RESEARCH

AVRAHAM AIZENBUD

I am working on representation theory and related questions of geometry and analysis. Most of my research until now can be divided into 2 parts:

(1) representation theory of reductive groups over local fields (e.g. the general linear group GL_n over the field of real numbers $\mathbb{R}$ or the fields of p-adic numbers $\mathbb{Q}_p$). I am particulary interested in Harmonic analysis on such groups.

(2) The theory of distributions and, in particular, invariant distributions.

In my work I use various tools from representation theory, algebraic geometry, analysis and algebra.

My results have applications in different areas of representation theory and the theory of automorphic forms.

Contents

1. Distributions

The theory of distributions and, in particular, the study of invariant distributions are important tools in representation theory. We are interested in the study of distributions both as a topic as itself and as a useful tool in representation theory.

One can divide the study of distributions into 2 cases:

- the Archimedean case
- the non-Archimedean case.

In both cases the space of distributions is described as the space of functionals on a certain space of test functions on certain geometric objects.

In the Archimedean case the underlying geometric objects are smooth manifolds. Often one would like to restrict the study to a narrower generality such as real algebraic manifolds or, more generally, Nash (i.e. smooth semi-algebraic) manifolds (see [Shi87] for description of the theory of Nash manifolds). The reason that one wants to consider this generality is that it allows to give some growth conditions on the distributions. Distributions satisfying those growth conditions are called Schwartz distributions, and the corresponding test functions are called Schwartz functions, see for more details [AG08].

In the non-Archimedean case the underlying geometric objects are l-spaces, i.e. locally compact totally disconnected Hausdorff spaces, and the test functions are locally constant compactly supported functions. They are also referred to as Schwartz functions. For the the description of the theory of Schwartz distributions over l-space see [BZ76]. Note that in this case there is no distinction between Schwartz distributions and general distributions. Often one would like to restrict study of distributions to a narrower generality such as analytic non-Archimedean manifolds (in the sense of Serre). This allows one to use tools that rely upon the existence of local model.

In both cases the space of Schwartz functions is denoted by $\mathcal{S}(X)$ and the space of Schwartz distributions is denoted by $\mathcal{S}^*(X)$, where X is the underlying geometric object.

1.1. **The wave front set.** The notion of singular support (a.k.a. the characteristic variety) plays an important role in the theory of invariant distributions over real manifolds.

The notion of singular support appears in the theory of D-modules. The singular support of a distribution ξ on a real algebraic manifold X is defined to be the singular support of the D-module it generates.

The singular support of ξ is a closed subset of the cotangent bundle T^*X. Roughly speaking, the singular support of ξ over each point $x \in X$ is a cone in the cotangent space $T^*_x(X)$ that measures in what directions the distribution ξ has singularities at x, in terms of the differential equations ξ satisfies.

In the non-Archimedean case the theory of D-modules is not available, since differential operators do not act on distributions. Therefore, the notion of singular support is also not directly available. However, in both cases one has a similar notion: the notion of the wave front set. The wave front set of a distribution ξ is also a closed subset of the cotangent bundle T^*X. Similarly to the the above, wave front set measures singularities of a distribution, but using local Fourier transform instead of D-modules.

In the Archimedean case, the singular support and the wave front set are closely related. In particular, the wave front set is a subset of the singular support. While the wave front is available in both cases, its study is much more difficult than the study of the singular support since we do not have the theory of D-modules for it.

Usually, the theory of distributions in the non-Archimedean case is simpler than in the Archimedean case. However, the lack of the theory of D-modules reverses the situation in some aspects.

We am interested in using wave front set in non-Archimedean setting to transfer results on distributions proved in the Archimedean case with the help of singular support.

1.1.1. *The Integrability Theorem.* An important theorem in the theory of D-modules is the Integrability Theorem. This theorem states that the singular support of a D-module is coisotropic (a.k.a integrable). This theorem was proved in [GQS71, KKS73, Mal79, Gab81]. The integrability roughly means that the singular support is large (provided it is not empty). In particular, the Integrability Theorem implies the Bernstein inequality that states that the dimension of the singular support is at least the dimension of the underlying space.

The Integrability Theorem has a direct application in the theory of distributions in the Archimedean case. Namely, it implies that the singular support of a distribution is coisotropic. However, this approach can not be used in the non-Archimedean case where one has to consider the wave front set instead. In [Aiz] we have studied the wave front set in the non-Archimedean case. We introduced the notion of "weakly co-isotropic" sub-variety of T^*X and proved the following partial analog of the Integrability Theorem.

Theorem I. *Let ξ be a distribution on analytic manifold X over non-archimedean field. Then the wave front set of ξ is weakly coisotropic subvariety of T^*X.*

This result is significantly weaker than the conjectural full analog of the Integrability Theorem. Its proof is also much simpler than the proof of the Integrability Theorem. However, for many applications such as those described in §§1.2, §§1.4, it can replace the Integrability Theorem. The reason is that nonempty weakly coisotropic variety contains a non-empty coisotropic variety (and therefore also a Lagrangian subvariety).

1.1.2. *Wave front set of a Fourier transform of algebraic measures.* Using the theory of D-modules, Bernstein has proved the following theorem:

Theorem 1.1.1 ([Ber72])**.** *Let p be a polynomial on a real vector space V. Consider $|p|$ as distribution on V and consider its Fourier transform $\mathcal{F}(|p|)$ as a distribution on V^*. Then $\mathcal{F}(|p|)$ is a smooth function on an open dense set $U \subset V^*$.*

In fact, Bernstein proved a much stronger result. He studied holonomic distributions. Holonomic distributions are distributions with Lagrangian singular support (this basically means that singular support is as small as it can be in view of the Integrability Theorem). Then he proved that the class of holonomic distributions is closed under various operations including the Fourier transform. This implies theorem 1.1.1, in view of the simple facts that a polynomial is a holonomic distribution and that holonomic distribution is smooth on an open dense set.

Again, this approach cannot be directly applied in the non-Archimedean case. In [AD] we proved an analog of this theorem. Namely, we introduced the notion of WF-holonomic distribution, i.e. a distribution whose wave front set is small in an appropriate sense. Then we prove the following theorem.

Theorem II. *Let p be a polynomial on a vector space V over non-Archimedean field. Consider $|p|$ as distribution on V and consider its Fourier transform $\mathcal{F}(|p|)$ as a distribution on V^*. Then $\mathcal{F}(|p|)$ is WF-holonomic. This implies that it is a smooth function on an open dense set $U \subset V^*$.*

Remark.

- *We proved analogous results for other measures of algebraic origin.*
- *The main tool used in the proof is Hironaka's desingularization theorem.*
- *Several years ago, another analog of theorem 1.1.1 which does not consider the wave front set, was proved using model theory (see* [HK06] *and* [CL05]*)*
- *Our approach has two advantages over the proofs in* [Ber72], [HK06] *and* [CL05]
  1. *It fits both the Archimedean and the non-Archimedean case.*
  2. *It gives bounds on the wave front set and thus on the open dense set U. These bounds are explicit in terms of resolution of singularities, which, today, has rather explicit descriptions (see e.g.* [Kol07]*).*

1.2. **Uncertainty Principles.** An uncertainty principle is a claim that certain restrictions on a distribution on a vector space are inconsistent with certain restrictions on its Fourier transform. The most well-known example of such principle is the Heisenberg uncertainty principal. This principle is quantitative. We are mostly interested in qualitative uncertainty principles. One can consider the Integrability Theorem as an example of such a principle. Namely, restrictions on the support of a distribution together with restrictions on the support of its Fourier transform gives restrictions on its singular support (resp. wave front set). These restrictions are often inconsistent with integrability.

In our work we use different kinds of uncertainty principles. Occasionally (see [AGS08, AG09a, Aiz, AGb]) we developed these principles further.

1.3. **Localization Principle.** In many cases one can localize the study of equivariant distributions. In the non-Archimedean case very general localization principle was proved in [Ber82]. Namely, given a map $p : X \to Y$, one can reduce the study of equivarint distributions on X to the study of equivariant distributions on each of the fibers $p^{-1}(y)$ for each $y \in Y$.

A complete analog of this result in the Archimedean case seems to be far from reach. However, in [AG09a, AGb, AOSa, AOSb] we proved several partial analogs of this theorem.

1.4. **Vanishing of equivariant distributions.** Many questions of representation theory can be reduced to proving that certain space of equivariant distributions vanishes. More precisely, we study a group G acting on a manifold X and a character χ of the group G, and we interested in showing that

$$\mathcal{S}^*(X)^{G,\chi} = 0.$$

Here $\mathcal{S}^*(X)^{G,\chi}$ is the space of distributions satisfying $g\xi = \chi(g)\xi$.

Most of the works [AGRS10, AGS08, AG09a, AG10b, AG09b, AGa, AS12, AOSa, AOSb] and a part of the work [Aiz] are devoted to different instances of this problem.

In most of these works we have followed the strategy described in [AG09a]. Namely, we linearize the space X. Then we prove, using an inductive argument, that any distribution in $\mathcal{S}^*(X)^{G,\chi}$ has to be supported in a small closed subset of X that we call the singular set. Then we use some more complicated geometric tools (such as the localization principle) along with non-geometric tools based on Fourier transform (such as various kinds of uncertainty principles) in order to conclude vanishing of $\mathcal{S}^*(X)^{G,\chi}$.

This strategy fits best for reductive groups. Unipotent groups require different methods as developed in [AGb, AOSa, AOSb].

In the works listed above we analyze many different spaces of equivariant distributions. In particular, we have established the following results

Theorem III.

(1) *Any distribution on $GL_{n+1}(F)$ which is invariant w.r.t. conjugation by $GL_n(F)$ is also invariant w.r.t. transposition. Here F is an arbitrary local field of characteristic zero and $GL_n(F)$ is embedded into $GL_{n+1}(F)$ in the standard way.*

(2) *The natural analog of (1) for the orthogonal and the unitary group holds in the non-Archimedean case; the Archimedean case was proved in [SZ].*

(3) *Any distribution on $GL_n(F)$ which is invariant w.r.t. conjugation by $P_n(F)$ is also invariant w.r.t. conjugation by $GL_n(F)$. Here F is Archimedean local field (i.e. $\mathbb{R}$ or $\mathbb{C}$) and P is the mirabolic subgroup (i.e. The subgroup of matrices with last row $(0 \cdots 01)$).*

(4) *For various pairs Symmetric pairs (i.e. pairs $G \supset H$ where H is the group of fixed points of some involution θ on G) we have proved that any distribution on G which is invariant w.r.t. the two sided action of H is invariant w.r.t. σ, where σ is the anti-involution of G given by the composition of θ with the inversion. These pairs include*

(a) $(GL_{n+k}(F), GL_n(F) \times GL_k(F))$ *for* $F = \mathbb{R}, \mathbb{C}$. *The non-Archimedean case was proved in* [JR96].
(b) $(GL_n(\mathbb{C}), GL_n(\mathbb{R}))$. *The non-Archimedean case was proved in* [Fli91].
(c) $(O_{n+k}(\mathbb{C}), O_n(\mathbb{C}) \times O_k(\mathbb{C}))$.
(d) $(GL_n(\mathbb{C}), O_n(\mathbb{C}))$.
(e) $(GL_{2n}(F), Sp_{2n}(F))$ *for* $F = \mathbb{R}, \mathbb{C}$. *The non-Archimedean case was proved in* [HR90].

(5) *Any distribution on* $GL_n(F)$ *which is* ψ*-equivariant w.r.t. the right action of a Klyachko subgroup* $H_{k,r}$ *and and the left action of a transposed Klyachko subgroup* $\overline{H_{k',r'}}$ *is zero. Here* ψ *is a generic character of* $H_{k,r} \times \overline{H_{k',r'}}$, *The field* F *is Archimedean and we assume that* $r \neq r'$. *For more detail on Klyachko subgroup see* [OS] *where the non-Archimedean case is proved.*

(6) *Any distribution on* $GL_n(F)$ *which is* ψ*-equivariant w.r.t. the action of* $H_{k,r} \times \overline{H_{k,r}}$, *is invariant w.r.t. transposition. Here* ψ *is a generic character of* $H_{k,r} \times \overline{H_{k,r}}$, *the field* F *is Archimedean, and we assume that* $min(r, k) = 1$.

The non-Archimedean case for any r, k *is proved in* [OS]

Remark.

- *The methods we introduced in these works have further applications in other works, see e.g.* [Say, JSZ1, JSZ2, SZ, Walb, Hen, Zha10].
- *Most of the above results are applied to proving the Gelfand property of various pairs (See* §§§*2.1.1* *below).*
- *The statement* (1) *was conjectured in the 1980-s by Bernstein and Rallis. We proved it in* [AGRS10] *for non-Archimedean* F *and in* [AG09b] *for Archimedean* F. *The Archimedean case was done independently in* [SZ]. *In* [Aiz] *we gave a uniform proof of this theorem for all local fields of characteristic* 0.
- *The statement of* (3) *results from* (1). *The non Archimedean counterpart of* (3) *was proved in* [Ber82]. *The statement* (3) *implies Kirillov's conjecture. In the Archimedean case Kirillov's conjecture was also proved before our work (in* [Sah89] *for the field of the complex numbers and in* [Bar03] *for any Archimedean* F*). Our proof of* (1) *provides an alternative proof of Kirillov's conjecture in both cases.*
- *The result* (2) *has important applications in number theory, see e.g.* [GP94, GGP, Wala].

1.5. **Comparison of orbital integrals.** As it was mentioned earlier, one of the aims of my research is the study of the space $\mathcal{S}^*(X)^G$ of G-invariant distributions on X. One can consider a dual question — the study of the space $\mathcal{S}(X)_G$ of the G co-invariants of the representation $\mathcal{S}(X)$. In the non-Archimedean case these two problems are equivalent since $\mathcal{S}^*(X)^G = (\mathcal{S}(X)_G)^*$. However, in the Archimedean case the second space contains slightly more information since $\mathcal{S}(X)_G$ does not have to be Hausdorff.

One way to study the space $\mathcal{S}(X)_G$ is via regular orbital integrals. Namely, in many cases we have a categorical quotient $X \to X//G$. Let $(X//G)_{reg}$ be the set of regular values of this map and X_{reg} be its preimage. Usually the set $(X//G)_{reg}$ consists of closed orbits of maximal dimension. The set X_{reg} is the union of these orbits. Let $\Omega_{X,G} : \mathcal{S}(X) \to C^\infty((X//G)_{reg})$ be the direct image (more precisely, it is the restriction to X_{reg} followed by the direct image). Clearly $\Omega_{X,G}$ factors through $\mathcal{S}(X)_G$. In many cases (more precisely, when we have "density of regular orbital integrals") $\Omega_{X,G}$ gives an isomorphism between $\mathcal{S}(X)_G$ and the image $\Omega_{X,G}(\mathcal{S}(X))$.

One can often make a similar construction for the twisted case (when we are given a character ψ of G and study the co-equvivariants $\mathcal{S}(X)_{G,\psi}$).

It is usually very hard to explicitly describe the space $\Omega_{X,G}(\mathcal{S}(X))$. However, it often happens that $\Omega_{X,G}(\mathcal{S}(X))$ is the same for different pairs X and G. Namely, suppose we have another group G' acting on a space X' so that the categorical quotients $X//G$ and $X'//G'$ are equal. Then one can ask whether the spaces $\Omega_{X,G}(\mathcal{S}(X))$ and $\Omega_{X',G'}(\mathcal{S}(X'))$ coincide. More generally, one can ask whether there exists a matching factor $\gamma \in C^\infty((X//G)_{reg}$ so that $\Omega_{X,G}(\mathcal{S}(X)) = \gamma \cdot \Omega_{X',G'}(\mathcal{S}(X'))$.

This phenomenon is very useful for the Langlands program, more specifically, for comparison of representations of different groups. Let G be a reductive group. The characters of representations of G are $Ad(G)$-invariant distributions on G. Harish-Chandra proved that these characters span (in an appropriate sense) the space of invariant distributions $\mathcal{S}^*(G)^{Ad(G)}$. Often one can relate the geometric quotient $G//Ad(G)$ with $G'//Ad(G')$ where G' is another group. In this case on can use the study of orbital integrals in order to compare $\mathcal{S}^*(G)^{Ad(G)}$ and $\mathcal{S}^*(G')^{Ad(G')}$. This gives a relation between the (complexified and completed) Grothendieck group of representations of G and the (complexified and completed) Grothendieck group of representations of G'. Often one can use the trace formula in order to deduce from this a relation between the sets of irreducible representations of G and of G'.

One can use similar considerations when studying “relative representation theory” (see §§2.1). In this case one will use the relative trace formula and will compare the categorical quotients $H_2\backslash\backslash G//H_1$ with $H_2'\backslash\backslash G'//H_1'$.

In the work [AGb], we establish the following equality of spaces of orbital integrals

Theorem IV. *Let N_n be the group of $n \times n$ upper-unipotent matrices. Let $N(\mathbb{R}) \times N(\mathbb{R})$ act on $GL_n(\mathbb{R})$ by $(n_1, n_2)(x) = n_1^t x n_2$. Let $A \subset GL_n$ be the set of diagonal matrices. Let $\tilde{\Omega}_1 : \mathcal{S}(GL_n(\mathbb{R})) \to C^\infty(A)$ be the map defined by*

$$\tilde{\Omega}_1(f)(a) = \alpha(a) \int_{n \in N(\mathbb{R}) \times N(\mathbb{R})} f(n(a))\psi(n)dn,$$

where ψ is a “non-degenerate” character of $N(\mathbb{R}) \times N(\mathbb{R})$ and α is some normalizing factor. Let S_n be the space of non-degenerate hermitian forms and let $N(\mathbb{C})$ act on it by $n(x) = n^t x n$. Let $\tilde{\Omega}_2 : \mathcal{S}(S_n) \to C^\infty(A)$ be a map defined similarly to $\tilde{\Omega}_1$.

Then $\mathrm{Im}(\tilde{\Omega}_2) = \mathrm{Im}(\tilde{\Omega}_1)$

The non-Archimedean counterpart of this theorem was done in [Jac03].

An application of these works can be found in [FLO].

1.6. **Homology of spaces of Schwartz sections.** Study of co-invariants $\mathcal{S}(X)_G$ can be generalized in two different directions.

(1) We can consider a G-equivariant vector bundle (or, in the non-Archimedean case, a sheaf) E over X and study the space of invariant Schwartz sections $\mathcal{S}(X, E)_G$. Note that this generalization includes the case of co-equivariants, since one can include any character of G into E. In the Archimedean case this generalization appears naturally when one decomposes a manifold into an open and a closed submanifolds. Then one needs studying the sections of normal bundles to the closed submanifold.
(2) We can study the homology $H_*(G, \mathcal{S}(X, E))$ instead of just $H_0(G, \mathcal{S}(X, E)) = \mathcal{S}(X, E)_G$. Again, this occurs naturally since when we decompose our manifold, we present the space $\mathcal{S}(X, E)$ as an extension of two G representations. If we are interested in co-invariants of an extension, we have to study higher homology of it components.

    It is not completely clear what type of group homology should be used in the Archimedean case. Therefore, it is often easier to study the Lie algebra homology $H_*(\mathfrak{g}, \mathcal{S}(X, E))$. This might give different answer if G is not contractible.

We believe that the study of $H_*(G, \mathcal{S}(X, E))$ is a fundamental mathematical question and an essential part of my work is directly related to this question.

As complicated as this problem may be in the general case, it is rather simple in the case when X is a single orbit. In this case the homology can be computed explicitly. This computation is often called the Shapiro lemma. It was clasically done in a different setting. In the non-Archimedean case it follows directly from the Grothendieck lemma (see e.g. [AAG]). In the Archimedean case this is more problematic since the category of G-representations is not abelian. In case that G is contractible, one can replace it by its Lie algebra. In this case we proved it in [AG10a] and some generalisation of it in [AGS].

2. Representation theory of reductive groups over local fields

Similarly to the theory of distributions, representation theory of reductive groups over local fields is also divided into the Archimedean and non-Archimedean case. In what follows we fix a local field F (i.e. a locally compact non-discrete topological field). The field F might be Archimedean (i.e. $\mathbb{R}$ or $\mathbb{C}$) or non-Archimedean.

Also, for an algebraic variety X, we will consider $X(F)$ as a topological space. In the non-Archimedean case, it will be an l-space and in the Archimedean case (for smooth X), it will be a real manifold. In case there is no possible confusion, we will denote $X(F)$ simply by X.

Let G denote an algebraic group defined over F (unless stated otherwise G will be reductive). By "representation of G" we will mean a representation of $G(F)$.

We are interested in the study of smooth G-representations. In the non-Archimedean case these are representations in a complex (usually ∞-dimensional) vector space such that each vector has an open stabilizer. In the Archimedean case this means a topological representation in a Frechet space V which is continuous w.r.t. each of the seminorms and such that for any vector $v \in V$ the action map $G \to V$ is differentiable. In the Archimedean case one also has to impose an additional condition — admissibility.

In the Archimedean case there is another category of representations — the category of Harish-Chandra modules. By Casselmann-Walach theorem (see [Wal92], §§§11.6.8) the category of Harish-Chandra modules is equivalent to the category of smooth admissible representations. However, certain constructions give different answers in those categories. We will be manly interested in the category of smooth admissible representations and we will view the category of Harish-Chandra modules as a tool for its study.

2.1. Harmonic analysis on spherical spaces.

One can view representation theory of a given group G as harmonic analysis on G, viewed as a space, with respect to the two-sided action of the group $G \times G$. For example, if $G(F)$ is a compact group, the Peter-Weyl Theorem states that the decomposition into direct sum of the space $L^2(G)$ with respect to the action of $G \times G$ is labeled by the irreducible representations of G. However, for non-compact G this point of view is limited, since not every type of representation theory of G can be studied in terms of harmonic analysis of a certain space of functions on G. In particular, it seems impossible to study in such a way general unitary representations. Nevertheless, many questions of representation theory can be reformulated in the language of harmonic analysis on G.

Let $H \subset G$ be a subgroup. Following the above point of view, we may consider harmonic analysis on G/H with respect to the action of G as a generalization of representation theory. In order to obtain the representation theory of a group G', one should substitute $G := G' \times G'$ and $H := \Delta G'$ – the diagonal copy of G. In what follows, we will refer to harmonic analysis on G/H as "relative representation theory" or "representation theory of the pair (G, H)".

It seems unpractical to consider harmonic analysis on G/H as a generalization of representation theory for arbitrary H, therefore we will always put some assumptions on H. One popular assumption is that H is a symmetric subgroup, i.e. there exists an involution on G such that H is the set of its fixed points. A weaker assumption is that H is a spherical subgroup, i.e. there exists a Borel subgroup B of G such that HB is open. In what follows, we will consider only those cases and we will refer to the pair (G, H) and the space G/H as symmetric or spherical.

We have studied several problems in harmonic analysis on spherical spaces G/H. Most of them are translations of known results of representation theory to the case of a pair. These problems are interesting not only as natural analogs of fundamental results, but they also have applications in classical questions of representation theory, such as classification of representations and constructions of models of representations, and in topics of the theory of automorphic forms such as study of automorphic periods and the relative trace formula.

2.1.1. *Gelfand Pairs.* The first fundamental result in representation theory is Schur's lemma. When we translate this result to the relative case we obtain the following property:

Definition 2.1.1. *A pair (G,H) consisting of a group and a subgroup is said to be a Gelfand pair if any irreducible representation π of G is "included" in $\mathcal{S}(G(F)/H(F))$ with multiplicity at most 1. Formally speaking, it means that* $\dim \operatorname{Hom}(\mathcal{S}(G(F)/H(F)), \pi) \leq 1$ *or, equivalently,* $\dim \operatorname{Hom}_{H(F)}(\pi|_{H(F)}, \mathbb{C}) \leq 1$.

If we replace the trivial representation with a character ψ, the pair is called twisted Gelfand pair w.r.t. ψ.

An overview of the theory of Gelfand pairs can be found, for example, in [vD08].

Note that if a pair (G, H) is a Gelfand pair, then the "decomposition" of $\mathcal{S}(G(F)/H(F))$ to irreducible representations is unique.

Gelfand pairs have various applications to classical questions of representation theory and harmonic analysis. These include the classification of representations and constructing canonical bases for irreducible representations and spaces of functions on homogeneous spaces. More recent applications of Gelfand pairs are in the theory of automorphic forms, for instance in the splitting of automorphic periods and in the relative trace formula. Some of these applications are described in [Gro91].

There is a stronger version of the notion of a Gelfand pair:

Definition 2.1.2. *A pair (G, H) is said to be a strong Gelfand pair if any irreducible representation π of $G(F)$, when restricted to $H(F)$, "includes" any irreducible representation of H with multiplicity at most 1. Formally speaking, this means that for any irreducible representation ρ of H, we have* $\operatorname{Hom}_{H(F)}(\pi|_{H(F)}, \rho) \leq 1$.

The notion of Gelfand pair and strong Gelfand pair are connected in the following way:

Proposition 2.1.3. *A pair (G, H) is a strong Gelfand pair if and only if the pair $(G \times H, \Delta H)$ is a Gelfand pair.*

The main tool to prove that a pair (G, H) is a Gelfand pair is the following criterion by Gelfand and Kazhdan ([GK75]).

Theorem 2.1.4. *Suppose that there exists an involutive anti-automorphism σ of G such that any distribution on G which is invariant with respect to the $H \times H$ two-sided action is invariant with respect to σ. Then the pair (G, H) is a Gelfand pair.*

This criterion implies an analogous criterion for strong Gelfand pairs.

Using the results discribed in §1.4 and the Gelfand-Kazhdan Criterion, we have proved the following theorem

Theorem V.

(1) *The following pairs are strong Gelfand pairs.*
  (a) $(GL_{n+1}(F), GL_n(F))$ *for arbitrary local field F of characteristic* 0. *The Archimedean case was proved independently in* [SZ].

   (b) *$(O(V \oplus F), O(V)$ when V is a quadratic space over arbitrary non-Archimedean local field F of characteristic 0 and O is the orthogonal group. The Archimedean case is proved in* [SZ].
   (c) *$(U(V \oplus E), U(V)$ when V is a hermitian space over a quadratic extension E/F of arbitrary local field of characteristic 0 and U is the unitary group. The Archimedean case is proved in* [SZ]
(2) *The following pairs are Gelfand pairs.*
   (a) *$(GL_{n+k}(F), GL_n(F) \times GL_k(F))$ for $F = \mathbb{R}, \mathbb{C}$. The non-Archimedean case was proved in* [JR96].
   (b) *$(GL_n(\mathbb{C}), GL_n(\mathbb{R}))$. The non-Archimedean case was proved in* [Fli91].
   (c) *$(O_{n+k}(\mathbb{C}), O_n(\mathbb{C}) \times O_k(\mathbb{C}))$.*
   (d) *$(GL_n(\mathbb{C}), O_n(\mathbb{C}))$.*
   (e) *$(GL_{2n}(F), Sp_{2n}(F))$ for $F = \mathbb{R}, \mathbb{C}$. The non-Archimedean case was proved in* [HR90].
   (f) *$(GL_{2n+1}(F), Con_{2n+1}(F))$ for $F = \mathbb{R}, \mathbb{C}$. Here $Con_{2n+}(F)$ is the contact group. The non-Archimedean case was proved in* [OS].
(3) *The pair $Mat_{2\times n}(F) \rtimes (GL_{n+2}(F), SL_2(F) \times N_n(F))$ is a twisted Gelfand pair w.r.t the generic character of $N_n(F)$. $F = \mathbb{R}, \mathbb{C}$. Here N_n is the group of upper unipotent matrices. The non-Archimedean case was proved in* [OS].

Apart of the Gelfand-Kazhdan method, there are other methods for proving the Gelfand property. One such method, that will be discussed in section 2.1.3, is to transfer results about the Gelfand property from zero characteristic to positive characteristic. This is useful, since some of the tools that we have used for proving the condition of the Gelfand-Kazhdan criterion are based on the Jordan decomposition and the Luna slice theorem which are problematic in positive characteristic.

Another useful method for proving the Gelfand property is averaging of functionals. An important construction in representation theory is averaging with respect to the action of a subgroup $H_1 \subset G$, of an H_2-invariant functional on a representation of G, where $H_2 \subset G$ is another subgroup. Many constructions in representation theory and automorphic forms (such as intertwining operators, periods of automorphic forms, certain L-factors, etc.) can be viewed as a special case of the above construction. Note that often this integral does not converge and one has to regularize it, usually using analytic continuation. One can use this construction in order to deduce the Gelfand property of one pair from the Gelfand property of another pair.

We implemented this method in the work [AGJ09] where we deduced the uniqueness of Shalika functionals in the Archimedean case from the Gelfand property of the pair $(GL_{n+k}, GL_n \times GL_k)$. Namely, we have proved the following theorem.

Theorem VI. *Let $P_{n,n} \subset GL_{2n}$ be the maximal parabolic subgroup corresponding to the partition (n, n) and let $p : P_{n,n} \to M_{n,n} = GL_n \times GL_n$ be the projection to its Levi factor. Let $GL_n \subset M_{n,n}$ be the diagonal and let $S := p^{-1}(GL_n)$. Let F be an Archimedean field. Let ψ be a generic additive character of $S(F)$. Then the pair $(GL_{2n}(F), S(F))$ is twisted Gelfand pair w.r.t. ψ.*

The non-Archimedean counterpart of this theorem was done in [JR96].

In addition, We believe that the methods that will be discussed in section 2.1.2 will be very helpful in proving the Gelfand property of spherical pairs.

2.1.2. *Cohen-Macaulay property.* There are several phenomena in representation theory and harmonic analysis which are hard to formalize exactly and on the first glance do not necessarily look related. We believe that these phenomena are related to a Cohen-Macaulay property.

In this section we will discuss only the non-Archimedean case. Let us shortly describe some of those phenomena:

Continuity: A fundamental question in representation theory of a pair (G,H) is the computation of the multiplicity $\dim \mathrm{Hom}(\mathcal{S}(G/H),\pi)$. Let π_s be a family of representations (i.e. π_s are parabolic induction of a family of characters χ_s). In general the function $m_s := \dim \mathrm{Hom}(\mathcal{S}(G/H),\pi_s)$ is upper-semicontinuous. Often it happens that it is continuous.

Density: Let $H_1, H_2 \subset G$ be two spherical subgroups. Consider the space of two-sided invariant distributions $\mathcal{S}^*(G)^{H_1\times H_2}$. Let $\mathcal{O}$ be the set of regular $H_1\times H_2$-double cosets in G. often we have density of orbital integrals, i.e.:

$$\mathcal{S}^*(G)^{H_1\times H_2} = \overline{\bigoplus_{Z\in\mathcal{O}} \mathcal{S}^*(O)^{H_1\times H_2}}.$$

Freeness: Let $K \subset G$ be a open compact subgroup. Sometimes the module $\mathcal{S}(G/H)^K$ is free over the center of the Hecke algebra $\mathcal{H}_K(G) \cong \mathcal{S}(G)^{K\times K}$.

Let us now recall the Cohen-Macaulay property:

Definition 2.1.5. *A finitely generated module M over a commutative algebra A (finitely generated over a field) is called Cohen-Macaulay of dimension d if there exists a polynomial subalgebra $B\subset A$ of rank d such that M is free and finitely generated over B.*

The following theorem is well known (see e.g. [BBG97] section 2).

Theorem 2.1.6. *A finitely generated module M over a commutative algebra A is Cohen-Macaulay of dimension d if and only if for any smooth and d-dimensional algebra B acting on M, commuting with the action of A, the module M is projective over B.*

This theorem implies that the notion of Cohen-Macaulay module is stable when one changes the algebra. Namely, we have the following corollary.

Corollary 2.1.7. *Let M be a module over a commutative algebras A and B. s.t. the actions of these algebras commute and M is finitely generated over both. Than M is Cohen-Macaulay over A if and only if M is Cohen-Macaulay over B.*

This flexibility allows one to define the notion of Cohen-Macaulay module in different abelian categories such as the category of G-representations.

We believe that the phenomena described above are all related to the following conjecture:

Conjecture 1. *The module $\mathcal{S}(G(F)/H(F))$ is a (locally) Cohen-Macaulay object in the category of smooth G-representations $\mathcal{M}(G)$.*

In the group situation this conjecture specializes to the following theorem.

Theorem 2.1.8 (Bernstein)**.** *The category $\mathcal{M}(G)$ is (locally) Cohen-Macaulay.*

This theorem follows from Bernstein's second adjointness theorem (see [Ber]). One can reformulate this theorem in the following way:

Theorem 2.1.9 (Bernstein)**.** *For any congruence subgroup $K_n \subset G(F)$, the Hecke algebra $\mathcal{H}_{K_n}(G)$, of double K_n invariant measures on $G(F)$, is a Cohen-Macaulay module over its center.*

In many cases (for example, the case $G = GL_n$) the Bernstein center is regular. In these cases the above theorem implies that $\mathcal{H}_{K_n}(G)$ is locally free over its center.

This theorem has also analog in the Archimedean case, see [BBG97].

In [AS] we proved Conjecture 1 for certain special cases and relate it to the phenomena above. Namely, we have proved the following theorem

Theorem VII.

- *Conjecture 1 holds for several rank one spherical spaces including $GL_{n+1}(F)/GL_n(F)$ and $GL(V)/U(V)$ where V is a 2 dimensional hermitian space.*
- *Conjecture 1 for a pair (G,H) implies that for any cuspidal representation ρ of a Levi $M \subset G$ the dimension $m_\chi = \dim \mathrm{Hom}(\mathcal{S}(G/H), i_M^G(\rho \otimes \chi))$ is a constant function on the regular locus of the variety $\{\chi \in \mathfrak{X}(M) | m_\chi \neq 0\}$. Here i_M^G is the parabolic induction and $\mathfrak{X}(M)$ is the variety of unramified characters of M.*
- *Conjecture 1 for a pair (G,H) implies that*
$$\mathcal{S}^*(G)^{\tilde{N}\times H} = \overline{\bigoplus_{Z\in\mathcal{O}} \mathcal{S}^*(O)^{\tilde{N}\times H}}.$$
Here $\tilde{N}$ is the unipotent radical of the minimal parabolic subgroup of G extended by the maximal compact subgroup of its Levi factor and $\mathcal{O}$ is a certain set of double cosets.
- *Conjecture 1 is equivalent to the fact that $\mathcal{S}(G/H)^K$ is a (locally) Cohen Macaulay module over the center of the Hecke algebra $\mathcal{H}_K(G)$ for any open compact group $K \subset G$. In case when this center is regular and a generic K-distinguished representation is H-distinguished this implies that $\mathcal{S}(G/H)^K$ is a locally free over $\mathcal{H}_K(G)$.*

2.1.3. *Spherical Pairs over Close Local Fields.* There are several methods that allow one to relate problems over fields of positive characteristic with problems over fields of zero characteristic. Most of these methods are based on approximating a field of zero characteristic with fields of positive characteristic. There is a different method developed in [Kaz86], which is based on approximating a local field of positive characteristic with local fields of zero characteristic.

In this work the following theorem is proved:

Theorem 2.1.10 (Kazhdan)**.** *Let G be a reductive group that splits over $\mathbb{Z}$. Let K_n be a congruence subgroup. Then the Hecke algebra $\mathcal{H}_{K_n}(G)$ does not change when we replace F with a "close enough" local field F'.*

This theorem means that the representation theory of $G(F)$, when F is a field of positive characteristic, can be approximated by the representation theory of $G(F')$, where F' is a field of zero characteristic.

In the work [AAG], we prove the following analog of this theorem.

Theorem VIII. *Let (G,H) be a spherical pair which satisfies certain assumptions. Then the Hecke module $\mathcal{S}(G(F)/H(F))^{K_n}$ does not change when we replace F with a "close enough" local field F'.*

This theorem allows us to deduce the following corollary from Theorem VII:

Corollary IX.
- *The pair $(GL_{n+1}(F), GL_n(F))$ is a strong Gelfand pair for a local field F of arbitrary characteristic.*
- *The pair $(GL_{n+k}(F), GL_n(F) \times GL_k(F))$ is a Gelfand pair for a local field F of characteristic different from 2.*

In the proof of Theorem VIII we used the fact that the Hecke module is finitely generated over the Hecke algebra. We also proved the following criterion for this fact:

Theorem X. *Let (G,H) be a spherical pair. Then the following are equivalent:*
- *For any irreducible (smooth) representation π of G we have $\dim \mathrm{Hom}_H(\pi, \mathbb{C}) < \infty$.*
- *$\mathcal{S}(G(F)/H(F))^{K_n}$ is finitely generated over $\mathcal{H}_{K_n}(G)$.*
- *The dimension $\dim \mathrm{Hom}_H(\pi, \mathbb{C}) < \infty$ is bounded for any Bernstein block.*

The proof in [AAG] relies on the theory of the Bernstein center (see [BD84]) and on certain smoothness analysis of some group schemes over local rings.

2.1.4. *Distinguished representations with respect to a symmetric subgroup.* Another fundamental question in representation theory of a pair (G,H) is characterization of H-distinguished representations of G, i.e. those representations π satisfying $\mathrm{Hom}_H(\pi|_H,\mathbb{C}) \neq 0$. In general, this question is very complicated and in some cases carries an arithmetical content (for one such case, see [GP94] and [Wala]). However, for symmetric pairs the following partial answer to this question is expected:

Conjecture 2. *Let (G,H) be a symmetric pair, and let θ be an involutive automorphism of G such that $H = G^\theta$. Let π be an irreducible H-distinguished representation of G. Then the L-packet of π is invariant with respect to the functor $\pi \mapsto \tilde{\pi} \circ \theta$, where $\tilde{\pi}$ denotes the smooth dual representation.*

In general, we expect that this conjecture will be difficult to prove since it involves the notion of L-packet which is a very sophisticated notion. However, for the case of GL_n, we hope that the conjecture will be simpler, since all the L-packets in this case are singletons. In the case when the symmetric pair in question satisfies the Gelfand-Kazhdan criterion (see Theorem 2.1.4), one can prove the following stronger version of this conjecture:

Theorem 2.1.11. *Let G be a reductive group and let θ be its involutive automorphism. Set $H := G^\theta$. Assume (G,H) satisfies the Gelfand-Kazhdan criterion. Let π be an H-distinguished irreducible representation of G. Then $\pi \cong \tilde{\pi} \circ \theta$.*

A special case of this theorem was proved in [JR96], as part of Theorem 1.1, and one can easily prove this theorem along the same lines.

More challenging is the case when the symmetric pair does not satisfy the Gelfand-Kazhdan criterion. In [AL] we have studied one such case. We have proved the following theorem.

Theorem XI. *Let $G = GL(V)$ where V is an hermitian space over $\mathbb{C}$. Let θ be the involution given by $\theta(x) = (\overline{x}^t)^{-1}$ and let σ be given by $\sigma(x) = \overline{x}^t$. In this case $H := G^\theta$ is the unitary group. Than for any H-distinguished G representation we have $\pi \cong \tilde{\pi} \circ \theta$*

The non-Archimedean case is done in [FLO].

2.2. **Derivatives of representations of GL_n.** The theory of derivatives of representation of GL_n was developed in [BZ77] in the non-Archimedean case. It became an important tool in representation theory of GL_n and particularly in the study of the internal structure of those representations, see e.g. [Zel80, Tad86]. A decade later the theory of derivatives was adapted to the study of unitary representations of GL_n in the Archimedean case, see [Sah89, Sah90, SS90]. In [AGS] we generalized this theory to arbitrary smooth representations.

The theory of derivatives is based on analysis of representation theory of the mirabolic group $P_n \subset GL_n$, i.e. the subgroup of matrices with the last row $(0 \cdots 01)$. In some sense, one can reduce the study the of representations of P_n to the study the of representations of GL_{n-1} and of P_{n-1}. By induction this assigns to any representation π of P_n a sequence of representations $D^i(\pi)$ of GL_{n-i}. These representations called the derivatives of π. One defines derivatives of representations of GL_n by restricting them to P_n and then taking derivatives. It turns out that the highest (nonzero) derivative plays a special role. A representation is of depth d if its d-th derivative is the highest (nonzero) derivative.

The above description can be made rigorous in the non-Archimedean case. However, the Archimedean case is aggravated by the following difficulties:

(1) There is no workable category of P_n representation
(2) The procedure of attaching a GL_{n-1} representation to a P_n representation can be thought of as a fiber. The fiber coincides with the stalk in the non-Archimedean case. However, this is not true in the Archimedean case.

If we are only interested in unitary representations, then the first difficulty can be overcome since there is nice a category of unitary representations of P_n. The second difficulty also turns out to be irrelevant. However, this approach will only give the higher derivative. This was done in [Sah89], and this highest derivative was called the adduced representation.

Due to (1) we were unable to directly use in [AGS] the picture described above, but rather defined the derivative using the action of the Lie algebra. We took into account (2) by replacing the "fiber" with the "jet" which is intermediate object between the fiber and the stalk (a similar construction was used by Casselmann in his version of Jacquet functor in the Archimedean case). We proved some basic properties of the derivative. In particular, we proved the following theorem.

Theorem XII. *Let $\mathcal{M}_\infty(G_n)$ denote the category of smooth admissible Fréchet representations of moderate growth and let $\mathcal{M}^d_\infty(G_n)$ denote the subcategory of representations of depth $\leq d$. Then*

1. *The d-th derivative defines a functor $\mathcal{M}^d_\infty(G_n) \to \mathcal{M}_\infty(G_{n-d})$.*
2. *The functor $D^d : \mathcal{M}^d_\infty(G_n) \to \mathcal{M}_\infty(G_{n-d})$ is exact.*
3. *Let $n = n_1 + \cdots + n_d$ and let χ_i be characters of G_{n_i}. Let $\pi = \chi_1 \times \cdots \times \chi_d \in \mathcal{M}_d(G_n)$ denote the corresponding monomial representation. Then*
$$D^d(\pi) \cong ((\chi_1)|_{G_{n_1-1}} \times \cdots \times (\chi_d)|_{G_{n_d-1}})$$
4. *If τ is an irreducible unitary representation of G_n and the corresponding smooth representation τ^∞ has depth d then the d-th derivative $D^d(\tau^\infty)$ coincides with the adduced representation $(A\tau)^\infty$*

We applied this theorem for the study of adduced representations and the study of degenerate Whittaker functionals which are generalization of the classical Whittaker functionals. More precisely,

- we completed the computation that was started in [Sah89, Sah90, SS90] of adduced representations of all irreducible unitary representations.
- We proved uniqueness of degenerate Whittaker functionals for unitary representations. The existence was proved in [GS].

3. Other topics of my research

3.1. **A quantum analog of Kostant theorem.** A fundamental result in representation theory of Lie algebras is Kostant's theorem saying that the algebra of polynomial functions on a reductive Lie algebra is free as a module over its invariants. In [AY11] we prove a quantum analog of this theorem for the general linear group. Namely, we proved the following theorem.

Theorem XIII. *The quantum deformation $\mathcal{O}_q(\mathfrak{gl}_n)$ of the algebra of polynomials on $\mathfrak{gl}_n$ is free over the subalgebra of invariants under the adjoint coaction of $\mathcal{O}(GL_q(n))$, for q not a root of unity or $q = 1$.*

There are other quantum versions of Kostant's theorem, see [JL94, Bau00]. Our result is stronger, however, unlike [JL94, Bau00], it is restricted to the $\mathfrak{gl}_n$ case. In fact, we can not even formulate our result for other cases since it is not clear what is a natural quantum deformation of the algebra of polynomials on a general reductive algebra.

3.2. **Morse theory.** Additionally, I am interested in differential topology and, in particular, Morse theory. Morse theory assigns a chain complex to a Riemannian manifold with a Morse function. The homology of this complex is the homology of the manifold.

In [AZ] we investigated the functorial properties of this assignment.

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Avraham Aizenbud, Massachusetts Institute of Technology Department of Mathematics Cambridge, MA 02139 USA.

E-mail address: aizenr@gmail.com

URL: http://math.mit.edu/~aizenr/