Optimal Hardy-type inequalities and the spectrum of the corresponding operator

Yehuda Pinchover

Department of Mathematics, Technion
32000 Haifa, ISRAEL

"Applications of Analysis: Game Theory, Spectral Theory and Beyond"

A workshop in honor of Yakar Kannai's 70th birthday

The Weizmann Institute of Science, Israel
December 24–27, 2012

Joint work with Baptiste Devyver and Martin Fraas

The setting

Consider a second-order elliptic operator P with real coefficients in divergence form

$$Pu := -\frac{1}{m(x)}\mathrm{div}\left[m(x)\Big(A(x)\nabla u + \tilde{b}(x)u\Big)\right] + b(x)\cdot\nabla u + c(x)u,$$

which is defined in a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ (or more generally, on a smooth noncompact manifold Ω of dimension n, $\mathrm{d}\nu := m\,\mathrm{d}x$).

The setting

Consider a second-order elliptic operator P with real coefficients in divergence form

$$Pu := -\frac{1}{m(x)}\operatorname{div}\left[m(x)\Big(A(x)\nabla u + \tilde{b}(x)u\Big)\right] + b(x)\cdot\nabla u + c(x)u,$$

which is defined in a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ (or more generally, on a smooth noncompact manifold Ω of dimension n, $\mathrm{d}\nu := m\,\mathrm{d}x$).

Prototype equations are given by the Laplace-Beltrami operator $-\Delta$ and the Schrödinger operator $-\Delta + V(x)$.

Agmon's problem

Problem (Agmon (1982))

Given a symmetric elliptic operator P in $\mathbb{R}^n$, find a continuous, nonnegative function W which is 'as large as possible' such that for some neighborhood of infinity Ω_R the following inequality holds

$$\int_{\Omega_R} P\varphi\,\overline{\varphi}\,\mathrm{d}\nu \geq \int_{\Omega_R} W(x)|\varphi|^2\,\mathrm{d}\nu \qquad \forall\,\varphi \in C_0^\infty(\Omega_R).$$

Agmon's problem

Problem (Agmon (1982))

Given a symmetric elliptic operator P in $\mathbb{R}^n$, find a continuous, nonnegative function W which is 'as large as possible' such that for some neighborhood of infinity Ω_R the following inequality holds

$$\int_{\Omega_R} P\varphi\,\overline{\varphi}\,\mathrm{d}\nu \geq \int_{\Omega_R} W(x)|\varphi|^2\,\mathrm{d}\nu \qquad \forall\,\varphi \in C_0^\infty(\Omega_R).$$

Agmon used such W to measure the decay of solutions of the equation $Pu = \lambda u$ in $\mathbb{R}^n$ via the celebrated Agmon's metric

$$\mathrm{d}s^2 := W(x)\sum_{i,j=1}^{n} a_{ij}(x)\,\mathrm{d}x_i\,\mathrm{d}x_j, \quad \text{where } \left[a_{ij}\right] := A^{-1}.$$

The decay is given in terms of W and a function h satisfying

$$|\nabla h(x)|_A^2 < W(x) \qquad \text{a.e. } \Omega.$$

Features of Hardy inequality $W(x) = \frac{C_H}{|x|^2}$

Let $\Omega^\star := \mathbb{R}^n \setminus \{0\}$. Consider the celebrated Hardy inequality

$$\int_{\Omega^\star} |\nabla\varphi|^2 \,\mathrm{d}x \geq \lambda \int_{\Omega^\star} \frac{C_H}{|x|^2} |\varphi(x)|^2 \,\mathrm{d}x \qquad \forall\varphi \in C_0^\infty(\Omega^\star), \tag{0.1}$$

where $\lambda \leq 1$ and $C_H := \left(\frac{n-2}{2}\right)^2$.

Features of Hardy inequality $W(x) = \frac{C_H}{|x|^2}$

Let $\Omega^\star := \mathbb{R}^n \setminus \{0\}$. Consider the celebrated Hardy inequality

$$\int_{\Omega^\star} |\nabla\varphi|^2 \,\mathrm{d}x \geq \lambda \int_{\Omega^\star} \frac{C_H}{|x|^2} |\varphi(x)|^2 \,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega^\star), \tag{0.1}$$

where $\lambda \leq 1$ and $C_H := \left(\frac{n-2}{2}\right)^2$. It has the following important features:

(a) $P = -\Delta - \frac{C_H}{|x|^2}$ is *critical* in $\Omega^\star$, i.e., for any $V(x) \gneqq \frac{C_H}{|x|^2}$ the inequality

$$\int_{\Omega^\star} |\nabla\varphi|^2 \,\mathrm{d}x \geq \int_{\Omega^\star} V(x) |\varphi(x)|^2 \,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega^\star)$$

is not valid. In particular, $\lambda = 1$ is the best constant for (0.1).

Features of Hardy inequality $W(x) = \frac{C_H}{|x|^2}$

Let $\Omega^\star := \mathbb{R}^n \setminus \{0\}$. Consider the celebrated Hardy inequality

$$\int_{\Omega^\star} |\nabla\varphi|^2 \,\mathrm{d}x \geq \lambda \int_{\Omega^\star} \frac{C_H}{|x|^2} |\varphi(x)|^2 \,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega^\star), \tag{0.1}$$

where $\lambda \leq 1$ and $C_H := \left(\frac{n-2}{2}\right)^2$. It has the following important features:

(a) $P = -\Delta - \frac{C_H}{|x|^2}$ is *critical* in $\Omega^\star$, i.e., for any $V(x) \gneqq \frac{C_H}{|x|^2}$ the inequality

$$\int_{\Omega^\star} |\nabla\varphi|^2 \,\mathrm{d}x \geq \int_{\Omega^\star} V(x) |\varphi(x)|^2 \,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega^\star)$$

is not valid. In particular, $\lambda = 1$ is the best constant for (0.1).

(b) $\lambda = 1$ is also optimal for test functions supported in any fixed neighborhood of either 0 or ∞.

Features of Hardy inequality $W(x) = \frac{C_H}{|x|^2}$

Let $\Omega^\star := \mathbb{R}^n \setminus \{0\}$. Consider the celebrated Hardy inequality

$$\int_{\Omega^\star} |\nabla\varphi|^2 \,\mathrm{d}x \geq \lambda \int_{\Omega^\star} \frac{C_H}{|x|^2} |\varphi(x)|^2 \,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega^\star), \tag{0.1}$$

where $\lambda \leq 1$ and $C_H := \left(\frac{n-2}{2}\right)^2$. It has the following important features:

(a) $P = -\Delta - \frac{C_H}{|x|^2}$ is *critical* in $\Omega^\star$, i.e., for any $V(x) \gneqq \frac{C_H}{|x|^2}$ the inequality

$$\int_{\Omega^\star} |\nabla\varphi|^2 \,\mathrm{d}x \geq \int_{\Omega^\star} V(x) |\varphi(x)|^2 \,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega^\star)$$

is not valid. In particular, $\lambda = 1$ is the best constant for (0.1).

(b) $\lambda = 1$ is also optimal for test functions supported in any fixed neighborhood of either 0 or ∞.

(c) The corresponding Rayleigh-Ritz variational problem

$$\inf_{\varphi \in \mathcal{D}^{1,2}(\Omega^\star)} \left\{ \frac{\int_{\Omega^\star} |\nabla\varphi|^2 \,\mathrm{d}x}{\int_{\Omega^\star} \frac{C_H}{|x|^2} |\varphi(x)|^2 \,\mathrm{d}x} \right\}$$

admits no minimizer.

Criticality theory

Definition

Let P be a general, second-order elliptic operator on a domain $\Omega \subset \mathbb{R}^n$ (or on a noncompact manifold Ω), $n \geq 2$.

- P *is nonnegative* ($P \geq 0$) *in* Ω if the equation $Pu = 0$ in Ω admits a global positive (super)solution.

Criticality theory

Definition

Let P be a general, second-order elliptic operator on a domain $\Omega \subset \mathbb{R}^n$ (or on a noncompact manifold Ω), $n \geq 2$.

- P *is nonnegative* ($P \geq 0$) *in* Ω if the equation $Pu = 0$ in Ω admits a global positive (super)solution.
- $P \geq 0$ in Ω is said to be *critical in* Ω if $P - W \not\geq 0$ in Ω for any $W \gneqq 0$. Otherwise, P is *subcritical in* Ω.

Criticality theory

Definition

Let P be a general, second-order elliptic operator on a domain $\Omega \subset \mathbb{R}^n$ (or on a noncompact manifold Ω), $n \geq 2$.

- P *is nonnegative* ($P \geq 0$) *in* Ω if the equation $Pu = 0$ in Ω admits a global positive (super)solution.
- $P \geq 0$ in Ω is said to be *critical in* Ω if $P - W \not\geq 0$ in Ω for any $W \gneqq 0$. Otherwise, P is *subcritical in* Ω.
- If $P \not\geq 0$ in Ω, then P is *supercritical in* Ω.

Criticality theory

Remarks

1. In the symmetric case, $P \geq 0$ iff the quadratic form associated to P is nonnegative on $C_0^\infty(\Omega)$ (i.e. $\int_\Omega P\varphi\overline{\varphi}\,\mathrm{d}\nu \geq 0 \quad \forall\,\varphi \in C_0^\infty(\Omega)$).

Criticality theory

Remarks

1. In the symmetric case, $P \geq 0$ iff the quadratic form associated to P is nonnegative on $C_0^\infty(\Omega)$ (i.e. $\int_\Omega P\varphi\overline{\varphi}\,\mathrm{d}\nu \geq 0 \quad \forall\,\varphi \in C_0^\infty(\Omega)$).
2. P is subcritical in Ω iff it admits a positive minimal Green function $G_P^\Omega(x,y)$ in Ω.

Criticality theory

Remarks

1. In the symmetric case, $P \geq 0$ iff the quadratic form associated to P is nonnegative on $C_0^\infty(\Omega)$ (i.e. $\int_\Omega P\varphi\overline{\varphi}\,\mathrm{d}\nu \geq 0 \quad \forall\,\varphi \in C_0^\infty(\Omega)$).
2. P is subcritical in Ω iff it admits a positive minimal Green function $G_P^\Omega(x,y)$ in Ω.
3. P is subcritical in Ω iff it admits a positive supersolution u in Ω which is not a solution. So, $P - W \geq 0$, where $W := Pu/u \gneqq 0$.

Criticality theory

Remarks

1. In the symmetric case, $P \geq 0$ iff the quadratic form associated to P is nonnegative on $C_0^\infty(\Omega)$ (i.e. $\int_\Omega P\varphi\overline{\varphi}\,\mathrm{d}\nu \geq 0 \quad \forall\,\varphi \in C_0^\infty(\Omega)$).
2. P is subcritical in Ω iff it admits a positive minimal Green function $G_P^\Omega(x,y)$ in Ω.
3. P is subcritical in Ω iff it admits a positive supersolution u in Ω which is not a solution. So, $P - W \geq 0$, where $W := Pu/u \gneqq 0$.
4. If P is critical in Ω, then the equation $Pu = 0$ admits a unique positive solution ψ in Ω, called the *Agmon's ground state of P in Ω*.

Optimal Hardy-weight: Features (a)–(c)

We assume that $x_0 = 0 \in \Omega$, and denote $\Omega^* := \Omega \setminus \{0\}$.

Definition

Let P be subcritical in Ω. We say that $W \geq 0$ is an *optimal Hardy-weight for P in Ω^** if $P - W$ has the following properties:

Optimal Hardy-weight: Features (a)–(c)

We assume that $x_0 = 0 \in \Omega$, and denote $\Omega^* := \Omega \setminus \{0\}$.

Definition

Let P be subcritical in Ω. We say that $W \geq 0$ is an *optimal Hardy-weight for P in Ω^** if $P - W$ has the following properties:

(a) $P - W$ is critical in Ω^*. In particular,

$$\max \left\{ \lambda \in \mathbb{R} \mid P - \lambda W \geq 0 \text{ in } \Omega^* \right\} = 1.$$

Optimal Hardy-weight: Features (a)–(c)

We assume that $x_0 = 0 \in \Omega$, and denote $\Omega^* := \Omega \setminus \{0\}$.

Definition

Let P be subcritical in Ω. We say that $W \geq 0$ is an *optimal Hardy-weight for P in Ω^** if $P - W$ has the following properties:

(a) $P - W$ is critical in Ω^*. In particular,

$$\max\left\{\lambda \in \mathbb{R} \mid P - \lambda W \geq 0 \text{ in } \Omega^*\right\} = 1.$$

(b) $\lambda = 1$ is also optimal for the inequality $P - \lambda W \geq 0$ in any fixed punctured neighborhood of either 0 or ∞ in Ω.

Optimal Hardy-weight: Features (a)–(c)

We assume that $x_0 = 0 \in \Omega$, and denote $\Omega^* := \Omega \setminus \{0\}$.

Definition

Let P be subcritical in Ω. We say that $W \geq 0$ is an *optimal Hardy-weight for P in Ω^** if $P - W$ has the following properties:

(a) $P - W$ is critical in Ω^*. In particular,

$$\max\left\{\lambda \in \mathbb{R} \mid P - \lambda W \geq 0 \text{ in } \Omega^*\right\} = 1.$$

(b) $\lambda = 1$ is also optimal for the inequality $P - \lambda W \geq 0$ in any fixed punctured neighborhood of either 0 or ∞ in Ω.

(c) Denote the ground states of $P - W$ and $P^\star - W$ in Ω^* by ψ and $\psi^\star$. Then $\psi\psi^\star$ is not $W\mathrm{d}\nu$-integrable in any fixed neighborhood of either 0 or ∞ (P is said to be *null-critical in $\Omega^\star$*).

Optimal Hardy-weight: Features (a)–(c)

We assume that $x_0 = 0 \in \Omega$, and denote $\Omega^* := \Omega \setminus \{0\}$.

Definition

Let P be subcritical in Ω. We say that $W \geq 0$ is an *optimal Hardy-weight for P in Ω^** if $P - W$ has the following properties:

(a) $P - W$ is critical in Ω^*. In particular,

$$\max\left\{\lambda \in \mathbb{R} \mid P - \lambda W \geq 0 \text{ in } \Omega^*\right\} = 1.$$

(b) $\lambda = 1$ is also optimal for the inequality $P - \lambda W \geq 0$ in any fixed punctured neighborhood of either 0 or ∞ in Ω.

(c) Denote the ground states of $P - W$ and $P^\star - W$ in Ω^* by ψ and $\psi^\star$. Then $\psi\psi^\star$ is not $W\mathrm{d}\nu$-integrable in any fixed neighborhood of either 0 or ∞ (P is said to be *null-critical in $\Omega^\star$*).

Aim: For general P and Ω find an optimal Hardy-weight W

The supersolution construction

Lemma (Supersolution construction)

Let v_j be two positive solutions (resp. supersolutions) of the equation $Pu = 0$, $j = 0, 1$, in a domain Ω, and let $v := v_1/v_0$. Then for any $0 \leq \alpha \leq 1$ the function

$$v_\alpha(x) := \big(v_1(x)\big)^\alpha \big(v_0(x)\big)^{1-\alpha} = \big(v(x)\big)^\alpha v_0(x)$$

is a positive solution (resp. supersolution) of the equation

$$\big[P - 4\alpha(1-\alpha)W(x)\big]u = 0 \qquad \text{in } \Omega.$$

Here

$$\boxed{W(x) := \frac{|\nabla v|_A^2}{4v^2} \geq 0, \quad \text{where} \quad |\xi|_A^2 := \xi \cdot A\xi\,.}$$

In particular, $P - W \geq 0$ in Ω.

Main result

Theorem

Let P be a subcritical operator in Ω, and let $G(x) := G_P^\Omega(x,0)$. Let u be a positive solution of the equation $Pu = 0$ in Ω satisfying

$$\lim_{x\to\infty} v(x) = 0, \qquad \text{where } v(x) := \frac{G(x)}{u(x)}, \text{ and}$$

∞ is the ideal point in the one-point compactification $\hat{\Omega}$ of Ω. Consider the supersolution $v_{1/2} := \sqrt{Gu}$.

Main result

Theorem

Let P be a subcritical operator in Ω, and let $G(x) := G_P^{\Omega}(x,0)$. Let u be a positive solution of the equation $Pu = 0$ in Ω satisfying

$$\lim_{x\to\infty} v(x) = 0, \qquad \text{where } v(x) := \frac{G(x)}{u(x)}, \text{ and}$$

∞ is the ideal point in the one-point compactification $\hat{\Omega}$ of Ω. Consider the supersolution $v_{1/2} := \sqrt{Gu}$.

Then the Hardy-weight $W := \frac{|\nabla v|_A^2}{4v^2}$ is an optimal Hardy-weight in $\Omega^\star$.

Main result

Theorem

Let P be a subcritical operator in Ω, and let $G(x) := G_P^{\Omega}(x,0)$. Let u be a positive solution of the equation $Pu = 0$ in Ω satisfying

$$\lim_{x\to\infty} v(x) = 0, \qquad \textit{where } v(x) := \frac{G(x)}{u(x)}, \textit{ and}$$

∞ is the ideal point in the one-point compactification $\hat{\Omega}$ of Ω. Consider the supersolution $v_{1/2} := \sqrt{Gu}$.

Then the Hardy-weight $W := \frac{|\nabla v|_A^2}{4v^2}$ is an optimal Hardy-weight in $\Omega^\star$.

Furthermore, in the symmetric case, let σ, (σ_{ess}) be the (essential) spectrum of the operator $\tilde{P} := W^{-1}P$ on $L^2(\Omega^\star, W\mathrm{d}\nu)$. Then

$$\sigma = \sigma_{\mathrm{ess}} = [1, \infty).$$

Main result

Theorem

Let P be a subcritical operator in Ω, and let $G(x) := G_P^\Omega(x,0)$. Let u be a positive solution of the equation $Pu = 0$ in Ω satisfying

$$\lim_{x\to\infty} v(x) = 0, \qquad \text{where } v(x) := \frac{G(x)}{u(x)}, \text{ and}$$

∞ is the ideal point in the one-point compactification $\hat{\Omega}$ of Ω. Consider the supersolution $v_{1/2} := \sqrt{Gu}$.

Then the Hardy-weight $W := \frac{|\nabla v|_A^2}{4v^2}$ is an optimal Hardy-weight in $\Omega^\star$.

Furthermore, in the symmetric case, let σ, (σ_{ess}) be the (essential) spectrum of the operator $\tilde{P} := W^{-1}P$ on $L^2(\Omega^\star, W\mathrm{d}\nu)$. Then

$$\sigma = \sigma_{\mathrm{ess}} = [1, \infty).$$

cf. Adimurthi-Sekar, Carron, Cowan, D'Ambrosio, Li-Wang, Cazacu-Zuazua, ...

On the condition $\lim_{x\to\infty} \frac{G(x)}{u(x)} = 0$

Remark

By a result of A. Ancona (2002), if P is symmetric, or more generally if $G_P^\Omega(x,y) \asymp G_P^\Omega(y,x)$, then a positive solution u of the equation $Pu = 0$ in Ω satisfying

$$\lim_{x\to\infty} \frac{G(x)}{u(x)} = 0$$

always exists.

Proof's outline

Surprisingly, the proof of the main theorem is similar to the following proof of the particular case of the classical Hardy inequality.
Let $P = -\Delta$ be the Laplace operator on $\Omega^\star := \mathbb{R}^n \setminus \{0\}$, where $n \geq 3$, and denote by $G(x) := |x|^{2-n}$ the corresponding positive minimal Green function.
Consider the positive superharmonic function in Ω

$$v_{1/2}(x) := |x|^{(2-n)/2} = G(x)^{1/2} = \sqrt{G(x)\mathbf{1}}.$$

By the supersolution construction, $W(x) = C_H|x|^{-2}$. So, we obtain the Hardy inequality

$$\int_{\Omega^\star} |\nabla\varphi|^2 \,\mathrm{d}x \geq \int_{\Omega^\star} \frac{C_H}{|x|^2}|\varphi(x)|^2 \,\mathrm{d}x \qquad \forall\varphi \in C_0^\infty(\Omega^\star).$$

Criticality

To prove that $W(x) = C_H|x|^{-2}$ is an optimal Hardy-weight, we analyze oscillatory properties of the corresponding radial Euler's equation

$$-u'' - \frac{n-1}{r}u' - \lambda\frac{C_H}{r^2}u = 0 \qquad r \in (0,\infty), \tag{0.2}$$

where $\lambda \in \mathbb{R}$. For $\lambda \neq 1$ two linearly independent solutions of (0.2) are given by

$$u_\pm(r) = \left(r^{(2-n)/2}\right)\left(r^{(2-n)/2}\right)^{\pm\sqrt{1-\lambda}},$$

while for $\lambda = 1$ two linearly independent solutions of (0.2) are expressed by

$$u_+(r) = r^{(2-n)/2}, \quad u_-(r) = \left(r^{(2-n)/2}\right)\log(r^{2-n}).$$

Criticality

To prove that $W(x) = C_H|x|^{-2}$ is an optimal Hardy-weight, we analyze oscillatory properties of the corresponding radial Euler's equation

$$-u'' - \frac{n-1}{r}u' - \lambda\frac{C_H}{r^2}u = 0 \qquad r \in (0,\infty), \tag{0.2}$$

where $\lambda \in \mathbb{R}$. For $\lambda \neq 1$ two linearly independent solutions of (0.2) are given by

$$u_\pm(r) = \left(r^{(2-n)/2}\right)\left(r^{(2-n)/2}\right)^{\pm\sqrt{1-\lambda}},$$

while for $\lambda = 1$ two linearly independent solutions of (0.2) are expressed by

$$u_+(r) = r^{(2-n)/2}, \quad u_-(r) = \left(r^{(2-n)/2}\right)\log(r^{2-n}).$$

For $\lambda < 1$ both solutions are positive, and therefore, the operator $P - \lambda C_H|x|^{-2}$ is subcritical in $\Omega^\star$. For $\lambda = 1$ only u_+ is positive, and moreover, it is dominated by $|u_-|$ at both ends $r = 0$ and $r = \infty$. Hence, u_+ is a ground state, and $P - C_H|x|^{-2}$ is critical in $\Omega^\star$ (Khas'minskiĭ criterion for recurrency).

Optimality near infinity and null-criticality

Finally, for $\lambda > 1$ the solution of (0.2) given by

$$\varphi_\xi(r) := \mathfrak{Re}\{u_+(r)\} = r^{(2-n)/2} \cos\left[\xi \log(r^{2-n})\right], \quad \text{where } \xi := \frac{\sqrt{\lambda - 1}}{2},$$

oscillates in compact sets near zero and near infinity, and therefore, the best possible constant for the validity of the Hardy inequality in any neighborhood of either the origin or infinity is also 1. In particular,

$$\inf\left\{\sigma(-C_H^{-1}|x|^2\Delta, \Omega^*)\right\} = \inf\left\{\sigma_{\mathrm{ess}}(-C_H^{-1}|x|^2\Delta, \Omega^*)\right\} = 1.$$

Optimality near infinity and null-criticality

Finally, for $\lambda > 1$ the solution of (0.2) given by

$$\varphi_\xi(r) := \mathfrak{Re}\{u_+(r)\} = r^{(2-n)/2} \cos\left[\xi \log(r^{2-n})\right], \quad \text{where } \xi := \frac{\sqrt{\lambda - 1}}{2},$$

oscillates in compact sets near zero and near infinity, and therefore, the best possible constant for the validity of the Hardy inequality in any neighborhood of either the origin or infinity is also 1. In particular,

$$\inf\left\{\sigma(-C_H^{-1}|x|^2\Delta, \Omega^*)\right\} = \inf\left\{\sigma_{\mathrm{ess}}(-C_H^{-1}|x|^2\Delta, \Omega^*)\right\} = 1.$$

Furthermore, since $\varphi_\xi \to \varphi_0$ as $\xi \to 0$, the orthogonality relation

$$\int_{\left\{-\frac{\pi}{2} < \xi \log(r^{2-n}) < 0\right\}} \varphi_\xi \varphi_{3\xi}\, r^{-2}\, \mathrm{d}r = 0$$

implies that $\varphi_0(r) = r^{(2-n)/2} \notin L^2(\Omega^\star, |x|^{-2}\, \mathrm{d}x)$, which shows the null-criticality of the Hardy operator $-\Delta - C_H|x|^{-2}$ in $\Omega^\star$.

The entire spectrum

The spectral representation of $\tilde{P} := C_H^{-1}|x|^2(-\Delta)$, restricted to the radial functions, is obtained by *Mellin's transform*, $\mathcal{M} : L^2(0,\infty) \longrightarrow L^2(\mathbb{R})$

$$\mathcal{M}f(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(r) r^{i\xi - \frac{1}{2}} dr.$$

In fact, the composition of the unitary operator

$$L^2\Big((0,\infty), r^{n-1}\frac{C_H}{r^2}\,\mathrm{d}r\Big) \to L^2(0,\infty); \quad f(r) \mapsto \frac{\sqrt{|n-2|}}{2} f(r^{1/(n-2)}),$$

and the Mellin transform, gives a unitary operator

$$\mathfrak{U} : L^2_{\mathrm{rad}}(\Omega^\star, W\,\mathrm{d}x) \cong L^2\Big((0,\infty), r^{n-1}\frac{C_H}{r^2}\,\mathrm{d}r\Big) \to L^2(\mathbb{R}),$$

which is a spectral representation for $\tilde{P}$ restricted to radial function: in this representation, $\tilde{P}$ is just the multiplication by $\lambda = 1 + 4\xi^2$. Indeed, this follows from the fact that

$$\Big(\tilde{P} - (4\xi^2 + 1)\Big) \left(r^{n-2}\right)^{i\xi - \frac{1}{2}} = 0.$$

Hence, $\sigma(\tilde{P}, \Omega^*) = \sigma_{\mathrm{ess}}(\tilde{P}, \Omega^*) = [1, \infty)$.

Proof in the general case

Loosely speaking, to obtain the general result, just replace in the above proof, the function $r^{(2-n)} = r^{(2-n)}/\mathbf{1}$ with the function $v(x) = \frac{G(x)}{u(x)}$, and the radial functions with the space of functions v that are proportional to u on the level sets of G/u (i.e. $v = uf(G/u)$, where $f : (0, \infty) \to \mathbb{C}$).

Proof in the general case

Loosely speaking, to obtain the general result, just replace in the above proof, the function $r^{(2-n)} = r^{(2-n)}/\mathbf{1}$ with the function $v(x) = \frac{G(x)}{u(x)}$, and the radial functions with the space of functions v that are proportional to u on the level sets of G/u (i.e. $v = uf(G/u)$, where $f : (0, \infty) \to \mathbb{C}$). In particular, let $\varphi(\xi, x) := \varphi_\xi(x) = u\,\left(\frac{G}{u}\right)^{1/2} \exp(\mathrm{i}\xi \log(G/u))$. We have

Theorem

In the symmetric case, $\mathcal{F} : L^2_{\mathrm{rad}}(\Omega^\star, W\mathrm{d}\nu) \to L^2(\mathbb{R}, \mathrm{d}\xi)$ *given by*

$$\mathcal{F}f(\xi) := \sqrt{\frac{2}{\pi}} \int_{\Omega^\star} f(x)\varphi(\xi, x)W(x)\mathrm{d}\nu(x) \qquad \xi \in \mathbb{R},$$

is a unitary operator, whose inverse is given by

$$\mathcal{F}^{-1}g(x) = \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}} g(\xi)\varphi(-\xi, x)\,\mathrm{d}\xi.$$

Furthermore,

$$\mathcal{F}\frac{1}{W}P\mathcal{F}^{-1}f(\xi) = (1 + 4\xi^2)f(\xi).$$

Application: Rellich-type inequality

Corollary

Assume that P is subcritical in Ω, symmetric in $L^2(\Omega, \mathrm{d}\nu)$. Let $W > 0$ be the obtained optimal Hardy weight. Then the induced Agmon metric is complete, and by Agmon, the following Rellich-type inequality holds true

$$\int_\Omega |u|^2 W(x)\,\mathrm{d}x \le \int_\Omega \frac{|Pu|^2}{W(x)}\,\mathrm{d}x \qquad \forall u \in C_0^\infty(\Omega^\star).$$

Application: Rellich-type inequality

Corollary

Assume that P is subcritical in Ω, symmetric in $L^2(\Omega, \mathrm{d}\nu)$. Let $W > 0$ be the obtained optimal Hardy weight. Then the induced Agmon metric is complete, and by Agmon, the following Rellich-type inequality holds true

$$\int_\Omega |u|^2 W(x)\,\mathrm{d}x \leq \int_\Omega \frac{|Pu|^2}{W(x)}\,\mathrm{d}x \qquad \forall u \in C_0^\infty(\Omega^\star).$$

Example (Ghoussoub-Moradifam (2011), Caldiroli-Musina (2012))

Take $\Omega^\star = \mathbb{R}^n \setminus \{0\},\ n \geq 3$ with the *optimal* Hardy-weight $W(x) := C_H|x|^{-2}$. Then for any $0 \leq \mu < 1$ the following Rellich-type inequality holds true (with the best constant)

$$\left(\frac{n-2}{2}\right)^4 (1-\mu^2)^2 \int_{\Omega^\star} \frac{|u(x)|^2}{|x|^{2+(n-2)\mu}}\,\mathrm{d}x \leq \int_{\Omega^\star} |\Delta u|^2 |x|^{2-(n-2)\mu}\,\mathrm{d}x \quad \forall u \in C_0^\infty(\Omega^\star).$$

Generalizations

1. Boundary singularities.

Generalizations

1. Boundary singularities.
2. Finitely many ends.

Generalizations

1. Boundary singularities.
2. Finitely many ends.
3. The quasilinear case.

Confucius (500 BCE)- The Analects, Section 1.2

The Master said:

At fifteen, I had my mind bent on learning.

Confucius (500 BCE)- The Analects, Section 1.2

The Master said:

At fifteen, I had my mind bent on learning.

At thirty, I stood firm.

Confucius (500 BCE)- The Analects, Section 1.2

The Master said:

At fifteen, I had my mind bent on learning.

At thirty, I stood firm.

At forty, I had no doubts.

Confucius (500 BCE)- The Analects, Section 1.2

The Master said:

At fifteen, I had my mind bent on learning.

At thirty, I stood firm.

At forty, I had no doubts.

At fifty, I knew the decrees of Heaven.

Confucius (500 BCE)- The Analects, Section 1.2

The Master said:

At fifteen, I had my mind bent on learning.

At thirty, I stood firm.

At forty, I had no doubts.

At fifty, I knew the decrees of Heaven.

At sixty, my ear was an obedient organ for the reception of truth.

Confucius (500 BCE)- The Analects, Section 1.2

The Master said:

At fifteen, I had my mind bent on learning.

At thirty, I stood firm.

At forty, I had no doubts.

At fifty, I knew the decrees of Heaven.

At sixty, my ear was an obedient organ for the reception of truth.

At seventy, I could follow what my heart desired, without transgressing what was right.

Mishnah (200 CE), Ethics of the Fathers (5.21)

He [Yehuda ben Teima] used to say:

. . .

Mishnah (200 CE), Ethics of the Fathers (5.21)

He [Yehuda ben Teima] used to say:

. . .

Forty [is the age] for understanding,

Mishnah (200 CE), Ethics of the Fathers (5.21)

He [Yehuda ben Teima] used to say:

...

Forty [is the age] for understanding,

Fifty [is the age] for [giving] counsel,

Mishnah (200 CE), Ethics of the Fathers (5.21)

He *[Yehuda ben Teima]* *used to say:*

...

Forty *[is the age]* *for understanding,*

Fifty *[is the age]* *for [giving] counsel,*

Sixty *[is the age]* *for sagacity,*

Mishnah (200 CE), Ethics of the Fathers (5.21)

He [Yehuda ben Teima] used to say:

...

Forty [is the age] for understanding,

Fifty [is the age] for [giving] counsel,

Sixty [is the age] for sagacity,

Seventy [is the age] for Seiva (hoariness)

Mishnah (200 CE), Ethics of the Fathers (5.21)

He [Yehuda ben Teima] used to say:

. . .

Forty [is the age] for understanding,

Fifty [is the age] for [giving] counsel,

Sixty [is the age] for sagacity,

Seventy [is the age] for Seiva (hoariness)

. . .

Mishnah (200 CE), Ethics of the Fathers (5.21)

He [Yehuda ben Teima] used to say:

. . .

Forty [is the age] for understanding,

Fifty [is the age] for [giving] counsel,

Sixty [is the age] for sagacity,

Seventy [is the age] for Seiva (hoariness)

. . .

They shall still bring forth fruit in old age
they shall stay fresh and flourishing (Psalms 92,15)

Mazal Tov!