Testing Euclidean Spanners

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Abstract

In this paper we develop a *property testing algorithm* for the problem of testing whether a directed geometric graph with bounded (out)degree is a $(1+\delta)$-spanner.

1 Introduction

Property testing is the computational task of deciding whether a given object has a predetermined property Π or is far away from every object with property Π. Thus, property testing can be viewed as a relaxation of a standard decision problem. The main goal of property testing is to develop randomized algorithms that perform this relaxed decision task by only looking at a small part of the input object, i.e. we want to develop algorithms whose running time is sublinear in the object's description size.

Property testing has been introduced by Rubinfeld and Sudan [30] and the study of combinatorial properties has been initiated by Goldreich, Goldwasser, and Ron [23]. Since then, property testing algorithms have been developed for properties of functions [22, 21, 11], properties of distributions [8, 7], algebraic properties [26], graph and hypergraph properties [23, 3, 14, 10], and geometric properties. In this paper we continue the study of property testing algorithms for *geometric properties*. Previous work on geometric property testing includes testing algorithms for convexity of polygons [18], convexity [29], geometric properties of point sets (for example convex position) and the Euclidean minimum spanning tree [16, 15, 17], and clusterability of point sets [1, 15].

Our contribution In this work, we develop property testing algorithms for Euclidean spanners. A weighted directed geometric graph (P, E) is a directed graph whose vertex set is a set of points in the Euclidean space $\mathbb{R}^d$ and whose edge weights (lengths) are given by the Euclidean distance of the vertices, i.e. edge $[p, q\rangle$ has length $\|p-q\|_2$. A graph is called $(1+\delta)$-spanner, if for every pair of vertices p, q the shortest path distance $d_G(p, q)$ in G is at most $(1+\delta)\cdot\|p-q\|_2$, i.e. the shortest path distance in G is a good approximation of the true distance of the points p and q. Euclidean spanners are a fundamental geometric graph structure as they can be used to approximately solve many geometric proximity problems, and they find applications, for example, in the area of mobile ad-hoc networks. Many different constructions of Euclidean spanners are known. Euclidian spanners with a linear number of edges can for example be constructed by using so-called Θ-graphs [13, 25] or structures based on the well-separated pair decomposition [12, 28]. Also techniques to construct spanners with bounded-degree are known [5]. For more details we refer to the book [28]. We investigate the question whether a given graph is a Euclidean spanner. The related question of computing the stretch factor of a given graph has recently been studied in [4, 19, 27]. Additionally, Ahn et al. [2] discuss the problem to find an edge whose removal leads to the smallest possible increase in the stretch factor, and Farshi et al. [20] consider the question which edge should be added to receive the best decrease in the stretch factor (both articles consider very special cases only).

We say that a geometric graph G is ϵ-far from a $(1+\delta)$-spanner, if one has to insert more than ϵn edges into G to make it a $(1+\delta)$-spanner. A property tester is a randomized algorithm that has to distinguish $(1+\delta)$-spanners from graphs which are ϵ-far from any $(1+\delta)$-spanner using a sublinear number of queries to the graph, which is assumed to be stored in adjacency list representation.

In this paper, we show that the property of being a $(1+\delta)$-spanner can be tested with $\tilde{\mathcal{O}}\left(\delta^{-3d}\epsilon^{-3}\sqrt{n}\right)$ queries for bounded (out)degree graphs.

2 Preliminaries

Let $G=(P,E)$ be a directed geometric graph with vertex set $P := \{1,\dots,n\}$ and edge set E, where the vertices of P are points in the $\mathbb{R}^d$ and $d \in \mathbb{N}$ is a constant. We use $[p,q\rangle$ to denote a directed edge from p to q. The length of an edge $[p,q\rangle$ is defined to be the Euclidean distance $\|p-q\|_2$ between p, q, i. e. the edge lengths are induced by the positions of vertices. We use $d_G(p,q)$ to denote the shortest path distance from p to q, i. e. $d_G(p,p) = 0$, $d_G(p,q) = \|p-q\|_2$, if $[p,q\rangle \in E$ and $d_G(p,q) := \min_{\text{paths } Q \text{ from } p \text{ to } q} \sum_{[p',q'\rangle \in Q} d_G(p',q')$ else. We assume that G has an outdegree of at most $D \in \mathbb{N}$ and that it is stored in the adjacency list model [24], i. e. we have access to a function $f_G : P \times \{1,\dots,D\} \to P \cup \{+\}$, where $f_G(p,i)$ returns the i-th neighbor of vertex p if p has at least i neighbors and $+$ otherwise.

Definition 1 *Let $\delta > 0$ be a parameter. A geometric graph G is called a $(1+\delta)$-spanner, if $d_G(p,q) \le (1+\delta)||q-p||_2$ for all pairs of vertices $[p,q\rangle \in P^2, p \neq q$.*

In this paper we will assume that $0 < \delta < 1$.

Definition 2 *Let G be a directed geometric graph and let $0 < \epsilon < 1$. A graph G is ϵ-*far *from being a $(1+\delta)$-spanner, if one has to insert more than ϵn edges to make G a $(1+\delta)$-spanner (note that there is no restriction to maintain the degree bound D). G is ϵ-close to being a $(1+\delta)$-spanner, if it is not ϵ-far from it.*

An algorithm $\mathcal{A}$ is called a *property tester with one-sided error* for the property of being a $(1+\delta)$-spanner, if for any directed geometric graph G it outputs

- *true* with a probability of 1, if G is a $(1+\delta)$-spanner
- *false* with probability at least 2/3, if G is ϵ-far from being a $(1+\delta)$-spanner

when it is given n, δ and ϵ as input and oracle access to f_G. The *query complexity* of $\mathcal{A}$ is the worst-case number of accesses to f_G it needs.

3 The algorithm

Our algorithm for testing geometric spanners works as follows. We first sample a set of $s = \tilde{\mathcal{O}}(\frac{\sqrt{n}}{\delta^{\mathcal{O}(1)}\epsilon^{\mathcal{O}(1)}})$ vertices $p_1, \ldots, p_s$ uniformly at random. Then we start a shortest path computation from each vertex using Dijkstra's algorithm until $O(\log n/(\epsilon\delta^{2d}))$ vertices have been visited. We call this traversal Dijkstra traversal. Finally, we check for every sample point p_i if there exists another sample point p_j within a distance of $W/(1+\delta)$ such that $(1+\delta)\cdot\|p_i - p_j\|_2 > d_{G'}(p_i, p_j)$, where G' is the graph induced by the Dijkstra traversals, i.e. G' contains exactly the vertices and edges that have been visited during all such traversals, and W is the maximal graph distance reached during the traversal.

UniformTester(n, G, δ, ϵ)
    Sample $s = \tilde{\mathcal{O}}(\delta^{-d}\epsilon^{-2}\sqrt{n})$ points $p_1, \ldots, p_s$ from P u.i.d. without replacement
    for $i \leftarrow 1$ **to** s
        Perform a Dijkstra traversal in G from p_i until
            $s' = \tilde{\mathcal{O}}(\delta^{-2d}\epsilon^{-1})$ nodes have been visited
        Let R be the set of vertices visited and let $W = \max_{q \in R} d_G(p_i, q)$
        forall points p_j such that $\|p_i - p_j\|_2 < \frac{1}{1+\delta} \cdot W$
            if the Dijkstra traversal did not reach p_j **or** $d_G(p_i, p_j) \geq (1+\delta)\|p_i - p_j\|_2$
                return false
    return true

Notice that this algorithm only tests small neighborhoods of certain points. At first glance it seems unlikely that the spanner property can be tested by local investigations. Consider the path depicted in Figure 1. If δ is chosen appropriately, then the spanner property might be fulfilled for all pairs (v_i, v_j) except for (v_1, v_9). This means that the violation cannot be found by sampling only parts of the path. Surprisingly, when distinguishing spanners and graphs that are ϵ-far from being a spanner, the situation is different. We show that a geometric graph cannot be ϵ-far from being a spanner if it does only contain 'global' violations of the spanner property like the one in Figure 1.

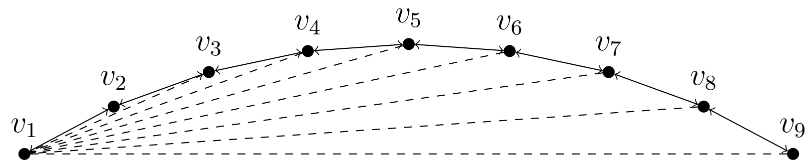

Figure 1: A curved path where the quotient of the distance in the graph and the euclidean distance grows for increasing i.

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