Representations of constitutions under incomplete information

Bezalel Peleg **Shmuel Zamir**

Center for the Study of Rationality
The Hebrew University, Jerusalem, Israel.

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The framework

- A set of players (*Society*), each of which has to choose a strategy that best serves his goal.
- The strategies chosen by all players determine the resulting *social state*.
- There is incomplete information among the players regarding the preference relations of each player on the set of possible social states.
- The constitution and the power structure are given by an *effectivity function*.
- A *decision scheme* assigns to any profile of declared preference relations, a probability distribution on the set of social states.

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Given a constitution (effectivity function), is there a decision scheme representing the constitution such that the induced incomplete information game has a *Bayesian-Nash-Equilibrium* (BNE) in pure strategies ?

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Example (After Gibbard (1974))

- The society : $N = \{1,2\}$.
- Each individual has two shirts, *white* (w) and *blue* (b), and has to wear exactly one of them.
- The set of *social states* is $A = \{ww, wb, bw, bb\}$.
- Each individual is free to choose the color of his/her shirt, then the *effectivity function* ,E, is:

$$E(\{1\}) = \{\{ww, wb\}^+, \{bw, bb\}^+\},$$

$$E(\{2\}) = \{\{ww, bw\}^+, \{wb, bb\}^+\},$$

  and $E(N) = P_0(A)$.
- Player 1 has two types: $T^1 = \{1_c, 1_n\}$ and player 2 has one type: $T^2 = \{2\}$.
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Example (continued.)

- Each type of a player has a *von-Neumann Morgenstern* utility function.
- Each player declares a *preference ordering* on the social states.
- Given the profile of declared preferences, a *decision scheme* chooses the social state (randomly).

Question:

Is there a pure strategy Bayes-Nash equilibrium of this game ?

Answer:

Yes, and we shall later exhibit one.

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The model

- Let $N = \{1, 2, \ldots, n\}$ be the set of *players* (voters).
- Let $A = \{a_1, a_2, \ldots, a_m\}$ be the set of *alternatives* (social states), $m \geq 2$.
- For a finite set D let $P(D) = \{D' | D' \subseteq D\}$ and $P_0(D) = P(D) \setminus \{\emptyset\}$.

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Effectivity function

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An *effectivity function* (EF) is a function $E : P(N) \to P(P_0(A))$ satisfying:

(i) $A \in E(S)$ for all $S \in P_0(N)$.

(ii) $E(\emptyset) = \emptyset$.

(iii) $E(N) = P_0(A)$.

Interpretation:

$B \in E(S)$ means that the coalition S has the legal right to see the final outcome in the set B.

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Properties of effectivity functions

- An effectivity function E is **monotonic** if:

$$[S \in P_0(N),\ S' \supseteq S, \text{ and } B' \supseteq B,\ \ B \in E(S)] \Rightarrow B' \in E(S').$$

- An effectivity function E is **superadditive** if:

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Social Choice Correspondence

- A *social choice correspondence* (SCC) is a function

$$H : W^N \to P_0(A),$$

where W is the set of *weak* (i.e., complete and transitive) orderings of A.
- Let $H : W^N \to P_0(A)$ be an SCC. A coalition $S \in P_0(N)$ is *effective* for $B \in P_0(A)$ if there exists $Q^S \in W^S$ such that for all $R^{N \setminus S} \in W^{N \setminus S}$, $H(Q^S, R^{N \setminus S}) \subseteq B$.
- The effectivity function of H, denoted by E^H, is given by $E^H(\emptyset) = \emptyset$ and for $S \in P_0(N)$,

$$E^H(S) = \{B \in P_0(A) | S \text{ is effective for } B\}.$$

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Definition

A social choice correspondence H is a *representation* of the effectivity function E if $E^H = E$.

Definition

- A *decision scheme* (DS) is a function $d : W^N \to \Delta(A)$.
- The Social Choice Correspondence associated with the decision scheme d, denoted by H_d, is defined by:

$$H_d(R^N) = \{x \in A | d(x, R^N) > 0\}.$$

- A decision scheme d is said to be a *representation* of the effectivity function E if $E^{H_d} = E$.

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The uniform core

For any weak preference relation on A, $R \in W$.

- Denote the strict preference by P.
- Denote the indifference relation by I, that is, xIy holds for $x, y \in A$ if xRy and yRx.
- Given a vector of preference relations R^N and a coalition $S \subseteq N$, we write $BP^S A \setminus B$ if $xP^i y$ for all $x \in B$, $y \in A \setminus B$ and $i \in S$.

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Definition

Let E be an effectivity function, $R^N \in W^N$ a profile of preference relations on A, and $S \in P_0(N)$ a non empty coalition.

- A set of alternatives $B \in E(S)$ *uniformly dominates* $A \setminus B$ via the coalition S at R^N if $BP^S A \setminus B$.
- In that case, for any alternative $x \in A \setminus B$ we also say that B *uniformly dominates* x via the coalition S.
- The *uniform core* of E and R^N, denoted by $C_{uf}(E, R^N)$ (or shortly $C_{uf}(R^N)$), is the set of all alternatives in A that are not uniformly dominated at R^N.

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Given an effectivity function E and a vector of preference relations R^N,

- An alternative $x \in A$ is *dominated* by $B \subseteq A$, $x \notin B$ via the coalition $S \in P_0(N)$, if $B \in E(S)$ and $B \; P^S \{x\}$.
- An alternative $x \in A$ is *not dominated* at (E, R^N) if there is no pair (S, B) of a coalition $S \in P_0(N)$ and a set of states B not containing x that dominates x via the coalition S.
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It follows from the definitions that the core is a subset of the uniform core.

Example (Based on the Condorcet Paradox)

Let $N = \{1,2,3\}$, $A = \{x,y,z\}$ and the effectivity function E given by:

$$E(S) = \begin{cases} P_0(A) & \text{if } |S| > 1 \\ \{A\} & \text{if } |S| = 1 \end{cases}$$

For the vector of preference relations:

$$R^N = \begin{array}{ccc} \underline{1} & \underline{2} & \underline{3} \\ x & z & y \\ y & x & z \\ z & y & x \end{array}$$

At (E,R^N) every alternative is dominated but not uniformly dominated. Hence, $C(E,R^N) = \emptyset$ while $C_{uf}(E,R^N) = A$.

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Let E be a monotonic and superadditive EF and let $R^N \in W^N$. Then the uniform core $C_{uf}(E, R^N)$ is non-empty.

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Example (Continued.)

- By Keiding and Peleg's theorem, $C_{uf}(E, \cdot)$ is a representation of E by a social choice correspondence.
- Convert this into a representation by a decision scheme by assigning the uniform distribution on $C_{uf}(E, R^N)$.
- For example, if $R^1 = (ww, wb, bw, bb)$ and
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- $C_{uf}(E, R^N) = \{ww, wb\}$, and hence,
- A decision scheme representing E satisfies:

$$d(ww, R^N) = d(wb, R^N) = 1/2$$

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Representation under complete information

- Given a society $N = \{1, 2, \ldots, n\}$,
- A set of social states $A = \{a_1, a_2, \ldots, a_m\}$,
- An effectivity function E,
- von-Neumann Morgenstern utility functions, $u^1, \ldots, u^n$, on $\Delta(A)$.

Theorem

Given a monotonic and superadditive effectivity function E, and vNM utility functions $(u^1, \ldots, u^n)$, then there is a decision scheme $d : W^N \to \Delta(A)$ such that,

- *The decision scheme d is a representation of the effectivity function E.*
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Incomplete information

An *information structure* (IS) is a $2n$-tuple $\mathscr{I} = (T^1, \ldots, T^n; p^1, \ldots, p^n)$ where T^i is the (finite) set of types of player $i \in N$, and for all $i \in N$ and $t^i \in T^i$, $p^i(\cdot|t^i)$ is a probability distribution on $\times_{j \neq i} T^j$.

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It is not assumed that the beliefs of the players $p^i(\cdot|t^i)$ are derived from a common prior that is, the game Γ is not necessarily a Harsanyi game.

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Definition

1. A generalized decision scheme (GDS) is a function $d : W^N \times T \to \Delta(A)$.
2. A strategy of player i (with respect to a GDS) is a pair (s^i, π^i) where $s^i : T^i \to W$ and $\pi^i : T^i \to \; T^i$.
   Denote by S^i the set of all such mappings and let $S = S^1 \times, \cdots, \times S^n)$.
   Equivalently, a strategy of player i is a mapping $\tilde{s}^i : T^i \to W \times T^i$.
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The Bayesian game

An information structure $\mathscr{I} = (T^1, \ldots, T^n; p^1, \ldots, p^n)$,
A vector of utility functions $(u^i)_{i \in N}$ where $u^i : A \times T \to \mathbb{R}$,
A generalized decision scheme $d : W^N \times T \to \Delta(A)$,defines a game of incomplete information:

$$\Gamma_d = (N; W, \ldots, W; \mathscr{I}; u^1, \ldots, u^n; d).$$

- The set of actions of player $i \in N$ of any possible type t^i is $W \times T^i$. The set of pure strategies of player i is $\tilde{S}^i$.
- The payoff to player t^i when the players play the pure strategy vector $\tilde{s} = (\tilde{s}^1, \ldots, \tilde{s}^n) \in \tilde{S}$ is $U^i(\tilde{s}|t^i)$ given by:

$$U^i_d(\tilde{s}|t^i) = \sum_{t^{-i} \in T^{-i}} p^i(t^{-i}|t^i) \sum_{x \in A} u^i(x; t) d(x; \tilde{s}^1(t^1), \ldots, \tilde{s}^n(t^n)).$$

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The Bayesian game

An information structure $\mathscr{I} = (T^1, \dots, T^n; p^1, \dots, p^n)$,
A vector of utility functions $(u^i)_{i \in N}$ where $u^i : A \times T \to \mathbb{R}$,
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Bayes Nash quilibrium

Definition

An n-tuple of strategies $\tilde{s}$ is a *Bayesian Nash equilibrium* (BNE) if for all $i \in N$, all $t^i \in T^i$ and all $(R^i, \hat{t}^i) \in W \times T^i$,

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Main result

Theorem

Let $E : P(N) \to P(P_0(A))$ be a monotonic and superadditive EF. Let $\mathscr{I} = (T^1, \ldots, T^n; p^1, \ldots, p^n)$ be an IS, and let $(u^1, \ldots, u^n)$, $u^i : A \times T \to \mathbb{R}$, be a vector of vNM utilities for the players. Then E has a representation by a generalized decision scheme $d : W^N \times T \to \Delta(A)$ such that the game $\Gamma_d = (N; W, \ldots, W; \mathscr{I}; (u^i)_{i \in N}; d)$ has a BNE in pure strategies.

Outline of the proof

Define the generalized decision scheme $d_1 : W^N \times T \to \Delta(A)$ by

$$d_1(R^N, t) = d_{uf}(R^N), \quad \forall (R^N, t) \in W^N \times T.$$

Consider the ex-ante game:

$$G_{d_1} = (N; S^1, \dots, S^n; h^1, \dots, h^n; d_1)$$

in which the payoff functions are:

$$h^i(s^1, \dots, s^n) = \sum_{t \in T} p^i(t) \sum_{x \in A} u^i(x, t) d_1(x; s(t)),$$

Note that in this game, the strategy sets are S^i rather than $\tilde{S}^i$ since $d_1(R^N, t)$ does not depend on t.

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Outline of the proof cont.

Let $(q(s))_{s\in S}$ be a correlated equilibrium (CE) of the game G_{d_1}. The equilibrium conditions are:

$$\sum_{s\in S} q(s)h^i(s) \geq \sum_{s\in S} q(s)h^i(s^{-i},\delta(s^i)),$$

which holds for all $i \in N$ and for all $\delta : S^i \to S^i$.
From this (by appropriate choice of δ) that:

$$\sum_{s\in S} q(s)U^i_{d_1}(s|t^i) \geq \sum_{s\in S} q(s)U^i_{d_1}(s^{-i},R^i|t^i),$$

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Define now a generalized decision scheme d by:

- $d(x; I^N, t) = \sum_{s\in S} q(s) d_1(x; s(t)),\ \forall x \in A,\ \forall t \in T.$
- $d(x; (I^{-i}, R^i), t) = \sum_{s\in S} q(s) d_1(x; s^{-i}(t^{-i}), R^i),$ for all $i \in N$, $R^i \in W$, $t \in T$, and $x \in A$.
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Outline of the proof cont.

Claim:

- This generalized decision scheme d is a representation of the effectivity function E.
  Basically because the uniform core d_{uf} is a representation of E (By Peleg and Keiding).
- The vector $\tilde{s}$ in which $\tilde{s}^i(t^i) = (I, t^i)$,
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Outline of the proof cont.

Deviation of player i of type t^i:

- Deviate from (I, t^i) to (R^i, t^i) where $R^i \neq I$.
  This is not profitable by the CE inequality:

$$\sum_{s \in S} q(s) U^i_{d_1}(s|t^i) \geq \sum_{s \in S} q(s) U^i_{d_1}(s^{-i}, R^i|t^i).$$

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$$\sum_{s\in S} q(s) U^i_{d_1}(s|t^i) \geq \sum_{s\in S} q(s) U^i_{d_1}(s^{-i}, s^i(\tilde{t}^i)|t^i).$$

- Deviate from (I, t^i) to $(R^i, \tilde{t}^i)$ where $R^i \neq I$ and $\tilde{t}^i \neq t^i$.
  This is not profitable as in the first case since:

$$d(x; (I^{-i}, R^i), (t^{-i}, \tilde{t}^i)) = d(x; (I^{-i}, R^i), t).$$

Definition

A preference relation $R \in W$ is *dichotomous* if there exist $B_1, B_2 \in P(A)$ such that $B_1 \neq \emptyset, B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = A$ such that xIy if $x, y \in B_i,\ i = 1,2$ and xPy if $x \in B_1,\ y \in B_2$. The set of all dichotomous preferences in W is denoted by W_δ.

Since a dichotomous preference relation is determined by a single subset $B \subseteq A$, the set of most preferred alternatives, we use the notation $R = \dfrac{B}{A \setminus B}$ for a generic dichotomous preference relation.

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Theorem

Let $E : P(N) \to P(P_0(A))$ be a monotonic and superadditive EF. Let $\mathscr{I} = (T^1, \ldots, T^n; p^1, \ldots, p^n)$ be an IS, and let $(u^1, \ldots, u^n)$ be a vector of utilities for the players. Then E has a representation by a generalized decision scheme $d : W_\delta^N \times T \to \Delta(A)$ such that the game $\Gamma = (N; W_\delta, \ldots, W_\delta; \mathscr{I}; (u^i)_{i \in N}; d)$ has a (pure strategy) BNE.

Example (back to Gibbard's example.)

Recall the information structure $\mathscr{I} = (T^1, p^2)$ where $T^1 = \{1_c, 1_n\}$ and $p^2(1_c) = p^2(1_n) = 1/2$. (player 2 has one type).

- $u^1(ww, 1_c) = u^1(bb, 1_c) = 1$ and $u^1(bw, 1_c) = u^1(wb, 1_c) = 0$ (1_c likes 'conformity').
- $u^1(a, 1_n) = u^1(a, 1_c) - 1$ for all $a \in A$ (1_n also likes 'conformity' but at a lower level of utilities).
- $u^2(a, 1_c) = -u^1(a, 1_c)$ and $u^2(a, 1_n) = -u^1(a, 1_n)$ for all $a \in A$ (the utility of player 2 is 'opposed' to that of player 1 whatever his type is).

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Example (continued.)

Consider the Bayesian game in which the players submit dichotomous preferences:

$$\Gamma_{\delta} = (N; W_{\delta}, W_{\delta}; \mathscr{I}; u^1, u^2; d_{uf})$$

In the strategic form of this game:

- Player 2 has 16 pure strategies (indexed by the subsets of A).
- Player 1 has 16^2 pure strategies.

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Example (The reduced game.)

Figure The restriction of the game Γ_δ.

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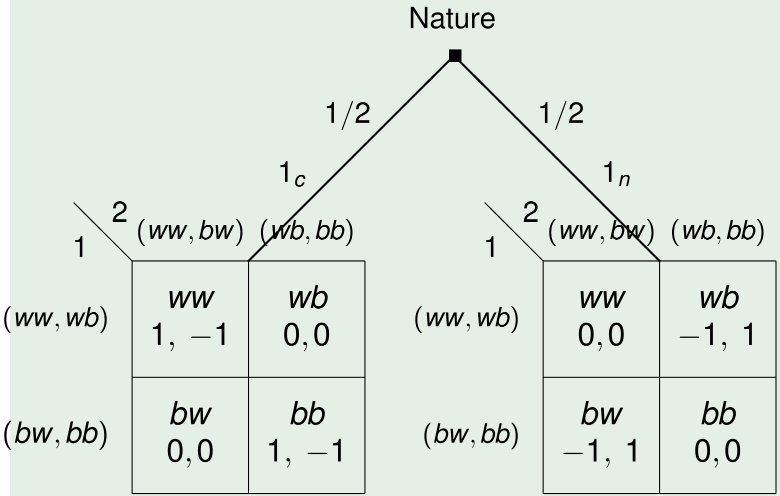

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Example (The reduced game cont.)

Here, the pure strategies are denoted by the upper-set in the dichotomous preference that is: $(ww, wb) \equiv \frac{ww, wb}{bw, bb}$ etc.

- A BNE of this restricted game is (s^1, s^2) where

$$s^1(1_c) = \frac{ww, wb}{bw, bb}, \quad s^1(1_n) = \frac{bw, bb}{bw, bb},$$

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Example (Cont.)

It turns out that in this simple example the BNE can be obtained from the game induced by a decision scheme (rather than a GDS):

- Define a decision scheme d that satisfies:

$$d(a;\hat{I}^N) = \frac{1}{4} \text{ for all } a \in A$$

  and

$$d(a;\hat{I}^{-i}, R^i) = \frac{1}{4} \text{ for all } a \in A \text{ and } i \in N$$

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Example (Two-person 2×2 games)

- Consider the game $G = (\{1,2\}; C^1, C^2; u^1, u^2)$ in which:
  - The players are 1 and 2.
  - The pure strategy sets are C^1 and C^2 respectively, satisfying $|C^i| = 2, i = 1,2$.
  - The utility functions are $u^i : C^1 \times C^2 \to \mathbb{R}$, $i = 1,2$.
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- The corresponding payoffs to a correlated strategy p is

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$$v^1 = \max_{\sigma^1 \in \Delta(C^1)} \min_{c^2 \in C^2} u^1(\sigma^1, c^2)$$

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Example (Two-person 2×2 games, cont.)

- A *correlated strategy* is a probability distribution p on $C = C^1 \times C^2$.
- The corresponding payoffs to a correlated strategy p is

$$u^i(p) = \sum_{c^1 \in C^1} \sum_{c^2 \in C^2} p(c) u^i(c^1, c^2), \ i = 1, 2.$$

- The security levels (in mixed strategies) of player 1 and player 2 are:

$$v^1 = \max_{\sigma^1 \in \Delta(C^1)} \min_{c^2 \in C^2} u^1(\sigma^1, c^2)$$

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Definition

A decision scheme $d : W_\delta^N \to \Delta(C)$ is *individually rational* (IR) (w.r.t. the game G) if each player $i \in N$ has a strategy $V^i \in W_\delta$ such that $u^i(d(V^i, R^{N\setminus\{i\}})) \geq v^i$ for all $R^{N\setminus\{i\}} \in W_\delta^{N\setminus\{i\}}$.

Proposition

Let $p \in \Delta(C)$. Then $u^i(p) \geq v^i$ for $i = 1, 2$, if and only if there exists a decision scheme $d : W_\delta^N \to \Delta(C)$ such that,

(i) *The decision scheme d is a representation of E^G, the EF of G.*

(ii) *The game $\Gamma = (N; W_\delta, W_\delta; u^1, u^2; d)$ has a Nash equilibrium $(R^1, R^2) \in W_\delta^N$ such that $d(\cdot\,, (R^1, R^2)) = p$.*

(iii) *The decision scheme d is individually rational.*

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Consider the prisoners' dilemma given in the following game:

$G =$

1 \ 2	c	d
C	$2, 2$	$-6, 3$
D	$3, -6$	$0, 0$

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Example (The prisoners' dilemma, Cont.)

Here $v^1 = v^2 = 0$ and the set of NE payoffs is given in Figure 1:

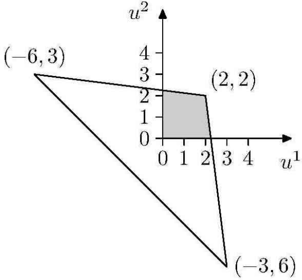

Figure 1: The NE payoffs in the prisoners' dilemma .

Recall that $(0,0)$ is the unique correlated equilibrium payoff.

Example (The prisoners' dilemma, Cont.)

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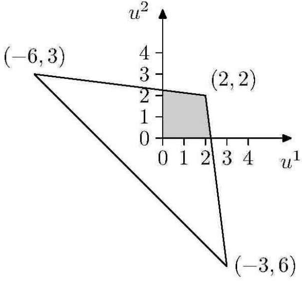

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