On the negative spectrum of the Schrödinger operator in the two-dimensional case: an overview

Workshop Kannai-70

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GENERAL SETTING ($\mathbb{R}^d$, any $d \geq 1$)

$$\mathbf{H}_V = -\Delta - V, \qquad V \geq 0$$

is the Schrödinger operator on $\mathbb{R}^d$ with the potential $-V$.
Precise definition – via the quadratic form

$$\mathbf{Q}_V[u] = \int (|\nabla u|^2 - V|u|^2)dx, \qquad u \in H^1(\mathbb{R}^d).$$

Under appropriate assumptions about V the operator $\mathbf{H}_V$ is well-defined and self-adjoint in $L^2(\mathbb{R}^d)$.

If $V(x) \to 0$ as $x \to \infty$, the positive spectrum is $[0, \infty)$, a.c.

The problem

The negative spectrum consists of eigenvalues of finite multiplicity, with the only possible accumulation point at $\lambda = 0$.

GENERAL PROBLEM, COMING FROM PHYSICS:
TO ESTIMATE THE NUMBER $N_-(\mathbf{H}_V)$ OF THESE EIGENVALUES (counted with multiplicities) IN TERMS OF V

Often, one inserts a large parameter $\alpha > 0$ (THE COUPLING CONSTANT) and studies the behavior of $N_-(\mathbf{H}_{\alpha V})$ as $\alpha \to \infty$.

The Weyl asymptptic formula

For nice potentials (say, $V \in C_0^\infty$)
Weyl's asymptotic formula is satisfied:

$$\boxed{\lim_{\alpha\to\infty} \alpha^{-d/2} N_-(\mathbf{H}_{\alpha V}) = \frac{v_d}{(2\pi)^d} \int V^{d/2} dx} \qquad \text{(W)}$$

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(v_d is the volume of the ball $|x| \le 1$.)

The growth $N_-(\mathbf{H}_{\alpha V}) = O(\alpha^{d/2})$
is called SEMI-CLASSICAL.

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The conditions guaranteeing the semi-classical behavior of $N_-(\mathbf{H}_V)$ and those guaranteeing (W) DEPEND ON THE DIMENSION.

WHAT IS SPECIAL ABOUT THE 2D-CASE???

Let us discuss other dimensions!

Dimensions $d > 2$

The central result is

CWIKEL (1977) – LIEB ((1976)– ROZENBLUM (1972) estimate:

For any $V \in L^{d/2}(\mathbb{R}^d),\ d > 2$,
the semi-classical estimate holds:

$$\boxed{N_-(\mathbf{H}_V) \le C(d) \int V^{d/2} dx} \qquad \text{(CLR)}$$

and the asymptotic formula (W) is satisfied.

Dimensions $d > 2$: comments

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2. The condition $V \in L^{d/2}$
is NECESSARY and SUFFICIENT
for the validity of both (CLR) and (W).

3. If $\int V^{d/2}dx$ is SMALL, then $N_-(\mathbf{H}_V) = 0$
(NO NEGATIVE EIGENVALUES!)

$d = 1$

$\Delta u = u''$

The most convenient estimate (SEMICLASSICAL): define

$I_k = (e^{k-1}, e^k)$ for $k \in \mathbb{N}$. Then

$$N_-(\mathbf{H}_V) \le 1 + C\sqrt{\int_{(-1,1)} V(x)dx} + C\sum_{k\in\mathbb{N}}\sqrt{\int_{|x|\in I_k} |x|V(x)dx}$$

(see survey M.S. arXiv:1203.1156)

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4. $N_-(\mathbf{H}_V) \geq 1$ for ANY non-trivial $V \geq 0$.

$d = 2$: main peculiarities

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3. Sobolev Embedding theorem with the limiting exponent (which is equal to infinity) fails.

$d = 2$: preliminary information

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2. ANALOG OF (CLR) FAILS FOR GENERAL POTENTIALS. AN INVERSE INEQUALITY HOLDS:

$$N_-(\mathbf{H}_V) \geq c \int V dx, \qquad c > 0$$

(A.Grigor'yan, Y.Netrusov, S-T. Yau, 2004; MR 2195408).

NO ANALOG FOR $d > 2$!

$d = 2$, general potentials – preliminaries. I

Orthogonal decomposition
of the Sobolev space $H^1(\mathbb{R}^2)$:

$\mathcal{F}_1$ – the subspace of functions $u(x) = f(|x|)$;

$$\mathcal{F}_2 = \mathcal{F}_1^{\perp}$$

(orthogonality in the metric of $H^1(\mathbb{R}^2)$)

$\mathcal{G}_1, \mathcal{G}_2$ – their closures in $L^2(\mathbb{R}^2)$.
They are orthogonal in L^2

$d = 2$, general potentials. II

Restrictions of the quadratic form $\mathbf{Q}_V[u] = \int(|\nabla u|^2 - V|u|^2)dx$ to $\mathcal{F}_1, \mathcal{F}_2$ generate operators on $\mathcal{G}_1, \mathcal{G}_2$ – say, $\mathbf{H}_V^{(1)}, \mathbf{H}_V^{(2)}$.

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For estimation of $N_-(\mathbf{H}_V)$ it is sufficient (and necessary!) to estimate $N_-(\mathbf{H}_V^{(1)}),\ N_-(\mathbf{H}_V^{(2)})$.
The latter estimates have a different nature and require different techniques

Estimate for $\mathbf{H}_V^{(1)}$

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Let $\{c_k\}_{k\in\mathcal{K}}$ be a number sequence. $\mathcal{K}$ can be any countable set. Suppose $c_k \to 0$, i.e., $\#\{k : |c_k| > \varepsilon\} < \infty$ for any $\varepsilon > 0$.

Define $\{c_n^*\}_{n\in\mathbb{N}}$ as the non-increasing rearrangement of $\{|c_k|\}$.

Space ℓ^1_w

DEFINITION

$$\{c_k\} \in \ell^1_w \Leftrightarrow \|\{c_k\}\|_{1,w} := \sup_n (nc_n^*) < \infty.$$

In other words, $c_n^* = O(n^{-1})$.

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Clearly, $\ell^1 \subset \ell^1_w$ and $\|\{c_k\}\|_{1,w} \le \|\{c_k\}\|_1$.

Estimate for $\mathbf{H}_V^{(2)}$. Space $L\log L$

ESTIMATE FOR $\mathbf{H}_V^{(2)}$ REQUIRES EMBEDDING THEOREM THAT FOR $d = 2$ INVOLVES THE ORLICZ SPACE $L\log L$.

The space $L\log L(\Omega)$ on a set $\Omega \subset \mathbb{R}^d$ is defined by the condition

$$\int_\Omega [(1 + |f(x)|)\log(1 + |f(x)|) - |f(x)|]\, dx < \infty$$

This is a Banach space but the above integral is NOT a norm.

Norms in $L \log L$

There are several standard ways to define a norm in $\mathfrak{B} = L \log L$. Of course, all these norms are mutually equivalent. I do not present their definitions here.

We use $\|f\|^{av}_{\mathfrak{B}(\Omega)}$ – a special norm, with an appropriate normalization depending on shape of Ω.
If Ω_1, Ω_2 are homothetic, f_1 is a function on Ω_1, and f_2 is its "transplantation" to Ω_2, then

$$\|f_1\|^{av}_{\mathfrak{B}(\Omega_1)} = \|f_2\|^{av}_{\mathfrak{B}(\Omega_2)}$$

(IMPORTANT FOR THE ESTIMATES!)

Basic estimate, I

Two partitions of $\mathbb{R}^2$ and two number sequences (depending on V) are used.

FIRST PARTITION:

$$\Omega_0 = \{|x| < 1\}; \qquad \Omega_k = \{2^{(k-1)} < |x| < 2^k\},\ k \in \mathbb{N}$$

(the unit disk and the annuli whose inner radii form a geometric series. The ratio (which is taken e) is indifferent – it affects only the value of the estimating constant).

DEFINE $\mathfrak{B}_k(V) = \|V\|^{av}_{\mathfrak{B}(\Omega_k)}$ (responsible for $\mathbf{H}^{(2)}_V$)

Basic estimate, II

SECOND PARTITION:

$$X_0 = \{|x| < e\}; \qquad X_k = \{e^{2^{k-1}} < |x| < e^{2^k}\}, \; k \in \mathbb{N}$$

(the disk X_0 and the annuli such that the LOGARITHMS of their inner radii form a geometric series).

$$\zeta_0(V) = \int_{X_0} V(x)dx, \qquad \zeta_k(V) = \int_{X_k} V(x)\log|x|dx, \; k \in \mathbb{N}.$$

(responsible for $\mathbf{H}_V^{(1)}$)

Basic estimate, III

THEOREM (M.S. 1994; MR 1276138)

SUPPOSE $\{\mathfrak{B}_k(V)\} \in \ell^1, \ \{\zeta_k(V)\} \in \ell^1_w.$

THEN THE SEMI-CLASSICAL ESTIMATE IS SATISFIED:

$$N_-(\mathbf{H}_V) \le 1 + C\left(\sum_k \mathfrak{B}_k(V) + \|\{\zeta_k(V)\}\|_{1,w}\right)$$

IF, IN ADDITION, $\zeta_n^*(V) = o(n^{-1})$,
THEN ALSO (W) IS VALID.

Comments

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(REASON: many people do not like
Orlicz spaces as something too exotic)

A general comment

Among the people working on the estimates, there are those looking for the SIMPLEST POSSIBLE ESTIMATE
and those
looking for the SHARPEST POSSIBLE ESTIMATE.

In the problem of estimating $N_-(\mathbf{H}_V)$ for $d > 2$ we are LUCKY, due to (CLR) that is BOTH SIMPLE and SHARP.

But for $d = 2$ ONE HAS TO CHOOSE...

Result by M.Birman – A. Laptev (1996; MR 1443037)

They used L^p-roughened estimate.
The main concern was ASYMPTOTIC BEHAVIOR OF $N_-(\mathbf{H}_{\alpha V})$.

UNDER SOME BROAD ASSUMPTIONS ABOUT V,

$$N_-(\mathbf{H}_{\alpha V}) \sim N_-(\mathbf{H}^{(1)}_{\alpha V}) + N_-(\mathbf{H}^{(2)}_{\alpha V})$$

Independence of contributions of $\mathbf{H}^{(1)}_{\alpha V}, \mathbf{H}^{(2)}_{\alpha V}$ to the asymptotics!

Competition between these contributions.
In particular, the potentials were constructed such that $N_-(\mathbf{H}_{\alpha V}) = O(\alpha)$ but (W) fails.

Radial potentials: result by A.Laptev (1998)

1998 is the date of preprint (Inst. Mittag -Leffler)
Journal publication of 2000; MR 1819649.

EQUIVALENT FORMULATION: Let $V(x) = F(|x|)$, i.e., the potential V depends only on $|x|$. Then

$$N_-(\mathbf{H}_V^{(2)}) \le C \int V(x)dx.$$

An extremely important and unexpected result!
Direct analog of (CLR) (for $\mathbf{H}_V^{(2)}$ the term 1 disappears)
A very elegant proof.

Radial potentials: integral estimate

K. Chadan, N.Khuri, A.Martin, T.-T.Wu (2002; MR 1952194): FOR A RADIAL POTENTIAL

$$\boxed{N_-(\mathbf{H}_V) \le 1 + C\int V(x)(1 + |\ln|x||)dx}.$$

In fact, they re-discovered Laptev's approach, and also made the next step – the passage from $\mathbf{H}_V^{(2)}$ to $\mathbf{H}_V$.

Back to the general potentials: refinement of M.S.-94

Below r, ϑ stand for the polar coordinates in $\mathbb{R}^2$.
Define $V_{\text{rad}}(r) = (2\pi)^{-1} \int_0^{2\pi} V(r,\vartheta)d\vartheta$;

$$V_{\text{nrad}}(r,\vartheta) = V(r,\vartheta) - V_{\text{rad}}(r).$$

Consider the space $\mathfrak{M} = L^1(\mathbb{R}_+; \mathfrak{B}(\mathbb{S}^1))$, with the norm

$$\|f\|_{\mathfrak{M}} = \int_0^\infty \|f(r,.)\|_{\mathfrak{B}(\mathbb{S}^1)} r dr$$

Define, for $k \in \mathbb{Z}$,

$$\eta_k(V) = \int_{e^{2^{k-1}} < |x| < e^{2^k}} V(x)|\ln|x||dx.$$

Refinement of M.S.-94: E. Shargorodsky 2012

$$\boxed{N_-(\mathbf{H}_V) \le 1 + C(\|\{\eta_k(V)\}\|_{1,w} + \|V_{\mathrm{nrad}}\|_{\mathfrak{M}})}$$

(E.Shargorodsky, arXiv: 1203:4833; see also Laptev and M.S, arXiv: 1201:3074)

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(E.Shargorodsky, arXiv: 1203:4833; see also Laptev and M.S, arXiv: 1201:3074)

THIS IS THE SHARPEST SEMI-CLASSICAL ESTIMATE KNOWN SO FAR.

COMMENT: In M.S.-94 $k \in \mathbb{N}$. Here $k \in \mathbb{Z}$:
small annuli around $x = 0$ are involved. The conditions allow singularity of V at $x = 0$.
M.S.-94 allows singularity only at infinity.

Radial potentials: necessary and sufficient condition

Let V depend only on $|x|$. Then the last term in the above estimate DISAPPEARS, and we get

$$\boxed{N_-(\mathbf{H}_V) \le 1 + C\|\{\eta_k(V)\}\|_{1,w}}$$

In fact, the condition

$$\{\eta_k(V)\} \in \ell^1_w$$

is NOT ONLY SUFFICIENT BUT ALSO NECESSARY for the semi-classical behavior of $N_-(\mathbf{H}_V)$ with radial V (Laptev, M.S., 2012; MR 2954515).

The latter condition, plus $\eta^*_n(V) = o(n^{-1})$, is NECESSARY AND SUFFICIENT for (W) (for such potentials)

The Chadan-Khuri-Martin-Wu estimate for radial potentials

$$\boxed{N_-(\mathbf{H}_V) \le 1 + C \int V(x)(1 + |\ln|x||)dx}$$

is a direct consequence of Laptev – M.S. 2012 (due to the estimate of ℓ^1_w-quasinorm through the norm in ℓ^1). But this was realized only recently.

ACTUALLY, EVERYTHING COULD BE DONE IN 1998 BUT THIS POSSIBILITY WAS MISSED.

A.Grigor'yan – N.Nadirashvili estimate (arXiv:1112.4986)

(with refinement due to E.Shargorodsky, arXiv:1203.4833)

LET $\mathfrak{B}_k = \mathfrak{B}_k(V) = \|V\|^{av}_{\mathfrak{B}(\Omega_k)}, \ k \geq 0;$

$$\zeta_k = \zeta_k(V) = \int_{e^{2^{k-1}}}^{e^{2^k}} V(x) \log|x| dx, \ k \in \mathbb{N};$$

$$\zeta_0 = \zeta_k(V) = \int_{|x|<e} V(x) dx.$$

THERE EXIST CONSTANTS $m_1, m_2 > 0$ SUCH THAT

$$\boxed{N_-(\mathbf{H}_V) \leq 1 + C \left(\sum_{\zeta_k > m_1} \sqrt{\zeta_k} + \sum_{\mathfrak{B}_k > m_2} \mathfrak{B}_k \right)}$$

(GN used L^p-roughening)

Comments.

Truncated sums on the right. If the series without truncation CONVERGE, we come to the semi-classical estimate as in M.S.-1994.

If the series DIVERGE, we come to a non-semi-classical estimate.

PROOF requires only a minor change in the original argument of M.S.-1994 – shown by E.Shargorodsky

Khuri-Martin-Wu conjecture

NON-INCREASING SPHERICAL REARRANGEMENT OF V is a function $V_*(|x|)$, such that for any $s > 0$

$$\text{meas}\{x \in \mathbb{R}^2 : V_*(|x|) > s\} = \text{meas}\{x \in \mathbb{R}^2 : V(|x|) > s\}.$$

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KMW-CONJECTURE (2002):

$$N_-(\mathbf{H}_V) \le 1 + C \int_{\mathbb{R}^2} V(x) \ln(2 + |x|) dx$$
$$+ C \int_{|x|<1} V_*(|x|) \ln \frac{1}{|x|} dx$$

KMW-CONJECTURE had been recently justified by E. Shargorodsky, arXiv:1203.4833.

He showed that KMW is a consequence of M.S.-1994, and that the latter is strictly sharper.

Some unsolved problems I

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1. The necessary AND sufficient condition for the semi-classical behavior of $N_-(\mathbf{H}_V)$ is unknown. It may happen that it is impossible to express it in terms of function spaces for V (as it is for $d = 1$).

2. What happens in the similar problem on a two-dimensional Riemannian manifold?
(Its structure "at infinity" counts).

Some unsolved problems II

3. Is it possible to obtain estimates invariant wrt MOTIONS in $\mathbb{R}^2$?

In all the known estimates the point $x = 0$ plays a special role and it is not clear how to avoid it, due to the necessity to separate the subspace of radial functions.

WARMEST CONGRATULATIONS TO YAKAR!