My own interest in high-dimensional expanders (HDX) is quite limited. In particular, my interest is restricted to the two-dimensional case, which I view as referring to d-regular expander graphs in which the subgraph induces by the d neighbors of each vertex induce an expander (i.e., a d-vertex graph with a normalized second eignenvalue that is bounded away from 1). I find the result of Izhar Oppenheim that asserts that a strong enough version of the latter ("local") condition implies the former ("global" expansion) condition very interesting.[1] The following exposition follows an oral explanation by Irit Dinur.

For a d-regular graph $G = ([n], E)$, we consider the stochastic matrix M that represents a random walk on this graph; that is, the $(i, j)^{\text{th}}$ entry of M equals $1/d$ if $\{i, j\} \in E$ and equals 0 otherwise. Letting $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the eingenvalues of M, which are all in $[-1, +1]$.

For $\lambda \in (0, 1)$, We say that G is an λ-expander if $|\lambda_i| \leq \lambda$ for every $i \in \{2, ..., n\}$. Recall that in such a case G is connected and non-bipartite (i.e., $\lambda_2 < 1$ and $\lambda_n > -1$), and that $\lambda_1 = 1$ is associated with the uniform eigenvector. We shall survey a proof of the following result, which arised in the context of the study of high dimensional expanders (HDX).

Theorem 1 (local expansion implies gloabl expansion): *Let $G = (V, E)$ be a connected regular graph and $\lambda \in (0, 0.5)$. Suppose that for every $v \in V$, the subgraph of G induced by the neighbors of v is a* (regular) *λ-expander. Furthermore, suppose that each of these subgraphs has the same number of edges and that each edge appears in the same number of subgraphs. Then, G is a $\frac{\lambda}{1-\lambda}$-expander.*

Proof: We upper-bound each of the eigenvalues of M by considering the corresponding eigenvector; that is, for $\gamma < 1$, we consider $f : V \to \mathbb{R}$ such that Mf (viewed as a vector) equals $\gamma \cdot f$. We first note that

$$\text{Exp}_{\{u,v\} \in E}\left[f(u) \cdot f(v)\right] = \frac{\langle f, Mf \rangle}{|V|} = \frac{\gamma}{|V|} \cdot \|f\|_2^2. \tag{1}$$

Using the furthermore hypothesis, we observe that a different way of selecting a random edge consists of (uniformly) selecting a random vertex $w \in V$ and (uniformly) selecting a random edge in the subgraph induced by the neighbours of w. Thus, denoting by E_w the set of edges in the subgraph of G induced by the neighbors of w, we have

$$\frac{\gamma}{|V|} \cdot \|f\|_2^2 = \text{Exp}_{w \in V}\left[\text{Exp}_{\{u,v\} \in E_w}\left[f(u) \cdot f(v)\right]\right]. \tag{2}$$

Letting $\Gamma(w)$ denote the set of nerighbours of w in G and $f_w : \Gamma(w) \to \mathbb{R}$ denote the restriction of f to $\Gamma(w)$, we apply the Expander Mixing Lemma (Lemma 2) to the inner expectation in Eq. (2), and obtain (for every $w \in V$)

$$\text{Exp}_{\{u,v\} \in E_w}\left[f_w(u) \cdot f_w(v)\right] \leq \text{Exp}_{u,v \in \Gamma(w)}\left[f_w(u) \cdot f_w(v)\right] + \lambda \cdot \frac{|\langle f_{w,\perp}, f_{w,\perp} \rangle|}{|\Gamma(w)|} \tag{3}$$

[1] This result is a special case of Theorem 1.4 in Oppenheim's paper "Local Spectral Expansion Approach to High Dimensional Expanders Part I: Descent of Spectral Gaps" (*Discrete Comput. Geom.*, Vol. 59.2, pages 293–330, 2018). There is a typo in the statement of Theorem 1.4: The hypothesis is that X is (λ, κ)-local-expander. We ignore κ and note that Opperheim's λ corresponds to our $1 - \lambda$ (since he consideres the normalized Laplacian of the graph whereas we consider the normalized adjacency matrix).

where $f_{w,\perp}$ denotes the projection of f_w on the subspace orthogonal to the all-1 vector. Now, we rewrite Eq. (3) as follows

$$\mathrm{Exp}_{\{u,v\}\in E_w}\left[f(u)\cdot f(v)\right] \;\leq\; \mathrm{Exp}_{u\in\Gamma(w)}[f_w(u)]^2 + \lambda\cdot\frac{\|f_{w,\perp}\|^2}{|\Gamma(w)|} \tag{4}$$

Next, we shall use the following two observations:

1. $\|f_{w,\perp}\|^2 = \|f_w\|^2 - \|f_{w,||}\|^2$, where $f_{w,||}$ denotes the projection of f_w in direction $(1,1,...,1)$.
2. $\|f_{w,||}\|^2 = \sum_{v\in\Gamma(w)}\mathrm{Exp}_{u\in\Gamma(w)}[f_w(u)]^2$, because $f_{w,||}(v) = \mathrm{Exp}_{u\in\Gamma(w)}[f_w(u)]$ for every $v\in\Gamma(w)$.

Combining these observations with Eq. (4), we get

$$\mathrm{Exp}_{\{u,v\}\in E_w}\left[f(u)\cdot f(v)\right] \;\leq\; \mathrm{Exp}_{u\in\Gamma(w)}[f_w(u)]^2 + \lambda\cdot\left(\frac{\|f_w\|^2}{|\Gamma(w)|} - \mathrm{Exp}[f_w]^2\right) \tag{5}$$

Next, rearranging Eq. (5), and combining it with Eq. (2) we get

$$\frac{\gamma}{|V|}\cdot\|f\|_2^2 \;\leq\; (1-\lambda)\cdot\mathrm{Exp}_{w\in V}\left[\mathrm{Exp}_{u\in\Gamma(w)}[f_w(u)]^2\right] + \lambda\cdot\mathrm{Exp}_{w\in V}\left[\frac{\|f_w\|^2}{|\Gamma(w)|}\right] \tag{6}$$

We now analyze the two terms that appear in Eq. (6).

The first term (equiv., $\mathrm{Exp}_{w\in V}[\mathrm{Exp}_{u,v\in\Gamma(w)}[f_w(u)\cdot f_w(v)]]$**).** A key observation is that *the distribution of (u,v) that results by selecting uniformly $w\in V$ and $u,v\in\Gamma(w)$ equals to the distribution of (u,v) obtained by selecting uniformly $u\in V$, $w\in\Gamma(u)$ and $v\in\Gamma(w)$*. Equivalently, once u is selected, the vertex v is determined by taking a two-step random walk from u. Hence,

$$\begin{aligned}
\mathrm{Exp}_{w\in V}\left[\mathrm{Exp}_{u,v\in\Gamma(w)}\left[f(u)\cdot f(v)\right]\right] &= \mathrm{Exp}_{u\in V}\left[f(u)\cdot\mathrm{Exp}_{w\in\Gamma(u),v\in\Gamma(w)}\left[f(v)\right]\right]\\
&= \frac{\langle f, M^2 f\rangle}{|V|}\\
&= \frac{\gamma^2}{|V|}\cdot\|f\|^2
\end{aligned}$$

where the last equality is due to $Mf=\gamma\cdot f$.

The second term (i.e., $\mathrm{Exp}_{w\in V}\left[\frac{\|f_w\|^2}{|\Gamma(w)|}\right]$**).** Here we observe that $\sum_{w\in V}\sum_{v\in\Gamma(w)}\frac{f_w(v)^2}{|\Gamma(w)|} = \sum_{v\in V}f(v)^2 = \|f\|^2$. Hence, $\mathrm{Exp}_{w\in V}\left[\frac{\|f_w\|^2}{|\Gamma(w)|}\right] = \|f\|^2/|V|$.

Plugging the derived equalities into Eq. (6), we get

$$\frac{\gamma}{|V|}\cdot\|f\|_2^2 \;\leq\; (1-\lambda)\cdot\frac{\gamma^2}{|V|}\cdot\|f\|^2 + \lambda\cdot\frac{\|f\|^2}{|V|} \tag{7}$$

which means that $\gamma \leq (1-\lambda)\cdot\gamma^2+\lambda$. Using $\lambda<1$, this simplies to $\gamma\leq(1+\gamma)\cdot\lambda$, and implies $\gamma\leq\lambda/(1-\lambda)$. The theorem follows. $\blacksquare$

Appendix: A general form of the Expander Mixing Lemma

For any function $f : V \to \mathbb{R}$, we denote by $f_{||}$ the projection of f in direction $(1, 1, ..., 1)$, and by $f_\perp$ the projection on the orthogonal subspace.

Lemma 2 (Expander Mixing Lemma, general form): *Let $G = (V, E)$ be a regular λ-expander, then for every two functions $g, h : V \to \mathbb{R}$ it holds that*

$$\left|\mathrm{Exp}_{\{u,v\}\in E}\left[g(u)\cdot h(v)\right] - \mathrm{Exp}_{u,v\in V}\left[g(u)\cdot h(v)\right]\right| \;\leq\; \lambda\cdot\frac{|\langle g_\perp, h_\perp\rangle|}{|V|}$$

In the popular case of Boolean functions, using $\langle g_\perp, h_\perp\rangle \leq \langle g, h\rangle \leq \|g\|\cdot\|h\|$ and letting $A = g^{-1}(1)$ and $B = g^{-1}(1)$, we get

$$\left|\frac{|\{(a,b)\in A\times B : \{a,b\}\in E\}|}{|E|} - \frac{|A|\cdot|B|}{|V|^2}\right| \;\leq\; \lambda\cdot\frac{\sqrt{|A|\cdot|B|}}{|V|}$$

Proof: Viewing functions as vectors, we use the notation Mf and note that $(Mf)(u) = \mathrm{Exp}_{v\in\Gamma(u)}[f(v)]$, where $\Gamma(u)$ is the set of neighbors of u in G. Hence,

$$\begin{aligned}\mathrm{Exp}_{\{u,v\}\in E}\left[g(u)\cdot h(v)\right] &= \mathrm{Exp}_{u\in V}\left[g(u)\cdot((Mh)(u)\right] && (8)\\ &= \frac{\langle g, Mh\rangle}{|V|} && (9)\end{aligned}$$

Recalling that $\langle g, h\rangle = \langle g_{||}, h_{||}\rangle + \langle g_\perp, h_\perp\rangle$, we get

$$\langle g, Mh\rangle = \langle g_{||}, h_{||}\rangle + \langle g_\perp, (Mh)_\perp\rangle \tag{10}$$

and note that $|\langle g_\perp, (Mh)_\perp\rangle| \leq \lambda\cdot|\langle g_\perp, h_\perp\rangle|$. Using $\sum_{v\in V} f_\perp(v) = 0$, we note that $f_{||}(u) = \mathrm{Exp}[f_{||}] = \mathrm{Exp}[f]$ for every $u \in V$, and $\langle g_{||}, h_{||}\rangle = |V|\cdot\mathrm{Exp}[g]\cdot\mathrm{Exp}[h]$ follows. Hence, Eq. (10) implies

$$\left|\langle g, Mh\rangle - |V|\cdot\mathrm{Exp}[g]\cdot\mathrm{Exp}[h]\right| \leq \lambda\cdot|\langle g_\perp, h_\perp\rangle|. \tag{11}$$

Combining Eq. (8) & (9) with Eq. (11), we get

$$\left|\mathrm{Exp}_{\{u,v\}\in E}\left[g(u)\cdot h(v)\right] - \mathrm{Exp}_{u\in V}[g(u)]\cdot\mathrm{Exp}_{v\in V}[h(v)]\right| \leq \lambda\cdot|\langle g_\perp, h_\perp\rangle|/|V|,$$

and the lemma follows. ■