WEYL DENOMINATOR IDENTITY FOR AFFINE LIE SUPERALGEBRAS WITH NON-ZERO DUAL COXETER NUMBER

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Abstract. Weyl denominator identity for the affinization of a basic Lie superalgebra with non-zero Killing form was formulated by V. Kac and M. Wakimoto and was proven by them for the defect one case. In this paper we prove this identity.

0. Introduction

Let $\mathfrak{g}$ be a basic Lie superalgebra with a non-zero Killing form. Let $\hat{\mathfrak{g}}$ be the affinization of $\mathfrak{g}$. Let $\mathfrak{h}$ (resp., $\hat{\mathfrak{h}}$) be the Cartan subalgebra in $\mathfrak{g}$ (resp., in $\hat{\mathfrak{g}}$) and let $(-,-)$ be the bilinear form on $\hat{\mathfrak{h}}^*$ which is induced by the Killing form on $\mathfrak{g}$. Let Δ (resp., $\hat{\Delta}$) be the root system of $\mathfrak{g}$ (resp., of $\hat{\mathfrak{g}}$). We set

$$\Delta^{\#} := \{\alpha \in \Delta_{\overline{0}} |\ (\alpha,\alpha) > 0\}.$$

Then $\Delta^{\#}$ is a root system of a simple Lie algebra. Let $\hat{\Delta}^{\#}$ be the affinization of $\Delta^{\#}$. Denote by $\hat{W}^{\#}$ (resp., $W^{\#}$) the subgroup of $GL(\hat{\mathfrak{h}})$ generated by the reflections $s_\alpha : \alpha \in \hat{\Delta}_{\overline{0}},\ (\alpha,\alpha) > 0$ (resp., $s_\alpha : \alpha \in \Delta^{\#}$). Then $W^{\#}$ is the Weyl group of $\Delta^{\#}$ and $\hat{W}^{\#}$ is the corresponding affine Weyl group. Recall that $\hat{W}^{\#} = W^{\#} \ltimes T$, where $T \subset \hat{W}^{\#}$ is the translation group, see [K2], Chapter 6. Let Π be a set of simple roots for $\mathfrak{g}$, and let $\hat{\Pi} = \Pi \cup \{\alpha_0\}$ be the corresponding set of simple roots for $\hat{\mathfrak{g}}$. Let Δ_+, $\hat{\Delta}_+$ be the corresponding sets of positive roots. We set

$$R := \frac{\prod_{\alpha\in\Delta_{+,0}}(1-e^{-\alpha})}{\prod_{\alpha\in\Delta_{+,1}}(1+e^{-\alpha})}, \quad \hat{R} := \frac{\prod_{\alpha\in\hat{\Delta}_{+,0}}(1-e^{-\alpha})}{\prod_{\alpha\in\hat{\Delta}_{+,1}}(1+e^{-\alpha})}.$$

Following [KW], we call R the *Weyl denominator* and $\hat{R}$ the *affine Weyl denominator*. The Weyl denominator identity conjectured by V. Kac and M. Wakimoto in [KW] can be written as

$$\hat{R}e^{\hat{\rho}} = \sum_{w\in T} w(Re^{\rho}),$$

where $\hat{\rho} \in \hat{\mathfrak{h}}^*$ is such that $2(\hat{\rho},\alpha) = (\alpha,\alpha)$ for each $\alpha \in \Pi$. The original form of this identity is given in formula (2). In this paper we prove this identity.

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In the paper [G] we proved the analog of Weyl denominator identity for finite-dimensional Lie superalgebras (also formulated and partially proven by Kac-Wakimoto). The proof of the present result makes use of this version of Weyl denominator identity.

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1. Kac-Moody superalgebras

The notions of a Kac-Moody superalgebras and its Weyl group were introduced in [S]. We recall some definitions below and then prove Lemmas 1.3.2,1.5.1. In the sequel, we will apply these lemma to the case of affine Lie superalgebras; in this case the lemmas can be also verified using the explicit description of root systems.

1.1. **Construction of Kac-Moody superalgebras.** Let $A=(a_{ij})$ be an $n\times n$-matrix over $\mathbb{C}$ and let τ be a subset of $I:=\{1,\ldots,n\}$. Let $\mathfrak{g}=\mathfrak{g}(A,\tau)=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ be the associated Lie superalgebra constructed as in [K1],[K2]. Recall that, in order to construct $\mathfrak{g}(A,\tau)$, one considers a realization of A, i.e. a triple $(\mathfrak{h},\Pi,\Pi^\vee)$, where $\mathfrak{h}$ is a vector space of dimension $n+corank A$, $\Pi\subset\mathfrak{h}^*$ (resp. $\Pi^\vee\subset\mathfrak{h}$) is a linearly independent set of vectors $\{\alpha_i\}_{i\in I}$ (resp. $\{\alpha_i^\vee\}_{i\in I}$), such that $\langle\alpha_i,\alpha_j^\vee\rangle=a_{ji}$, and constructs a Lie superalgebra $\tilde{\mathfrak{g}}(A,\tau)$ on generators $e_i,f_i,\mathfrak{h}$, subject to relations:

$$[\mathfrak{h},\mathfrak{h}]=0,\quad [h,e_i]=\langle\alpha_i,h\rangle e_i,\quad [h,f_i]=-\langle\alpha_i,h\rangle f_i,\quad \text{for } i\in I, h\in\mathfrak{h},\quad [e_i,f_j]=\delta_{ij}\alpha_i^\vee,$$
$$p(e_i)=p(f_i)=\overline{1} \text{ if } i\in\tau,\quad p(e_i)=p(f_i)=\overline{0} \text{ if } i\not\in\tau,\quad p(\mathfrak{h})=\overline{0}.$$

Then $\mathfrak{g}(A,\tau)=\tilde{\mathfrak{g}}(A,\tau)/J=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$, where J is the maximal ideal of $\tilde{\mathfrak{g}}(A,\tau)$, intersecting $\mathfrak{h}$ trivially, and $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) is the subalgebra generated by the images of the e_i's (resp. f_i's). We obtain the triangular decomposition $\mathfrak{g}(A)=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}$.

Let Δ be the set of roots of $\mathfrak{g}(A)$, i.e. $\Delta=\{\alpha\in\mathfrak{h}^*|\alpha\neq 0\ \&\ \mathfrak{g}_\alpha\neq 0\}$, $\Delta_+=\{\alpha\in\mathfrak{h}^*|\mathfrak{n}_\alpha\neq 0\}$, $\Delta_-=\{\alpha\in\mathfrak{h}^*|\mathfrak{n}_{-,\alpha}\neq 0\}$. One has $\Delta=\Delta_+\coprod\Delta_-$, $\Delta_-=-\Delta_+$.

We say that a simple root α_i is *even* (resp., *odd*) if $i\not\in\tau$ (resp., $i\in\tau$) and that α_i is *isotropic* if $a_{ii}=0$. One readily sees that if $i\in\tau$ (i.e., e_i,f_i are odd), then $[e_i,e_i],[f_i,f_i]\in J$ iff $a_{ii}=0$. Therefore for a simple root α one has $2\alpha\in\Delta$ iff α is a non-isotropic and odd.

Note that, multiplying the i-th row of the matrix A by a non-zero number corresponds to multiplying e_i and $\alpha_i^\vee$ by this number, thus giving an isomorphic Lie superalgebra. Hence we may assume from now on that $a_{ii}=2$ or 0 for all $i\in I$.

1.1.1. We consider the case when the Cartan matrix $A=(a_{ij})$ is such that

(1) $a_{ii}\in\{0,2\}$ for all $i\in I$ and $a_{ij}=0$ forces $a_{ji}=0$;

(2) if $i\not\in\tau$, then $a_{ii}=2$ and $a_{ij}\in\mathbb{Z}_{\leq 0}$ for $j\neq i$;

(3) if $i\in\tau$ and $a_{ii}=2$, then $a_{ij}\in 2\mathbb{Z}_{\leq 0}$ for $j\neq i$.

In this case $\operatorname{ad} e_i,\operatorname{ad} f_i$ act locally nilpotently for each $i\in I$.

1.2. **Weyl group.** Recall a notion of odd reflections, see [S]. Let Π be a set of simple roots and Δ_+ be the corresponding set of positive roots. Fix a simple regular isotropic root $\beta \in \Pi$ and set $s_\beta(\Pi) := \{s_\beta(\alpha)|\ \alpha \in \Pi\}$, where

$$\text{for } \alpha \in \Pi \quad \begin{array}{lll} s_\beta(\alpha) = -\alpha, & s_\beta(\alpha^\vee) = \alpha^\vee & \text{if } \alpha = \beta, \\ s_\beta(\alpha) = \alpha, & s_\beta(\alpha^\vee) = \alpha^\vee & \text{if } a_{\alpha\beta} = 0, \alpha \neq \beta, \\ s_\beta(\alpha) = \alpha + \beta, & s_\beta(\alpha^\vee) = a_{\alpha\beta}\beta^\vee + a_{\beta\alpha}\alpha^\vee & a_{\alpha\beta} \neq 0, a_{\alpha\alpha} + 2a_{\alpha\beta} = 0, \\ s_\beta(\alpha) = \alpha + \beta, & s_\beta(\alpha^\vee) = 2\frac{a_{\alpha\beta}\beta^\vee + a_{\beta\alpha}\alpha^\vee}{a_{\beta\alpha}(a_{\alpha\alpha} + 2a_{\alpha\beta})} & a_{\alpha\beta}, a_{\alpha\alpha} + 2a_{\alpha\beta} \neq 0. \end{array}$$

One has $\langle \alpha, \alpha^\vee \rangle \in \{0, 2\}$ for each $\alpha \in s_\beta(\Pi)$.

By [S], Sect. 3, $s_\beta(\Pi)$ is a set of simple roots for Δ and the corresponding set of positive roots is $s_\beta(\Delta_+) := \Delta_+ \setminus \{\beta\} \cup \{-\beta\}$. The Cartan matrix corresponding to $s_\beta(\Pi)$ is $(\langle \alpha^\vee, \alpha' \rangle)_{\alpha, \alpha' \in s_\beta(\Pi)}$.

1.2.1. We assume that $\mathfrak{g}(A)$ is such that *for any chain of odd reflections, the corresponding Cartan matrix satisfies the conditions (1)-(3) of 1.1.1.* By [S] Section 6, the finite-dimensional Kac-Moody superalgebras and their affinizations satisfy this assumption; other examples and classification are given in [S],[HS].

1.2.2. Let Θ be the collection of all possible sets of simple roots obtained from Π by finite sequences of odd reflections.

1.2.3. **Definition.** An even root $\alpha \in \Delta$ is called *principal* if $\alpha \in \Pi'$ or $\frac{1}{2}\alpha \in \Pi'$ for some $\Pi' \in \Theta$.

1.2.4. For each principal root α we fix $\alpha^\vee$ as follows: we choose $\Pi' \in \Theta$ such that $\alpha \in \Pi'$ or $\frac{1}{2}\alpha \in \Pi'$; in first case, we take $\alpha^\vee \in (\Pi')^\vee$ and in the second case we take $\alpha^\vee := (\frac{1}{2}\alpha)^\vee/2$, where $(\frac{1}{2}\alpha)^\vee \in (\Pi')^\vee$. Thanks to the assumption 1.2.1, $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$ for each $\beta \in \Pi'$. Thus for each principal root α one has $\langle \Delta, \alpha^\vee \rangle \subset \mathbb{Z}$.

The matrix A is called *symmetrizable* if for some invertible diagonal matrix D the product DA is a symmetric matrix. If A is symmetrizable, then $\mathfrak{g}(A)$ admits a non-degenerate invariant bilinear form and the restriction of this form induces a non-degenerate bilinear form $(-,-)$ on $\mathfrak{h}^*$. In this case, for each principal root α the coroot $\alpha^\vee$ is given by the formula $\langle \mu, \alpha^\vee \rangle = \frac{2(\mu,\alpha)}{(\alpha,\alpha)}$ for any $\mu \in \mathfrak{h}^*$.

1.2.5. Take $\Pi' \in \Theta$. Recall that if $\alpha \in \Pi'$ is odd and is such that $\langle \alpha, \alpha^\vee \rangle \neq 0$, then 2α is a root. Thus $\alpha \in \Pi'$ is principal iff $\langle \alpha, \alpha^\vee \rangle \neq 0$.

Since the odd reflections do not change the set of even positive roots, all principal roots are positive.

For a principal root α let $s_\alpha \in GL(\mathfrak{h}^*)$ be the reflection $\mu \mapsto \mu - \langle \mu, \alpha^\vee \rangle \alpha$. If $\alpha \in \Pi$, then $s_\alpha(\Delta_+(\Pi) \setminus \{\alpha\}) = \Delta_+(\Pi) \setminus \{\alpha\}$. If $\alpha/2 \in \Pi$, then $s_\alpha(\Delta_+(\Pi) \setminus \{\alpha, \alpha/2\}) = \Delta_+(\Pi) \setminus \{\alpha, \alpha/2\}$.

1.2.6. **Definition.** The *Weyl group* W is the subgroup of $GL(\mathfrak{h}^*)$ generated by the reflections s_α with respect to the principal roots. Clearly, $\det s_\alpha = -1$ so $\det w = \pm 1$ for each $w \in W$. Denote by $\mathrm{sgn} : W \to \{\pm 1\}$ the group homomorphism $\mathrm{sgn}(w) := \det w$.

One has $W\Delta = \Delta$.

1.2.7. *Remark.* Let $\mathfrak{g}$ be a finite-dimensional Kac-Moody superalgebra. By [S], $\mathfrak{g}$ satisfies the assumption 1.2.1. Let Δ be the root system of $\mathfrak{g}$. In this case $\mathfrak{g}_0$ is a reductive Lie algebra so $\Delta_{\overline{0}}$ is a root system of finite type. The set of principal roots in Δ is a set of simple roots in $\Delta_{\overline{0}}$ (corresponding to the set of positive roots $\Delta_{\overline{0}} \cap \Delta_+$). In particular, the Weyl group of $\mathfrak{g}$ coincides with the Weyl group of $\Delta_{\overline{0}}$.

Consider the case when $\mathfrak{g} \neq \mathfrak{gl}(n,n)$. The affinization $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ is a Kac-Moody superalgebra, satisfying the assumption 1.2.1 (see [S]). Let $\hat{\Delta}$ be the root system of $\hat{\mathfrak{g}}$. In this case $\hat{\Delta}_{\overline{0}}$ is a disjoint union of affine root systems (which are the affinizations of irreducible components of $\Delta_{\overline{0}}$) and the set of principal roots in $\hat{\Delta}$ is a set of simple roots in $\hat{\Delta}_{\overline{0}}$ (corresponding to the set of positive roots $\hat{\Delta}_{\overline{0}} \cap \hat{\Delta}_+$). In particular, the Weyl group of $\hat{\mathfrak{g}}$ coincides with the Weyl group of $\hat{\Delta}_{\overline{0}}$, so it is the direct product of affine Weyl groups.

1.2.8. In the sequel we will use the following lemma.

Lemma. *For any $w \in W$ the set $R(w) := \Delta_+ \cap w^{-1}\Delta_-$ is finite.*

Proof. Let α be a principal root, i.e. $\alpha \in \Pi'$ or $\frac{1}{2}\alpha \in \Pi'$ for some $\Pi' \in \Theta$; let Δ'_+ be the corresponding set of the positive roots. By above, $\Delta'_+ \cap s_\alpha \Delta'_- \subset \{\alpha, \frac{1}{2}\alpha\}$. Therefore $\Delta_+ \cap s_\alpha(\Delta_-) \subset \{\alpha, \frac{1}{2}\alpha\} \cup \big(\Delta_+ \setminus \Delta'_+\big)$. Since $\Pi' \in \Theta$, the set $\Delta_+ \setminus \Delta'_+$ is finite. Hence $R(s_\alpha)$ is finite as well.

Now let $w = s_\alpha y$, where $y \in W$ is such that $R(y)$ is finite. One has

$$R(w) \subset R(y) \cup \{\gamma \in \Delta_+ |\ y\gamma \in \Delta_+ \cap s_\alpha(\Delta_-)\} \subset R(y) \cup y^{-1}R(s_\alpha),$$

so $R(w)$ is finite. The claim follows. $\square$

1.3. Set

$$Q^+ = \sum_{\alpha \in \Pi} \mathbb{Z}_{\geq 0}\alpha, \quad P := \{\lambda \in \mathfrak{h}^* |\ \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Pi \text{ s.t. } \langle \alpha, \alpha^\vee \rangle \neq 0\}.$$

Clearly, P is an additive subgroup of $\mathfrak{h}^*$. The conditions on the Cartan matrix in 1.1.1 ensure that $\Delta \subset P$; in particular, $P - Q^+ \subset P$. Introduce the standard partial order on P by $\mu \leq \nu$ if $(\nu - \mu) \in Q^+$. Introduce the height function $\mathrm{ht} : Q^+ \to \mathbb{Z}_{\geq 0}$ by $\mathrm{ht}(\sum_{\alpha \in \Pi} m_\alpha \alpha) := \sum_{\alpha \in \Pi} m_\alpha$.

1.3.1. Choose ρ such that $\langle\rho,\alpha^\vee\rangle=\frac{1}{2}\langle\alpha,\alpha^\vee\rangle$ for each $\alpha\in\Pi$. For each $\Pi'\in\Theta$ set $\rho_{\Pi'}:=\rho+\sum_{\beta\in\Delta_+(\Pi)\setminus\Delta_+(\Pi')}\beta$. One readily sees that $\langle\rho_{\Pi'},\alpha^\vee\rangle=\frac{1}{2}\langle\alpha,\alpha^\vee\rangle$ for each $\alpha\in\Pi'$. The assumption 1.2.1 ensures that $\langle\beta,\alpha^\vee\rangle\in\mathbb{Z}$ for each $\alpha\in\Pi'$ such that $\langle\alpha,\alpha^\vee\rangle\neq 0$. We conclude that $\langle\rho,\alpha^\vee\rangle\in\mathbb{Z}$ for each $\alpha\in\Pi'$ such that $\langle\alpha,\alpha^\vee\rangle\neq 0$ and each $\Pi'\in\Theta$. In particular, $W\rho\subset(\rho+\sum_{\alpha\in\Delta}\mathbb{Z}\alpha)$.

1.3.2. **Lemma.** *Let Π_+ be the set of principal roots satisfying $\langle\rho,\alpha^\vee\rangle\geq 0$ and let W_+ be the subgroup of W generated by the reflections $\{s_\alpha,\alpha\in\Pi_+\}$.*

(i) One has $\rho-w\rho\in Q^+$ for any $w\in W_+$.

(ii) If $w=s_{\alpha_{i_1}}\dots s_{\alpha_{i_r}}$ is a reduced decomposition of $w\in W_+$, then

$$\mathrm{ht}(\rho-w\rho)\geq|\{j:\langle\rho,\alpha_{i_j}\rangle\neq 0\}|.$$

(iii) The stabilizer of ρ in W_+ is generated by the reflections $\{s_\alpha|\ \alpha\in\Pi_+\ \&\ \langle\rho,\alpha\rangle=0\}$.

Proof. By [S], Cor. 4.10, W is the Weyl group of a Kac-Moody algebra, whose set of simple roots coincides with the set of principal roots in Δ. Therefore W_+ is the Weyl group of a Kac-Moody algebra, whose set of simple roots coincides with Π_+. For $w\in W_+$, let $l(w)$ be the length of w. Write $w=w's_\alpha$, where $l(w)>l(w')$ and $\alpha\in\Pi_+$. By [J], A.1, the inequality $l(w)>l(w')$ implies that $w'\alpha$ is a non-negative linear combination of elements of Π_+, so $w'\alpha\in\Delta_+$. One has

$$\rho-w\rho=\rho-w'\rho+\langle\rho,\alpha\rangle w'\alpha.$$

By 1.3.1, $\langle\rho,\alpha\rangle\in\mathbb{Z}$. Since $\alpha\in\Pi_+$, one has $\langle\rho,\alpha\rangle\in\mathbb{Z}_{\geq 0}$ so $\langle\rho,\alpha\rangle w'\alpha\in Q^+$ and $\langle\rho,\alpha\rangle w'\alpha=0$ iff $\langle\rho,\alpha\rangle=0$. The assertions (i), (ii) follow by induction on the length of w; (iii) follows from (ii). □

1.4. **The algebra $\mathcal{R}$.** Call a Q^+-*cone* a set of the form $(\lambda-Q^+)$, where $\lambda\in\mathfrak{h}^*$.

For a formal sum of the form $Y:=\sum_{\nu\in P}b_\nu e^\nu,\ b_\nu\in\mathbb{Q}$ define the *support* of Y by $\mathrm{supp}(Y):=\{\nu|\ b_\nu\neq 0\}$. Let $\mathcal{R}$ be a vector space over $\mathbb{Q}$, spanned by the sums of the form $\sum_{\nu\in Q^+}b_\nu e^{\lambda-\nu}$, where $\lambda\in P,\ b_\nu\in\mathbb{Q}$. In other words, $\mathcal{R}$ consists of the formal sums $Y=\sum_{\nu\in P}b_\nu e^\nu$ with the support lying in a finite union of Q^+-cones. Note that for any non-zero $Y\in\mathcal{R}$ the support of Y has a maximal element (with respect to the order introduced in 1.3).

Clearly, $\mathcal{R}$ has a structure of commutative algebra over $\mathbb{Q}$. One has If $Y\in\mathcal{R}$ is such that $YY'=1$ for some $Y'\in\mathcal{R}$, we write $Y^{-1}:=Y'$.

1.4.1. *Action of the Weyl group.* For $w\in W$ set $w(\sum_{\nu\in P}b_\nu e^\nu):=\sum_{\nu\in P}b_\nu e^{w\nu}$. One has $wY\in\mathcal{R}$ iff $w(\mathrm{supp}\,Y)$ is a subset of a finite union of Q^+-cones.

Let W' be a subgroup of W. Let $\mathcal{R}_{W'}:=\{Y\in\mathcal{R}|\ wY\in\mathcal{R}$ for each $w\in W'\}$. Clearly, $\mathcal{R}_{W'}$ is a subalgebra of $\mathcal{R}$.

1.4.2. *Infinite products.* An infinite product of the form $Y=\prod_{\alpha\in X}(1+a_\alpha e^{-\alpha})^{r(\alpha)}$, where $a_\alpha\in\mathbb{Q},\ \ r(\alpha)\in\mathbb{Z}_{\geq 0}$ and $X\subset\Delta$ is such that the set $X\setminus\Delta_+$ is finite, can be naturally viewed as an element of $\mathcal{R}$; clearly, this element does not depend on the order of factors. Let $\mathcal{Y}$ be the set of such infinite products. For any $w\in W$ the infinite product

$$wY:=\prod_{\alpha\in X}(1+a_\alpha e^{-w\alpha})^{r(\alpha)},$$

is again an infinite product of the above form, since the set $w\Delta_+\setminus\Delta_+=-(w\Delta_-\cap\Delta_+)$ is finite by Lemma 1.2.8. Hence $\mathcal{Y}$ is a W-invariant multiplicative subset of $\mathcal{R}_W$.

It is easy to see that the elements of $\mathcal{Y}$ are invertible in $\mathcal{R}$: using the geometric series we can expand Y^{-1} (for example, for $\alpha\in\Delta_+$ one has $(1-e^{\alpha})^{-1}=-e^{-\alpha}(1-e^{-\alpha})^{-1}=-\sum_{i=1}^{\infty}e^{-i\alpha}$).

1.4.3. *The subalgebra $\mathcal{R}'$.* Denote by $\mathcal{R}'$ the localization of $\mathcal{R}_W$ by $\mathcal{Y}$. By above, $\mathcal{R}'$ is a subalgebra of $\mathcal{R}$. Observe that $\mathcal{R}'\not\subset\mathcal{R}_W$: for example, $(1-e^{-\alpha})\in\mathcal{R}'$, but $(1-e^{-\alpha})^{-1}=\sum_{j=0}^{\infty}e^{-j\alpha}\not\in\mathcal{R}_W$. We extend the action of W from $\mathcal{R}_W$ to $\mathcal{R}'$ by setting $w(Y^{-1}Y'):=(wY)^{-1}(wY')$.

An infinite product of the form $Y=\prod_{\alpha\in X}(1+a_\alpha e^{-\alpha})^{r(\alpha)}$, where a_α, X are as above and $r(\alpha)\in\mathbb{Z}$ lies in $\mathcal{R}'$ and $wY=\prod_{\alpha\in X}(1+a_\alpha e^{-w\alpha})^{r(\alpha)}$. One has

$$\operatorname{supp}(Y)\subset\lambda'-Q^+,\quad\text{where }\lambda':=-\sum_{\alpha\in X\setminus\Delta_+:a_\alpha\neq 0}r_\alpha\alpha.$$

1.4.4. Let W' be a subgroup of W. For $Y\in\mathcal{R}'$ we say that Y *is W'-invariant (resp., W'-skew-invariant)* if $wY=Y$ (resp., $wY=\operatorname{sgn}(w)Y$) for each $w\in W'$.

Let $Y=\sum a_\mu e^\mu\in\mathcal{R}_{W'}$ be W'-skew-invariant. Then $a_{w\mu}=(-1)^{\operatorname{sgn}(w)}a_\mu$ for each μ and $w\in W'$. In particular, $W'\operatorname{supp}(Y)=\operatorname{supp}(Y)$, and, moreover, for each $\mu\in\operatorname{supp}(Y)$ one has $\operatorname{Stab}_{W'}\mu\subset\{w\in W'|\ \operatorname{sgn}(w)=1\}$. The condition $Y\in\mathcal{R}_{W'}$ is essential: for example, for $W'=\{\mathrm{id},s_\alpha\}$, the expression $Y:=e^\alpha-e^{-\alpha}$ is W'-skew-invariant, so $Y^{-1}=e^{-\alpha}(1-e^{-2\alpha})^{-1}$ is also W'-skew-invariant, but $\operatorname{supp}(Y^{-1})=-\alpha,-3\alpha,\dots$ is not s_α-invariant.

Take $Y=\sum a_\mu e^\mu\in\mathcal{R}_{W'}$ and set $\sum_{w\in W'}\operatorname{sgn}(w)wY=:\sum b_\mu e^\mu$. One has $b_\mu=\sum_{w\in W'}\operatorname{sgn}(w)a_{w\mu}$ so $b_\mu=\operatorname{sgn}(w)b_{w\mu}$ for each $w\in W'$. We conclude that

$$Y\in\mathcal{R}_{W'}\ \&\ \sum_{w\in W'}\operatorname{sgn}(w)wY\in\mathcal{R}\implies\left\{\begin{array}{l}\sum_{w\in W'}\operatorname{sgn}(w)wY\in\mathcal{R}_{W'};\\ \sum_{w\in W'}\operatorname{sgn}(w)wY\text{ is }W'\text{-skew-invariant;}\\ \operatorname{supp}(\sum_{w\in W'}\operatorname{sgn}(w)wY)\text{ is }W'\text{-stable.}\end{array}\right.$$

1.5. For each $\Pi'\in\Theta$ (see 1.2.2) introduce the following elements of $\mathcal{R}$:

$$R(\Pi')_0:=\prod_{\alpha\in\Delta_+(\Pi')\cap\Delta_{\overline{0}}}(1-e^{-\alpha}),\qquad R(\Pi')_1:=\prod_{\alpha\in\Delta_+(\Pi')\cap\Delta_{\overline{1}}}(1+e^{-\alpha}),\quad R(\Pi'):=\frac{R(\Pi')_0}{R(\Pi')_1}.$$

We set

$$R_0 := R(\Pi)_0, \quad R_1 := R(\Pi)_1, \quad R := R(\Pi).$$

One readily sees from 1.3.1 that $R(\Pi')e^{\rho_{\Pi'}} = Re^{\rho}$ for any $\Pi' \in \Theta$.

1.5.1. **Lemma.** *Re^{ρ} is a W-skew-invariant element of $\mathcal{R}'$.*

Proof. By 1.4.2, $R_0, R_1 \in \mathcal{Y}$ so $Re^{\rho} \in \mathcal{R}'$. Let α be a principal root. If $\alpha \in \Pi'$, then $s_\alpha(\Delta'_+ \setminus \{\alpha\}) = \Delta'_+ \setminus \{\alpha\}$. If $\alpha/2 \in \Pi'$, then $s_\alpha(\Delta'_+ \setminus \{\alpha, \alpha/2\}) = \Delta'_+ \setminus \{\alpha, \alpha/2\}$. In both cases $s_\alpha(R(\Pi')e^{\rho_{\Pi'}}) = -R(\Pi')e^{\rho_{\Pi'}}$. By 1.5, $R(\Pi')e^{\rho_{\Pi'}} = Re^{\rho}$. The claim follows. □

2. Proof of the denominator identity

We retain notation of Sect. 0.

Fix triangular decomposition of the reductive Lie algebra $\mathfrak{g}_0$. By [S], any two sets of simple roots of $\mathfrak{g}$, which are compatible with a triangular decomposition of the reductive Lie algebra $\mathfrak{g}_0$, are connected by a chain of odd reflections. By 1.5, both sides of the denominator identity $\hat{R}e^{\hat{\rho}} = \sum_{w\in T} w(Re^{\hat{\rho}})$ do not change if we substitute Π by $s_\beta\Pi$, where β is a simple odd root of $\mathfrak{g}$. Hence it is enough to prove the denominator identity for one choice of Π; this is done in this section.

2.1. **Another form of denominator identity.** Let us recall the denominator identity for $\mathfrak{g}$ (see [KW],[G] for a proof).

Recall that $S \subset \Delta_{\overline{1}}$ is called a *maximal isotropic set of roots* if S is a basis of a maximal isotropic subspace in $\mathfrak{h}^*$ with respect to the form $(-,-)$. By [KW], there exists a maximal isotropic set of roots S and each such S is a subset of a set of simple roots (for each S there exists Π such that $S \subset \Pi$).

Fix a maximal isotropic set of roots S and a set of simple roots Π such that $S \subset \Pi$. The denominator identity for $\mathfrak{g}$ takes the following form:

$$Re^{\rho} = \sum_{w\in W^\#} \mathrm{sgn}(w) w\Big(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big), \tag{1}$$

where $\rho \in \mathfrak{h}^*$ is such that $2(\rho,\alpha) = (\alpha,\alpha)$ for each $\alpha \in \Pi$. Note that $\hat{\rho}-\rho$ is $W^\#$-invariant. Using (1) we obtain

$$\begin{aligned}\sum_{y\in T} y(Re^{\hat{\rho}}) &= \sum_{y\in T} y\big(e^{\hat{\rho}-\rho}Re^{\rho}\big) = \sum_{y\in T} y\big(e^{\hat{\rho}-\rho}\sum_{w\in W^\#}\mathrm{sgn}(w)w\big(\tfrac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\big)\big)\\ &= \sum_{y\in T} y\big(\sum_{w\in W^\#}\mathrm{sgn}(w)w\big(\tfrac{e^{\hat{\rho}}}{\prod_{\beta\in S}(1+e^{-\beta})}\big)\big) = \sum_{w\in \hat{W}^\#} w\big(\tfrac{e^{\hat{\rho}}}{\prod_{\beta\in S}(1+e^{-\beta})}\big).\end{aligned}$$

Hence the denominator identity for $\hat{\mathfrak{g}}$ can be rewritten as

$$\hat{R}e^{\hat{\rho}} = \sum_{w\in \hat{W}^\#} \mathrm{sgn}(w) w\Big(\frac{e^{\hat{\rho}}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big). \tag{2}$$

We set
$$Y:=\sum_{w\in\hat{W}^{\#}}\operatorname{sgn}(w)w\Big(\frac{e^{\hat{\rho}}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big).$$

2.2. **Notation.** We set
$$\Delta_2=\{\alpha\in\Delta_{\overline{0}}|\ (\alpha,\alpha)<0\}.$$
Then $\Delta_{\overline{0}}=\Delta^{\#}\coprod\Delta_2$, and $\Delta^{\#},\Delta_2$ are root systems of semisimple Lie algebras.

Denote by δ the minimal imaginary root in $\hat{\Delta}$. Let Π be a set of simple roots for Δ_+ and $\theta\in\Delta_+$ be a maximal root. Recall that $\hat{\Pi}=\Pi\cup\{\delta-\theta\}$ is the set of simple roots for $\hat{\Delta}_+=\cup_{s=1}^{\infty}\{s\delta+\Delta\}\cup\Delta_+$.

For $\mathfrak{g}\neq B(n,n)$, we fix a set of simple roots Π for Δ such that

$$\begin{array}{ll}(i) & \Pi \text{ contains a maximal isotropic set of roots } S;\\ (ii) & \forall\alpha\in\Pi\ \ (\alpha,\alpha)\geq 0;\\ (iii) & \theta\in\Delta^{\#},\end{array}$$

see 3.2 for a choice of Π. For $\mathfrak{g}=B(n,n)$ we choose Π as in 3.2; in this case the properties (i) and (ii) hold, but θ is isotropic. Note that in all cases $(\theta,\theta)\geq 0$. Combining with (ii), we get $(\hat{\rho},\beta)=(\beta,\beta)/2\geq 0$ for all $\beta\in\hat{\Pi}$. Set $\hat{Q}^+:=\sum_{\alpha\in\Pi}\mathbb{Z}_{\geq 0}\alpha$. We obtain
$$(\hat{\rho},\hat{Q}^+)\geq 0,\quad \frac{(\hat{\rho},\alpha)}{(\alpha,\alpha)}\geq 0\ \text{ for }\alpha\in\hat{\Delta}^{\#}.$$

2.3. **Support of $\hat{R}e^{\hat{\rho}}$.** Set
$$U:=\{\mu\in\hat{\rho}-\hat{Q}^+|\ (\mu,\mu)=(\hat{\rho},\hat{\rho})\}.$$
From representation theory we know that the character of the trivial $\hat{\mathfrak{g}}$-module is a linear combination of the characters of Verma $\hat{\mathfrak{g}}$-modules $M(\lambda)$, where $\lambda\in-\hat{Q}$ and $(\lambda+\hat{\rho},\lambda+\hat{\rho})=(\hat{\rho},\hat{\rho})$ (since $\hat{\mathfrak{g}}$ admits the Casimir element). Therefore $1=\sum_{\lambda\in U-\hat{\rho}}a_{\lambda}\hat{R}^{-1}e^{\lambda}$ that is
$$\operatorname{supp}(\hat{R}e^{\hat{\rho}})\subset U.$$

2.4. **Support of Y.** Expanding the summands of Y we obtain
$$\operatorname{supp}\Big(\frac{e^{w\hat{\rho}}}{\prod_{\beta\in S}(1+e^{-w\beta})}\Big)\subset\{w\hat{\rho}-\hat{Q}^+\}\cap\{w\hat{\rho}+\sum_{\beta\in S}\mathbb{Z}w\beta\}.$$
Since $(\hat{\rho},S)=(S,S)=0$ this implies

$$\operatorname{supp}\Big(\frac{e^{w\hat{\rho}}}{\prod_{\beta\in S}(1+e^{-w\beta})}\Big)\subset\{\mu\in w\hat{\rho}-\hat{Q}^+|\ (\mu,\mu)=(\hat{\rho},\hat{\rho})\}. \tag{3}$$

2.4.1. **Lemma.** *(i) Y is a well-defined element of $\mathcal{R}$ (see 1.4 for the notation);*

(ii) $\mathrm{supp}(Y)\subset U$*;*

(iii) for $\mathfrak{g}\neq B(n,n)$ the coefficient of $e^{\hat\rho}$ in Y is equal to 1.

Proof. By [S], the set of principal roots of $\hat\Delta_+$ is the set of simple roots of $\hat\Delta_{+,0}$. By 2.2, $\hat W^{\#}$ is a subgroup of the group W_+ introduced in Lemma 1.3.2. By above, $\mathrm{supp}\big(\frac{e^{w\hat\rho}}{\prod_{\beta\in S}(1+e^{-w\beta})}\big)\subset w\hat\rho-\hat Q^+$. In the light of Lemma 1.3.2 for (i) it is enough to show that $H_r:=\{w\in\hat W^{\#}|\ \mathrm{ht}(\hat\rho-w\hat\rho)\leq r\}$ is finite for each r.

Let Σ be the set of simple roots of $\hat\Delta^{\#}_+$. Set $\Sigma_0:=\{\alpha\in\Sigma|\ (\hat\rho,\alpha)=0\}$. By Lemma 1.3.2, $H_0=\mathrm{Stab}_{\hat W^{\#}}\hat\rho$ is the subgroup of $\hat W^{\#}$ generated by the reflections $\{s_\alpha:\alpha\in\Sigma_0\}$ and any $w\in H_r$ is of the form $w_1s_{\beta_1}w_2s_{\beta_2}w_3\dots s_{\beta_r}w_{r+1}$, where $w_j\in H_0$ and $\beta_j\in\Sigma\setminus\Sigma_0$. This means that the finiteness of H_0 implies the finiteness of H_r for $r\geq 0$ and that H_0 is the Weyl group of the Dynkin diagram corresponding to Σ_0. Hence for (i) it is enough to verify that the Dynkin diagram of Σ_0 is of finite type. This can be shown as follows. Observe that Σ is an indecomposable Dynkin diagram of affine type. Since $\Sigma_0\subset\Sigma$, it is enough to verify that $\Sigma_0\neq\Sigma$. Since $(\hat\rho,\delta)=h^\vee\neq 0$, there exists $\beta\in\hat\Pi$ such that $(\hat\rho,\beta)\neq 0$ that is $(\beta,\beta)\neq 0$. By 2.2, $(\alpha,\alpha)\geq 0$ for all $\alpha\in\hat\Pi$, so $(\beta,\beta)>0$. Hence β or 2β belongs to $\hat\Delta^{\#}$. Therefore $(\hat\rho,\hat\Delta^{\#})\neq 0$ so $\Sigma_0\neq\Sigma$. This establishes (i).

Combining (3) and Lemma 1.3.2 (i), we obtain $\mathrm{supp}(Y)\subset U$, thus (ii).

Let us show that the coefficient of $e^{\hat\rho}$ in Y is 1 for $\mathfrak{g}\neq B(n,n)$. Indeed, by above, $\hat\rho\in\mathrm{supp}\big(\frac{e^{w\hat\rho}}{\prod_{\beta\in S}(1+e^{-w\beta})}\big)$ forces $w\in H_0$. By 2.2, for $\mathfrak{g}\neq B(n,n)$ one has $\theta\in\Delta^{\#}$ so $\alpha_0\in\Sigma\setminus\Sigma_0$ and thus $H_0\subset W^{\#}$. Therefore the coefficient of $e^{\hat\rho}$ in Y is equal to the coefficient of $e^{\hat\rho}$ in the expression

$$\sum_{w\in W^{\#}}\mathrm{sgn}(w)w(\frac{e^{\hat\rho}}{\prod_{\beta\in S}(1+e^{-\beta})})=e^{\hat\rho-\rho}\sum_{w\in W^{\#}}\mathrm{sgn}(w)w(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}).$$

Using the denominator identity (1) we get $\sum_{w\in W^{\#}}\mathrm{sgn}(w)w(\frac{e^{\hat\rho}}{\prod_{\beta\in S}(1+e^{-\beta})})=Re^{\hat\rho}$. Clearly, the coefficient of $e^{\hat\rho}$ in $Re^{\hat\rho}$ is equal to 1. This establishes (iii). □

2.4.2. **Lemma.** *For $\mathfrak{g}=B(n,n)$ the coefficient of $e^{\hat\rho}$ in Y is equal to* 1.

Proof. Expanding the expression $w(\frac{e^{\hat\rho}}{\prod_{\beta\in S}(1+e^{-\beta})})$, we see that the coefficient of $e^{\hat\rho}$ in Y is equal to the sum $\sum_{w\in H}\mathrm{sgn}(w)$, where

$$H:=\{w\in\hat W^{\#}|\ w\hat\rho=\hat\rho\ \&\ wS\subset\hat\Delta_+\}.$$

Take $w\in H$ and write $w=t_\mu y$, where $y\in W^{\#}$ and $t_\mu\in T$ (see Sect. 0 for notation) is given by

$$t_\mu(\lambda)=\lambda+(\lambda,\delta)\mu-((\lambda,\mu)+\frac{(\mu,\mu)}{2}(\lambda,\delta))\delta \ \text{ for } \lambda\in\hat{\mathfrak{h}}^*.$$

Retain notation of 3.2. One has $S=\{\delta_i-\varepsilon_i\}_{i=1}^n$ and $w(\delta_i-\varepsilon_i)=\delta_i-y\varepsilon_i+(y^{-1}\mu,\varepsilon_i)\delta$, because $(\mu,\Delta_2)=0$, see 2.2 for notation. The condition $wS\subset\hat{\Delta}_+$ gives $(y^{-1}\mu,\varepsilon_i)\geq 0$ for $i=1,\dots,n$. On the other hand,

$$w\hat{\rho}=y\hat{\rho}+h^\vee\mu-((\hat{\rho},y^{-1}\mu)+\frac{(\mu,\mu)}{2}h^\vee)\delta.$$

Since μ and $\hat{\rho}-y\hat{\rho}$ lie in the $\mathbb{Q}$-span of $\Delta^{\#}$, the condition $w\hat{\rho}=\hat{\rho}$ gives $(\hat{\rho},y^{-1}\mu)+\frac{(\mu,\mu)}{2}h^\vee=0$. One has $(\mu,\mu)\geq 0$ since μ lies in the $\mathbb{Q}$-span of $\Delta^{\#}$. Since $h^\vee>0$, we get

$$0\geq(\hat{\rho},y^{-1}\mu)=(\rho,y^{-1}\mu)=(\frac{1}{2}\sum_{i=1}^n\varepsilon_i,y^{-1}\mu).$$

Using the above inequalities $(y^{-1}\mu,\varepsilon_i)\geq 0$, we conclude that $0=(\hat{\rho},y^{-1}\mu)=\frac{(\mu,\mu)}{2}h^\vee$ that is $\mu=0$. Therefore $w=y\in W^{\#}$. Now we can obtain the statement using the argument of the proof of Lemma 2.4.1 (iii), or, by observing that $y\hat{\rho}=\hat{\rho}$ implies that y permutes $\{\varepsilon_i\}_{i=1}^n$ and then $yS\subset\hat{\Delta}_+$ forces $y=\{\mathrm{id}\}$. $\square$

2.5. Assume that the denominator identity does not hold so $\hat{R}e^{\hat{\rho}}-Y\neq 0$.

The coefficient of $e^{\hat{\rho}}$ in $\hat{R}e^{\hat{\rho}}$ is equal to 1. From Lemmas 2.4.1, 2.4.2 we get

$$\operatorname{supp}\big(\hat{R}e^{\hat{\rho}}-Y\big)\subset U\setminus\{\hat{\rho}\}. \tag{4}$$

Let $\hat{\rho}^{\#}$ be the standard element for the root system $\hat{\Delta}^{\#}=(\hat{\Delta}^{\#}\cap\hat{\Delta}_+)\coprod(\hat{\Delta}^{\#}\cap\hat{\Delta}_-)$. Set

$$X:=\hat{R}_1e^{\hat{\rho}^{\#}-\hat{\rho}}(\hat{R}e^{\hat{\rho}}-Y).$$

By the above assumption $X\neq 0$.

By 1.4.2, $\hat{R}_0,\hat{R}_1e^{\hat{\rho}^{\#}-\hat{\rho}}\in\mathcal{R}_W$, where W is the Weyl group of $\hat{\mathfrak{g}}$. By Lemma 1.5.1, $\hat{R}e^{\hat{\rho}},\hat{R}_0e^{\hat{\rho}^{\#}}$ are $\hat{W}^{\#}$-skew-invariant elements of $\mathcal{R}'$. Therefore for each $w\in\hat{W}^{\#}$ one has

$$\frac{\hat{R}_0e^{\hat{\rho}^{\#}}}{\hat{R}_1e^{\hat{\rho}^{\#}-\hat{\rho}}}=\hat{R}e^{\hat{\rho}}=\operatorname{sgn}(w)w(\hat{R}e^{\hat{\rho}})=\operatorname{sgn}(w)\frac{w(\hat{R}_0e^{\hat{\rho}^{\#}})}{w(\hat{R}_1e^{\hat{\rho}^{\#}-\hat{\rho}})}=\frac{\hat{R}_0e^{\hat{\rho}^{\#}}}{w(\hat{R}_1e^{\hat{\rho}^{\#}-\hat{\rho}})}.$$

Thus $\hat{R}_1e^{\hat{\rho}^{\#}-\hat{\rho}}$ is a $\hat{W}^{\#}$-invariant element of $\mathcal{R}_W$. Therefore

$$\hat{R}_1e^{\hat{\rho}^{\#}-\hat{\rho}}Y=\sum_{w\in\hat{W}^{\#}}\operatorname{sgn}(w)w\big(\hat{R}_1e^{\hat{\rho}^{\#}}\prod_{\beta\in S}(1+e^{-\beta})^{-1}\big)$$

and so

$$X=\hat{R}_0e^{\hat{\rho^{\#}}}-\sum_{w\in\hat{W}^{\#}}\operatorname{sgn}(w)wZ,\ \text{ where } Z:=e^{\hat{\rho}^{\#}}\prod_{\beta\in\hat{\Delta}_{\bar{1}+}\setminus S}(1+e^{-\beta}).$$

Since $\hat{\rho}^{\#}-\hat{\rho}\in \operatorname{supp}(\hat{R}_1 e^{\hat{\rho}^{\#}-\hat{\rho}})\subset \hat{\rho}^{\#}-\hat{\rho}-\hat{Q}^+$, we obtain from (4)

$$\max\operatorname{supp}(X)=\hat{\rho^{\#}}-\hat{\rho}+\max\operatorname{supp}\big(\hat{R}e^{\hat{\rho}}-Y\big)\subset \hat{\rho^{\#}}-\hat{\rho}+(U\setminus\{\hat{\rho}\}),$$

that is

$$\max\operatorname{supp}(X)\subset\{\mu\in\hat{\rho^{\#}}-(\hat{Q}^+\setminus\{0\})|\ (\mu-\hat{\rho^{\#}}+\hat{\rho},\mu-\hat{\rho^{\#}}+\hat{\rho})=(\hat{\rho},\hat{\rho})\}. \tag{5}$$

2.5.1. Recall [J], A.1.18, [KT], that for any $\lambda\in\hat{\mathfrak{h}}^*$ the stabilizer $\operatorname{Stab}_{\hat{W}^{\#}}\lambda$ is either trivial or contains a reflection s_α. Let us call $\lambda\in P$ *regular* if $\operatorname{Stab}_{\hat{W}^{\#}}\lambda=\{\mathrm{id}\}$. Say that the orbit $\hat{W}^{\#}\lambda$ is regular if λ is regular (so the orbit consists of regular points).

By 1.4.2, $Z\in\mathcal{R}_W$. One has $\sum_{w\in\hat{W}^{\#}}\operatorname{sgn}(w)wZ=\hat{R}_1e^{\hat{\rho^{\#}}-\hat{\rho}}Y\in\mathcal{R}$, because $Y\in\mathcal{R}$. In the light of 1.4.4, $\sum_{w\in\hat{W}^{\#}}\operatorname{sgn}(w)wZ$ belongs to $\mathcal{R}_{\hat{W}^{\#}}$ and is $\hat{W}^{\#}$-skew-invariant. By Lemma 1.5.1, $\hat{R}_0e^{\hat{\rho}^{\#}}\in\mathcal{R}_W$ is $\hat{W}^{\#}$-skew-invariant. Hence X belongs to $\mathcal{R}_{\hat{W}^{\#}}$ and is $\hat{W}^{\#}$-skew-invariant. Using 1.4.4 we conlclude that $\operatorname{supp}(X)$ is a union of regular $\hat{W}^{\#}$-orbits.

2.5.2. Take $\nu\in\max\operatorname{supp}X$. Then ν is a maximal element in a regular $\hat{W}^{\#}$-orbit and, by (5), $\nu\in\hat{\rho}^{\#}-(\hat{Q}^+\setminus\{0\})$.

One has $\hat{Q}^+\subset\mathbb{Q}\hat{\Delta}^{\#}+\mathbb{Q}M$, where $M=\Delta_2$ (see 2.2 for notation) if Δ is not of the type $A(m,n),C(n)$, and, for the types $A(m,n),C(n)$ one has $M=\Delta_2\cup\{\xi\}$, where $\xi\in\mathbb{Q}\Delta$ is such that $(\xi,\xi)<0$, $(\xi,\Delta_{\overline{0}})=0$. The element ξ is given in 3.2.

Write $\nu=\hat{\rho^{\#}}+\nu_1+\nu_2$, where $\nu_1\in\mathbb{Q}\hat{\Delta}^{\#}$ and $\nu_2\in\mathbb{Q}M$. Since $\hat{W}^{\#}\nu_2=\nu_2$, the vector $\nu-\nu_2$ is also a maximal element in a regular $\hat{W}^{\#}$-orbit. For each simple root α of $\hat{\Delta}^{\#}_+$ one has

$$\langle\nu-\nu_2,\alpha^\vee\rangle=\langle\nu,\alpha^\vee\rangle\in\langle\hat{\rho}^{\#},\alpha^\vee\rangle-\langle\hat{Q}^+,\alpha^\vee\rangle\subset\mathbb{Z},$$

since $\langle\hat{\rho}^{\#},\alpha^\vee\rangle=1$ and $\langle\hat{\Delta},\alpha^\vee\rangle\subset\mathbb{Z}$, by 1.2.4. In the light of Lemma 3.1.1, $\nu-\nu_2=\hat{\rho}^{\#}+\nu_1\in\hat{\rho}^{\#}-\mathbb{Q}\delta$ so $\nu_1=-s\delta$ for some $s\in\mathbb{Q}$.

Substituting $\nu_1=-s\delta$ and using (5) we get

$$(\hat{\rho}-s\delta+\nu_2,\hat{\rho}-s\delta+\nu_2)=(\hat{\rho},\hat{\rho}). \tag{6}$$

By 2.2, $(\hat{\rho},-s\delta+\nu_2)\leq 0$, since $-s\delta+\nu_2\in-\hat{Q}^+$. Therefore $(\nu_2,\nu_2)\geq 0$. Recall that the form $(-,-)$ is negatively definite on Δ_2 and so is negatively definite on M. Thus $(\nu_2,\nu_2)\geq 0$ gives $\nu_2=0$. Now the formula (6) gives $s=0$ (because $(\hat{\rho},\delta)=h^\vee\neq 0$). Hence $\nu=\hat{\rho}^{\#}$, a contradiction.

3. Appendix

3.1. The following lemma is used in 2.5.2. Let $\mathfrak{g}$ be an affine Lie algebra, let Π be its set of simple roots. Let W be the Weyl group of $\mathfrak{g}$. Define regular W-orbits as in 2.5.1.

3.1.1. **Lemma.** *If $\lambda\in\sum_{\alpha\in\Pi}\mathbb{Q}\alpha$ is such that $\lambda+\rho$ is a maximal element in a regular W-orbit and $\langle\lambda,\alpha^\vee\rangle\in\mathbb{Z}$ for any $\alpha\in\Pi$, then $\lambda\in\mathbb{Q}\delta$, where δ is the minimal imaginary root.*

Proof. For each $\alpha\in\Pi$ set $k_\alpha:=\langle\lambda,\alpha^\vee\rangle$. Since $\lambda+\rho$ is a maximal element in a regular W-orbit one has $s_\alpha(\lambda+\rho)=\lambda+\rho-k_\alpha\alpha\not\geq\lambda+\rho$, so $k_\alpha\not\in\mathbb{Z}_{\leq 0}$. By the assumption, k_α is an integer, so $k_\alpha>0$.

Write $\lambda=\sum_{\alpha\in\Pi}x_\alpha\alpha,\ x_\alpha\in\mathbb{Q}$. Since $\langle\lambda,\alpha^\vee\rangle>0$ for each $\alpha\in\Pi$, we have $Ax\geq 0$, where A is the Cartan matrix of $\mathfrak{g}$ and $x=(x_\alpha)_{\alpha\in\Pi}$. From [K2], Thm. 4.3 it follows that $\sum_{\alpha\in\Pi}x_\alpha\alpha\in\mathbb{Q}\delta$ as required. □

3.2. **Basic Lie superalgebras.** The basic Lie superalgebras with a non-zero Killing form, which are not Lie algebras, are $A(m,n),m\not=n;B(m,n);C(n);D(m,n),m\not=n+1;F_4;G_3$. Below for each of these root systems we give an example of Π satisfying the conditions (i), (ii) of 2.2. In all cases $\Delta^\#$ lies in the lattice spanned by $\{\varepsilon_i\}_{i=1}^{\max(m,n)}$ and Δ_2 lies in the lattice spanned by $\{\delta_i\}_{i=1}^{\min(m,n)}$.

Retain notation of 2.2. In all cases except $B(m,m)$ one has $\theta\in\Delta^\#$ (the condition (iii) in 2.2); for $B(m,m)$ one has $(\theta,\theta)=0$.

In all cases except $A(m,n),C(n)$ one has $\mathbb{Q}\Delta=\mathbb{Q}\Delta_{\overline{0}}$; for $A(m,n),C(n)$ one has $\mathbb{Q}\Delta=\mathbb{Q}\Delta_{\overline{0}}+\mathbb{Q}\xi$, where ξ is described below.

3.2.1. *Case $A(m,n),m\not=n$.* Since $A(m,n)\cong A(n,m)$, we may assume that $m>n$. The roots are $\Delta_{\overline{0}}=\{\varepsilon_i-\varepsilon_j\}_{1\leq i\not=j\leq m}\cup\{\delta_i-\delta_j\}_{1\leq i\not=j\leq n}$, $\Delta_{\overline{1}}=\{\pm(\varepsilon_i-\delta_j)\}_{1\leq i\leq n}^{1\leq j\leq m}$. Set

$$\Pi:=\{\varepsilon_1-\delta_1,\delta_1-\varepsilon_2,\varepsilon_2-\delta_2,\ldots,\delta_n-\varepsilon_{n+1},\varepsilon_{n+1}-\varepsilon_{n+2},\ldots,\varepsilon_{m-1}-\varepsilon_m\}$$

and $S:=\{\varepsilon_i-\delta_i\}$. One has $\theta=\varepsilon_1-\varepsilon_m\in\Delta^\#$. We take $\xi=\frac{\sum_{i=1}^m\varepsilon_i}{m}-\frac{\sum_{j=1}^n\delta_j}{n}$. One has $(\xi,\Delta_{\overline{0}})=0$.

3.2.2. *Case $B(m,n),m<n$.* The roots are $\Delta_{\overline{0}}=\{\pm\varepsilon_i\pm\varepsilon_j;\pm 2\varepsilon_i\}_{1\leq i\not=j\leq n}\cup\{\pm\delta_i\pm\delta_j;\pm\delta_i\}_{1\leq i\not=j\leq m}$, $\Delta_{\overline{1}}=\{\pm\varepsilon_i\pm\delta_j;\pm\varepsilon_i\}_{1\leq i\leq n}^{1\leq j\leq m}$. We take

$$\Pi:=\{\varepsilon_1-\delta_1,\delta_1-\varepsilon_2,\varepsilon_2-\delta_2,\ldots,\varepsilon_m-\delta_m,\delta_m-\varepsilon_{m+1},\varepsilon_{m+1}-\varepsilon_{m+2},\ldots,\varepsilon_{n-1}-\varepsilon_n,\varepsilon_n\}$$

and $S:=\{\varepsilon_i-\delta_i\}_{i=1}^m$. One has $\theta=2\varepsilon_1\in\Delta^\#$.

3.2.3. *Case $B(n,n)$.* The roots are as above. We take

$$\Pi:=\{\delta_1-\varepsilon_1,\varepsilon_1-\delta_2,\ldots,\delta_n-\varepsilon_n,\varepsilon_n\}$$

and $S:=\{\delta_i-\varepsilon_i\}$. One has $\theta=\delta_1+\varepsilon_1$.

3.2.4. *Case* $B(m,n), m \geq n+1$. The roots are $\Delta_{\overline{0}} = \{\pm\varepsilon_i \pm \varepsilon_j; \pm\varepsilon_i\}_{1\leq i\neq j\leq m} \cup \{\pm\delta_i \pm \delta_j; \pm 2\delta_i\}_{1\leq i\neq j\leq n}$, $\Delta_{\overline{1}} = \{\pm\varepsilon_i \pm \delta_j; \pm\varepsilon_i\}_{1\leq i\leq n}^{1\leq j\leq m}$. We take for $m = n+2$

$$\Pi := \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1, \delta_1 - \varepsilon_3, \ldots, \delta_n - \varepsilon_{n+2}, \varepsilon_{n+2}\},$$

and for $m > n+2$

$$\Pi := \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1, \delta_1 - \varepsilon_3, \ldots, \delta_n - \varepsilon_{n+2}, \varepsilon_{n+2} - \varepsilon_{n+3}, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m\}.$$

Then $S := \{\varepsilon_i - \delta_i\}_{i=1}^n$ lies in Π. One has $\theta = \varepsilon_1 + \varepsilon_2 \in \Delta^{\#}$.

3.2.5. *Case* $C(m)$. The roots are $\Delta_{\overline{0}} = \{\pm\varepsilon_i \pm \varepsilon_j; \pm 2\varepsilon_i\}_{1\leq i\neq j\leq m}$, $\Delta_{\overline{1}} = \{\pm\varepsilon_i \pm \delta_1\}_{1\leq j\leq m}$ Set

$$\Pi := \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m - \delta_1, \varepsilon_m + \delta_1\}.$$

One has $\theta = 2\varepsilon_1 \in \Delta^{\#}$. Observe that $\Delta_2 = \emptyset$. We take $\xi := \delta_1$.

3.2.6. *Case* $D(m,n), n \geq m$. The roots are $\Delta_{\overline{0}} = \{\pm\varepsilon_i \pm \varepsilon_j; \pm 2\varepsilon_i\}_{1\leq i\neq j\leq n} \cup \{\pm\delta_i \pm \delta_j\}_{1\leq i\neq j\leq m}$, $\Delta_{\overline{1}} = \{\pm\varepsilon_i \pm \delta_j\}_{1\leq i\leq n}^{1\leq j\leq m}$. We take

$$\Pi := \{\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \varepsilon_2 - \delta_2, \ldots, \delta_m - \varepsilon_m, \varepsilon_m - \varepsilon_{m+1}, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}$$

and $S := \{\varepsilon_i - \delta_i\}_{i=1}^m$. One has $\theta = 2\varepsilon_1 \in \Delta^{\#}$.

3.2.7. *Case* $D(m,n), m > n+1$. The roots are $\Delta_{\overline{0}} = \{\pm\varepsilon_i \pm \varepsilon_j\}_{1\leq i\neq j\leq n} \cup \{\pm\delta_i \pm \delta_j; \pm 2\delta_i\}_{1\leq i\neq j\leq m}$, $\Delta_{\overline{1}} = \{\pm\varepsilon_i \pm \delta_j\}_{1\leq i\leq n}^{1\leq j\leq m}$. For $m = n+2$ set

$$\Pi := \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1, \delta_1 - \varepsilon_3, \ldots, \delta_n - \varepsilon_{n+2}, \delta_n + \varepsilon_{n+2}\},$$

for $m > n+2$ set

$$\Pi := \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1, \delta_1 - \varepsilon_3, \ldots, \delta_n - \varepsilon_{n+2}, \varepsilon_{n+2} - \varepsilon_{n+3}, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_{m-1} + \varepsilon_m\}.$$

Then Π contains $S := \{\varepsilon_i - \delta_i\}_{i=1}^n$. One has $\theta = \varepsilon_1 + \varepsilon_2 \in \Delta^{\#}$.

3.2.8. *Case* $F(4)$. We choose

$$\Pi := \{(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \delta_1)/2; (-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \delta_1)/2; (-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \delta_1)/2; \varepsilon_1 - \varepsilon_2\}.$$

In this case S can be any odd simple root; one has $\theta = \varepsilon_3 - \varepsilon_2 \in \Delta^{\#}$.

3.2.9. *Case* $G(3)$. For $G(3)$ the roots are expressed in terms of linear functions $\varepsilon_1, \varepsilon_2, \varepsilon_3$, corresponding to G_2, $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$, and δ_1, corresponding to A_1; we choose $\Pi = \{\delta_1 - \varepsilon_2, \varepsilon_3 - \delta_1, -\varepsilon_3 - \varepsilon_1\}$. In this case S can be any odd simple root; one has $\theta = \varepsilon_3 - \varepsilon_1 \in \Delta^{\#}$.

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