# Rigorous Quantitative Analysis of Multigrid[1]

# I. Constant Coefficients Two-Level Cycle with $L_2$ Norm

**Achi Brandt**[2]

Department of Applied Mathematics & Computer Science
The Weizmann Institute of Science
Rehovot 76100, Israel

## Abstract

Exact numerical convergence factors for any multigrid cycle can be predicted by local mode (Fourier) analysis. For general linear elliptic PDE systems with piecewise smooth coefficients in general domains discretized by uniform grids, it is proved that, in the limit of small meshsizes, these predicted factors are indeed obtained, provided the cycle is supplemented with a proper processing at and near the boundaries. That processing, it is proved, costs negligible extra computer work. Apart from mode analysis, a Coarse Grid Approximation (CGA) condition is introduced which is both necessary and sufficient for the multigrid algorithm to work properly. The present part studies the $L_2$ convergence in one cycle, for equations with constant coefficients. In the sequel[1] [P2], extensions is discussed to many cycles (asymptotic convergence), to more levels with arbitrary cycle types ($V$, $W$, etc.), and to FMG algorithms. Various error norms and their relations to the orders of the inter-grid transfer operators are analyzed. Global mode analysis, required to supplement the local analysis in various border cases, is developed and partial relaxation sweeps are systematically introduced into both analysis and practice.

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Content

# 1. Introduction

Since the early days of multigrid development, the "local mode analysis" (LMA), based on heuristic local Fourier decomposition of the error function, has been the chief tool for the practical design, precise quantitative understanding, and even debugging, of the various multigrid processes. Although rigorously justified in very special cases only, the easily-computable predictions of that analysis have turned out precise for quite general PDE boundary value problems discretized on uniform grids, with quite general domains and boundary conditions. In several important cases, however, the predicted LMA convergence factors were not obtained, presumably due to the influence of boundaries, which are usually not accounted for by the local analysis; domains with reentrant corners are a notorious example (and see others in Sec. 11.1 of [P2]).

The purpose of this article is to give a general rigorous framework to the local mode analysis on one hand, and to the treatment of boundaries on the other hand. Using essentially the weakest possible assumptions, it will be proved, for general linear elliptic systems in general domains, that the convergence factors predicted by the local mode analysis *can* be obtained. That is, the predicted factors *are* indeed obtained, provided a proper treatment, costing negligible extra work (when the meshsize is sufficiently small), is applied at and near boundaries.

The convergence factors thus proven are not just qualitative; they are *quantitatively sharp*: they are exactly obtained (or arbitrarily closely approached) by the worst local mode. By comparison, almost all other multigrid theories (see, e.g., [H], [MMB], [BP] and references therein), for all their great theoretical interest, give estimates which are not quantitative (containing unspecified constants) or very unrealistic, hence their use in *practical* algorithmic design is limited, and sometimes even misleading (see discussion in [G, §14]). Furthermore, these other theories, except for those restricted to variational problems, require an unknown, sufficiently large number of relaxation sweeps per cycle to guarantee convergence (thus actually analyzing much inferior algorithms). Quantitatively realistic (sometimes even sharp) two-level convergence estimates for general symmetric problems, not necessarily on regular grids, were derived in [AMGT], and for one special case also in [Brs], but the estimates in [AMGT] — unlike the present theory — cannot be improved by adding more relaxation sweeps per cycle, and cannot be generalized to $V$ cycles. The algebraic theory of relaxation developed in [AMGT] is used here as an auxiliary tool.

The main objective of the present article is to introduce a new *approach*, a general *blueprint*, for the rigorous analysis of multigrid process and, in particular, for the rigorous justification of the local mode analysis[3]. This approach is capable of analyzing a multitude of situations (different types of algorithms, cycles, error

[3] Since the appearance of the present article in [RL], another work [RS] has appeared with

norms, domains, equations, boundary conditions, discretizations, etc.). However, trying to include all these into one article would be very laborious, both to write and to read. Thus, for simplicity and clarity we instead choose to confine our first detailed presentation here to **an example**: *the analysis of the two-level one-cycle $L_2$ convergence factor for general systems with constant coefficients and pointwise relaxation*, full generality being kept in terms of the shape of the (bounded) domain, the boundary conditions and the type of interior equations (satisfying some necessary conditions). Extensions to variable coefficients and non-uniform finite elements, to block relaxation schemes, to other error norms (enabling other inter-grid transfer orders), to many cycles (asymptotic convergence), to more levels ($V$ and $W$ cycles) and to FMG algorithms — are then discussed, in the form of comments, in [P2].

For our showcase example (the two-grid one-cycle $L_2$ theory with pointwise relaxation) all the assumptions made by the theory are clearly listed and discussed in a special section (Sec. 6; they are summarized in Sec. 6.6). Instead of showing what concrete systems satisfy these assumptions, we only show that the assumptions are *necessary*, in the sense that they must be satisfied by any proper coarsening algorithm (as defined in Sec. 6.1). So what the presented theory shows is *conditional*; namely, *if* the multigrid algorithm works properly *at all*, then its "ideal" convergence rate is exactly predicted by the local mode analysis. "Ideal" here means that sometimes negligibly-costing corrections should be made in the cycle to obtain this rate. Why this ideal rate is the important one to predict, and when and how these corrections should actually be implemented in practice, is discussed in [P2] (Secs. 11.1–11.2). *One* way of implementing such corrections, *not necessarily the most practical one* but enough to prove that the associated cost is negligible, is given in Sec. 7 below.

Since the assumptions in our analysis are shown to be *necessary*, the question which concrete systems satisfy them is thereby neatly separated into an independent piece of research; it is by and large outside the scope of the present article. We only make in this regard the following general observations:

(i) The assumptions are all *qualitative*, i.e., employing undetermined constants. Our results, on the other hand, are *quantitative*, and even *sharply quantitative*; i.e., we explicitly derive the worst possible convergence rate (in the limit of small meshsize). This rate, it is also shown, is usually independent of boundary shapes and boundary conditions.

(ii) No restriction is made to positive definite and variational formulations. General *systems* (not just scalar equations) are considered.

(iii) Except for its stability, no assumption is made about the relaxation scheme. Thus, whether the relaxation is good or bad is not *assumed* by the

similar objectives, related through different results, and quite different proofs.

present theory, but *calculated* by it. This is in contrast to other multigrid theories which are restricted to special relaxation schemes for special types of problems.

(iv) Most assumptions are obvious stability requirements plus a weak ellipticity assumption. The only less obvious assumptions are the Coarse Grid Approximation (CGA) condition and the interpolation order supposition. They are discussed in more details in Secs. 6.1 and 6.2, respectively. We show there that the CGA is a necessary condition and is therefore weaker than similar-looking assumptions made (for the corresponding norm) by other multigrid theories, and that the present theory can deal exactly with the interpolation orders of interest.

(v) The assumptions can be checked *aposteriori*. In Sec. 11.5 of [P2] we will discuss when and how to do it, and how to correct the algorithm if the assumptions are not satisfied. Such aposteriori checking, made possible by the sharp quantitative aspect of the local mode analysis, is more practical than the attempt to ascertain in advance, by rigorous analysis, that the necessary assumptions indeed are satisfied. In many cases the latter is too complicated to be done, especially by the practitioner, even if he limits himself exactly to the narrow classes of problems and algorithms analyzed in the literature. In point of fact, the practitioner seldom bothers about this. Offered to him here is a practical way to know *when* he needs to bother, and how then to go about it.

The theory in Sec. 6.1.2 and in Sec. 10.1.3 of [P2], together with a concrete example we do briefly discuss (Sec. 6.1.3), are enough to establish that, *even without mode analysis* (and possibly for cases where the latter is inapplicable at all because of unstructured grids or unstructured relaxation ordering), the CGA condition introduced here (Sec. 6.1), unlike various "Approximation Properties" formulated elsewhere, is the precise tool to separate coarsening questions from the choice of relaxation. The CGA condition is a necessary and sufficient condition for the coarsening to be "proper", i.e., to be able, when combined with *sufficiently many* sweeps of *the most suitable* pointwise relaxation scheme, to produce $h$-independent *two-grid* convergence. We furthermore prove that as soon as this condition is satisfied, the local mode analysis can sharply calculate the ideal quantitative efficiency of *any* given cycle ($V$ or $W$), with *any* given number of relaxation sweeps and with *any* given pointwise relaxation scheme.

The various error norms that can be analyzed are discussed in Sec. 10.1 of [P2]. Each norm implies what orders can be used in each of the intergrid transfer operators: general rules are specified in Sec. 10.1.6; they generally follow rules previously published in [G, §4.3] and also in [PH], with some extensions. Certain transfer orders give rise to "border" cases, in which the local mode analysis should be supplemented by a certain new type of "global mode analysis", described in Sec. 10.2.2.

The practical implications of our analysis, including practical ways for con-

structing and debugging multigrid solvers, are generally reviewed in Sec. 11 of [P2]. A major emphasis is on the importance and practicality of adding partial (local) relaxation passes to the multigrid algorithm. Although anticipated before [MOC, App. A.9] and occasionally used [Bai], this tool of partial relaxation has never before been incorporated systematically into multigrid theory and practice.

As a *practical* tool for calculating smoothing rates, a kind of local mode analysis was first used in the early 1970s to obtain the first multigrid algorithms of modern efficiency [P72]. Previous approaches to two-grid ([B25], [B15], [B5], [B19], [B17], [B16]) or multi-grid [B6] algorithms had been based on the "acceleration" point of view, lacking the concept of smoothing as a local process accurately predictable by local mode analyses. The absence of *quantitative* insights led to erroneous algorithmic concepts (e.g., large coarse/fine meshsize ratios) and to inferior efficiency. The convergence rate per computer operation estimated in [B6], for example, is $10^4$ below that of modern algorithms, and was not considered practical.

The first description of the local mode analysis for relaxation (with extended tables of smoothing factors for both pointwise and linewise schemes) appeared in [MOC]; the first two-level mode analysis—in [BD]; the first "semi" and "partial" smoothing factors (corresponding to algorithms with semi coarsening and with several frequency-shifted coarsenings)—in [Sta]; and the smoothing factors for colored (e.g., red-black) relaxation were introduced in [G82] (following less satisfactory definitions in [Sta]). [G82] also reviews all the previous mode analysis work. A good and extensive exposition of the local mode analysis, with many more results, is given in Part III of [ST].

As a *rigorous* tool to obtain *realistic* multigrid convergence factors, mode analysis was previously used in App. C of [MOC]. But the estimates there are still not fully *sharp*, and they are limited to the Poisson equation in rectangular domains.

## 2. Bigrid Cycle

The basis and/or the model for any theoretical treatment of multigrid solvers is the analysis of the multigrid cycle in the simplest case where only two grids are involved. We assume for simplicity that these are two uniform square grids, with meshsizes $h$ and $H = 2h$. The fine grid system of equations (no distinction is yet made between interior equations and boundary conditions) is written as

$$A^h u^h = f^h \tag{2.1}$$

where $u^h$ and $f^h$ are real or complex vector-valued functions on $\Omega^h$, the intersection of the lattice $\{x = (\alpha_1, \ldots, \alpha_d)h \mid \alpha_i \text{ integers}\}$ with the bounded prob-

lem domain $\Omega \subseteq \mathbb{R}^d$. More generally, $\Omega^h$ may in fact be *staggered*; that is, $u^h = (u^{h,1}, \ldots, u^{h,q})^T$ and $f^h = (f^{h,1}, \ldots, f^{h,q})^T$, and each $u^{h,j}$ and each $f^{h,j}$ are scalar-valued functions defined on different uniform grids, $\Omega^{h,1,j}$ and $\Omega^{h,2,j}$ respectively, where each $\Omega^{h,k,j}$ is the intersection of the problem domain $\Omega$ with a translated lattice

$$\{x = (\alpha_1, \ldots, \alpha_d)h + s^{h,k,j} \mid \alpha_i \text{ are integers}\} \tag{2.2}$$

for some fixed shift $s^{h,k,j} = (s_1^{h,k,j}, \ldots, s_d^{h,k,j})$, with $|s_i^{h,k,j}| \leq h/2$, $(i = 1, \ldots, d)$. A similar staggered grid may be introduced with meshsize $H = 2h$, and the functions defined on it will be denoted $u^H$ and $f^H$. The coarse-to-fine interpolation of solutions (or approximate solutions, or correction to solutions) is denoted $I_H^h$; e.g., $u^h = I_H^h u^H$. The fine-to-coarse transfer of right-hand sides (or residuals) is denoted $I_h^H$; e.g., $f^H = I_h^H f^h$. An operator (matrix) $A^H$ is given on grid $2h$ which approximates $A^h$: it may be constructed similarly to $A^h$ by discretizing the same differential problem, or it may be defined by the Galerkin approximation $A^H = I_h^H A^h I_H^h$. The orders of $I_H^h$ and $I_h^H$, the approximation of $A^h$ by $A^H$ and other assumptions will be discussed below (Sec. 6).

With this notation, the *bigrid cycle* can be defined. Reserving the notation $u^h$ for the exact solution of (2.1), we will replace the superscript $h$ by other superscripts to denote various approximations to $u^h$. The cycle starts with a given approximation $u^A$, and improves it by the following three steps.

(i) *Pre-relaxation.* $\nu_1$ relaxation sweeps are first performed using the fine grid equations (2.1). Typical is the Gauss-Seidel relaxation sweep, where the discrete equations are scanned in some prescribed order, each one in its turn being satisfied by changing a corresponding unknown. This is natural for equations derivable from a variational (e.g., minimization) principle, where each equation indeed corresponds to one unknown. For more general cases, more general types of relaxation schemes exist. A general way for constructing efficient relaxation schemes for general discretized PDE systems is described in [G84, §3.7]; the description is for the *interior* relaxation, not near boundaries, but that is essentially all one actually needs (cf. Sec. 4 below).

(ii) *Coarse grid correction* (CGC). Denoting by $u^B$ the approximate solution obtained at the end of Step (i), and by $v^B = u^h - u^B$ the corresponding error, a coarse grid approximation to $v^B$, $v^H$, is calculated by solving the coarse grid equations

$$A^H v^H = I_h^H (f^h - A^h u^B). \tag{2.3}$$

Then $v^H$ is used to correct the fine grid approximation:

$$u^C = u^B + I_H^h v^H \tag{2.4}$$

(iii) *Post-relaxation.* Starting with $u^C$, $\nu_2$ additional relaxation sweeps are performed, yielding the final approximation $u^D$.

In assessing the efficiency of this cycle, one should of course disregard the work involved in solving (2.3), because in the true multigrid cycle these equations are solved approximately, by employing recursively one or two similar cycles at the coarser level.

**Notation.** The linear relaxation operator will be denoted by $R$ and the CGC operator by $S$. That is, $v^B = R^{\nu_1} v^A$, $v^C = S v^B$ and $v^D = R^{\nu_2} v^C$, where $v^A = u^h - u^A$, $v^B = u^h - u^B$, $v^C = u^h - u^C$ and $v^D = u^h - u^D$.

The *cycle convergence factor* is defined to be

$$\lambda = \sup \frac{\|v^D\|}{\|v^A\|}, \tag{2.5}$$

where the sup is taken over all possible initial approximations $u^A \neq u^h$. The norm in (2.5) is the $\ell_2$ norm, which will be used throughout most of the presentation here, although other norms may often be more appropriate (see Sec. 10.1 of [P2]). Our purpose is to calculate $\lambda$. Another possible purpose, to which we will refer in [P2] (Secs. 10.1.1 and 10.3), is to calculate the *asymptotic convergence factor per cycle*, defined by

$$\lambda' = \lim_{n \to \infty} \sup (\|v^{[n]}\| / \|v^{[0]}\|)^{1/n}, \tag{2.6}$$

where $v^{[n]}$ is the error after $n$ applications of the cycle, so in particular $v^{[0]} = v^A$ and $v^{[1]} = v^D$.

The coefficient $C$ will be used throughout the article to denote a generic constant: not necessarily the same on different occurrences, independent of $h$. The notation $C_\ell$ will be used in case the constant depends on the integer $\ell$.

# 3. Formal Mode Analysis in Entire Space

## 3.1 One level modes

In case $\Omega = \mathbb{R}^n$ (and hence also in case of a rectangular domain with periodic boundary conditions, where the problem can be extended to the entire space $\mathbb{R}^n$), and assuming "constant coefficient" operators $A^h$ and $A^H$ and a "consistently ordered" relaxation scheme (terms to be defined below), the convergence factors $\lambda$ and $\lambda'$ can easily be calculated by a mode (Fourier) analysis. That is, the initial error $v^A$, and similarly the error $v^h$ at any other stage, can be written as a linear

combination (integral; or sum, in case of periodic boundary conditions) of Fourier modes

$$v_\alpha^h = \int_{|\theta|\le\pi} \hat{v}^h(\theta) e^{i\theta\alpha} d\theta, \tag{3.1}$$

and the change of the coefficients $\hat{v}^h(\theta)$ under each of the processes in the cycle can explicitly be calculated, yielding an explicit calculation of $\lambda$ (or $\lambda'$).

The notation in (3.1) is as follows: $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a vector of integers, $v_\alpha^h = (v_\alpha^{h,1}, \ldots v_\alpha^{h,q})^T$ with $v_\alpha^{h,j} = v^{h,j}(\alpha h + s^{h,1,j})$, $\theta = (\theta_1, \ldots, \theta_d)$, $\theta\alpha = \theta_1\alpha_1 + \cdots + \theta_d\alpha_d$, $|\theta| = \max(|\theta_1|, \ldots, |\theta_d|)$ and

$$\hat{v}^h(\theta) = (2\pi)^{-d} \sum_\beta v_\beta^h e^{-i\theta\beta}, \tag{3.2}$$

$\sum_\beta$ denoting summation over the entire grid of integer vectors $\beta = (\beta_1, \ldots, \beta_d)$. As implied by (3.2), $\hat{v}^h(\theta) = \hat{v}^h(\theta_1, \ldots, \theta_d)$ can naturally be extended as a $2\pi$-*periodic* function, i.e.,

$$\hat{v}^h(\theta) = \hat{v}^h(\theta_1 + 2\pi, \theta_2, \ldots, \theta_d) = \cdots = \hat{v}^h(\theta_1, \ldots, \theta_{d-1}, \theta_d + 2\pi) \tag{3.3}$$

for any $\theta \in \mathbb{R}^d$. To be precise, the integration meant in (3.1), and similarly below, is over one cell of this period, e.g. the cell

$$-\pi \le \theta_j < \pi, \qquad (j = 1, \ldots, d). \tag{3.4}$$

We assume that the decomposition (3.1) *exists*, and that the *Parseval identity*

$$\int_{|\theta|\le\pi} \|\hat{v}^h(\theta)\|^2 d\theta = (2\pi)^{-d} \sum_\alpha \|v_\alpha^h\|^2$$

holds. This is true for a wide class of error functions. We do not prove it here, because in this section the development is purely formal; a proof will be given later (Sec. 8.1), when dealing with real, bounded domains $\Omega$.

We use the notation $A^h$ and $A^H$ for the discrete fine-grid and coarse-grid operators, respectively, including both interior and boundary operations. In case of the infinite domain they of course coincide with the *interior operators*, which we denote by $L^h$ and $L^H$, respectively. Grid values of $r^h = L^h v^h$ will be denoted by

$$\begin{aligned} r_\alpha^h &= r^h(\alpha h) \\ &= \big(r^{h,1}(\alpha h + s^{h,2,1}), \ldots, r^{h,q}(\alpha h + s^{h,2,q})\big)^T, \end{aligned}$$

where in case of a staggered grid $r^h(\alpha h)$ is only a short-hand writing of the latter equality. That $L^h$ has "constant coefficients" means that it has the (block-Toeplitz) form

$$(L^h u^h)_\alpha = \sum_\gamma a_\gamma^h(h) u_{\alpha+\gamma}^h, \tag{3.5}$$

where $\sum_\gamma$ is a summation over integer vectors $\gamma = (\gamma_1, \ldots, \gamma_d)$, and each $a^h_\gamma(h)$ is a $q \times q$ matrix which explicitly depends on the meshsize $h$ (using it in the denominator of divided differences). Hence, with the Fourier decomposition (3.1) for $v^h$, it is easy to see that if $r^h = L^h v^h$ and

$$r^h_\alpha = \int_{|\theta| \le \pi} \hat{r}^h(\theta) e^{i\theta\alpha} d\theta \tag{3.6}$$

then $\hat{r}^h(\theta) = \hat{L}^h(\theta, h)\hat{v}^h(\theta)$, where

$$\hat{L}^h(\theta, h) = \sum_\gamma a^h_\gamma(h) e^{i\theta\gamma}. \tag{3.7}$$

$\hat{L}^h(\theta, h)$, which is an easily computed (easily programmed) $q \times q$ matrix of functions of $\theta$, is called the *symbol* of $L^h$.

For example, the simplest discretization of the Laplacian $L = \sum_i \partial^2/\partial x_i^2$ has the form (3.5) with $a^h_0(h) = -2dh^{-2}$, $a^h_\gamma(h) = h^{-2}$ if $\gamma^2 = \gamma_1^2 + \cdots + \gamma_d^2 = 1$, and $a^h_\gamma(h) = 0$ if $\gamma^2 > 1$. Hence, (3.7) yields

$$\hat{L}^h(\theta, h) = 2h^{-2} \sum_{i=1}^{d} (\cos\theta_i - 1) \tag{3.7a}$$

We thus see that the operator $A^h = L^h$ does not couple different modes: For each $\theta$, the Fourier coefficient $\widehat{L^h v^h}(\theta)$ depends only on $\hat{v}^h(\theta)$ for the same $\theta$. This property also holds for the operation of relaxation, provided the relaxation scheme is "consistently ordered", i.e., for any $\alpha$ and $\gamma$ the point $\alpha$ is relaxed *after* the point $\alpha - \gamma$ if and only if it is relaxed *before* $\alpha + \gamma$. We will however soon extend the class of relaxation schemes treated by our analysis to include some important schemes which are not strictly consistently ordered, such as red-black schemes.

## 3.2 Bigrid analysis

On grid $H = 2h$, the Fourier mode $\exp(i\theta x/h) = \exp(i2\theta x/H)$ "aliases" (coincides) with any other mode $\exp(i\theta' x/h) = \exp(i2\theta' x/H)$ for which $\theta = \theta' (\mathrm{mod}\ \pi)$, i.e., for which each $(\theta_j - \theta'_j)/\pi$ is an integer, $(j = 1, \ldots, d)$. Such modes, or such $\theta$ and $\theta'$, we will call *harmonics* of each other. Every intergrid transfer, either $I^H_h$ or $I^h_H$, must couple each component with all its harmonics. If the transfer has constant coefficients (i.e., it repeats itself at each coarse grid cell), as we will assume, then it will couple *only* harmonics. We will therefore consider simultaneously each set of harmonics in the range (3.4).

For this purpose, let $\tau_1^j \ldots \tau_d^j$ be the binary representation of the integer $j$, $0 \le j < 2^d$, i.e., $\tau_k^j = 0$ or 1 and $j = \sum_{k=1}^d 2^{d-k}\tau_k^j$; and let $\tau^j = (\tau_1^j, \ldots, \tau_d^j)\pi$. Each set of harmonics in the range (3.4) includes one frequency $\theta$ in the range

$$-\frac{\pi}{2} \le \theta_k < \frac{\pi}{2}, \qquad (k = 1, \ldots, d) \tag{3.8}$$

which we will call *the lowest harmonic*. For each such $\theta$, its set of harmonics in the range (3.4) is the set of $D = 2^d$ components

$$\{\theta^j = \theta + \tau^j (\mathrm{mod}\ 2\pi); \quad j = 0, 1, \ldots, D-1\}, \tag{3.9}$$

including the lowest harmonic $\theta = \theta^0$ itself.

On the coarse grid each $\theta^j$ component $\exp(i\theta^j x/h) = \exp(i2\theta^j x/H)$ appears as a $\theta^0$ component $\exp(i2\theta^0 x/H)$. Hence, operating with a constant-coefficient fine-to-coarse transfer $I_h^H$ on any fine-grid residual such as $r^h$ in (3.6) gives

$$(I_h^H r^h)_\alpha = I_h^H r^h(\alpha H) = \int^0 \sum_{j=0}^{D-1} \hat{I}_h^H(\theta^j)\hat{r}^h(\theta^j) e^{i2\theta^0 \alpha} d\theta^0, \tag{3.10}$$

where $\int^0$ denotes, here and below, integration over the domain (3.8), with $\theta^j$ relating to $\theta^0$ through (3.9), and $\hat{I}_h^H(\theta)$ is a $q \times q$ matrix of easily programmed functions of $\theta$, called the *symbol* of $I_h^H$. Similarly, the correction interpolation $I_H^h$ will transfer the coarse grid solution

$$v_\alpha^H = v^H(\alpha H) = \int^0 \hat{v}^H(\theta^0) e^{i2\theta^0 \alpha} d\theta^0 \tag{3.11}$$

to the fine grid correction

$$(I_H^h v^H)_\alpha = I_H^h v^H(\alpha h) = \int^0 \sum_{j=0}^{D-1} \hat{I}_H^h(\theta^j)\hat{v}^H(\theta^0) e^{i\theta^j \alpha} d\theta^0, \tag{3.12}$$

where the $q \times q$ matrix of functions $\hat{I}_H^h(\theta)$ is called the symbol of $I_H^h$. Note that if $2^d I_h^H$ is the adjoint of $I_H^h$, as is often the case, then $2^d \hat{I}_h^H(\theta) = \hat{I}_H^h(\theta)^\dagger$ (superscript $\dagger$ denoting conjugate transposition). Most often both these symbols are diagonal and simple. For $I$-order multipolynomial interpolation, for example,

$$\hat{I}_H^h(\theta)_{k,\ell} = \delta_{k,\ell} \prod_{i=1}^d \varphi_I(\cos\theta_i), \quad (k, l = 1, \ldots, q) \tag{3.13}$$

where $\varphi_2(\xi) = (1+\xi)/2$, $\varphi_4(\xi) = (2+3\xi-\xi^3)/4$, etc.

Since harmonics are coupled anyway, we will allow them to be coupled also by the relaxation process. We thus extend the concept of consistently ordered relaxation to any relaxation operator $R$ such that, operating with it on the error function (3.1) will give an error function of the form

$$(Rv^h)_\alpha = \int^0 \sum_{j,k=0}^{D-1} \hat{R}_j(\theta^k)\hat{v}^h(\theta^k)e^{i\theta^j\alpha}d\theta^0. \tag{3.14}$$

This will include red-black (and even $2^d$-colored) relaxation schemes. $\hat{R}_j(\theta)$ are easily computed $q \times q$ matrices. For example, for lexicographically ordered Gauss-Seidel relaxation of the simplest discrete Laplacian (described above), it is easy to see (cf. [MOC,§3.1] or [G84,§1.2] or [ST,§3.2]) that $\hat{R}_j(\theta^k) = \delta_{jk}\hat{R}(\theta^k)$, $(0 \le j,k < D)$, where

$$\hat{R}(\theta) = \hat{R}(\theta_1,\ldots,\theta_d) = \sum_{\nu=1}^{d} e^{i\theta_\nu}/(2d - \sum_{\nu=1}^{d} e^{-i\theta_\nu}). \tag{3.14a}$$

The special case where the relaxation does *not* couple different components, i.e., where $\hat{R}_j(\theta^k) = 0$ for $k \neq j$, will be called *uncolored relaxation*. This includes all lexicographically-ordered schemes, and is indeed equivalent to requiring the relaxation to be *strictly consistently ordered*, by which we mean that, for any vector of integers $\alpha = (\alpha_1,\ldots,\alpha_d)$, a relaxation step at $x$ precedes a similar step at $x+\alpha h$ if and only if it succeeds the step at $x-\alpha h$ (whenever all three points are in $\Omega^h$). The more general, *colored* relaxation should satisfy a similar condition, but only for vectors $\alpha$ of *even* integers.

The symbol $\hat{L}^H(\theta,H)$ of the coarse grid operator can be defined similarly to $\hat{L}^h(\theta,h)$ above (but applied to *coarse* grid functions $v^H$, and hence to Fourier modes $\exp(i\theta x/H)$, instead of the modes $\exp(i\theta x/h)$ used in (3.1)). Often, $L^H$ and $L^h$ are identical discretizations of the same differential operator, using only different meshsizes, in which case $\hat{L}^H(\theta,H) = \hat{L}^h(\theta,2h)$. Note that when $L^H$ is applied to a *fine* grid function $v^h$ (with the *fine* grid expansion (3.1)), its symbol is $\hat{L}^H(2\theta,H)$.

We can now describe, in terms of Fourier transforms in which harmonics are being blocked together, the entire bigrid cycle. To this end we introduce the following *block-matrix notation*: for any fine-grid error function $v^h$ (and similarly for any residual function; each being a *vector* of $q$ functions), with the Fourier transform (3.1), $\check{v}^h(\theta^0)$ will denote the $qD$-long vector $(\hat{v}^h(\theta^0)^\dagger,\ldots,\hat{v}^h(\theta^{D-1})^\dagger)^\dagger$. In particular, the initial error $v^A$ has the block-Fourier decomposition

$$v^A_\alpha = \int^0 E_\alpha \check{v}^A(\theta)e^{i\theta\alpha}d\theta, \tag{3.15}$$

where $E_\alpha$ is the $q \times qD$ matrix

$$E_\alpha = (e^{i\tau^0\alpha} I_q, \ldots, e^{i\tau^{D-1}\alpha} I_q), \tag{3.16}$$

$I_q$ being the $q \times q$ identity matrix. The error at the cycle end has a similar decomposition

$$v_\alpha^D = \int^0 E_\alpha \check{v}^D(\theta) e^{i\theta\alpha} d\theta, \tag{3.17}$$

and by the definition of the cycle we obtain the relation

$$\check{v}^D(\theta) = M(\theta)\check{v}^A(\theta), \qquad \left(-\frac{\pi}{2} \le \theta_i < \frac{\pi}{2}, i = 1, \ldots, d\right) \tag{3.18}$$

where the *cycle symbol* $M(\theta)$ is the $qD \times qD$ matrix

$$M(\theta) = \check{R}(\theta)^{\nu_2} [\check{I} - \check{I}_H^h(\theta)\check{L}^H(\theta)^{-1}\check{I}_h^H(\theta)\check{L}^h(\theta)]\check{R}(\theta)^{\nu_1}. \tag{3.19}$$

The block matrices in (3.19) are defined by

$$\begin{array}{rll} & & \underline{\text{full matrix dimension}} \\ \check{R}(\theta)_{JK} &= \hat{R}_{J-1}(\theta^{K-1}) & qD \times qD \\ \check{L}^h(\theta)_{JK} &= \delta_{JK}\hat{L}^h(\theta^{J-1}, h) & qD \times qD \\ \check{I}_h^H(\theta)_J &= \hat{I}_h^H(\theta^{J-1}) & q \times qD \\ \check{L}^H(\theta) &= \hat{L}^H(2\theta, H) & q \times q \\ \check{I}_H^h(\theta)_K &= \hat{I}_H^h(\theta^{K-1}) & qD \times q \\ \check{I}_{JK} &= \delta_{JK} I_q & qD \times qD \end{array} \tag{3.19a}$$

where $\theta^0, \ldots, \theta^{D-1}$ are related to $\theta$ by (3.9), and where $J$ and $K$ are *block* indices, pointing to the $q \times q$ block occupying columns $(J-1)q+1, (J-1)q+2, \ldots, Jq$ and rows $(K-1)q+1, (K-1)q+2, \ldots, Kq$. Whenever appearing $J, K = 1, \ldots, D$; the full matrix dimension shown on the right also indicates their range.

For example, in the two dimensional ($d = 2$) scalar ($q = 1$) case of the Laplace equation with lexicographic Gauss-Seidel relaxation, with bilinear coarse-to-fine interpolation and the common (full-weighting) fine-to-coarse transfer of residuals, the above symbols are given by

$$\check{R}(\theta) = \check{R}(\theta_1, \theta_2) = \begin{pmatrix} \hat{R}(\theta_1, \theta_2) & 0 & 0 & 0 \\ 0 & \hat{R}(\theta_1, \theta_2 + \pi) & 0 & 0 \\ 0 & 0 & \hat{R}(\theta_1 + \pi, \theta_2) & 0 \\ 0 & 0 & 0 & \hat{R}(\theta_1 + \pi, \theta_2 + \pi) \end{pmatrix}$$

$$\check{L}^h(\theta) = \check{L}^h(\theta_1, \theta_2) = \begin{pmatrix} \hat{L}^h(\theta_1, \theta_2) & 0 & 0 & 0 \\ 0 & \hat{L}^h(\theta_1, \theta_2 + \pi) & 0 & 0 \\ 0 & 0 & \hat{L}^h(\theta_1 + \pi, \theta_2) & 0 \\ 0 & 0 & 0 & \hat{L}^h(\theta_1 + \pi, \theta_2 + \pi) \end{pmatrix}$$

$$\check{I}_h^H(\theta) = \Bigg( \frac{1+\cos\theta_1}{2} \cdot \frac{1+\cos\theta_2}{2},\ \frac{1+\cos\theta_1}{2} \cdot \frac{1-\cos\theta_2}{2},\ \frac{1-\cos\theta_1}{2} \cdot \frac{1+\cos\theta_2}{2},$$
$$\frac{1-\cos\theta_1}{2} \cdot \frac{1-\cos\theta_2}{2} \Bigg)$$
$$\check{L}^H(\theta) = 2H^{-2}(\cos 2\theta_1 + \cos 2\theta_2 - 2)$$
$$\check{I}_H^h(\theta) = \check{I}_h^H(\theta)^T$$

and $\check{I}$ is the $4 \times 4$ identity matrix, where $\hat{R}(\theta)$ and $\hat{L}^h(\theta)$ are defined by (3.7a) and (3.14a) with $d = 2$.

From (2.5), (2.6), (3.15), (3.17), (3.18) and the Parseval identity it formally follows that

$$\lambda = \sup_{\theta \neq 0} \|M(\theta)\| \tag{3.20}$$

and

$$\lambda' = \sup_{\theta \neq 0} \sigma\big(M(\theta)\big), \tag{3.21}$$

where $\sigma(M)$ is the spectral radius of $M$ (i.e., its largest absolute eigenvalue), $\|M\|$ is its $\ell_2$ operator norm (hence $\|M\| = \sigma(MM^T)^{1/2}$) and each sup is over the range (3.8), with $\theta = 0$ being excluded. This exclusion is important since most often $\check{L}^H(0)$ is singular; the sup is normally still finite since $\check{I}_h^H(0)\check{L}^h(0)$ is suitably singular (rank deficient) too. For some bigrid cycles, $\lambda$ may be infinite; on a bounded domain, as we will see, such cycles can still be used, provided that $\lambda'$ is still finite (cf. Sec. 10.1.6 in [P2]).

It is easy to program the matrix function $M(\theta)$, hence to calculate (3.20) and (3.21) (see Sec. 10.3 in [P2]). Our task will be to prove that, for sufficiently small meshsizes and with a proper treatment of boundaries, the values of $\lambda$ and $\lambda'$ (defined by (2.5) and (2.6)) on any *bounded* domain are still given (or approximated as closely as one wishes) by (3.20) and (3.21).

### 3.3 Orders of operators

In terms of the operator symbols introduced above, we can define various concepts of operator order to which we will refer later. (Similar definitions were introduced in [G, §4.3]. The description here, especially (3.35) below, is more detailed and more accurate.) In these definitions it is assumed that (2.1) in the interior approximates $q$ differential equations in $q$ unknown functions. We denote by $m_{k\ell}$ the highest order of differentiation of the $\ell$-th unknown function in the $k$-th equation ($k, \ell = 1, \ldots, q$). This means that

$$|\tilde{L}^h(\theta, h)_{k,\ell}| \leq C\left(\frac{|\theta|}{h}\right)^{m_{k\ell}} \leq Ch^{-m_{k\ell}}. \tag{3.22}$$

**3.3.1 Principal terms.** The *order $m$ of the finite-difference operator $L^h$* is defined as the order of the corresponding differential operator $L$, i.e., as the highest order of differentiation in $\det L$ (the determinant of $L$, $L$ being written as a $q \times q$ matrix of differential operators). In terms of the symbols, $m$ is the highest power of $h^{-1}$ in $\det \hat{L}^h(\theta, h)$. For scalar ($q = 1$) equations, $m = m_{11}$.

The *principal terms* of $L^h$ are those that correspond to the principal terms of $L$, i.e., all the terms that contribute to $O(h^{-m})$ terms in $\det \hat{L}^h(\theta, h)$. This implies that to the $k$-th row of $\hat{L}^h(\theta, h)$ we can assign a "*row-order*" $m_{(k)} \geq 0$, and to its $\ell$-th column a "*column order*" $m^{(\ell)} \geq 0$, such that

$$m_{(k)} + m^{(\ell)} \geq m_{k\ell}, \tag{3.23a}$$

with equality in case of a principal term, and such that

$$\sum_{k=1}^{q} m_{(k)} + \sum_{\ell=1}^{q} m^{(\ell)} = m. \tag{3.23b}$$

Thus, in each $\hat{L}^h(\theta, h)_{k,\ell}$ the principal terms are $O(h^{-(m_{(k)}+m^{(\ell)})})$, and other terms are of lower order (in $h^{-1}$). Note that there may be a certain freedom in the definition of $m_{(k)}$ and $m^{(l)}$; namely, if there exists an integer $c \neq 0$ such that $m_{(k)} - c \geq 0$ and $m^{(l)} + c \geq 0$ for all $1 \leq k, l \leq q$, then one can replace all $m_{(k)}$ by $m_{(k)} - c$ and all $m^{(l)}$ by $m^{(l)} + c$. In case of a scalar equation we define

$$m_{(1)} = m^{(1)} = m/2.$$

For elliptic systems

$$\det \hat{L}^h(\theta, h) \geq C^{-1} \left( \frac{|\theta|}{h} \right)^m.$$

The same holds for $\hat{L}^H(\theta, H)$. Hence, by the Cramer rule and by (3.22) and (3.23), for elliptic systems

$$(\check{L}^H(\theta)^{-1})_{k,\ell} \leq C \left( \frac{h}{|\theta|} \right)^{m_{(\ell)}+m^{(k)}}. \tag{3.24}$$

The *approximation order of $L^h_{k,l}$ by $L^H_{k,l}$* is $\tilde{p}_{kl}$, defined by

$$\check{L}^H(\theta)_{k,l} = \hat{L}^h(\theta, h)_{k,l} + O\left( |\theta|^{\tilde{p}_{kl}} (\tfrac{|\theta|}{h})^{m_{kl}} \right),$$

where, here and below, the $O(\cdots)$ symbol refers to behavior at small $h$ *and* small $\theta$. Usually, $\tilde{p}_{kl}$ is the lowest among the approximation orders of $L^h_{k,l}$ and $L^H_{k,l}$. The *approximation order of* $L^h$ *by* $L^H$ is $\tilde{p} = \min \tilde{p}_{kl}$. For elliptic systems the above inequalities and Cramer rule yields

$$(\check{L}^H(\theta)^{-1})_{k,l} = (\hat{L}^h(\theta)^{-1})_{k,l} + O\left( |\theta|^{\tilde{p}} (\frac{h}{|\theta|})^{m^{(k)}+m_{(l)}} \right) \tag{3.25}$$

**Example 3.1.** In the *two-dimensional Stokes equations*, $u = (U, V, P)$ and the differential equations are

$$\begin{aligned} -\Delta U + P_x &= 0 \\ -\Delta V + P_y &= 0 \\ U_x + V_y &= 0, \end{aligned} \tag{3.26}$$

where $\Delta$ is the Laplacian and subscripts denote partial differentiations. In this case $q = 3$, $m = 4$ and $m_{(1)} = m_{(2)} = m^{(1)} = m^{(2)} = 1$, $m_{(3)} = m^{(3)} = 0$. To fit the framework of this paper this system should be discretized $h$-elliptically. This can be done either on a staggered grid, as in [G84, §18.2], or on a non-staggered grid, as in [CM3, App. C]. In case of the *staggered* discretization, the symbol will be

$$\hat{L}^h(\theta, h) = \hat{L}^h(\theta) = \begin{pmatrix} s_1^2 + s_2^2 & 0 & is_1 \\ 0 & s_1^2 + s_2^2 & is_2 \\ is_1 & is_2 & 0 \end{pmatrix} \tag{3.27}$$

where $s_j = 2h^{-1} \sin(\theta_j/2)$. For the *non-staggered* case

$$\hat{L}^h(\theta, h) = \hat{L}^h(\theta) = \begin{pmatrix} s_1^2 + s_2^2 & 0 & ic_1 s_1 \\ 0 & s_1^2 + s_2^2 & ic_2 s_2 \\ ic_1 s_1 & ic_1 s_1 & \beta h^2 (s_1^2 + s_2^2) \end{pmatrix} \tag{3.28}$$

where $c_j = \cos(\theta_j/2)$ and $1/16 \le \beta \le 1/12$. In each case $\check{L}^H(\theta) = \hat{L}^H(2\theta, H)$ is obtained from $\hat{L}^h(\theta)$ by replacing $s_j$ by $s'_j = h^{-1} \sin\theta_j$ and $c_j$ by $c'_j = \cos\theta$.

Note that in both systems all the terms are principal: there are no lower order terms as will appear in the Navier-Stokes system.

The Stokes system is an example of a system that cannot properly be analyzed by the simple $L_2$ theory: since $m^{(1)} = m^{(2)} \neq m^{(3)}$, the error norm used for $P$ must be different from the one used for $U$ and $V$ (see Secs. 3.3.5 and 6.1.1 in [P2]). Hence, although in the present section (Sec. 3) we keep the formalism general (for later use; see Sec. 10.1.5 in [P2]), the $L_2$ theory presented in this part will be applicable only for cases where $m^{(1)} = \cdots = m^{(q)}$, such as the following.

**Example 3.2.** *The Cauchy-Riemann equations* are

$$\begin{aligned} U_x + V_y &= f_1 \\ U_y - V_x &= f_2. \end{aligned} \tag{3.29}$$

Here $m = 2$, and we can define $m_{(1)} = m_{(2)} = 1$, $m^{(1)} = m^{(2)} = 0$. Alternative, we could define $m_{(1)} = m_{(2)} = 0$, $m^{(1)} = m^{(2)} = 1$.

**3.3.2 Relaxation orders.** It is straightforward to apply Fourier analysis to relaxation. Most relaxation schemes are found to be governed by the following orders of magnitude.

By an analog of (3.24), during a full relaxation sweep the changes in $v^{h,k}$ are $O(\sum_i h^{m^{(k)}+m_{(i)}}\hat{r}^{h,i}(\theta))$. The contribution of the $\theta$ Fourier component of $v^{h,l}$ to $\hat{r}^{h,i}(\theta)$ is $O((|\theta|/h)^{m_{il}})\hat{v}^{h,l}(\theta)$. For *uncolored relaxation* this implies that, in terms of the block notation (3.19a),

$$(\check{R}(\theta)_{1,1})_{k,l} = \delta_{kl} + O\Big(\sum_i h^{m^{(k)}+m_{(i)}-m_{il}}|\theta|^{m_{il}}\Big) \qquad (3.31a)$$

$$(\check{R}(\theta)_{J,J})_{k,l} = O\Big(\sum_i h^{m^{(k)}+m_{(i)}-m_{il}}\Big), \quad (2 \le J \le d) \qquad (3.31b)$$

$$\text{and} \quad \check{R}(\theta)_{J,K} = 0 \quad \text{for} \quad J \ne K. \qquad (3.31c)$$

For *colored relaxation*, colorful additional relations enter. Ignoring here full detailed description of all possible situations, the main effects can be summarized by the following modifications of (3.31b–c):

$$(\check{R}(\theta)_{J,1})_{k,l} = O\Big(|\theta|^{r_1^J}\sum h^{m^{(k)}+m_{(i)}-m_{il}}|\theta|^{m_{il}}\Big), \quad (2 \le J \le D) \qquad (3.31d)$$

where typically $0 \le r_1^J \le \infty$; and

$$(\check{R}(\theta)_{J,K})_{k,l} = O\Big(|\theta|^{r_K^J}\sum_i h^{m^{(k)}+m_{(i)}-m_{il}}\Big), (2 \le K \le D) \qquad (3.31e)$$

where typically $1 \le r_K^J \le \infty$ for $J \ge 2$ and $0 \le r_K^1 \le \infty$. One can refer to $r_K^J$ as the *$K$ to $J$ harmonic feeding order.*

**3.3.3 Inter-grid transfers.** It is assumed for simplicity that the residuals of each of the $q$ equations are transferred to the coarse grid separately from those of other equations, and we denote the "*order of restriction*" of the residuals of the $k$-th equation by $m_k$, and its "*secondary order*" by $\bar{m}_k$ $(k = 1, \ldots, q)$. By this we mean that

$$\hat{I}_h^H(\theta^0 + \tau^j)_{k\ell} = \delta_{k\ell}O(|\theta^0|^{m_k}), \qquad (j = 1, \ldots, D-1) \qquad (3.32a)$$

and

$$\hat{I}_h^H(\theta^0)_{k\ell} = \delta_{k\ell}[1 + O(|\theta^0|^{\bar{m}_k})]. \qquad (3.32b)$$

Similarly, assuming the interpolation of the $q$ functions are performed separately from each other, we denote the "*order of the interpolation*" of the $\ell$-th function by $m^\ell$ ($\ell = 1, \ldots, q$). This means that

$$\hat{I}_H^h(\theta^0 + \tau^j)_{k\ell} = \delta_{k\ell} O(|\theta^0|^{m^\ell}), \qquad (j = 1, \ldots, D-1) \tag{3.33a}$$

and

$$\hat{I}_H^h(\theta^0)_{k\ell} = \delta_{k\ell}[1 + O(|\theta^0|^{m^\ell}]. \tag{3.33b}$$

In case $2^d I_h^H$ is the adjoint of $I_H^h$ then $m_k = \bar{m}_k = m^k$. The $I$-th order multipolynomial interpolation (3.13) has the order $m^\ell = I$.

**3.3.4 Coarse grid correction operator.** The coarse grid correction (CGC) operator $S$ has the symbol

$$\check{S}(\theta) = \check{I} - \check{I}_H^h(\theta)\check{L}^H(\theta)^{-1}\check{I}_h^H(\theta)\check{L}^h(\theta). \tag{3.34}$$

Based on (3.24), (3.25) and (3.32b), it is easy to see that

$$\left(\check{L}^H(\theta)^{-1}\hat{I}_h^H(\theta)L^h(\theta)\right)_{k,l} = I_q + O\left(|\theta|^{\overline{m}} \sum_i \left(\frac{h}{|\theta|}\right)^{m^{(k)}+m_{(i)}-m_{il}}\right).$$

where $\overline{m} = \min[\tilde{p}, \overline{m}^1, \ldots, \overline{m}^q]$. With the above definitions of orders, and using for $\check{S}$ block indices as in (3.19a), it follows from (3.22), (3.25), (3.32) and (3.33) that for small $|\theta|$

$$\left(\check{S}(\theta)_{1,1}\right)_{k,\ell} = \delta_{kl} O(|\theta|^{m^k}) + O\left(|\theta|^{\overline{m}} \sum_i \left(\frac{h}{|\theta|}\right)^{m^{(k)}+m_{(i)}-m_{il}}\right) \tag{3.35a}$$

$$\left(\check{S}(\theta)_{1,K}\right)_{k,\ell} = O\left(\sum_i h^{m^{(k)}+m_{(i)}-m_{il}} |\theta|^{-m^{(k)}-m_{(i)}+m_i}\right) \tag{3.35b}$$

$$\left(\check{S}(\theta)_{J,1}\right)_{k,\ell} = \delta_{k\ell} O(|\theta|^{m^k}) + O\left(|\theta|^{m^k+\overline{m}} \sum_i \left(\frac{h}{|\theta|}\right)^{m^{(k)}+m_{(i)}-m_{il}}\right) \tag{3.35c}$$

$$\left(\check{S}(\theta)_{J,K}\right)_{k,\ell} = \delta_{JK}\delta_{k\ell} + O\left(\sum_i h^{m^{(k)}+m_{(i)}-m_{il}} |\theta|^{m^k-m^{(k)}-m_{(i)}+m_i}\right), \tag{3.35d}$$

for $2 \le J, K \le D$ and $1 \le k, \ell \le q$.

By (3.35a), the condition

$$\tilde{m} \underset{\text{def}}{=} \min[\tilde{p}, \bar{m}_1, \ldots, \bar{m}_q, m^1, \ldots, m^q] > 0 \tag{3.36}$$

is necessary for the CGC to effectively reduce smooth components; $\tilde{m}$ will be called the *CGC reduction order*.

**3.3.5 Orders for full efficiency**. Since for principal terms $m_{il} = m_{(i)} + m^{(l)}$, and since for some Fourier components $|\theta| \gg h$, various terms in (3.35) will be unbounded unless $m^{(k)} \geq m^{(l)}$. Exchanging the roles of $k$ and $l$ yields the reverse inequality, hence the condition

$$m^{(1)} = m^{(2)} = \cdots = m^{(q)} \tag{3.37}$$

is necessary for the $L_2$ theory to be applicable (cf. Secs. 3.3.1 and 6.1.1). Other norms and their relation to $m^{(k)}$ are discussed in Sec. 10.1.6 of [P2].

(3.37), (3.23a) and (3.35b) imply that a necessary condition for $\check{S}(\theta)_{1,K}$ to be small is either $m_i > m_{i*}$ or at least $m_i \geq m_{i*}$, where $m_{i*} = \max_l m_{il}$ is *the order of the $i$-th equation*. The *strict* inequality is clearly required to *guarantee* that the cycle performance is essentially free of the effects of the CGC amplification of high frequencies (see Sec. 6.2). A generalization of these requirements to other norms is discussed in Sec. 10.1.6 of [P2].

# 4. Treatment of Boundaries: General Approach

It is well known that the efficiency of the bigrid (and other multigrid) cycles may strongly be affected by the shape of the boundary curve (e.g., existence of reentrant corners), by the type and coefficients of the boundary conditions, by the boundary position relative to both the fine grid and the coarse grid, and by the discretization and solution processes (relaxation and inter-grid transfers) employed at and near the boundary. With this enormous variety, we believe it is unproductive in a general *quantitative* theory to rigorously analyze any one or other particular boundary situation. We will instead show that, in the limit $h \to 0$, the details of the boundary processes are never important, since, on one hand, they employ negligible amounts of computations, and, on the other hand, they can in a simple way always be *chosen* so that the overall efficiency (e.g., the convergence factor per cycle) is just the efficiency dictated by the *interior* processes, i.e., the efficiency calculated by (3.20) or (3.21).

Aside from simplicity and generality, two related reasons lead to this approach. First, we aim at an *exactly quantitative* analysis, i.e., calculating the actual numerical value (not just upper bounds) of $\lambda$ and $\lambda'$. This would be too difficult to do when all the details of complicated boundary situations should be taken into account. Secondly, from a practical point of view, our approach gives the more important information: it tells us what efficiency one *should* be getting

once the boundary processes have been *properly* adjusted. The analysis below also shows *one general way* of making this adjustment, although in practice, in each particular case some other ways may be more convenient or more effective at large values of $h$. (See the treatment of reentrant corners in [Bai, §4] and in [ZN]. For generality, in our treatment here, we do *not* exploit any smoothness properties of the boundary shape and the boundary operators.)

Thus, our approach is to allow the analyzed bigrid cycle to be modified near boundaries (and similarly also near other singular curves, such as interfaces, material discontinuities, super-element boundaries, etc.; cf. Secs. 9.1, 9.4 and 11.2 in [P2]), provided the work involved is negligible. The general way we propose to modify the cycle (in case one seeks the $\ell_2$ convergence discussed here) is to add a certain number of Kaczmarz [SK] relaxation passes over the boundary conditions and over the interior equations in some small neighborhood of the boundary (see details in Sec. 7). the reason for choosing this particular relaxation scheme is that it is "locally strongly reducing" (cf. Sec. 10.1.2) in the $L_2$ norm. We will therefore present now Kaczmarz relaxation and its relevant properties.

## 5. Kaczmarz Relaxation. Partial Sweeps

Consider the general system of linear (real or complex) algebraic equations

$$Au = f \quad \text{or} \quad \sum_{j=1}^{n} a_{ij}u_j = f_i, \qquad (i = 1, \ldots, n). \tag{5.1}$$

Given any approximation $u^*$, a *Kaczmarz relaxation step for the $i$-th equation* is defined as the replacement of $u^*$ by the vector closest to it on the $i$-th hyperplane (the hyperplane of solutions to the $i$-th equation). This means that each $u_k^*$ is replaced by $u_k^* + \beta_i \bar{a}_{ik}$, where $\bar{a}_{ik}$ is the complex conjugate of $a_{ik}$, $\beta_i = r_i^* / \sum_{j=1}^n |a_{ij}|^2$ and $r_i^* = f_i - \sum_{j=1}^n a_{ij}u_j^*$. This $r_i^*$, the residual of the $i$-th equation just before relaxing it, is called the $i$-th *dynamic* residual. Note that the residual of the equation just *after* relaxing it is zero, of course.

Suppose a Kaczmarz relaxation pass is made over the first $m$ equations ($1 \le m \le n$); i.e., for each of those equations in its turn in the natural order, a Kaczmarz relaxation step is performed. Denote the solution vectors before and after this relaxation pass by $u^0$ and $u^1$, respectively; the corresponding error vectors by $v^0 = u - u^0$ and $v^1 = u - u^1$; the corresponding residual vectors by $r^0 = f - Au^0 = Av^0$ and $r^1 = f - Au^1 = Av^1$; and the corresponding *normalized* residual vectors by $\hat{r}^0$ and $\hat{r}^1$, where

$$\hat{r}_i^s = r_i^s \Big/ \left( \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}, \qquad (i = 1, \ldots, n; s = 0, 1, *). \tag{5.2}$$

Also, denote by $\delta_i^*$ the $\ell_2$ norm of the solution change while relaxing the $i$-th equation, and note that

$$\begin{aligned} |\delta_i^*|^2 &= |\beta_i|^2 \sum_{k=1}^{n} |a_{ik}|^2 \\ &= |\tilde{r}_i^*|^2, \qquad (i = 1, \ldots, m), \end{aligned} \tag{5.3}$$

where $\tilde{r}_i^*$ is the *normalized dynamic* residual (defined in (5.2)). Using this notation and the $\ell_2$ norm $\|v\|^2 = \sum_i |v_i|^2$, we can formulate the following general property of Kaczmarz relaxation.

**Theorem 5.1.**

$$\begin{aligned} \|v^0\|^2 - \|v^1\|^2 &= \sum_{i=1}^{m} |\delta_i^*|^2 = \sum_{i=1}^{m} |\tilde{r}_i^*|^2 \\ &\geq \max\left( \gamma_0 \sum_{i=1}^{m} |\tilde{r}_i^0|^2, \quad \gamma_1 \sum_{i=1}^{m} |\tilde{r}_i^1|^2 \right) \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} \gamma_0 &= [(1+\gamma_-)(1+\gamma_+)]^{-1} \\ \gamma_1 &= (\gamma_-\gamma_+)^{-1} \\ \gamma_- &= \max_{2 \leq i \leq m} \left( \sum_{j=1}^{i-1} \sum_{\lambda=1}^{n} |a_{i\lambda} \bar{a}_{j\lambda}| \right) \Big/ \sum_{\lambda=1}^{n} |a_{i\lambda}|^2 \\ \gamma_+ &= \max_{1 \leq i \leq m-1} \left( \sum_{j=i+1}^{m} \sum_{\lambda=1}^{n} |a_{i\lambda} \bar{a}_{j\lambda}| \right) \Big/ \sum_{\lambda=1}^{n} |a_{i\lambda}|^2. \end{aligned} \tag{5.5}$$

**Proof.** The proof is analogous to the proof of Theorem 3.3 in [AMGT], which is the special case $m = n$. To see that the proof carries over to the case $m < n$, observe that the proofs of Theorems 3.1 and 3.2 in [AMGT] are easily modified to this case. ∎

The coefficients $\gamma_0$ and $\gamma_1$ can be interpreted as rough measures for the independence of the relaxed equations. In usual discretized PDEs, with proper scaling (see Remarks 5.1), they are always $O(1)$ (most often $\gamma_0 \approx 4$, $\gamma_1 \approx 1$). The theorem hence essentially says that the convergence is fast as long as the average normalized residual *of the relaxed equations* is comparable to the average error, averages being meant in the $\ell_2$ sense. Note that this can hold even when the average of *all* the normalized residuals is much smaller. The theorem thus spells out the significance

of *partial* relaxation, and even gives a useful criterion for deciding *where* to relax (see Sec. 11.2.1 in [P2]).

**Remark 5.1: Equation rescaling.** Note that Kaczmarz relaxation, as well as $v^0$, $v^1$, $\tilde{r}^*$, $\hat{r}^0$ and $\hat{r}^1$ appearing in (5.4), remain unchanged when each equation is rescaled (multiplied through by a constant). We can thus decrease $\gamma_\pm$ (thereby increasing $\gamma_1$ and $\gamma_2$) in the theorem to any values obtainable by such rescaling. It is then easy to see that for any local operator all $\gamma_i$ are $O(1)$. For example, this is obtained by rescaling so that $\sum_{j=1}^{n} |a_{ij}|^2 = 1$ for all $i$, yielding the *normalized* equation. In addition, $\gamma_i$ will be finite for many *non*-local operators.

# 6. Assumptions

Listed below are all our assumptions about the mathematical properties of the various multigrid ingredients: relaxation, $A^h$, $A^H$, $I_h^H$ and $I_H^h$. The discussion in this section will explain that each of these assumptions is *necessary*; the main content of our theory is of course to show, in subsequent sections, that the assumptions are *sufficient* — sufficient for achieving the predicted convergence factor (3.20). Except perhaps for the CGA (see discussion in Sec. 6.1), all other suppositions will be easily verifiable in any case of interest. They are all *qualitative* assumptions, in the sense that their main constants are arbitrary and unspecified. Indeed, the very point of this article is that such qualitative and necessary assumptions yield a *quantitative* and even *precise* prediction of convergence factors.

An *un*necessary assumption which for simplicity we do introduce at this first part of our presentation is that all interior processes (at distance greater than $O(h)$ from the boundary) have **constant coefficients.** This includes $L^h$, $L^H$, $I_h^H$ and $I_H^h$, as well as the **consistent ordering** of relaxation (extended as in Sec. 3). It is indeed only under this assumption that the local mode analysis is straightforwardly *defined*. The removing of this assumption, and the corresponding extension of the local mode analysis, are discussed in Sec. 9 of [P2].

The more important (less obvious) assumptions are described first (Secs. 6.1 and 6.2). All assumptions are summarized in Sec. 6.6.

## 6.1 Coarse grid approximation (CGA)

A necessary condition for the multigrid process to work properly is that errors which would converge slowly under *any* relaxation scheme should be well approximated on the coarse grid. The slower an error is reduced by any possible relaxation scheme, the better must its coarse grid approximation be. When such a condition

is *not* satisfied, the convergence factor of the bigrid cycle cannot be made uniformly (for all meshsizes) as small as one wishes by adding more relaxation sweeps (increasing $\nu_1 + \nu_2$). Moreover, if such a condition is unsatisfied at all multigrid levels, the $V$ cycle cannot be expected to produce convergence factors bounded away from 1 independently of the number of levels.

The basic relaxation schemes are the *pointwise* (local-processing) ones — in the sense that non-local *block* schemes, such as line relaxation, employ at each of their steps a solution process which could itself use a multigrid algorithm, with a pointwise scheme for its relaxation, hence the entire process may be interpreted as based on a pointwise relaxation scheme with semi coarsening (see [G, §4.2.1]). For simplicity *we will therefore refer below to pointwise schemes only* and defer discussion of non-local block schemes to[P2]. Now, any pointwise relaxation scheme introduces changes to the solution which are based on the size of the local residuals relative to the size of the coefficients of the corresponding equations, hence *any* such scheme *must* exhibit slow convergence when an error vector $v$ develops for which the normalized residuals $\tilde{r}$ are small, i.e., when $\|\tilde{r}\| \ll \|v\|$; see Sec. 5 above.

Unlike the *geometric notation* of Secs. 2 and 3, we have used in Sec. 5 the *algebraic notation*, where the unknowns $u$ are arranged in one long vector, and the fine-grid operator $A^h$ is correspondingly arranged as a big matrix $A$, with $(Au)_i = \Sigma a_{ij} u_j$. Keeping this notation, and motivated by the above discussion, we now introduce the *normalized operator* $\tilde{A}^h$, corresponding to the matrix $\tilde{A}$ defined by

$$(\tilde{A}u)_i = (Au)_i / \left( \sum_j |a_{ij}|^2 \right)^{1/2}. \tag{6.1}$$

Thus $\tilde{A}^h$ is the fine-grid operator normalized so that the sum of the squares of its coefficients at each point is 1. For example, if $A^h$ is a discretization of a second order differential equation with Neumann boundary conditions written in terms of divided differences, then $\tilde{A}^h$ is obtained from $A^h$ by multiplying each interior equation by an $O(h^2)$ factor, and each boundary equation by an $O(h)$ factor. When $v^h$ is the error function, then $r^h = A^h v^h$ is the residual function and $\tilde{r}^h = \tilde{A}^h v^h$ is the normalized residual function. Thus, in this notation, any pointwise relaxation scheme must exhibit slow convergence for any error $v^h$ for which $\|\tilde{A}^h v^h\| / \|v^h\|$ is small: the smaller this ratio, the slower the convergence. The multigrid process can therefore function properly only if for such errors the coarse grid correction (CGC) is sufficiently good: the smaller that ratio, the better the CGC. This leads to the following condition on the CGC operator $S$ (defined in Sec. 2).

**Coarse Grid Approximation (CGA) condition.** *For any $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$, independent of the meshsize, such that, if $\|\tilde{A}^h v^h\| \le \delta \|v^h\|$ then $\|S v^h\| \le \epsilon \|v^h\|$.*

We will *assume* this condition to hold. This will enable us to avoid dealing with many different cases and with details of boundary conditions. Indeed, the main point in this paper is to show that *quantitatively sharp* convergence factors $(\lambda, \lambda')$ can be derived from *qualitative* assumptions, such as this CGA assumption. The CGA, and in fact much stronger conditions, are normally assumed or proved in any $L_2$ multigrid theory, without having formerly yielded any realistic convergence constants at all, let alone the sharp constants of the local mode analysis. For example we prove below (Sec. 6.1.3) that the CGA condition is much weaker than the "Approximation Property" used in other $L_2$-norm theories.

More importantly, we can formally prove (see Sec. 6.1.2) that the above CGA condition is indeed a *necessary* condition for an $L_2$-norm multigrid convergence. Hence it is justified to assume it here, reserving for a separate study its verification for some or other concrete systems or under various possible hypotheses. The possible number of such systems or such hypotheses is very large — it includes all the various possibilities of discretizing near boundaries, for example. Indeed, it is exactly here, in the CGA verification, and *only* here, that the boundary details (its shape, its conditions, and their fine and coarse discretizations) matter. What the present theory will show, then, is that they matter only *qualitatively*: once satisfying the CGA condition, the quantitative ideal convergence speed does not depend on boundary details.

Note that by Theorem 6.1 below, the CGA condition is already enough to establish $h$-independent convergence for multigrid cycles with enough relaxation sweeps per cycle, where the relaxation can be either Kaczmarz, or (in positive-definite cases) Gauss-Seidel in arbitrary ordering, or under-Jacobi, or any other "reducing" relaxation scheme (cf. (6.2) and the remark following it). Thus, the rest of the assumptions below (Secs. 6.2–6.5) are only required in order to establish that the ideal obtainable efficiency (for *any* given type of relaxation and *any* given number of sweeps per cycle) is actually exactly the one calculated by (3.20).

**6.1.1 Limitation.** The $L_2$ CGA condition stated here cannot hold in many non-scalar ($q > 1$) cases. For instance, in case of the Stokes system (Example 3.1), an error $v^h$ which consists of no error in $U$ and $V$ and a highly oscillating error in $P$ cannot be much reduced by any coarse grid correction (because it is highly oscillatory), although it does satisfy $\|\tilde{A}^h v^h\| \leq O(h)\|v^h\|$. This shows that the error norm associated with $P$ should be different than those of $U$ and $V$. Hence such a system should be treated later, when other norms are discussed (see Secs. 10.1.1, 10.1.5 and 10.1.6 in [P2]).

**6.1.2 Necessity proof.** To formally show that the above CGA condition is necessary (and in fact, under stability requirements, also sufficient) for a multigrid algorithm to work properly (in one cycle, with a local relaxation scheme and in terms of the $\ell_2$ norm), we first introduce the following definitions.

We will say that a relaxation scheme $R$ is of a *local type* (*in the $L_2$ sense*) if (with the appropriate rescaling of Sec. 3.3.1) for any $v^h$

$$\|Rv^h - v^h\| \le C\|\tilde{A}^h v^h\|.$$

It is easy to see that every *local* relaxation (in the sense that, at each gridpoint $x$, $(Rv^h)(x) - v^h(x)$ is mainly determined by the residuals $A^h v^h(y)$ at points $y$ such that $|x - y| \le Ch$) must be of local type. Otherwise non-smooth errors (whose possible presence must be assumed when only local information is used) will diverge. Indeed, all familiar stable pointwise relaxation schemes are trivially of local type.

A CGC operator $S$ will be called *proper* (*in the $L_2$ sense*) if there exists a local-type relaxation scheme $R$ (possibly different from the relaxation scheme in the analyzed cycle) in combination with which $S$ gives "proper" multigrid cycles, in the sense that, for any $\varepsilon > 0$ there exists an integer $\nu(\varepsilon)$ independent of $h$ such that

$$\|SR^{\nu(\varepsilon)}\| \le \varepsilon.$$

(A weaker properness requirement is discussed in Sec. 6.1.5).

A CGC operator $S$ will be called *stable* (*in the $L_2$ sense*) if $\|S\| \le C$, i.e., $\|Sv^h\| \le C\|v^h\|$ for any $v^h$.

With these definitions, we can state the following.

**Theorem 6.1**. *If the CGC operator $S$ is stable, then it is proper if and only if it satisfies the CGA condition.*

**Proof**. Suppose the CGA condition is not satisfied by $S$. This means that there exists an $\varepsilon > 0$ for which the following holds: for any $\delta > 0$ there exists a $v^h = v^{h,\delta}$ such that

$$\|\tilde{A}^h v^h\| \le \delta\|v^h\| \quad \text{and} \quad \|Sv^h\| \ge 4\varepsilon\|v^h\|. \tag{6.1a}$$

For any $R$ of local type it can hence be shown by induction on $\ell$ that both

$$\|R^\ell v^h - v^h\| \le C_\ell \delta\|v^h\| \tag{6.1b}$$

and

$$\|\tilde{A}^h R^\ell v^h\| \le C_\ell \delta\|v^h\|. \tag{6.1c}$$

Hence, if $S$ is stable, for any $\nu$,

$$\|SR^\nu v^h - Sv^h\| \le C_\nu \delta\|v^h\|, \tag{6.1d}$$

so that, having chosen $\delta < 2\varepsilon/C_\nu$, we get by (6.1a)

$$\|SR^\nu v^h\| \ge \|Sv^h\| - C_\nu\delta\|v^h\| \ge 2\varepsilon\|v^h\|. \tag{6.1e}$$

This means that for any given integer $\nu$, one can choose $\delta$ sufficiently small so that $v^h = v^{h,\delta}$ satisfies both (6.1a) and (6.1e), which clearly shows that $S$ is not proper.

Conversely, if $S$ does satisfy the CGA condition, then using Theorem 5.1 it is trivial to show (cf. the proof of Theorem 6.2) that together with the Kaczmarz relaxation it gives proper multigrid cycles. ∎

In fact, if $S$ satisfies the CGA condition it will give proper multigrid cycles with any relaxation scheme $R$ which is "*reducing*" (*in the* $L_2$ *norm*), by which it is meant that, for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists an $h$-independent positive integer $\ell$ such that, for any $v^h$

$$\text{either} \quad \|R^\ell v^h\| \le \varepsilon_1 \|v^h\| \quad \text{or} \quad \|\tilde{A}^h R^\ell v^h\| \le \varepsilon_2 \|R^\ell v^h\|. \tag{6.2}$$

If $A$ is symmetric positive definite, then Gauss-Seidel, SOR, Jacobi under-relaxation and other popular schemes are easily shown to be reducing in this sense (and even in a stronger sense: see Sec. 10.1.4 in [P2]).

**6.1.3 Comparison to other sufficient conditions**. Since the CGA condition is necessary, it is clearly weaker than (or at least as weak as) any other sufficient condition proved in various $L_2$ theories. The "Approximation Property" in [HH, §3.1], for example, in the present notation can be written as

$$\|S(\tilde{A}^h)^{-1}\| \le C.$$

This clearly implies

$$\|Sv^h\| = \|S(\tilde{A}^h)^{-1}\tilde{A}^h v^h\| \le C\|\tilde{A}_h v^h\|$$

which is the CGA condition with $\delta(\varepsilon) = C\varepsilon$. The general CGA condition, however, allows the decrease in $\delta$ (as a function of $\varepsilon$) to be arbitrarily fast; e.g., $\delta = \varepsilon^{1/\gamma}$, with arbitrarily small $\gamma > 0$. Thus, the CGA condition is already implied by

$$\|Sv^h\| \le C\|\tilde{A}^h v^h\|^\gamma \|v^h\|^{1-\gamma} \qquad (\gamma > 0). \tag{6.3}$$

To show that (6.3) holds even in some cases for which the Approximation Property (i.e., the case $\gamma = 1$) fails, consider a scalar constant coefficient PDE of order $m$ discretized in the infinite space. In terms of the symbols introduced in Sec. 3.3.4, the Approximation Property requires

$$\|\check{S}(\theta)_{0j}\| \le C|\theta|^m, \qquad (j = 0, 1, \ldots, D-1),$$

while (6.3) is already implied by

$$\|\check{S}(\theta)_{0j}\| \le C|\theta|^{\gamma'}, \qquad (j = 0, 1, \ldots, D-1), \tag{6.3a}$$

for *any* $\gamma' > 0$, as can easily be established by the Parseval identity and the Hölder integral inequality.

It is easy to produce cases with $m = 2$ (e.g., Poisson equation) in which (6.3a) holds for $\gamma' = 1$ but not for $\gamma' > 1$. For example, such a case is trivially produced if in any multigrid process for which the Approximation Property does hold the interpolation $I_H^h$ is replaced by a first order one, e.g., zero degree polynomial interpolation, for which the error is $O(|\theta|)$).

**6.1.4 Weaker conditions.** The CGA condition as stated above is indeed *necessary* for the multigrid algorithm to work properly only if the latter does not employ local relaxation (partial sweeps). Since we consider here algorithms which do allow partial sweeps — the CGA condition can be weakened correspondingly. This section briefly describes such weaker conditions and their implications. (It can be skipped on first reading, together with Sec. 8.1.1.) We will make use of the maximum norm for normalized residuals

$$\|\tilde{r}^h\|_* = \max_{\alpha \in \Omega^h} |\tilde{r}^h_\alpha|.$$

The relaxation schemes that will be used in partial sweeps will naturally be of the successive displacement type, such as Gauss-Seidel or Kaczmarz. Each step of such schemes change the solution according to the values of residuals at the neighborhood of some gridpoint $\alpha$, whereby an error vector $v^h$ is changed into a new vector which we denote $R_\alpha v^h$. The *step* is called *local-type* if $\|R_\alpha v^h - v^h\| \leq C\|\tilde{A}^h v^h\|_*$.

The coarse grid correction operator $S$ is called *$L_2$-P-proper* (or *proper for Partial relaxation, in the $L_2$ sense*; or briefly *P-proper*) if for any $\varepsilon > 0$ there exists an integer $\nu(\varepsilon)$ independent of $h$ such that, any initial error $v^A$ can be relaxed, by at most $h^{-d}\nu(\varepsilon)$ local-type relaxation steps, to yield an error $v^B$ for which $\|Sv^B\| \leq \varepsilon\|v^A\|$.

The following is essentially the weakest CGA-type condition which can still guarantee $P$-properness.

**First Partial Coarse Grid Approximation (PCGA1) condition.** *For any $\varepsilon > 0$ there exists $\delta' = \delta'(\varepsilon)$ independent of the meshsize such that, if $\|\tilde{A}^h v^h\|_* \leq \delta' h^{d/2}\|v^h\|$ then $\|Sv^h\| \leq \varepsilon\|v^h\|$.*

**Theorem 6.2.** *If $S$ is stable and satisfies the PCGA1 condition, then $S$ is P-proper.*

**Proof.** Starting with an initial error $v^A$, the applied relaxation steps are Kaczmarz steps (cf. Sec. 5), each done for an equation $\alpha$ for which $|\tilde{A}^h v^*_\alpha| \geq C^{-1}\|\tilde{A}^h v^*\|_*$, where $v^*$ is the error function just before the step. (The implementation of

such steps is discussed in Sec. 11.2 of [P2].) This step reduces $\|v^*\|^2$ at least by $C^{-1}\|\tilde{A}v^*\|_*^2$, hence at most $C\delta'(\varepsilon)^{-2}h^{-d}$ such steps can be done before $\|\tilde{A}^h v^*\|_* \le \delta'(\varepsilon)h^{d/2}\|v^A\|$ is obtained. Thus $\nu(\varepsilon) = C\delta'(\varepsilon)^{-2}$. ∎

Although PCGA1 is sufficient for $P$-properness, it is not quite sufficient for obtaining the multigrid efficiency predicted by local mode analysis. For that, the following slightly stronger condition (still much weaker than the CGA condition) will be required.

**Second Partial Coarse Grid Approximation (PCGA2) condition**. *There exists a function $\gamma(h) > 0$ such that $\lim_{h\to 0}\gamma(h) = \infty$, for which the following holds: for any $\varepsilon > 0$ there exists $\delta'' = \delta''(\varepsilon)$ independent of the meshsize such that, if*

$$\|\tilde{A}v^h\|_* \le \delta'' h^{d/2}\gamma(h)\|v^h\| \quad \textit{and} \quad \|\tilde{A}^h v^h\| \le \delta''\|v^h\|,$$

*then $\|Sv^h\| \le \varepsilon\|v^h\|$.*

It is this condition that will be used in the subsequent theory (Sec. 8.1.1). The difference between PCGA1 and PCGA2 arises only at some very special border cases.

A further weakening of the above conditions can be made by using partial relaxation sweeps *after* the coarse grid correction. This weakening can simply be done by replacing $S$ in the condition by $S^+$, where $S^+v^h$ is any error that can be obtained from $Sv^h$ by negligibly costing partial sweeps. The resulting conditions will respectively be denoted PCGA1$^+$ and PCGA2$^+$. In the present $L_2$ theory, it is unlikely to have significant difference between $S^+$ and $S$: It is unusual for $\|Sv^h\|$ to be substantially reducible by local relaxation. But when higher Sobolev norms are used (see Sec. 10.1 in [P2]), the use of $S^+$ does sometimes make a difference: the value of $\|Sv^h\|$ (and in particular the value of its theoretical bounds) is often dominated by large difference quotients of $Sv^h$ in small neighborhoods of some singularities, which can easily be reduced by local relaxation.

**6.1.5 Weak properness**. A CGC operator $S$ will be called *weakly proper*, if there exist two local-type relaxation schemes $R_1$ and $R_2$, and for any $\varepsilon > 0$ there exist two $h$-independent integer $\nu_1(\varepsilon)$ and $\nu_2(\varepsilon)$ such that

$$\|R_2^{\nu_2(\varepsilon)} S R_1^{\nu_1(\varepsilon)}\| \le \varepsilon.$$

**Remark 6.1**. In the $L_2$ norm considered here, weak properness is seldom strictly weaker than properness, since in most cases (i.e., when suitable interpolation orders are used) the post-relaxation cannot reduce the $L_2$ norm of the error by more than some fixed factor.

**Theorem 6.3.** *If $S$ is weakly proper and stable and also "residual-stable", i.e.,*

$$\|A^h S v^h\| \le C\|A^h v^h\|,$$

*then $S$ satisfies the CGA condition.*

**Remark 6.2.** The residual-stability is often violated near structural singularities, but it is then normally satisfied with $S^+$ (see Sec. 6.1.4) instead of $S$. For such cases the theorem will show that at least the weaker CGA condition (with $S^+$ instead of $S$) is necessary, even for weak properness.

**Proof.** If the CGA condition is *not* satisfied then there exists an $\varepsilon > 0$ such that, for any $\delta > 0$ there exists an error $v^h = v^{h,\delta}$ satisfying (6.1a), hence also (6.1b), (6.1c) and, for any $\nu > 0$, (6.1d) and (6.1e). By the residual stability and (6.1c), and then (6.1e),

$$\begin{aligned}\|\tilde{A} S R^\nu v^h\| &\le C_\nu \delta \|v^h\| \\ &\le C_\nu \delta (2\varepsilon)^{-1} \|S R^\nu v^h\|.\end{aligned}$$

Hence, for any $R'$ of local type it can be shown by induction on $\ell$ that both

$$\|R'^\ell S R^\nu v^h - S R^\nu v^h\| \le C_\ell C_\nu \delta \varepsilon^{-1} \|S R^\nu v^h\|$$

and

$$\|\tilde{A}^h R'^\ell S R^\nu v^h\| \le C_\ell C_\nu \delta \varepsilon^{-1} \|S R^\nu v^h\|.$$

Hence, together with (6.1e)

$$\begin{aligned}\|R'^{\nu'} S R^\nu v^h\| &\ge (1 - C_{\nu'} C_\nu \delta/\varepsilon) \|S R^\nu v^h\| \\ &\ge 2\varepsilon (1 - C_{\nu'} C_\nu \delta/\varepsilon) \|v^h\|.\end{aligned}$$

Given any $\nu$ and $\nu'$ we choose any $\delta \le (2C_{\nu'} C_\nu)^{-1}\varepsilon$, thus obtaining $\|R'^{\nu'} S R^\nu v^h\| \ge \varepsilon \|v^h\|$, which together with (6.1a) shows that $S$ is not weakly proper. ∎

## 6.2 Inter-grid transfer orders

Another necessary condition for a multigrid cycle to work satisfactorily is that suitable orders are used for the coarse-to-fine correction interpolation operator $I_H^h$ and for the fine-to-coarse residual transfer $I_h^H$. The required orders depend on the norm we are using ($\ell_2$ here), and on what we mean by "satisfactorily". If all one wants is multigrid "properness" ($h$-independent convergence for a cycle with

sufficiently many relaxation sweeps) then we need no condition additional to the CGA (see Sec. 6.1.2). The mode analysis can of course tell us what orders are necessary for the CGA condition to hold: from (3.35b) it is clear that $m_i \geq m_{i*}$ is certainly necessary (for $\ell_2$ convergence in one cycle). But if one is actually interested in a *quantitative* control of the cycle efficiency, one should always use inter-grid transfer orders high enough to ensure that the convergence factor is no longer constrained by them. This means to require (see Sec. 3.3.5)

$$m_i > m_{i*}, \tag{6.4}$$

ensuring that the cycle efficiency is not determined by the size of (3.35b).

At first glance assumption (6.4) seems too strong, since in most applications $m_i = m_{i*}$ is used. This is because usually one is not interested in the $\ell_2$ convergence of one isolated cycle, but in the asymptotic convergence when many cycles are performed, or the performance of an FMG algorithm, or in the convergence by another norm (e.g., the energy norm). In all these cases, $m_i$ lower than allowed by (6.4) can be used (see Secs. 10.1.6, 10.3, 10.4 in [P2]). For the $\ell_2$ convergence in one cycle, $m_i = m_{i*}$ is a "border case", hence requires, in addition to the local mode analysis, a supplementary analysis to determine its quantitative efficiency (see Sec. 10.2 in [P2]; sometimes this analysis shows that the prediction of the local mode analysis still holds). Since our show-case first presentation below is for the simpler (non-border) situation, we must assume for now (6.4). This indeed is the order one *should* generally use if his interest is really in the $\ell_2$ convergence in one cycle.

No other explicit assumptions about transfer orders are needed now. Implicitly, implied by the CGA condition, we will of course necessarily satisfy (3.36).

## 6.3 Properties of $A^h$

It will be assumed that $A^h$ is a **local** operator. Namely, if we relate our algebraic notation to geometric locations through

$$(A^h u^h)(x_i) = \sum_j a_{ij} u^h(y_j) \tag{6.5}$$

then

$$a_{ij} = 0 \quad \text{for} \quad |x_i - y_j| > Ch. \tag{6.6}$$

For the interior operator $L^h$, defined by (3.5), this means that $a^h_\gamma(h) = 0$ for $|\gamma| > C$.

More precisely, it is enough to assume a somewhat weaker assumption. It will only be needed that $\gamma_1$ is finite (see Remark 5.1) and that (6.7) in the following lemma is satisfied.

**Lemma 6.1.** *If $A^h$ is a local operator, then for any vector-valued grid function $v$ and scalar continuum function $\varphi$*

$$\|\tilde{A}^h(\varphi v) - \varphi \tilde{A}^h v\| \le C\|v\| \max_{|x-y|\le Ch} |\varphi(x) - \varphi(y)|. \tag{6.7}$$

*where $\tilde{A}^h$ is the normalized operator (cf. (6.1)).*

**Proof.** By (6.5) and then (6.6) and the Cauchy-Schwarz inequality,

$$\begin{aligned} |[A^h(\varphi v)](x_i) - \varphi(x_i)A^h v(x_i)|^2 &= \left|\sum_j a_{ij}[\varphi(y_j) - \varphi(x_i)]v(y_j)\right|^2 \\ &\le \max_{|x-y|\le Ch} |\varphi(x) - \varphi(y)|^2 \sum_j |a_{ij}|^2 \sum_{|x_i - y_j|\le Ch} |v(y_j)|^2. \end{aligned}$$

Hence, dividing through by $\sum_j |a_{ij}|^2$ and then summing over $i$, (6.7) is obtained, since each $|v(x_j)|^2$ will appear in the sum in at most $q(2C+1)^d$ terms. ∎

It will also be assumed that the interior operator $L^h$ **approximates a differential operator** (of the first or higher order). More precisely, defining the normalized interior operator $\tilde{L}^h$ as the interior part of the normalized operator $\tilde{A}^h$ (cf. (6.1) and (3.5)), and denoting its symbol by $\hat{\tilde{L}}^h(\theta)$ (cf. (3.7)), it will be assumed that

$$\|\hat{\tilde{L}}^h(\theta)\| \to 0 \quad \text{as} \quad \max(h, |\theta|) \to 0, \tag{6.8}$$

where, as usual, $\|\cdot\|$ is the $\ell_2$ matrix norm. Indeed, it is easy to see that if $L^h$ is a difference approximation to a $q \times q$ differential operator (where each individual discrete equation approximates one of the $q$ differential equations), then

$$|\hat{\tilde{L}}^h(\theta)_{ij}| \le C(|\theta|^{m_{ij}} h^{m_{i*} - m_{ij}} + h^{m_{i*}}), \tag{6.9}$$

where $m_{i*} = \max_\ell m_{i\ell}$.

For some (e.g., Hermitian) finite element formulations, some of the unknown grid functions which do not correspond to continuum unknowns may have to be properly scaled for (6.8) to hold. Anyway, we use (6.8) only to show that the contribution of $v^f$ to $\bar{v}^d$ is small (see Sec. 8.4), which should be true for any reasonable type of approximation.

## 6.4 Stability

An obvious requirement that should be imposed is the stability of all the cycle processes in the norms of interest — $L_2$ norms in the present theory. Thus, we

will assume that, for any fine-grid error $v^h$ and residual $r^h$ and for any coarse-grid $v^H$ and $r^H$, the following holds.

$$\|Rv^h\| \leq C\|v^h\| \tag{6.10}$$

$$\|(A^H)^{-1}I_h^H A^h v^h\| \leq C\|v^h\| \tag{6.11}$$

$$\|I_h^H r^h\| \leq C\|r^h\| \tag{6.12}$$

$$\|(A^H)^{-1}r^H\| \leq C\|r^H\| \tag{6.13}$$

$$\|I_H^h v^H\| \leq C\|v^H\| \tag{6.14}$$

In addition we will also assume stability of relaxation in terms of the *residual* $L_2$ norm, i.e.,

$$\|\tilde{A}^h Rv^h\| \leq C\|\tilde{A}^h v^h\|. \tag{6.15}$$

For explicit processes, such as $A^h$, $I_h^H$, $I_H^h$ and Jacobi-type relaxation, these stability requirements are easily established. For non-explicit relaxation schemes, such as Gauss-Seidel and Kaczmarz, stability (6.10) or (6.15) can usually easily be checked by a version of local mode analysis (applied to the interior process, and adopting if necessary an approach similar to the one described in Sec. 4 above for handling boundaries): First stability is checked for the marching, point after point, within one line; then for the marching, line after line, within a plane (relating *line* Fourier decompositions of the errors); etc. It is, incidentally, quite important to check this stability, because sometimes in the search for schemes with minimal smoothing factors one can unwittingly run into unstable ones.

Usually then, the only stability requirements which are not easily verified are (6.11) and (6.13). A vast literature treats the latter, so here we can indeed simply *assume* it. Usually, in fact, results stronger than (6.13) are proved, in that the used Sobolev norms are higher on the left-hand than on the right-hand side. Furthermore, a condition much weaker than (6.13) would be enough, as can easily be seen by examining the places in Sec. 8 where (6.13) is used, i.e., in the final derivation of (8.21) and (8.30). But already condition (6.13) should anyway be satisfied by any usable discretization, whether or not a multigrid solver is applied to it. Condition (6.11) is listed here only for convenience: it is actually implied by the CGA assumption. (Because, if $v_*^h$ violates (6.11) with sufficiently large $C$, a suitably small multiple of it can be added to $v^h$ in the CGA condition and cause the latter to be violated.)

The stability assumptions together with the CGA assumption imply the following.

**Lemma 6.2.** *For any $\epsilon > 0$ and $c_* > 0$ there exists $\delta_1(\epsilon, c_*) > 0$ independent of $h$ such that, if the error at the beginning of the cycle satisfies*

$$\|v^A\| \leq c_* \tag{6.16}$$

*and*

$$\|\tilde{A}^h v^A\| \leq \delta_1(\epsilon, c_*), \tag{6.17}$$

*then the error at the end of the cycle satisfies*

$$\|v^D\| \leq \epsilon.$$

**Proof.** The stability requirements (6.10, 11, 14 and 15) imply that there exist $h$-independent constants $C_1$, $C_2$, $C_3$ and $C_4$ such that $\|v^B\| \leq C_1\|v^A\|$, $\|\tilde{A}^h v^B\| \leq C_2\|\tilde{A}^h v^A\|$, $\|v^D\| \leq C_3\|v^B\|$ and $\|v^D\| \leq C_4\|v^C\|$. Define

$$\delta_1(\epsilon, c_*) = \frac{\epsilon}{C_2 C_3}\delta(\epsilon_1), \quad \text{where} \quad \epsilon_1 = \frac{\epsilon}{c_* C_1 C_4}$$

and where $\delta(\epsilon_1)$ is the function defined by the CGA condition (cf. Sec. 6.1). If $\|v^B\| \leq \epsilon/C_3$ then the lemma trivially follows. If not, then by (6.17)

$$\|\tilde{A}^h v^B\| \leq C_2\|\tilde{A}^h v^A\| \leq C_2\delta_1 \leq \frac{C_2 C_3 \delta_1}{\epsilon}\|v^B\| = \delta(\epsilon_1)\|v^B\|.$$

By the CGA assumption it thus follows that $\|v^C\| \leq \epsilon_1\|v^B\|$, and hence

$$\|v^D\| \leq \epsilon_1 C_4\|v^B\| \leq \epsilon_1 C_1 C_4\|v^A\| \leq \epsilon_1 C_1 C_4 c_* = \epsilon. \quad \blacksquare$$

In case one wants to assume only the weaker CGA condition PCGA2 (Sec. 6.1.4), the stability residuals in relaxation should also be expressed in the *maximum* norm, supplementing (6.15) with the assumption

$$\|\tilde{A}^h R v^h\|_* \leq C\|\tilde{A}^h v^h\|_*. \tag{6.18}$$

For any reasonable relaxation scheme, the verification of this condition is trivial. With a proof essentially the same as for Lemma 6.2, one can then show the following.

**Lemma 6.3.** *Assuming PCGA2, for any $\varepsilon > 0$ and $c_* > 0$ there exists $\delta_1'(\varepsilon, c_*) > 0$ independent of $h$ such that, if $\|v^A\| \leq c_*$ and $\|\tilde{A}^h v^A\|_* \leq \delta_1'(\varepsilon, c_*)h^{d/2}\gamma(h)$ and $\|\tilde{A}^h v^A\| \leq \delta_1'(\varepsilon, c_*)$, then $\|v^D\| \leq \varepsilon$.*

## 6.5 Mode-analysis expressions. Smoothing factors

For each explicit and local process, each term in its associated symbol (cf. Sec. 3) is of course a linear combination of trigonometric functions of the form

$$e^{i\theta\alpha} = e^{i(\alpha_1\theta_1 + \cdots + \alpha_d\theta_d)}, \qquad |\alpha| = \max_j |\alpha_j| \leq C. \tag{6.19}$$

This is therefore true for each term in the matrices $\check{L}^h(\theta)$, $\check{I}_h^H(\theta)$, $\check{L}^H(\theta)$ and $\check{I}_H^h(\theta)$, and also $\check{R}(\theta)$ in case $R$ is a Jacobi-type relaxation. For more general relaxation schemes, $\check{R}(\theta)$ can be expressed as a product of such matrices (whose terms are linear combinations of trigonometric functions) and inverses of such matrices. Hence each term in $\check{R}(\theta)$ is a rational function of trigonometric functions (6.19). We will assume each of those terms to be *bounded*. This assumption, which can easily be checked and trivially holds in all familiar schemes, is equivalent to requiring

$$\|\check{R}(\theta)\| \le C, \qquad \left(\text{uniformly in } |\theta| \le \frac{\pi}{2}\right), \tag{6.20}$$

which largely overlaps assumption (6.10) above. (More precisely: (6.20) implies (6.10) in the interior. The latter, however, unlike (6.20), would allow $\|\check{R}(\theta)\|$ to be unbounded for $|\theta| \to 0$ in a way which is not characteristic to usual relaxation schemes. The theory below allows a condition weaker, but more complicated to state, than (6.20).)

A more delicate condition should be required from $\check{L}^H(\theta)$. Since its inverse will be used, we should roughly require that $\check{L}^H(\theta)$ is nonsingular for $\theta \neq 0$, which is a sort of **ellipticity** requirement. More precisely, we can use the weaker condition

$$\det \check{L}^H(\theta) \neq 0 \quad \text{for} \quad |\theta| \ge h^{1-\sigma_*} \tag{6.21}$$

with a certain sufficiently small $\sigma_* > 0$. This condition expresses *ellipticity on scale* $H^{\sigma_*}$ (see [G, §2]). Such conditions are easily checked and trivially satisfied by all discretized elliptic systems; but we can substantially further weaken them here.

Observe that $\det \check{L}^H(\theta)$ is a polynomial in trigonometric functions (6.19), hence, for small $|\theta|$ it is approximately a polynomial in $\theta$, and its derivatives with respect to $\theta$ are again polynomials in $\theta$. Ellipticity therefore yields that, for any non-negative integer $\ell$,

$$|\partial^\ell (\check{L}^H(\theta)^{-1})_{jk}| \le C_\ell h^{\kappa'} |\theta|^{-\kappa-\ell}, \qquad \left(0 < |\theta| \le \frac{\pi}{2}\right) \tag{6.22}$$

where $\partial^\ell$ is any $\ell$-order derivative with respect to $\theta = (\theta_1, \dots, \theta_d)$, and $\kappa$ and $\kappa'$ are independent of $\ell$. In fact we will use the much weaker condition

$$\left|\partial^\ell \left(\check{L}^H(\theta)^{-1} \check{I}_h^H(\theta) L^h(\theta)\right)_{jk}\right| \le C_\ell h^{\kappa'} |\theta|^{-\kappa-\ell} \qquad \left(h^{1-\sigma_*} \le |\theta| \le \frac{\pi}{2}\right) \tag{6.23}$$

where $\kappa$ and $\kappa'$ are independent of $\ell$ and $\sigma_* > 0$ sufficiently small. (The value $\sigma_* = 1/(2 \max_i m_{i*})$ will be shown suitable in the proof below.) This condition

is much weaker than ellipticity because it would normally hold for non-elliptic systems as well, especially under assumption (6.4).

Thus, ellipticity is not explicitly used here. It is, however, related to the CGA and (6.13) assumptions, and to the size of $\lambda$ (defined by (3.20)) that can be obtained. Real extensions to non-elliptic problems will be discussed in Sec. 10.5 of [P2].

Finally, we need to express in terms of symbols another assumption which in fact results form the CGA and (6.4) assumptions. The former implies that sufficiently smooth components are reduced as far as one wishes by means of the CGC step, while the latter, together with (6.13), imply that *harmonics* of sufficiently smooth components are practically unchanged by that step (cf. (3.35)). Hence, the two assumptions together imply that, for $|\theta| \to 0$, the CGC action on the error components $E_\alpha \check{v}^B(\theta) e^{i\theta\alpha}$ (i.e., the component $\theta$ and its harmonics) practically gives $E_\alpha \check{T}(\theta)\check{v}^B(\theta)e^{i\theta\alpha}$, where $\check{T}(\theta)$ is the $qD \times qD$ matrix defined by

$$\begin{aligned} \check{T}(\theta)_{JK} &= \delta_{JK} T(\theta + \tau^{J-1}) Iq \\ T(\theta) = T(\theta_1, \ldots, \theta_d) &= \begin{cases} 0 & \text{if } -\frac{\pi}{2} \le \theta_j < \frac{\pi}{2} \text{ for } j = 1, \ldots, d \\ 1 & \text{otherwise.} \end{cases} \end{aligned} \tag{6.24}$$

Thus, as $|\theta| \to 0$, $M(\theta) \approx \tilde{M}_\nu(\theta) \underset{\text{def}}{=} \check{R}(\theta)^{\nu_2} \check{T}(\theta) \check{R}(\theta)^{\nu_1}$, and, in view of (3.20), we can make it our assumption that

$$\limsup_{\max(|\theta|,h)\to 0} \|\check{R}(\theta)^{\nu_2} \check{T}(\theta) \check{R}(\theta)^{\nu_1}\| \le \lambda. \tag{6.25}$$

Although actually resulting from other assumptions, (6.25) can separately be checked.

Incidentally, $\check{T}(\theta)$ represents the ideal performance of the CGC step: for sufficiently good $I_h^H$, $A^H$ and $I_H^h$ one would expect $M(\theta) \approx \tilde{M}_\nu(\theta)$ for all $|\theta| \le \pi/2$. In that case $\lambda' \approx \bar{\mu}_\nu^\nu$, where

$$\bar{\mu}_\nu = \sup_{|\theta| \le \pi/2} \left(\sigma(\tilde{M}_\nu(\theta))\right)^{1/\nu}. \tag{6.26}$$

This $\bar{\mu}_\nu$ is the familiar "*smoothing factor*" [G, §3.1], which can thus be used as a rough efficiency predictor. It is a very useful predictor: first, because it is simpler to calculate than $\lambda$ or $\lambda'$ (especially when relaxation is *strictly* consistently ordered, hence uncolored, in which case $\bar{\mu}_\nu$ does not depend on $\nu$, and its calculation is reduced to computing $q \times q$ instead of $Dq \times Dq$ matrices). Also, and more important, $\bar{\mu}_\nu$ tells you what is the *ideal* performance one *can* obtain with a given relaxation scheme. Hence it allows a precise *separate* design of optimal relaxation. This design can usually further be reduced to the design of relaxation schemes

for simple scalar equations (the factors of the principal determinant of the matrix operator; see [G84, §3.7]).

### 6.6 Summary

In summary, for the present $L_2$ theory of one bigrid cycle we assume that the CGA condition is satisfied, as well as (6.4), (6.7), (6.8), (6.10), (6.12), (6.13), (6.14), (6.15), (6.20), (6.23) and (6.25). They are all necessary conditions ((6.4) is only necessary for the present type of theory; cf. Sec. 6.2 and Sec. 10.2 in [P2]). They are all easy to check for any given concrete case, except for the CGA condition (see the long discussion in Sec. 6.1) and (6.15) (see discussion in Sec. 6.4).

The CGA condition can be weakened to the PCGA2 or PCGA2$^+$ condition (cf. Sec. 6.1.4), by adding assumption (6.18).

## 7. Modified Cycle and Main Theorem

As explained in Sec. 4, the bigrid cycle which we will actually analyze is a modification, involving negligible extra work, of the cycle (i)–(iii) defined in Sec. 2. To define the modified cycle we introduce a constant $0 < \sigma_1 < 1$ (whose value will further be specified later) and a small "distance from the boundary" $\rho = h^{\sigma_1}$, and for any $\rho' > 0$ we denote

$$\Omega_{\rho'} = \{x \mid x \in \Omega, |x - y| > \rho' \quad \text{for any} \quad y \notin \Omega\}. \tag{7.1}$$

$$\hat{\Omega}^h_{\rho'} = \Omega^h - \Omega^h_{\rho'}. \tag{7.2}$$

*The steps of the modified cycles are the following.* First, $K_0$ passes of Kaczmarz relaxation are performed in $\hat{\Omega}^h_{3\rho}$. Then steps (i), (ii) and (iii) of the unmodified cycle (Sec. 2) are carried out. (Partial post-relaxation could be added between steps (ii) and (iii) if $S^+$ is used instead of $S$ in the CGA condition. See Sec. 6.1.4.)

The number $K_0$ of boundary passes, to be specified below, will be independent of $h$, hence the total extra work involved will be at most $O(h^{\sigma_1})$ compared to the work in any of the other steps of the cycle. $K_0$ will in fact depend only on local properties of the fine-grid operator $A^h$ (actually only on the quantity $\gamma_1$ defined in Theorem 5.1 and Remark 5.1) and on the accuracy $\epsilon$ to which we want to approximate the mode-analysis convergence factors by the modified-cycle convergence factor. This is the content of our main theorem.

**Theorem 7.1**. *Let $\lambda$ be the mode-analysis convergence factor* (3.20). *Then, under the assumptions listed in Sec.* 6.6, *for any $\epsilon > 0$ there is $K_0 = K_0(\epsilon)$, independent*

*of $h$, such that*

$$\frac{\|\bar{v}\|}{\|v\|} \leq \lambda + \epsilon, \tag{7.3}$$

*where $v$ and $\bar{v}$ are the error functions (vectors) before and after the application of the modified cycle, and $\|\cdot\|$ is the $\ell_2$* norm.

# 8. Proof

## 8.1 Cutting away the boundary

Let $K_1 = (\gamma_1 \delta_2)^{-1}$ for some constant $\delta_2 = \delta_2(\epsilon)$ which will be selected below, where $\gamma_1$ is defined in Remark 5.1. Denote by $v^{[k]}$ the error vector obtained after $k$ Kaczmarz boundary passes, and by $\|\tilde{A}v^{[k]}\|_{3\rho}$ the $\ell_2$ norm of $\tilde{A}v^{[k]}$ confined to the relaxed domain $\hat{\Omega}^h_{3\rho}$, where $A = A^h$ and the normalized operator $\tilde{A}$ is defined in (6.1). Then, for some $1 \leq k \leq K_1$,

$$\|\tilde{A}v^{[k]}\|^2_{3\rho} \leq \delta_2 \|v\|^2, \tag{8.1}$$

because otherwise, by Theorem 5.1, each Kaczmarz sweep would reduce the square error norm by more than $\gamma_1 \delta_2 \|v\|^2$, so the $K_1$ sweeps would reduce it below 0. Thus, taking $K_0 \leq K_1$ to be that $k$ for which $\|\tilde{A}v^{[k]}\|_{3\rho}$ is minimal, and denoting $v^{[K_0]}$ by $v^a$, we get

$$\|\tilde{A}v^a\|^2_{3\rho} \leq \delta_2 \|v\|^2 \quad \text{and} \quad \|v^a\| \leq \|v\|. \tag{8.2}$$

The second inequality is the result of the error-decreasing property of Kaczmarz relaxation (see Theorem 5.1).

We next note that, instead of calculating the final error $\bar{v}$ that results from applying Steps i), ii) and iii) starting with $v^a$ as the initial error, it is enough to calculate the final error $\bar{v}^b$ resulting from an initial error

$$v^b = \varphi v^a, \tag{8.3}$$

where $\varphi$ is a $C^\infty$ function defined on $\mathbb{R}^d$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 0$ for $x \notin \Omega_\rho$, $\varphi(x) = 1$ for $x \in \Omega_{2\rho}$ and, for any $x$ and $y$,

$$|\varphi(x) - \varphi(y)| \leq C|x - y|/\rho. \tag{8.4}$$

Indeed, from (8.4) and Lemma 6.1 it follows that

$$\begin{aligned} \|\tilde{A}(v^b - v^a)\| &= \|\tilde{A}(1 - \varphi)v^a\|_{3\rho} \\ &\leq \|\tilde{A}v^a\|_{3\rho} + C\frac{h}{\rho}\|v^a\| \\ &\leq C(\delta_2(\epsilon)^{1/2} + h^{1-\sigma_1})\|v\| \end{aligned} \tag{8.5}$$

the last inequality resulting from (8.2). Hence, for $h$ sufficiently small and choosing $\delta_2(\epsilon) \le [\delta_1(\epsilon/2,1)/(2C)]^2$, where $\delta_1$ is defined in Lemma 6.2, by (8.2) and Lemma 6.2 we get that

$$\|\bar{v} - \bar{v}^b\| \le \frac{\epsilon}{2}\|v\|. \tag{8.6}$$

**8.1.1 Modifications for weaker CGA.** In case the CGA condition is not satisfied, but the PCGA2 condition (Sec. 6.1.4) together with (6.18) are satisfied, the $K_0$ Kaczmarz boundary *passes* are replaced by a sequence of Kaczmarz *steps* (cf. Sec. 5), each done for an equation $\alpha$ for which $|A^h v^*_\alpha| = \|A^h v^*\|_{*3\rho}$, or at least $|A^h v^*_\alpha| \ge C^{-1}\|A^h v^*\|_{*3\rho}$, where $\|r\|_{*3\rho} = \max_{\alpha\in\hat{\Omega}^h_{3\rho}} |r_\alpha|$ and $v^*$ is the error function just before the step (hence $A^h v^*$ is the current vector of residuals). (For practical aspects of this, see Sec. 11.2 in [P2].) Each such step reduces $\|v^*\|^2$ at least by $C^{-1}\|\tilde{A}v^*\|^2_*$ (cf. Sec. 5), hence their total number cannot exceed $K_2 = C\|v\|^2/\delta_h^2$ before $\|\tilde{A}v^*\|_{*3\rho} \le \delta_h$ is obtained. Choosing $\delta_h = \delta_2 h^{d/2}\gamma(h)\|v\|$, we get $K_2 \le C\gamma(h)^{-2}h^{-d}\delta_2^{-2}$, so that, for sufficiently small $h$, the total work in the Kaczmarz steps is negligible compared to the overall work of the cycle, and they produce an error $v^a$ for which

$$\|\tilde{A}v^a\|_{*3\rho} \le \delta_2 h^{d/2}\gamma(h)\|v\|.$$

This trivially entails (8.2) as well. The rest of the proof remains basically unchanged, with Lemma 6.3 replacing Lemma 6.2. Note that $\gamma(h)$ is needed in this proof to ensure that $K_2 \ll h^{-d}$, hence the weaker condition PCGA1 could not replace PCGA2.

## 8.2 Separating away fringe components

The error function $v^b$, which vanishes outside $\Omega^h_\rho$, can now be extended to the entire space $\mathbb{R}^d$, by defining $v^b(x) = 0$ for $x \notin \Omega^h$. This extended function has the Fourier decomposition, similar to (3.1),

$$v^b_\alpha = \int_{|\theta|\le\pi} \hat{v}^b(\theta)e^{i\theta\alpha}d\theta \tag{8.7}$$

where

$$\hat{v}^b(\theta) = (2\pi)^{-d}\sum_\beta v^b_\beta e^{-i\theta\beta}, \tag{8.8}$$

$\sum_\beta$ denoting summation over the entire grid of integer vectors $\beta = (\beta_1,\ldots,\beta_d)$. As implied by (8.8), $\hat{v}^b(\theta)$ is $2\pi$-periodic (cf. (3.3)), and the range of integration

meant in (8.7) is over one period, e.g., over the range (3.4). The *existence* of the Fourier transform (8.7) is trivial, since $v^b$ has a bounded support and therefore $\hat{v}^b(\theta)$, defined by (8.8), is smooth and therefore also it is permitted, upon substituting (8.8) into the right-hand side of (8.7), to exchange the order of summation and integration. The integration then clearly vanishes unless $\alpha = \beta$, hence the summation yields the left-hand side of (8.7).

Numerical processes which are fully local, such as Jacobi relaxation sweeps, will operate separately on each Fourier component (i.e., they will produce a new error function, whose new value for the Fourier amplitude $\hat{v}(\theta)$ will depend only on the old value of $\hat{v}(\theta)$ for the same $\theta$). This is because such operators can directly be extended as constant-coefficient operators to the entire domain, at least for error functions vanishing near the boundaries. Other type of point-by-point relaxation sweeps, such as Gauss-Seidel and Kaczmarz, can also be so extended with negligible error (as we will see below), at least for error functions vanishing outside $\Omega_\rho^h$ where $\rho \gg h$. The inter-grid transfers $I_h^H$ and $I_H^h$ are truely local operators, but since they connect grid $h$ with grid $2h$ they couple harmonics (see Sec. 3). The only truely non-local operation in the multigrid cycle is $(A^H)^{-1}$ — the solution, in Step (ii), of the coarse-grid equations. The main idea of our proof is the observation that even $(A^H)^{-1}$ can be regarded as a local operation, provided it is confined to error functions vanishing outside $\Omega_\rho^h$ (except for possible residues smaller than any power of $h$) and having non-vanishing Fourier components $\hat{v}(\theta)$ only in the range $|\theta| \gg h/\rho$. Since $\rho = h^{\sigma_1} \gg h$, this range includes all components except for very smooth ones. Our next step in the proof is therefore to separate from $v^b$ those very smooth components, for which special estimates (using their smoothness rather than mode analysis) will then be applied.

The general form of $v^0$, the very smooth part to be separated from $v^b$, will be

$$v_\alpha^0 = \int_{|\theta| \le \pi} \psi(\theta) \hat{v}^b(\theta) e^{i\theta\alpha} d\theta \tag{8.9}$$

where $\psi$ is a $2\pi$-periodic function (cf. (3.3)) such that

$$\psi(\theta) = 1 \quad \text{for} \quad |\theta| \le \eta_0 \quad \text{and} \quad \psi(\theta) = 0 \quad \text{for} \quad \eta_1 \le |\theta| \le \pi \tag{8.10}$$

and where $h^{1-\sigma_1} \ll \eta_0 < \eta_1$. The choice of $\eta_0$, $\eta_1$ and other properties of $\psi$ will be described below. Since in the bigrid mode analysis each component $\theta$ is coupled to its "harmonics" $\theta^j = \theta + \tau^j (\mathrm{mod}\ 2\pi)$ ($j = 1, \ldots, D-1$; see Sec. 3), we should also separate from $v^b$ the harmonics of the very smooth components, i.e., the functions

$$v_\alpha^j = \int \psi(\theta - \tau^j) \hat{v}^b(\theta) e^{i\theta\alpha} d\theta, \qquad (j = 1, \ldots, D-1). \tag{8.11}$$

On the coarse grid these functions become very smooth, so their analysis, as we will see, will be a combination of mode analysis (in relaxation) and the use of smoothness-dependent estimates (in the coarse grid correction).

The remaining function

$$v^c = v^b - \sum_{j=0}^{D-1} v^j, \tag{8.12}$$

which represents the bulk of the error, will be analyzed by mode analysis. To do this, however, we need that $v^c$, like $v^b$, (very nearly) vanishes in a sufficiently large neighborhood of the boundary (so that the deviation of the multigrid processes from having constant coefficient over the entire space will have a negligible effect). This is obtained by choosing sufficiently smooth $\psi$. Specifically, we choose $\eta_1 = \eta_0 + \eta_2$ with

$$\eta_2 = O(h^{1-\sigma_2}), \quad \eta_0 = O(h^{1-\sigma_3}), \quad \text{and} \quad 1 > \sigma_3 > \sigma_2 > \sigma_1, \tag{8.13}$$

and then construct $\psi(\theta) \in C^\infty$ such that

$$0 \le \psi(\theta) \le 1 \quad \text{and} \quad |\partial^\ell \psi(\theta)| \le C_\ell \eta_2^{-\ell} \tag{8.14}$$

for any $\theta$ and any $\ell$-order derivative $\partial^\ell$, ($\ell = 0, 1, 2, \ldots$; the exact values of $\eta_2$, $\eta_0$, $\sigma_2$ and $\sigma_3$ will be further specified later). As we will see below this entails that $v^c$ practically vanishes outside $\Omega^h_{\rho/2}$.

Since all our cycle processes are linear we can apply them separately to $v^c$ and to $v^d = \sum_{j=0}^{D-1} v^j$ and estimate their separate contributions to the final error $\bar{v}$. We denote these contributions by $\bar{v}^c$ and $\bar{v}^d$, respectively.

## 8.3 Estimating the main post-cycle error $\bar{v}^c$

By (8.12), (8.9), (8.11) and similarly to (3.15), the main error $v^c$ can be decomposed as

$$v^c_\alpha = \int^0 \psi_0(\theta) E_\alpha \check{v}^b(\theta) e^{i\alpha\theta} d\theta \tag{8.15}$$

where $\psi_0(\theta) = 1 - \psi(\theta)$ and $\check{v}^b(\theta)$ is the vector

$$\check{v}^b(\theta) = (\hat{v}^b(\theta^0)^\dagger, \ldots, \hat{v}^b(\theta^{D-1})^\dagger)^\dagger. \tag{8.16}$$

Applying the bigrid processes to $v^c$ *in the infinite domain* would successively produce the functions $v^{c1}, v^{c2}, \ldots, v^{c6}$ defined by

$$v^{ct}_\alpha = \int^0 \psi_0(\theta) E^t_\alpha M_t(\theta) \check{v}^b(\theta) e^{i\theta\alpha} d\theta, \qquad (t = 1, \ldots, 6) \tag{8.17}$$

where $M_1(\theta) = \check{R}(\theta)^{\nu_1}$, $M_2(\theta) = \check{L}^h(\theta)M_1(\theta)$, $M_3(\theta) = \check{I}_h^H(\theta)M_2(\theta)$, $M_4(\theta) = \check{L}^H(\theta)^{-1}M_3(\theta)$, $M_5(\theta) = \check{I}_H^h(\theta)M_4(\theta)$ and $M_6(\theta) = \check{R}(\theta)^{\nu_2}[M_1(\theta) - M_5(\theta)] = M(\theta)$ (cf. (3.19)) and $E_\alpha^1 = E_\alpha^2 = E_\alpha^5 = E_\alpha^6 = E_\alpha$ while $E_\alpha^3 = E_\alpha^4 = I_q$. We have to prove that, with negligible differences, the same functions will be produced by applying the corresponding *real* processes, in the *real* domain $\Omega^h$ with its *real* discrete boundary conditions. Since the boundary conditions for the error functions are always homogeneous, it is enough to prove that each $v^{ct}$ practically vanishes outside $\Omega^h_{\rho/2}$. Indeed, its $j$-th component, $v^{ct,j}$, is a function which, by (8.17), has the general form

$$v_\alpha^{ct,j} = \sum_\beta \sum_{k=1}^q v_\beta^{b,k} \int_{|\theta|\le\pi} \psi_*(\theta) M_{tjk}(\theta,h) e^{i\theta(\alpha-\beta)} d\theta, \qquad (j=1,\dots,q) \quad (8.18)$$

where $\psi_*(\theta) = 1 - \psi(\theta^0)$, $\theta^0$ denoting, here and below in this section, the lowest harmonic of $\theta$, i.e., $-\frac{\pi}{2} \le \theta_i^0 < \frac{\pi}{2}$ and $(\theta_i - \theta_i^0)/\pi$ is an integer $(i = 1,\dots,d)$. By (8.14) and (8.10),

$$\begin{aligned} &0 \le \psi_*(\theta) \le 1, \quad |\partial^\ell \psi_*(\theta)| \le C_\ell \eta_2^{-\ell} \\ \text{and} \quad &\psi_*(\theta) = 0 \quad \text{for} \quad |\theta^0| \le \eta_0. \end{aligned} \qquad (8.19)$$

By the assumptions in Sec. 6.5, each $M_{tjk}(\theta,h)$ is a rational function of trigonometric functions of $\theta$, unbounded only at $\tau^0,\dots,\tau^{D-1}$, satisfying

$$|\partial^\ell M_{tjk}(\theta,h)| \le C_\ell h^{-\kappa'} |\theta^0|^{-\kappa-\ell}, \qquad \left(h^{1-\sigma_*} \le |\theta^0| \le \frac{\pi}{2}\right) \quad (8.20)$$

where $\partial^\ell$ is any $\ell$-order (partial) derivative with respect to $\theta = (\theta_1,\dots,\theta_d)$, and $\kappa$ and $\kappa'$ are independent of $\ell$. Choosing for each $\beta$ an index $1 \le j_\beta \le d$ such that $|\alpha_{j_\beta} - \beta_{j_\beta}| = |\alpha - \beta| \underset{\text{def}}{=} \max_i |\alpha_i - \beta_i|$ and then integrating by parts $\ell$ times with respect to $\theta_{j_\beta}$ the integral in (8.18), one obtains

$$\begin{aligned} |v_\alpha^{ct,j}| &\le C \sum_{\beta,k} |v_\beta^{b,k}| \, |\alpha-\beta|^{-\ell} \max_{|\theta|\le\pi} \left| \frac{\partial^\ell}{\partial \theta_{j_\beta}^\ell} [\psi_*(\theta) M_{tjk}(\theta,h)] \right| \\ &\le C_\ell \sum_{\beta,k} |v_\beta^{b,k}| \, |\alpha-\beta|^{-\ell} \sum_{0\le\ell'\le\ell} \eta_2^{-\ell'} h^{-\kappa'} \eta_0^{-\kappa-(\ell-\ell')} \end{aligned}$$

the second inequality resulting from (8.19), (8.20) and from assuming $\sigma_* < \sigma_3$ (hence $h^{1-\sigma_*} < \eta_0$, hence the applicability of (8.20) for any $|\theta^0| > \eta_0$). Now, for any $\alpha \notin \Omega^h_{\rho/2}$ and any $\beta$ such that $v_\beta^{b,k} \ne 0$ (hence $\beta \in \Omega^h_\rho$) we clearly have $|\alpha h - \beta h| \ge \rho/2 = h^{\sigma_1}/2$. Hence, for $\ell > d/2$, applying Cauchy-Schwarz

inequality and using magnitudes (8.13) for $\eta_2$ and $\eta_0$, the inequality above yields

$$\begin{aligned}|v_\alpha^{ct,j}| &\le C_\ell \|v^b\| \max_{0\le\ell'\le\ell} h^{(\sigma_1-1)d/2-\kappa'-(1-\sigma_3)\kappa+(\sigma_3-\sigma_1)(\ell-\ell')+(\sigma_2-\sigma_1)\ell'} \\ &\le C_\ell h^{-\kappa''+(\sigma_2-\sigma_1)\ell}\|v^b\|\end{aligned}$$

where $\kappa''$ is independent of $\ell$, and where $\sigma_3-\sigma_1$ has been replaced by $\sigma_2-\sigma_1$ using (8.13).

Thus, for $\alpha \notin \Omega^h_{\rho/2}$, since $\ell$ is arbitrary, we see that $|v_\alpha^{ct,j}|$ is smaller than any power of $h$. Hence, each $v^{ct}$ satisfies the homogeneous boundary conditions with an error smaller than any power of $h$. Due to the stability of the bigrid processes (6.10)–(6.14), we can therefore conclude that, for any desired $\ell$,

$$\|v^{c6}-\bar{v}^c\| \le C_\ell h^\ell \|v^b\|. \tag{8.21}$$

In addition, by (3.19)–(3.20) and the Parseval identity,

$$\|v^{c6}\| \le \lambda \|v^c\|, \tag{8.22}$$

which together with (8.21) yields the required estimate for $\bar{v}^c$.

## 8.4 Estimating fringe post-cycle error $\bar{v}^d$

Applying step (i) ($\nu_1$ relaxation sweeps) to $v^d$ one obtains $v^e$, defined by

$$v_\alpha^e = \int^0 \psi(\theta) E_\alpha \check{R}(\theta)^{\nu_1} \check{v}^b(\theta) e^{i\theta\alpha} d\theta, \tag{8.23}$$

and, as before, it is immaterial whether the relaxation incorporates the boundary conditions or not, since $v^e$ practically vanishes outside $\Omega^h_{\rho/2}$. Observing that $\tau^0 = 0$ in (3.16) and writing $E_\alpha = E + E'_\alpha$, where $E = (I_q, 0, \ldots, 0)$, the smooth part of $v^e$ is $v^f$, defined by

$$v_\alpha^f = \int^0 \psi(\theta) E \check{R}(\theta)^{\nu_1} \check{v}^b(\theta) e^{i\theta\alpha} d\theta. \tag{8.24}$$

Since $v^f$ practically vanishes near the boundary and its transform, by (8.24) and (8.10), contains Fourier components $e^{i\theta\alpha}$ only in the range $|\theta| \le \eta_0 = h^{1-\sigma_3}$, our assumption (6.8) implies that, for $h$ sufficiently small and any desired constant $\epsilon_0 > 0$,

$$\|\tilde{A}^h v^f\| \le \delta(\epsilon_0)\|v^f\|. \tag{8.25}$$

Hence, by the CGA assumption, applying CGC (the coarse grid correction step) to $v^f$ will give contribution smaller than $\epsilon_0\|v^f\|$. Since clearly $\|v^f\| \le \|v^e\| \le C\|v^b\| \le C\|v\|$ (the second inequality resulting from (6.20)), by (6.10) the contribution $\bar{v}^f$ of $v^f$ to $\bar{v}^d$ satisfies

$$\|\bar{v}^f\| \le C\epsilon_0\|v\| \le \frac{\epsilon}{12}\|v\|, \tag{8.26}$$

where the second inequality is obtained by a suitable choice of $\epsilon_0$.

The non-smooth part of $v^e$ is $v^g = v^e - v^f$, given by

$$\begin{aligned} v^g_\alpha &= \int^0 \psi(\theta) E'_\alpha \check{R}(\theta)^{\nu_1} \check{v}^b(\theta) e^{i\theta\alpha} d\theta \\ &= \int^0 \psi(\theta) E_\alpha \check{T}(\theta) \check{R}(\theta)^{\nu_1} \check{v}^b(\theta) e^{i\theta\alpha} d\theta. \end{aligned} \tag{8.27}$$

(see definition of $\check{T}(\theta)$ in (6.24)). Since the only non-vanishing Fourier components in (8.27) are of the form $\exp i(\theta + \tau^j)$, with $|\theta| \le \eta_0$ and $j \ge 1$, it follows from (3.22) and (3.32a) that

$$\begin{aligned} \|(I_h^H L^h v^g)_i\| &\le C\eta_0^{m_i} h^{-m_{i*}} \|v^g\| \\ &\le C h^{1-\sigma_3 m_i} \|v^g\|, \end{aligned} \tag{8.28}$$

the second inequality resulting from (8.13) and (6.4). Hence, by (6.20)

$$\|I_h^H L^h v^g\| \le C h^{1-\sigma_3 m_*} \|v^b\|, \tag{8.29}$$

where $m_* = \max_i m_i$. Since, for reasons as before, $I_h^H L^h v^g$ practically vanishes near the boundary (being smaller than any power of $h$ in $\Omega^h_{\rho/2}$), it would be obtained whether $A^h$ and $I_h^H$ are applied with or without the real boundary conditions ($A^h = L^h$ in the later case). We now apply to it the *real* (including boundary conditions) $(A^H)^{-1}$. Using (6.13), (6.14) and (6.10) we conclude from (8.29) that

$$\|R^{\nu_2} I_H^h (A^H)^{-1} I_h^H L^h v^g\| \le C h^{1-\sigma_3 m_*} \|v^b\|. \tag{8.30}$$

Choosing $\sigma_3 < 1/m_*$ and a sufficiently small $h$, the right-hand side of (8.30) is smaller than $\frac{\epsilon}{12}\|v\|$ and hence, by (8.26),

$$\|\bar{v}^d - R^{\nu_2} v^g\| \le \frac{\epsilon}{6}\|v^b\|. \tag{8.31}$$

By (8.27), and whether boundary conditions are used in $R$ or not, we have

$$R^{\nu_2} v^g = \int^0 \psi(\theta) E_\alpha \check{R}(\theta)^{\nu_2} \check{T}(\theta) \check{R}(\theta)^{\nu_1} \check{v}^b(\theta) e^{i\theta\alpha} d\theta.$$

Hence, using (8.14), (6.25) and the Parseval identity,

$$\|R^{\nu_2} v^g\| \le \lambda \|v^d\|, \tag{8.32}$$

yielding, together with (8.31), the required bound on $\bar{v}^d$.

## 8.5 Summary

The Fourier components of $v^{c6}$ are only in the range $\pi/2 \ge |\theta| \ge \eta_0$ and its harmonics, while those of $R^{\nu_2} v^g$ are in the range $|\theta| \le \eta_1 = \eta_0 + \eta_2$ and its harmonics. The only overlap (in terms of lowest harmonics) is thus in the range $\eta_0 \le |\theta| \le \eta_1$, whose volume is less than $O(\eta_2/\eta_0) = O(h^{\sigma_3 - \sigma_2})$ compared to the volume of either $\{|\theta| \le \eta_0\}$ or $\{|\theta| \ge \eta_1\}$. Hence, choosing any $\sigma < \sigma_{32} < \sigma_3 - \sigma_2$ and writing $\eta_0 = \eta_* h^{1-\sigma_3}$, we can choose $1 \le \eta_* \le 2$ such that the components in the overlap contribute at most $O(h^{\sigma_{32}})$ to either $\|v^{c6}\|^2$ or $\|R^{\nu_2} v^g\|^2$, so that

$$\|v^{c6} + R^{\nu_2} v^g\|^2 \le (\|v^{c6}\|^2 + \|R^{\nu_2} v^g\|^2)(1 + C h^{\sigma_{32}}). \tag{8.32a}$$

Using (8.21) and (8.31) we hence have, for sufficiently small $h$,

$$\begin{aligned}
\|\bar{v}^b\|^2 &= \|\bar{v}^c + \bar{v}^d\|^2 \\
&\le \left( \|v^{c6} + R^{\nu_2} v^g\| + \frac{\epsilon}{5} \|v^b\| \right)^2 \\
&\le (\|v^{c6}\|^2 + \|R^{\nu_2} v^g\|^2)(1 + C h^{\sigma_{32}}) \\
&\quad + 2\frac{\epsilon}{5} \|v^b\| (\|v^{c6}\| + \|R^{\nu_2} v^g\|) + \frac{\epsilon^2}{25} \|v^b\|^2.
\end{aligned}$$

Hence, by (8.22) and (8.32)

$$\|\bar{v}^b\|^2 \le \lambda^2 (\|v^c\|^2 + \|v^d\|^2)(1 + C h^{\sigma_{32}}) + \frac{2\epsilon}{5} \|v^b\| \lambda (\|v^c\| + \|v^d\|) + \frac{\epsilon^2}{25} \|v^b\|^2. \tag{8.33}$$

Furthermore, since in (8.9) and (8.11) $0 \le \psi \le 1$, the Parseval identity yields

$$\|v^b\|^2 = \|v^c + v^d\|^2 \ge \|v^c\|^2 + \|v^d\|^2, \tag{8.34}$$

and therefore

$$\|v^c\| \le \|v^b\|, \quad \|v^d\| \le \|v^b\|. \tag{8.35}$$

For sufficiently small $h$, by (8.33), (8.34) and (8.35)

$$\begin{aligned}
\|\bar{v}^b\| &\le \lambda^2 \|v^b\|^2 (1 + C h^{\sigma_{32}}) + \left( \frac{4\epsilon}{5} \lambda + \frac{\epsilon^2}{25} \right) \|v^b\|^2 \\
&\le \left( \lambda + \frac{\epsilon}{2} \right)^2 \|v^b\|^2 \qquad (8.36) \\
&\le \left( \lambda + \frac{\epsilon}{2} \right)^2 \|v\|^2,
\end{aligned}$$

the last inequality resulting from (8.3) and (8.2). Thus, using (8.6), we obtain (7.3). ∎

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