THE GELFAND-KAZHDAN CRITERION AS A NECESSARY AND SUFFICIENT CRITERION

AVRAHAM AIZENBUD

Contents

Abstract. We show that under certain conditions the Gelfand-Kazhdan criterion for the Gelfand property is a necessary condition. We work in the generality of finite groups, however part of the argument carries over to p-adic and real groups.

1. Introduction

In this note we study the Gelfand-Kazhdan criterion for the Gelfand property (see [GK75]) and show that under some conditions it is not only a sufficient condition but also a necessary one. We discus mostly finite groups, but we hope that some of these methods can be pushed to the generality of p-adic groups (and even Lie groups). The Gelfand-Kazhdan criterion was originally developed as a version of the Gelfand trick that is valid for p-adic groups and not only for compact groups. However, even for finite groups the Gelfand-Kazhdan criterion is slightly more informative than the Gelfand trick.

The main result of this note is the following:

Theorem 1. *Let G be a finite group, $\theta : G \to G$ be an involution and $H \subset G$ be a θ-stable subgroup. Assume that for any $x \in G$ there exists $g \in G$ s.t. $gx^{-1}g^{-1} = \theta(x)$. Then the following are equivalent:*

1. *(G, H) is a Gelfand pair.*
2. *For any $x \in G$ there are $h_1, h_2 \in H$ s.t. $h_1x^{-1}h_2 = \theta(x)$.*

We also have slightly more general version of this theorem.

Theorem 2. *Let G be a finite group, $\theta : G \to G$ be an involution and $H \subset G$ be a θ-stable subgroup. Then the following are equivalent:*

1. *(G, H) is a Gelfand pair and any H-distinguished representation π of G (i.e. a representation satisfying $(\pi)^H \neq 0$) satisfies $\pi \circ \theta \cong \pi^*$.*
2. *For any $g \in G$ there are $h_1, h_2 \in H$ s.t. $h_1g^{-1}h_2 = \theta(g)$*

This theorem implies the previous one.

We can generalize this Theorem further:

Theorem 3. *Let G be a finite group, $\theta : G \to G$ be an involution and $H \subset G$ be a subgroup. Then the following are equivalent:*

Date: January 31, 2020.

(1) *(G,H) is a Gelfand pair and any H-distinguished representation possesses a symmetric non-zero bilinear form B satisfying $B(\pi(g)v,w)=B(v,\pi(\theta(g^{-1}))w)$.*
(2) *For any $g\in G$ there are $h_1,h_2\in H$ s.t. $h_1g^{-1}\theta(h_2)=\theta(g)$.*

Theorem 3 implies Theorem 2 by [Vin06, Proposition 3].

The last theorem follows from Proposition 4 below which is a reinterpretation of the original proof of the Gelfand-Kazhdan criterion. In order to formulate this proposition, we need to recall the definition of the twisted Frobenius–Schur indicator.

Notation 1. *Let $\pi\in \mathrm{irr}(G)$ and let $\theta: G\to G$ be an involution. We denote*

$$\varepsilon_\theta(\pi)=\begin{cases}0, & \pi\not\cong\pi^*\circ\theta,\\ 1, & \pi \text{ possesses a non-zero symmetric bilinear form } B \text{ satisfying}\\ & B(\pi(g)v,w)=B(v,\pi(\theta(g^{-1}))w),\\ -1, & \pi \text{ possesses a non-zero anti-symmetric bilinear form } B \text{ satisfying}\\ & B(\pi(g)v,w)=B(v,\pi(\theta(g^{-1}))w).\end{cases}$$

Proposition 4. *Let G be a finite group, $\theta: G\to G$ be an involution and $H\subset G$ be a subgroup. Let V be the space of functions on G which are left invariant w.r.t. H, right invariant w.r.t. $\theta(H)$ and anti-invariant w.r.t. $\sigma:=\theta\circ\mathrm{inv}$, where $\mathrm{inv}: G\to G$ is the inversion. Then*

$$V\cong\left(\bigoplus_{\varepsilon_\theta(\pi)=0}\pi^H\otimes(\pi^*)^{\theta(H)}\right)^{S_2,\mathrm{sign}}\oplus\left(\bigoplus_{\varepsilon_\theta(\pi)=1}\Lambda^2(\pi^H)\right)\oplus\left(\bigoplus_{\varepsilon_\theta(\pi)=-1}Sym^2(\pi^H)\right),$$

where the action of S_2 on $\bigoplus_{\varepsilon_\theta(\pi)=0}\pi^H\otimes(\pi^*)^{\theta(H)}$ is given by the involution $s(v\otimes w)\mapsto w\otimes v$ where $v\otimes w\in\pi^H\otimes(\pi^*)^{\theta(H)}$ and $w\otimes v\in(\pi^*)^{\theta(H)}\otimes\pi^H\cong(\pi^*\circ\theta)^H\otimes(\pi^*\circ\theta)$.

Corollary 5. *Using the notations above we have,*

$$\#\{O\in H\backslash G/\theta(H):\sigma(O)\neq O\}=\sum_{\pi\in\mathrm{irr}(G)}\dim(\pi^H)\left(\dim(\pi^{\theta(H)})-\varepsilon_\theta(\pi)\right).$$

1.1. **The case of p-adic and real groups.** Some of the arguments above work also for l-groups and even real reductive groups. First of all, as in the original Gelfand-Kazhdan criterion, the second condition in all three theorems should be replaced by a condition on distributions. Similarly, the space V as above should be replaced by a space of distributions.

The proof of [Vin06] works also for l-groups (see [Vin06, Lemma 3]), and the same argument seems to work for real reductive groups. Thus, the main difference is in Proposition 4. Proposition 4 does not work as is in those cases. However, the construction of the spherical (a.k.a. relative) character gives an embedding

$$\nu:\left(\bigoplus_{\varepsilon_\theta(\pi)=0}(\pi^*)^H\otimes(\tilde\pi^*)^{\theta(H)}\right)^{S_2,\mathrm{sign}}\oplus\left(\bigoplus_{\varepsilon_\theta(\pi)=1}\Lambda^2((\pi^*)^H)\right)\oplus\left(\bigoplus_{\varepsilon_\theta(\pi)=-1}Sym^2((\pi^*)^H)\right)\to V.$$

Using this, the implication (2) $\Rightarrow$ (1) of Theorems 1, 2 and 3 follows. In fact, this is a reformulation of the classical proof of the Gelfand-Kazhdan criterion, and its extension that was proven in [JR96]. It is reasonable to expect that in many cases ν has dense image. If this is the case, then Theorems 1, 2 and 3 hold in the p-adic and real settings. Namely, consider the following:

Definition. *Let $H_1, H_2 \subset G$ be subgroups of an l-group or of a real reductive group and let $\mathcal{S}^*(G)$ denote the space of Schwartz distributions on G. We say that (G, H_1, H_2) satisfies spectral density if the space spanned by spherical characters of irreducible (admissible) representations of G w.r.t. H_1, H_2 is dense in $\mathcal{S}^*(G)^{H_1 \times H_2}$.*

We prove that a weaker property is satisfied in the p-adic case for many cases in [AGS15, Theorems C and D].

The argument above show the following:

Theorem 6. *If G is an l-group (or real reductive group) and H is a subgroup s.t. (G, H, H) satisfies spectral density, then Theorems 1, 2 and 3 hold for G, H with the above mentioned changes.*

2. Proof of Proposition 4

Let $X = G/H$, and set $H' = \theta(H)$ and $X' = G/H'$. Let σ be the involution of $X \times X'$ given by $([g],[h]) \mapsto ([\theta(h)],[\theta(g)])$. We have

$$\mathbb{C}[X] = \bigoplus_{\pi \in \mathrm{irr}(G)} \pi \otimes (\pi^*)^H \quad \text{and} \quad \mathbb{C}[X'] = \bigoplus_{\pi \in \mathrm{irr}(G)} \pi \otimes (\pi^*)^{H'}.$$

Thus

$$W := \mathbb{C}[G]^{H \times H'} \cong \mathbb{C}[X \times X']^{\Delta G} \cong \bigoplus_{\pi \in \mathrm{irr}(G)} (\pi^*)^H \otimes (\pi)^{H'},$$

where ΔG denotes the diagonal embedding of G into $G \times G$. Its remains to understand the action of σ on W. For this let us first analyze the action of σ on $\mathbb{C}[X \times X']$. We have

$$\begin{aligned}
\mathbb{C}[X \times X'] &\cong \bigoplus_{\pi,\tau \in \mathrm{irr}(G)} \pi \otimes (\pi^*)^H \otimes \tau \otimes (\tau^*)^{H'} \cong \bigoplus_{\pi,\tau \in \mathrm{irr}(G)} (\pi \circ \theta) \otimes (\pi^*)^{H'} \otimes \tau \otimes (\tau^*)^{H'} \\
&\cong \bigoplus_{\pi,\tau \in \mathrm{irr}(G)} (\pi \circ \theta) \otimes \tau \otimes (\pi^*)^{H'} \otimes (\tau^*)^{H'} \\
&\cong \left(\bigoplus_{\pi \in \mathrm{irr}(G)} (\pi \circ \theta) \otimes \pi \otimes (\pi^*)^{H'} \otimes (\pi^*)^{H'} \right) \oplus \left(\bigoplus_{\pi \not\cong \tau \in \mathrm{irr}(G)} (\pi \circ \theta) \otimes \tau \otimes (\pi^*)^{H'} \otimes (\tau^*)^{H'} \right).
\end{aligned}$$

The action of σ on

$$\bigoplus_{\pi \not\cong \tau \in \mathrm{irr}(G)} (\pi \circ \theta) \otimes \tau \otimes (\pi^*)^{H'} \otimes (\tau^*)^{H'}$$

is given by interchanging the summand corresponding to (π, τ) with the summand corresponding to (τ, π). The action of σ on

$$\bigoplus_{\pi \in \mathrm{irr}(G)} (\pi \circ \theta) \otimes \pi \otimes (\pi^*)^{H'} \otimes (\pi^*)^{H'}$$

is by acting on each summand separately, and is given by

$$v \otimes w \otimes \alpha \otimes \beta \mapsto w \otimes v \otimes \beta \otimes \alpha.$$

Now, let us restrict this action to $((\pi \circ \theta) \otimes \pi \otimes (\pi^*)^{H'} \otimes (\pi^*)^{H'})^{\Delta G} \cong ((\pi \circ \theta) \otimes \pi)^{\Delta G} \otimes (\pi^*)^{H'} \otimes (\pi^*)^{H'}$. We see that the space $(\pi \circ \theta) \otimes \pi)^G$ is either 0 or 1-dimensional, and in the latter case, the action of σ on it is given by $\varepsilon_\theta(\pi)$.

Finally we use the fact that $V = \{w \in W : \sigma(w) = -w\}$ and obtain the required identity.

References

[AGS15] Avraham Aizenbud, Dmitry Gourevitch, and Eitan Sayag. $\mathfrak{z}$-finite distributions on p-adic groups. *Adv. Math.*, 285:1376–1414, 2015.

[GK75] I. M. Gel'fand and D. A. Kajdan. Representations of the group $\mathrm{GL}(n, K)$ where K is a local field. In *Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971)*, pages 95–118, 1975.

[JR96] Hervé Jacquet and Stephen Rallis. Uniqueness of linear periods. *Compositio Math.*, 102(1):65–123, 1996.

[Vin06] C. Ryan Vinroot. Involutions acting on representations. *J. Algebra*, 297(1):50–61, 2006.

Avraham Aizenbud, Faculty of Mathematical Sciences, Weizmann Institute of Science, Rehovot, Israel

Email address: aizenr@gmail.com

URL: http://aizenbud.org