\varphi_+\| \cdot \left(\int_m^\infty < A(\lambda)\tilde{w}(\cdot,\lambda), \tilde{w}(\cdot,\lambda) > d\lambda\right)^{1/2}.$$

$$| < A(\lambda)f, g > |$$
$$\leq C^2 \cdot \lambda(\lambda^2 - m^2)^{-\frac{1}{2}} \cdot \mathrm{Min}\left((\lambda^2 - m^2)^{s-\frac{1}{2}}, 1\right) \cdot \|f\|_{\mathcal{L}^{2,s}}\|g\|_{\mathcal{L}^{2,s}}.$$

So that

$$\left|\langle u_+, w\rangle\right| \leq (2\pi)^{1/2}\|\varphi_+\| \cdot \left(\int_m^\infty < A(\lambda)\tilde{w}(\cdot,\lambda), \tilde{w}(\cdot,\lambda) > d\lambda\right)^{1/2}$$

$$\leq (2\pi)^{1/2}\|\varphi_+\| C^2 \cdot \int_m^\infty \lambda(\lambda^2 - m^2)^{-\frac{1}{2}} \operatorname{Min}\left((\lambda^2 - m^2)^{s-\frac{1}{2}}, 1\right) \cdot \|\tilde{w}(\cdot,\lambda)\|^2_{\mathcal{L}^{2,s}} d\lambda.$$

Take $s = 1$ and use the Plancherel theorem

$$\left|\langle u_+, w\rangle\right| \leq (2\pi)^{1/2} C(m^2+1)^{1/2}\|\varphi_+\| \left(\int_{\mathbf{R}} \|w(\cdot,t)\|^2_{\mathcal{L}^{2,1}} dt\right)^{1/2}.$$

Let $f(x,t) \in \mathcal{L}^2(\mathbf{R}^{n+1})$, $w(x,t) = (1+|x|^2)^{-\frac{1}{2}} f(x,t)$,

$$\int_{\mathbf{R}} \|w(\cdot,t)\|^2_{\mathcal{L}^{2,1}} dt = \|f\|^2_{\mathcal{L}^2(\mathbf{R}^{n+1})},$$

$$\left|\langle (1+|x|^2)^{-\frac{1}{2}} u_+, f\rangle\right| \leq (2\pi)^{1/2} C \cdot (m^2+1)^{1/2} \|\varphi_+\| \cdot \|f\|_{\mathcal{L}^2(\mathbf{R}^{n+1})}.$$ **Q.E.D**

BACKGROUND for LAP–Limiting Absorption Principle

- Eidus, Ikebe, Jäger, Kato, Kuroda (1960's),
- Agmon, Boutet de Monvel-Berthier, Hörmander, Mochizuki, Mourre, Saito (1970's).
- "Low-order" perurbations of constant-coefficient elliptic operators.
- $N-$body (Perry, Sigal, Simon).
- LAP–periodic case (divergence-form)–Murata and Tsuchida (2006).
  (1-D case by Weidmann in 1970's).

Physical models (second-order elliptic operators with "top order" variable coefficients):

- "acoustic" or "elastic" propagators in nonuniform media. ("layered media"− beginning in the 1980's)

  MBA, Croc, Dermenjian, Guillot, Kadowaki, Perthame, Vega and Weder.

- Divergence-form with smooth coefficients (**semiclassical**),

  Melrose, Ralston, Robert, Sjöstrand (away from thresholds)

- Bony, Bouclet, Häfner (near thresholds, 2008).

- Divergence-form non-smooth coefficients,

  MBA, (2008-2010)