MULTIPLICITY ONE THEOREMS

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Abstract. In the local, characteristic 0, non-archimedean case, we consider distributions on $GL(n+1)$ which are invariant under conjugation by $GL(n)$. We prove that such distributions are invariant by transposition. This implies multiplicity at most one for restrictions from $GL(n+1)$ to $GL(n)$.
Similar Theorems are obtained for orthogonal or unitary groups.

Introduction

Let $\mathbb{F}$ be a non-archimedean local field of characteristic 0. Let W be a vector space over $\mathbb{F}$ of finite dimension $n+1 \geqslant 1$ and let $W = V \oplus U$ be a direct sum decomposition with $\dim V = n$. Then we have an imbedding of $GL(V)$ into $GL(W)$. Our goal is to prove the following Theorem:

Theorem (1). *If π (resp. ρ) is an irreducible admissible representation of $GL(W)$ (resp. of $GL(V)$) then*

$$\dim\left(\mathrm{Hom}_{GL(V)}(\pi_{|GL(V)}\,,\,\rho)\right) \leqslant 1.$$

We choose a basis of V and a non-zero vector in U thus getting a basis of W. We can identify $GL(W)$ with $GL(n+1,\mathbb{F})$ and $GL(V)$ with $GL(n,\mathbb{F})$. The transposition map is an involutive anti-automorphism of $GL(n+1,\mathbb{F})$ which leaves $GL(n,\mathbb{F})$ stable. It acts on the space of distributions on $GL(n+1,\mathbb{F})$.

Theorem 1 is a Corollary of

Theorem (2). *A distribution on $GL(W)$ which is invariant under conjugation by $G = GL(V)$ is invariant by transposition.*

One can raise a similar question for orthogonal and unitary groups. Let $\mathbb{D}$ be either $\mathbb{F}$ or a quadratic extension of $\mathbb{F}$. If $x \in \mathbb{D}$ then $\overline{x}$ is the conjugate of x if $\mathbb{D} \neq \mathbb{F}$ and is equal to x if $\mathbb{D} = \mathbb{F}$.

Let W be a vector space over $\mathbb{D}$ of finite dimension $n+1 \geqslant 1$. Let $\langle .,. \rangle$ be a non-degenerate hermitian form on W. This form is bi-additive and

$$\langle dw, d'w' \rangle = d\,\overline{d'}\langle w, w' \rangle, \quad \langle w', w \rangle = \overline{\langle w, w' \rangle}.$$

Given a $\mathbb{D}$-linear map u from W into itself, its adjoint u^* is defined by the usual formula

$$\langle u(w), w' \rangle = \langle w, u^*(w') \rangle.$$

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Choose a vector e in W such that $\langle e, e\rangle \neq 0$; let $U = \mathbb{D}e$ and $V = U^{\perp}$ the orthogonal complement. Then V has dimension n and the restriction of the hermitian form to V is non-degenerate.

Let M be the unitary group of W that is to say the group of all $\mathbb{D}$-linear maps m of W into itself which preserve the hermitian form or equivalently such that $mm^* = 1$. Let G be the unitary group of V. With the p-adic topology both groups are of type lctd (locally compact, totally discontinuous) and countable at infinity. They are reductive groups of classical type.

The group G is naturally imbedded into M.

Theorem (1'). *If π (resp ρ) is an irreducible admissible representation of M (resp of G) then*

$$\dim\left(\mathrm{Hom}_G(\pi_{|G}, \rho)\right) \leq 1.$$

Choose a basis $e_1, \dots e_n$ of V such that $\langle e_i, e_j\rangle \in \mathbb{F}$. For

$$w = x_0 e + \sum_1^n x_i e_i$$

put

$$\overline{w} = \overline{x}_0 e + \sum_1^n \overline{x_i}\, e_i.$$

If u is a $\mathbb{D}$-linear map from W into itself, let $\overline{u}$ be defined by

$$\overline{u}(w) = \overline{u(\overline{w})}.$$

Let σ be the anti-involution $\sigma(m) = \overline{m}^{-1}$ of M; Theorem 1' is a consequence of

Theorem (2'). *A distribution on M which is invariant under conjugation by G is invariant under σ.*

The structure of our proof. Let us describe briefly our proof. In section 1 we recall why Theorem 2 (2') implies Theorem 1(1'). This idea goes to back Gelfand-Kazhdan ([GK75]).

For the proofs of Theorems 2 and 2' we systematically use two classical results : Bernstein's localization principle and a variant of Frobenius reciprocity which we call Frobenius descent. For the convenience of the reader they are both recalled in section 2.

Then we proceed with $\mathrm{GL}(n)$. The proof is by induction on n; the case $n = 0$ is trivial. In general we first linearize the problem by replacing the action of $\mathrm{GL}(V)$ on $\mathrm{GL}(W)$ by the action on the Lie algebra of $\mathrm{GL}(W)$. As a $\mathrm{GL}(V)$-module this Lie algebra is isomorphic to a direct sum $\mathfrak{g} \oplus V \oplus V^* \oplus \mathbb{F}$ with $\mathfrak{g}$ the Lie algebra of $G = \mathrm{GL}(V)$ and V^* the dual space of V. The group G acts trivially on $\mathbb{F}$, by the adjoint action on its Lie algebra and the natural actions on V and V^*. The component $\mathbb{F}$ plays no role. Let u be a linear bijection of V onto V^* which transforms some basis of V into its dual basis. The involution may be taken as

$$(X, v, v^*) \mapsto (u^{-1}\,{}^t X\, u, u^{-1}(v^*), u(v)).$$

We have to show that a distribution T on $\mathfrak{g} \oplus V \oplus V^*$ which is invariant under G and skew relative to the involution is 0.

In section 3 we prove that the support of such a distribution is contained in the set of singular elements. On the $\mathfrak{g}$ side, using Harish-Chandra descent we get that the support of T must be contained in $(\mathfrak{z}+\mathcal{N}) \times (V \oplus V^*)$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and $\mathcal{N}$ the cone of nilpotent elements in $\mathfrak{g}$. On the $V \oplus V^*$ side we show that the support must be contained in $\mathfrak{g} \times \Gamma$ where Γ is the cone $\langle v, v^* \rangle = 0$ in $V \oplus V^*$. On $\mathfrak{z}$ the action is trivial so we are reduced to the case of a distribution on $\mathcal{N} \times \Gamma$.

In section 4 we consider such distributions. The end of the proof is based on two facts. First, viewing the distribution as a distribution on $\mathcal{N} \times (V \oplus V^*)$ its partial Fourier transform relative to $V \oplus V^*$ has the same invariance properties and hence must also be supported on $\mathcal{N} \times \Gamma$. This implies in particular a homogeneity condition on $V \oplus V^*$. The idea of using Fourier transform in this kind of situation goes back at least to Harish-Chandra ([HC99]) and is conveniently expressed using a particular case of the Weil or oscillator representation.

For $(v, v^*) \in \Gamma$, let X_{v,v^*} be the map $x \mapsto \langle x, v^* \rangle v$ of V into itself. The second fact is that the one parameter group of transformations

$$(X, v, v^*) \mapsto (X + \lambda X_{v,v^*}, v, v^*)$$

is a group of (non-linear) homeomorphisms of $[\mathfrak{g}, \mathfrak{g}] \times \Gamma$ which commute with G and the involution. It follows that the image of the support of our distribution must also be singular. This allows us to replace the condition $\langle v, v^* \rangle = 0$ by the stricter condition $X_{v,v^*} \in \operatorname{Im} \operatorname{ad} X$.

Using the stratification of $\mathcal{N}$ we proceed one nilpotent orbit at a time, transferring the problem to $V \oplus V^*$ and a fixed nilpotent matrix X. The support condition turns out to be compatible with direct sum so that it is enough to consider the case of a principal nilpotent element. In this last situation the key is the homogeneity condition coupled with an easy induction.

The orthogonal and unitary cases are proved in a similar vein. In section 5 we reduce the support to the singular set. Here the main difference is that we use Harish-Chandra descent directly on the group. Note that the Levi subgroups have components of type GL so that Theorem 2 has to be used. Finally in section 6 we consider the case of a distribution whose support is contained in the set of singular elements; the proof is along the same lines as in section 4.

Remarks. As for the archimedean case, partial analogs of the results of this paper were obtained in [AGS08a, AGS08b, vD08]. Recently, the full analogs were obtained in [AG08] and [SZ08].

Let us add some comments on the Theorems themselves. First note that Theorem 2 gives an independent proof of a well known theorem of Bernstein: choose a basis $e_1, \ldots, e_n$ of V, add some vector e_0 of W to obtain a basis of W and let P be the isotropy of e_0 in $\mathrm{GL}(W)$. Then Theorem B of [Ber84] says that a distribution on $\mathrm{GL}(W)$ which is invariant under the action of P is invariant under the action of $\mathrm{GL}(W)$. Now, by Theorem 2 such a distribution is invariant under conjugation by the transpose of P and the group of inner automorphisms is generated by the images of P and its transpose. This Theorem implies Kirillov's conjecture which states that any unitary irreducible representation of $\mathrm{GL}(W)$ remains irreducible when restricted to P.

The occurrence of involutions in multiplicity at most one problems is of course nothing new. The situation is fairly simple when all the orbits are stable by the involution thanks to Bernstein's localization principle and constructivity theorem ([BZ76, GK75]). In our case this is not true : only generic orbits are stable. Non-stable orbits may carry invariant measures but they do not extend to the ambient space (a similar situation is already present in [Ber84]).

An illustrative example is the case $n = 1$ for GL. It reduces to $\mathbb{F}^*$ acting on $\mathbb{F}^2$ as $(x, y) \mapsto (tx, t^{-1}y)$. On the x axis the measure $d^*x = dx/|x|$ is invariant but does not extend invariantly. However the symmetric measure

$$f \mapsto \int_{\mathbb{F}^*} f(x,0)d^*x + \int_{\mathbb{F}^*} f(0,y)d^*y$$

does extend.

As in similar cases (for example [JR96]) our proof does not give a simple explanation of why all invariant distributions are symmetric. The situation would be much better if we had some kind of density theorem. For example in the GL case let us say that an element (X, v, v^*) of $\mathfrak{g} \oplus V \oplus V^*$ is regular if $(v, Xv, \ldots X^{n-1}v)$ is a basis of V and $(v^*, \ldots, {}^tX^{n-1}v^*)$ is basis of V^*. The set of regular elements is a non-empty Zariski open subset; regular elements have trivial isotropy subgroups. The regular orbits are the orbits of the regular elements; they are closed, separated by the invariant polynomials and stable by the involution (see [RS07]). In particular they carry invariant measures which, the orbits being closed, do extend and are invariant by the involution. It is tempting to conjecture that the subspace of the space of invariant distributions generated by these measures is weakly dense. This would provide a better understanding of Theorem 2. Unfortunately if true at all, such a density theorem is likely to be much harder to prove.

Assuming multiplicity at most one, a more difficult question is to find when it is one. Some partial results are known.

For the orthogonal group (in fact the special orthogonal group) this question has been studied by B. Gross and D. Prasad ([GP92, Pra93]) who formulated a precise conjecture. An up to date account is given by B. Gross and M. Reeder ([GR06]). In a different setup, in their work on "Shintani" functions A. Murase and T. Sugano obtained complete results for $GL(n)$ and the split orthogonal case but only for spherical representations ([Kat03, Mur96]). Finally we should mention, Hakim's publication [Hak03], which, at least for the discrete series, could perhaps lead to a different kind of proof.

Multiplicity one theorems have important applications to the relative trace formula, to automorphic descent, to local and global liftings of automorphic representations, and to determinations of L-functions. In particular, multiplicity at most one is used as a hypothesis in the work [GPSR97] on the study of automorphic L-functions on classical groups. At least for the last two authors, the original motivation for this work came in fact from [GPSR97].

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1. Theorem 2(2') implies Theorem 1(1')

A group of type lctd is a locally compact, totally disconnected group which is countable at infinity. We consider smooth representations of such groups. If (π, E_π) is such a representation then (π^*, E_π^*) is the smooth contragradient. Smooth induction is denoted by Ind and compact induction by ind. For any topological space T of type lctd, $\mathcal{S}(T)$ is the space of functions locally constant, complex valued, defined on T and with compact support. The space $\mathcal{S}'(T)$ of distributions on T is the dual space to $\mathcal{S}(T)$.

Proposition 1.1. *Let M be a lctd group and N a closed subgroup, both unimodular. Suppose that there exists an involutive anti-automorphism σ of M such that $\sigma(N) = N$ and such that any distribution on M, biinvariant under N, is fixed by σ. Then, for any irreducible admissible representation π of M*

$$\dim\left(\mathrm{Hom}_M(ind_N^M(1), \pi)\right) \times \dim\left(\mathrm{Hom}_M(ind_N^M(1), \pi^*)\right) \leq 1.$$

This is well known (see for example [Pra90]).

Remark. *There is a variant for the non-unimodular case; we will not need it.*

Corollary 1.1. *Let M be a lctd group and N a closed subgroup, both unimodular. Suppose that there exists an involutive anti-automorphism σ of M such that $\sigma(N) = N$ and such that any distribution on M, invariant under conjugation by N, is fixed by σ. Then, for any irreducible admissible representation π of M and any irreducible admissible representation ρ of N*

$$\dim\left(\mathrm{Hom}_N(\pi_{|N}, \rho^*)\right) \times \dim\left(\mathrm{Hom}_N((\pi^*)_{|N}, \rho)\right) \leq 1.$$

Proof. Let $M' = M \times N$ and N' be the closed subgroup of M' which is the image of the diagonal embedding of N in M'. The map $(m,n) \mapsto mn^{-1}$ of M' onto M defines a homeomorphism of M'/N' onto M. The inverse map is $m \mapsto (m,1)N'$. On M'/N' left translations by N' correspond to the action of N on M by conjugation. We have a bijection between the space of distributions T on M invariant under the action of N by conjugation and the space of distributions S on M' which are biinvariant under N'. Explicitly

$$\langle S, f(m,n)\rangle = \langle T, \int_N f(mn,n)dn\rangle.$$

Suppose that T is invariant under σ and consider the involutive anti-automorphism σ' of M' given by $\sigma'(m,n) = (\sigma(m), \sigma(n))$. Then

$$\langle S, f \circ \sigma'\rangle = \langle T, \int_N f(\sigma(n)\sigma(m), \sigma(n))dn\rangle.$$

Using the invariance under σ and for the conjugation action of N we get

$$\begin{aligned}\langle T, \int_N f(\sigma(n)\sigma(m), \sigma(n))dn\rangle &= \langle T, \int_N f(\sigma(n)m, \sigma(n))dn\rangle \\ &= \langle T, \int_N f(mn,n)dn\rangle \\ &= \langle S, f\rangle.\end{aligned}$$

Hence S is invariant under σ'. Conversely if S is invariant under σ' the same computation shows that T is invariant under σ. Under the assumption of the corollary we can now apply Proposition 1.1 and we obtain the inequality

$$\dim\Big(\mathrm{Hom}_{M'}(ind_{N'}^{M'}(1), \pi \otimes \rho)\Big) \times \dim\Big(\mathrm{Hom}_{M'}(ind_{N'}^{M'}(1), \pi^* \otimes \rho^*)\Big) \leq 1.$$

We know that $Ind_{N'}^{M'}(1)$ is the smooth contragredient representation of $ind_{N'}^{M'}(1)$; hence

$$\mathrm{Hom}_{M'}(ind_{N'}^{M'}(1), \pi^* \otimes \rho^*) \approx \mathrm{Hom}_{M'}(\pi \otimes \rho, Ind_{N'}^{M'}(1)).$$

Frobenius reciprocity tells us that

$$\mathrm{Hom}_{M'}\big(\pi \otimes \rho, Ind_{N'}^{M'}(1)\big) \approx \mathrm{Hom}_{N'}\big((\pi \otimes \rho)_{|N'}, 1\big).$$

Clearly

$$\mathrm{Hom}_{N'}\big((\pi \otimes \rho)_{|N'}, 1\big) \approx \mathrm{Hom}_N\big(\rho, (\pi_{|N})^*\big) \approx \mathrm{Hom}_N(\pi_{|N}, \rho^*).$$

Using again Frobenius reciprocity we get

$$\mathrm{Hom}_N\big(\rho, (\pi_{|N})^*\big) \approx \mathrm{Hom}_M\big(ind_N^M(\rho), \pi^*\big).$$

In the above computations we may replace ρ by ρ^* and π by π^*. Finally

$$\begin{aligned}
\mathrm{Hom}_{M'}(ind_{N'}^{M'}(1), \pi^* \otimes \rho^*) &\approx \mathrm{Hom}_N(\rho, (\pi_{|N})^*) \\
&\approx \mathrm{Hom}_N(\pi_{|N}, \rho^*) \\
&\approx \mathrm{Hom}_M(ind_N^M(\rho), \pi^*). \\
\mathrm{Hom}_{M'}(ind_{N'}^{M'}(1), \pi \otimes \rho) &\approx \mathrm{Hom}_N(\rho^*, ((\pi^*)_{|N})^*) \\
&\approx \mathrm{Hom}_N((\pi^*)_{|N}, \rho) \\
&\approx \mathrm{Hom}_M(ind_N^M(\rho^*), \pi).
\end{aligned}$$

□

Consider the case $M = GL(W)$ and $N = GL(V)$ in the notation of the intrduction. In order to use Corollary 1.1 to infer Theorem 1 from Theorem 2 it remains to show that

(1) $$\mathrm{Hom}_N((\pi^*)_{|N}, \rho) \approx \mathrm{Hom}_N(\pi_{|N}, \rho^*)$$

Let E_π be the space of the representation π and let E_π^* be the smooth dual (relative to the action of $GL(W)$). Let E_ρ be the space of ρ and E_ρ^* be the smooth dual for the action of $GL(V)$. We know, [BZ76, section 7] that the contragredient representation π^* in E_π^* is isomorphic to the representation $g \mapsto \pi({}^t g^{-1})$ in E_π. The same is true for ρ^*. Therefore an element of $\mathrm{Hom}_N(\pi_{|N}, \rho^*)$ may be described as a linear map A from E_π into E_ρ such that, for $g \in N$

$$A\pi(g) = \rho({}^t g^{-1})A.$$

An element of $\mathrm{Hom}_N((\pi^*)_{|N}, \rho)$ may be described as a linear map A' from E_π into E_ρ such that, for $g \in N$

$$A'\pi({}^t g^{-1}) = \rho(g)A'.$$

This yields (1).

Similarly, we prove that Theorem 2' implies Theorem 1'. With the notation of the introduction, this would follow from Corollary 1.1 provided that

(2) $$\mathrm{Hom}\left(\pi^*_{|G}, \rho\right) \approx \mathrm{Hom}\left(\pi_{|G}, \rho^*\right).$$

To show (2) we use the following result of [MVW87, Chapter 4]. Choose $\delta \in GL_{\mathbb{F}}(W)$ such that $\langle \delta w, \delta w' \rangle = \langle w', w \rangle$. If π is an irreducible admissible representation of M, let π^* be its smooth contragredient and define π^δ by

$$\pi^\delta(x) = \pi(\delta x \delta^{-1}).$$

Then π^δ and π^* are equivalent. We choose $\delta = 1$ in the orthogonal case $\mathbb{D} = \mathbb{F}$. In the unitary case, fix an orthogonal basis of W, say $e_1, \ldots, e_{n+1}$, such that $e_2, \ldots, e_{n+1}$ is a basis of V; put $\langle e_i, e_i \rangle = a_i$. Then

$$\langle \sum x_i e_i, \sum y_j e_j \rangle = \sum a_i x_i \overline{y_i}.$$

Define δ by

$$\delta\left(\sum x_i e_i\right) = \sum \overline{x_i} e_i.$$

Note that $\delta^2 = 1$.

Let E_π be the space of π. Then, up to equivalence, π^* is the representation $m \mapsto \pi(\delta m \delta^{-1})$. If ρ is an admissible irreducible representation of G in a vector space E_ρ then an element A of Hom $\left(\pi^*_{|G}, \rho\right)$ is a linear map from E_π into E_ρ such that

$$A\pi(\delta g \delta^{-1}) = \pi(g)A, \quad g \in G.$$

In turn the contragredient ρ^* of ρ is equivalent to the representation $g \mapsto \rho(\delta g \delta^{-1})$ in E_ρ. Then an element B of Hom $\left(\pi_{|G}, \rho^*\right)$ is a linear map from E_π into E_ρ such that

$$B\pi(g) = \rho(\delta g \delta^{-1})B, \quad g \in G.$$

As $\delta^2 = 1$ the conditions on A and B are the same. Thus (2) follows.

From now on we concentrate on Theorems 2 and 2'.

2. Some tools

We shall state two theorems which are systematically used in our proof.

If X is a Hausdorff totally disconnected locally compact topological space (lctd space in short) we denote by $\mathcal{S}(X)$ the vector space of locally constant functions with compact support of X into the field of complex numbers $\mathbb{C}$. The dual space $\mathcal{S}'(X)$ of $\mathcal{S}(X)$ is the space of distributions on X with the weak topology. All the lctd spaces we introduce are countable at infinity.

If a lctd topological group G acts continuously on a lctd space X then it acts on $\mathcal{S}(X)$ by

$$(gf)(x) = f(g^{-1}x)$$

and on distributions by

$$(gT)(f) = T(g^{-1}f)$$

The space of invariant distributions is denoted by $\mathcal{S}'(X)^G$. More generally, if χ is a character of G we denote by $\mathcal{S}'(X)^{G,\chi}$ the space of distributions T which transform according to χ that is to say $gT = \chi(g)T$.

The following result is due to Bernstein [Ber84], section 1.4.

Theorem 2.1 (Localization principle)**.** *Let $q : Z \to T$ be a continuous map between two topological spaces of type lctd. Denote $Z_t := q^{-1}(t)$. Consider $\mathcal{S}'(Z)$ as $\mathcal{S}(T)$-module. Let M be a closed subspace of $\mathcal{S}'(Z)$ which is an $\mathcal{S}(T)$-submodule. Then $M = \overline{\bigoplus_{t\in T}(M \cap \mathcal{S}'(Z_t))}$.*

Corollary 2.1. *Let $q : Z \to T$ be a continuous map between topological spaces of type lctd. Let a lctd group H act on Z preserving the fibers of q. Let μ be a character of H. Suppose that for any $t \in T$, $\mathcal{S}'(q^{-1}(t))^{H,\mu} = 0$. Then $\mathcal{S}'(Z)^{H,\mu} = 0$.*

The second theorem is a variant of Frobenius reciprocity ([Ber84, section 1.5] and [BZ76, sections 2.21-2.36]).

Theorem 2.2 (Frobenius descent)**.** *Suppose that a unimodular lctd topological group H act transitively on a lctd topological space Z. Let $\varphi : E \to Z$ be an H-equivariant map of* *lctd topological spaces. Let $x \in Z$. Assume that the stabilizer $Stab_H(x)$ is unimodular. Let $W = \varphi^{-1}(x)$ be the fiber of x. Let χ be a character of H. Then*

(1) *There exists a canonical isomorphism* $\mathrm{Fr} : \mathcal{S}'(E)^{H,\chi} \to \mathcal{S}'(W)^{Stab_H(x),\chi}$ *given by*

$$\langle \mathrm{Fr}(\xi), f\rangle = \int_Z \chi(g_z)\langle \xi, g_z f\rangle dz,$$

where dz denotes the Haar measure on Z, and $g_z \in H$ is an element such that $g_z z = x$.

(2) *For any distribution $\xi \in \mathcal{S}'(E)^{H,\chi}$,* $\mathrm{Supp}(\mathrm{Fr}(\xi)) = \mathrm{Supp}(\xi) \cap W$.

In particular, consider the case where H acts transitively on Z and W is a finite dimensional vector space over $\mathbb{F}$ with a nondegenerate bilinear form B. Assume that H acts on W linearly preserving B. Let $\mathrm{Fr} : \mathcal{S}'(Z \times W)^{H,\chi} \to \mathcal{S}'(W)^{Stab_H(x)}$ be the Frobenius isomorphism with respect to the projection map $Z \times W \to Z$. Let $\mathcal{F}_B$ be the Fourier transform in the W-coordinate. We have

Proposition 2.1. *For any $\xi \in \mathcal{S}'(Z \times W)^{H,\chi}$, we have* $\mathcal{F}_B(\mathrm{Fr}(\xi)) = \mathrm{Fr}(\mathcal{F}_B(\xi))$

This Proposition will be used in sections 4 and 6.

Finally as $\mathbb{F}$ is non-archimedean, a distribution which is 0 on some open set may be identified with a distribution on the (closed) complement. This will be used throughout this work.

3. Reduction to the singular set : the GL(n) case

Consider the case of the general linear group. From the decomposition $W = V \oplus \mathbb{F}e$ we get, with obvious identifications

$$\mathrm{End}(W) = \mathrm{End}(V) \oplus V \oplus V^* \oplus \mathbb{F}.$$

Note that $\mathrm{End}(V)$ is the Lie algebra $\mathfrak{g}$ of G. The group G acts on $\mathrm{End}(W)$ by $g(X, v, v^*, t) = (gXg^{-1}, gv, {}^tg^{-1}v^*, t)$. As before choose a basis $(e_1, \dots, e_n)$ of V and let $(e_1^*, \dots, e_n^*)$ be the dual basis of V^*. Define an isomorphism u of V onto V^* by $u(e_i) = e_i^*$. On $GL(W)$ the involution σ is $h \mapsto u^{-1}{}^th^{-1}u$. It depends upon the choice of the basis but the action on the space of invariant distributions does not depend upon this choice.

It will be convenient to introduce an extension $\widetilde{G}$ of G of degree two. Let $\mathrm{Iso}(V, V^*)$ be the set of isomorphisms of V onto V^*. We define $\widetilde{G} = G \cup \mathrm{Iso}(V, V^*)$. The group law, for $g, g' \in G$ and $u, u' \in \mathrm{Iso}(V, V^*)$ is

$$g \times g' = gg', \;\; u \times g = ug, \;\; g \times u = {}^tg^{-1}u, \;\; u \times u' = {}^tu^{-1}u'.$$

Now from $W = V \oplus \mathbb{F}e$ we obtain an identification of the dual space W^* with $V^* \oplus \mathbb{F}e^*$ with $\langle e^*, V\rangle = (0)$ and $\langle e^*, e\rangle = 1$. Any u as above extends to an isomorphism of W onto W^* by defining $u(e) = e^*$. The group $\widetilde{G}$ acts on $GL(W)$:

$$h \mapsto ghg^{-1}, \;\; h \mapsto {}^t(uhu^{-1}), \quad g \in G, \, h \in GL(W), \; u \in \mathrm{Iso}(V, V^*)$$

and also on $\mathrm{End}(W)$ with the same formulas.

Let χ be the character of $\widetilde{G}$ which is 1 on G and -1 on $\mathrm{Iso}(V, V^*)$. Our goal is to prove that $\mathcal{S}'(GL(W))^{\widetilde{G},\chi} = (0)$.

Proposition 3.1. *If $\mathcal{S}'(\mathfrak{g} \oplus V \oplus V^*)^{\widetilde{G},\chi} = (0)$ then $\mathcal{S}'(GL(W))^{\widetilde{G},\chi} = (0)$.*

Proof. We have $\mathrm{End}(W) = \Big(\mathrm{End}(V) \oplus V \oplus V^*\Big) \oplus \mathbb{F}$ and the action of $\widetilde{G}$ on $\mathbb{F}$ is trivial. Thus $\mathcal{S}'(\mathfrak{g} \oplus V \oplus V^*)^{\widetilde{G},\chi} = (0)$ implies that $\mathcal{S}'(\mathrm{End}(W))^{\widetilde{G},\chi} = (0)$. Let $T \in \mathcal{S}'(GL(W))^{\widetilde{G},\chi}$. Let $h \in GL(W)$ and choose a compact open neighborhood K of $\mathrm{Det}\ h$ such that $0 \notin K$. For $x \in \mathrm{End}(W)$ define $\varphi(x) = 1$ if $\mathrm{Det}\, x \in K$ and $\varphi(x) = 0$ otherwise. Then φ is a locally constant function. The distribution $(\varphi_{|GL(W)})T$ has a support which is closed in $\mathrm{End}(W)$ hence may be viewed as a distribution on $\mathrm{End}(W)$. This distribution belongs to $\mathcal{S}'(\mathrm{End}(W))^{\widetilde{G},\chi}$ so it must be equal to 0. It follows that T is 0 in the neighborhood of h. As h is arbitrary we conclude that $T = 0$. $\square$

Our task is now to prove that $\mathcal{S}'(\mathfrak{g} \oplus V \oplus V^*)^{\widetilde{G},\chi} = (0)$. We shall use induction on the dimension n of V. The action of $\widetilde{G}$ is, for $X \in \mathfrak{g}$, $v \in V$, $v^* \in V^*$, $g \in G$, $u \in \mathrm{Iso}(V, V^*)$

$$(X, v, v^*) \mapsto (gXg^{-1}, gv, {}^tg^{-1}v^*), \ \ (X, v, v^*) \mapsto ({}^t(uXu^{-1}), {}^tu^{-1}v^*, uv).$$

The case $n = 0$ is trivial.

We suppose that V is of dimension $n \geq 1$, assuming the result up to dimension $n - 1$ and for all $\mathbb{F}$. If $T \in \mathcal{S}'(\mathfrak{g} \oplus V \oplus V^*)^{\widetilde{G},\chi}$ we are going to show that its support is contained in the "singular set". This will be done in two stages.

On $V \oplus V^*$ let Γ be the cone $\langle v^*, v\rangle = 0$. It is stable under $\widetilde{G}$.

Lemma 3.1. *The support of T is contained in $\mathfrak{g} \times \Gamma$.*

Proof. For $(X, v, v^*) \in \mathfrak{g} \oplus V \oplus V^*$ put $q(X, v, v^*) = \langle v^*, v\rangle$. Let Ω be the open subset $q \neq 0$. We have to show that $\mathcal{S}'(\Omega)^{\widetilde{G},\chi} = (0)$. By Bernstein's localization principle (Corollary 2.1) it is enough to prove that, for any fiber $\Omega_t = q^{-1}(t)$, $t \neq 0$, one has $\mathcal{S}'(\Omega_t)^{\widetilde{G},\chi} = (0)$.

G acts transitively on the quadric $\langle v^*, v\rangle = t$. Fix a decomposition $V = \mathbb{F}\varepsilon \oplus V_1$ and identify $V^* = \mathbb{F}\varepsilon^* \oplus V_1^*$ with $\langle\varepsilon^*, \varepsilon\rangle = 1$. Then $(X, \varepsilon, t\varepsilon^*) \in \Omega_t$ and the isotropy subgroup of $(\varepsilon, t\varepsilon^*)$ in $\widetilde{G}$ is, with an obvious notation $\widetilde{G}_{n-1}$. By Frobenius descent (Theorem 2.2) there is a linear bijection between $\mathcal{S}'(\Omega_t)^{\widetilde{G},\chi}$ and the space $\mathcal{S}'(\mathfrak{g})^{\widetilde{G}_1,\chi_1}$ and this last space is (0) by induction. $\square$

Let $\mathfrak{z}$ be the center of $\mathfrak{g}$ that is to say the space of scalar matrices. Let $\mathcal{N} \subseteq [\mathfrak{g}, \mathfrak{g}]$ be the nilpotent cone in $\mathfrak{g}$.

Proposition 3.2. *If $T \in \mathcal{S}'(\mathfrak{g} \oplus V \oplus V^*)^{\widetilde{G},\chi}$ then the support of T is contained in $(\mathfrak{z} + \mathcal{N}) \times \Gamma$. If $\mathcal{S}'(\mathcal{N} \times \Gamma)^{\widetilde{G},\chi} = (0)$ then $\mathcal{S}'(\mathfrak{g} \oplus V \oplus V^*)^{\widetilde{G},\chi} = (0)$.*

Proof. Let us prove that the support of such a distribution T is contained in $(\mathfrak{z} + \mathcal{N}) \times (V \oplus V^*)$. We use Harish-Chandra's descent method. For $X \in \mathfrak{g}$ let $X = X_s + X_n$ be the Jordan decomposition of X with X_s semisimple and X_n nilpotent. This decomposition commutes with the action of $\widetilde{G}$. The centralizer $Z_G(X)$ of an element $X \in \mathfrak{g}$ is unimodular

([SS70, page 235]) and there exists an isomorphism u of V onto V^* such that ${}^tX = uXu^{-1}$ (any matrix is conjugate to its transpose). It follows that the centralizer $Z_{\widetilde{G}}(X)$ of X in $\widetilde{G}$, a semi direct product of $Z_G(X)$ and S_2, is also unimodular.

Let E be the space of monic polynomials of degree n with coefficients in $\mathbb{F}$. For $p \in E$, let $\mathfrak{g}_p$ be the set of all $X \in \mathfrak{g}$ with characteristic polynomial p. Note that $\mathfrak{g}_p$ is fixed by $\widetilde{G}$. By Bernstein localization principle (Theorem 2.1) it is enough to prove that if p is not $(T-\lambda)^n$ for some λ then $\mathcal{S}'(\mathfrak{g}_p \times V \times V^*)^{\widetilde{G},\chi} = (0)$.

Fix p. We claim that the map $X \mapsto X_s$ restricted to $\mathfrak{g}_p$ is continuous. Indeed let $\widetilde{\mathbb{F}}$ be a finite Galois extension of $\mathbb{F}$ containing all the roots of p. Let

$$p(\xi) = \prod_1^s (\xi - \lambda_i)^{n_i}$$

be the decomposition of p. Recall that if $X \in \mathfrak{g}_p$ and $V_i = \mathrm{Ker}(X-\lambda_i)^{n_i}$ then $V = \oplus V_i$ and the restriction of X_s to V_i is the multiplication by λ_i. Then choose a polynomial R, with coefficients in $\widetilde{\mathbb{F}}$ such that for all i, R is congruent to λ_i modulo $(\xi - \lambda_i)^{n_i}$ and $R(0) = 0$. Clearly $X_s = R(X)$. As the Galois group of $\widetilde{\mathbb{F}}$ over $\mathbb{F}$ permutes the λ_i we may even choose $R \in \mathbb{F}[\xi]$. This implies the required continuity.

There is only one semi-simple orbit γ_p in $\mathfrak{g}_p$ and it is closed. We use Frobenius descent (Theorem 2.2) for the map $(X, v, v^*) \mapsto X_s$ from $\mathfrak{g}_p \times V \times V^*$ to γ_p.

Fix $a \in \gamma_p$; its fiber is the product of $V \oplus V^*$ by the set of nilpotent elements which commute with a. It is a closed subset of the centralizer $\mathfrak{m} = \mathfrak{Z}_{\mathfrak{g}}(a)$ of a in $\mathfrak{g}$. Let $M = Z_G(a)$ and $\widetilde{M} = Z_{\widetilde{G}}(a)$.

Following [SS70] let us describe these centralizers. Let P be the minimal polynomial of a ; all its roots are simple. Let $P = P_1 \dots P_r$ be the decomposition of P into (distinct) irreducible factors, over $\mathbb{F}$. If $V_i = \mathrm{Ker}\, P_i(a)$, then $V = \oplus V_i$ and $V^* = \oplus V_i^*$. An element x of G which commutes with a is given by a family $\{x_1, \dots, x_r\}$ where each x_i is a linear map from V_i to V_i, commuting with the restriction of a to V_i. Now $\mathbb{F}[\xi]$ acts on V_i, by specializing ξ to $a_{|V_i}$ and P_i acts trivially so that, if $\mathbb{F}_i = \mathbb{F}[\xi]/(P_i)$, then V_i becomes a vector space over $\mathbb{F}_i$. The $\mathbb{F}$-linear map x_i commutes with a if and only if it is $\mathbb{F}_i$-linear.

Fix i. Let ℓ be a non-zero $\mathbb{F}$-linear form on $\mathbb{F}_i$. If $v_i \in V_i$ and $v_i' \in V_i^*$ then $\lambda \mapsto \langle \lambda v_i, v_i' \rangle$ is an $\mathbb{F}$-linear form on $\mathbb{F}_i$, hence there exists a unique element $S(v_i, v_i')$ of $\mathbb{F}_i$ such that $\langle \lambda v_i, v_i' \rangle = \ell(\lambda S(v_i, v_i'))$. One checks trivially that S is $\mathbb{F}_i$-linear with respect to each variable and defines a non degenerate duality, over $\mathbb{F}_i$ between V_i and V_i^*. Here $\mathbb{F}_i$ acts on V_i^* by transposition, relative to the $\mathbb{F}$-duality $\langle .,. \rangle$,of the action on V_i. Finally if $x_i \in \mathrm{End}_{\mathbb{F}_i} V_i$, its transpose, relative to the duality $S(.,.)$ is the same as its transpose relative to the duality $\langle .,. \rangle$.

Thus M is a product of linear groups and the situation (M, V, V^*) is a composite case, each component being a linear case (over various extensions of $\mathbb{F}$).

Let u be an isomorphism of V onto V^* such that ${}^ta = uau^{-1}$ and that, for each i, $u(V_i) = V_i^*$. Then $u \in \widetilde{M}$ and $\widetilde{M} = M \cup uM$.

Suppose that a does not belong to the center of $\mathfrak{g}$. Then each V_i has dimension strictly smaller than n and we can use the inductive assumption. Therefore $\mathcal{S}'(\mathfrak{m} \oplus V \oplus V^*)^{\widetilde{M},\chi} = (0)$. However the nilpotent cone $\mathcal{N}_{\mathfrak{m}}$ in $\mathfrak{m}$ is a closed subset so $\mathcal{S}'(\mathcal{N}_{\mathfrak{m}} \times V \times V^*)^{\widetilde{M},\chi} = (0)$.

Together with Lemma 3.1 this proves the first assertion of the Proposition.

If a belongs to the center then $\widetilde{M} = \widetilde{G}$ and the fiber is $(a + \mathcal{N}) \times V \times V^*$. This implies the second assertion. □

Remark 1. *Strictly speaking the singular set is defined as the set of all (X, v, v^*) such that for any polynomial P invariant under $\widetilde{G}$ one has $P(X, v, v^*) = P(0)$. Thus, in principle, we also need to consider the polynomials $P(X, v, v^*) = \langle v^*, X^p v\rangle$ for $p > 0$. In fact, one can show that the support of the distribution T is contained in the singular set in the strict sense (i.e., the above polynomials vanish on the support). As this is not needed in the sequel we omit the proof.*

4. End of the proof for GL(n)

In this section we consider a distribution $T \in \mathcal{S}'(\mathcal{N} \times \Gamma)^{\widetilde{G},\chi}$ and prove that $T = 0$. The following observation will play a crucial role.

Choose a non-trivial additive character ψ of $\mathbb{F}$. On $V \oplus V^*$ we have the bilinear form

$$\big((v_1, v_1^*), (v_2, v_2^*)\big) \mapsto \langle v_1^*, v_2\rangle + \langle v_2^*, v_1\rangle$$

Define the Fourier transform of a function φ on $V \oplus V^*$ by

$$\widehat{\varphi}(v_2, v_2^*) = \int_{V\oplus V^*} \varphi(v_1, v_1^*)\, \psi(\langle v_1^*, v_2\rangle + \langle v_2^*, v_1\rangle)\, dv_1 dv_1^*$$

where $dv_1 dv_1^*$ is the self-dual Haar measure.

This Fourier transform commutes with the action of $\widetilde{G}$; hence the (partial) Fourier transform $\widehat{T}$ of our distribution T has the same invariance properties and the same support conditions as T itself.

Let $\mathcal{N}_i$ be the union of nilpotent orbits of dimension at most i. We will prove, by descending induction on i, that the support of any $(\widetilde{G}, \chi)$-equivariant distribution on $[\mathfrak{g}, \mathfrak{g}] \times \Gamma$ must be contained in $\mathcal{N}_i \times \Gamma$. Suppose we already know that, for some i, the support must be contained in $\mathcal{N}_i \times \Gamma$. We must show that, for any nilpotent orbit $\mathcal{O}$ of dimension i, the restriction of the distribution to $\mathcal{O} \times \Gamma$ is 0.

If $v \in V$ and $v^* \in V^*$ we call X_{v,v^*} the rank one map $x \mapsto \langle v^*, x\rangle v$. Let

$$\nu_\lambda(X, v, v^*) = (X + \lambda X_{v,v^*}, v, v^*), \quad (X, v, v^*) \in \mathfrak{g} \times \Gamma,\ \ \lambda \in \mathbb{F}.$$

Then ν_λ is a one parameter group of homeomorphisms of $\mathfrak{g} \times \Gamma$ and note that $[\mathfrak{g}, \mathfrak{g}] \times \Gamma$ is invariant. The key observation is that ν_λ *commutes with the action of* $\widetilde{G}$. Therefore the image of T by ν_λ transforms according to the character χ of $\widetilde{G}$. Its support is contained in $[\mathfrak{g}, \mathfrak{g}] \times \Gamma$ and hence must be contained in $\mathcal{N} \times \Gamma$ and in fact in $\mathcal{N}_i \times \Gamma$. This means that if (X, v, v^*) belongs to the support of T then, for all λ, $(X + \lambda X_{v,v^*}, v, v^*)$ must belong to $\mathcal{N}_i \times \Gamma$.

The orbit $\mathcal{O}$ is open in $\mathcal{N}_i$. Thus if $X \in \mathcal{O}$ the condition $X + \lambda X_{v,v^*} \in \mathcal{N}_i$ implies that, at least for $|\lambda|$ small enough, $X + \lambda X_{v,v^*} \in \mathcal{O}$. It follows that X_{v,v^*} belongs to the tangent space to $\mathcal{O}$ at the point X ; this tangent space is the image of $\operatorname{ad} X$.

Define $Q(X)$ to be the set of all pairs (v, v^*) such $X_{v,v^*} \in \operatorname{Im} \operatorname{ad} X$.

By the discussion above, it is enough to prove the following Lemma:

Lemma 4.1. *Let $T \in \mathcal{S}'(\mathcal{O} \times V \times V^*)^{\widetilde{G},\chi}$. Suppose that the support of T and of $\widehat{T}$ are contained in the set of triplets (X, v, v^*) such that $(v, v^*) \in Q(X)$. Then $T = 0$.*

Note that the trace of X_{v,v^*} is $\langle v^*, v\rangle$ and that $X_{v,v^*} \in \operatorname{Im} \operatorname{ad} X$ implies that its trace is 0. Therefore $Q(X)$ is contained in Γ.

We proceed in three steps. First we transfer the problem to $V \oplus V^*$ and a fixed nilpotent endomorphism X. Then we show that if Lemma 4.1 holds for (V_1, X_1) and (V_2, X_2) then it holds for the direct sum $(V_1 \oplus V_2, X_1 \oplus X_2)$. Finally, decomposing X into Jordan blocks we are left with the case of a principal nilpotent element for which we give a direct proof, using Weil representation.

Consider the map $(X, v, v^*) \mapsto X$ from $\mathcal{O} \times V \times V^*$ onto $\mathcal{O}$. Choose $X \in \mathcal{O}$ and let C (resp $\widetilde{C}$) be the stabilizer in G (resp. in $\widetilde{G}$) of an element X of $\mathcal{O}$; both groups are unimodular, hence we may use Frobenius descent (Theorem 2.2).

Now we have to deal with a distribution, which we still call T, which belongs to $\mathcal{S}'(V \oplus V^*)^{\widetilde{C},\chi}$ such that both T and its Fourier transform are supported by $Q(X)$ (Proposition 2.1). Let us say that X is *nice* if the only such distribution is 0. We want to prove that all nilpotent endomorphisms are nice.

Lemma 4.2. *Suppose that we have a decomposition $V = V_1 \oplus V_2$ such that $X(V_i) \subseteq V_i$. Let X_i be the restriction of X to V_i. Then if X_1 and X_2 are nice, so is X.*

Proof. Let $(v, v^*) \in Q(X)$ and choose $A \in \mathfrak{g}$ such that $X_{v,v^*} = [A, X]$. Decompose $v = v_1 + v_2,\ v^* = v_1^* + v_2^*$ and put

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}.$$

Writing X_{v,v^*} as a 2 by 2 matrix and looking at the diagonal blocks one gets that $X_{v_i,v_i^*} = [A_{i,i}, X_i]$. This means that

$$Q(X) \subseteq Q(X_1) \times Q(X_2).$$

For $i = 1, 2$ let C_i be the centralizer of X_i in $GL(V_i)$ and $\widetilde{C}_i$ the corresponding extension by S_2. Let T be a distribution as above and let $\varphi_2 \in \mathcal{S}(V_2 \oplus V_2^*)$. Let T_1 be the distribution on $V_1 \oplus V_1^*$ defined by $\varphi_1 \mapsto \langle T, \varphi_1 \otimes \varphi_2\rangle$. The support of T_1 is contained in $Q(X_1)$ and T_1 is invariant under the action of C_1 . We have

$$\langle \widehat{T}_1, \varphi_1\rangle = \langle T_1, \widehat{\varphi}_1\rangle = \langle T, \widehat{\varphi_1} \otimes \varphi_2\rangle = \langle \widehat{T}, \check{\varphi}_1 \otimes \widehat{\varphi_2}\rangle.$$

Here $\check{\varphi}_1(v_1, v_1^*) = \varphi_1(-v_1, -v_1^*)$. By assumption the support of $\widehat{T}$ is contained in $Q(X)$ so that the support of $\widehat{T}_1$ is supported in $-Q(X_1) = Q(X_1)$. Because (X_1) is nice this implies that T_1 in invariant under $\widetilde{C}_1$.

Extend the action of $\widetilde{C}_1$ to $V \oplus V^*$ trivially. We obtain that T is invariant with respect to $\widetilde{C}_1$. Similarly it is invariant under $\widetilde{C}_2$. Since the actions of $\widetilde{C}_1$ and $\widetilde{C}_2$ together with the action of C generate the action of $\widetilde{C}$ we obtain that T must be invariant under $\widetilde{C}$ and hence must be 0. $\square$

Decomposing X into Jordan blocks we still have to prove Lemma 4.1 for a principal nilpotent element. We need some preliminary results.

Lemma 4.3. *The distribution T satisfies the following homogeneity condition:*

$$\langle T, f(tv, tv^*)\rangle = |t|^{-n}\langle T, f(v, v^*)\rangle.$$

Proof. We use a particular case of Weil or oscillator representation. Let E be a vector space over $\mathbb{F}$ of finite dimension m. To simplify assume that m is even. Let q be a non-degenerate quadratic form on E and let b be the bilinear form

$$b(e, e') = q(e + e') - q(e) - q(e').$$

Fix a continuous non-trivial additive character ψ of $\mathbb{F}$. We define the Fourier transform on E by

$$\widehat{f}(e') = \int_E f(e)\psi(b(e, e'))de$$

where de is the self dual Haar measure.

There exists ([RS07]) a representation π of $SL(2, \mathbb{F})$ in $\mathcal{S}(E)$ such that:

$$\begin{aligned}
\pi\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} f(e) &= \psi(uq(e))f(e)\\
\pi\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix} f(e) &= \frac{\gamma(q)}{\gamma(tq)}|t|^{m/2}f(te)\\
\pi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} f(e) &= \gamma(q)\widehat{f}(e)
\end{aligned}$$

where $\gamma(\cdot)$ is a certain roots of unity, which is 1 if (E, q) is a sum of hyperbolic planes.

We have a contragredient action in the dual space $\mathcal{S}'(E)$.

Suppose that T is a distribution on E such that T and $\widehat{T}$ are supported on the isotropic cone $q(e) = 0$. This means that

$$\langle T, \pi\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} f\rangle = \langle T, f\rangle, \quad \langle \widehat{T}, \pi\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} f\rangle = \langle \widehat{T}, f\rangle.$$

Using the relation

$$\langle \widehat{T}, \varphi\rangle = \langle T, \overline{\gamma(q)}\pi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} f\rangle$$

the second relation is equivalent to

$$\langle T, \pi\begin{pmatrix}1 & 0\\ -u & 1\end{pmatrix} f\rangle = \langle T, f\rangle.$$

The matrices
$$\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}1 & 0\\ u & 1\end{pmatrix} \quad u \in \mathbb{F}$$
generate the group $SL(2,\mathbb{F})$. Therefore the distribution T is invariant by $SL(2,\mathbb{F})$. In particular
$$\langle T, f(te)\rangle = \frac{\gamma(tq)}{\gamma(q)}|t|^{-m/2}\langle T, f\rangle$$
and $T = \gamma(q)\widehat{T}$.

Remark 2. *Note that (for even m) $\gamma(tq)/\gamma(q)$ is a character of t and non-zero distributions which are invariant under $SL(2,\mathbb{F})$ do exist. In the case where m is odd one obtains a representation of the two-fold covering of $SL(2,\mathbb{F})$ and we obtain the same homogeneity condition. However $\gamma(tq)/\gamma(q)$ is not a character; hence no non-zero T can exist.*

In our situation we take $E = V \oplus V^*$ and $q(v,v^*) = \langle v^*, v\rangle$. Then
$$b\Big((v_1, v_1^*), (v_2, v_2^*)\Big) = \langle v_1^*, v_2\rangle + \langle v_2^*, v_1\rangle.$$
The Fourier transform commutes with the action of $\widetilde{G}$. Both T and $\widehat{T}$ are supported on $Q(X)$ which is contained in Γ. As $\gamma(tq) = 1$ for all t this proves the Lemma and also that $T = \widehat{T}$. □

Remark 3. *The same type of argument could have been used for the quadratic form* $\mathrm{Tr}(XY)$ *on* $\mathfrak{sl}(V) = [\mathfrak{g},\mathfrak{g}]$. *This would have given a short proof for even n and a homogeneity condition for odd n.*

Now we find $Q(X)$.

Lemma 4.4. *If X is principal then $Q(X)$ is the set of pairs (v, v^*) such that for $0 \le k < n$, $\langle v^*, X^k v\rangle = 0$.*

Proof. Choose a basis $(e_1, \ldots, e_n)$ of V such that $Xe_1 = 0$ and $Xe_j = e_{j-1}$ for $j \ge 2$. Consider the map $A \mapsto XA - AX$ from the space of n by n matrices into itself. This map is anti-symmetric with respect to the Killing form and hence its image is the orthogonal complement to its kernel. A simple computation shows that the kernel of this map, that is to say the Lie algebra $\mathfrak{c}$ of the centralizer C, is the space of polynomials (of degree at most $n-1$) in X. Therefore
$$Q(X) = \{(v,v^*) | X_{v,v^*} \in \mathrm{Im}\,\mathrm{ad}\,X\} = \{(v,v^*)|\forall\, 0 \le k < n,\ \mathrm{Tr}(X_{v,v^*}X^k) = 0\} =$$
$$= \{(v,v^*)|\forall\, 0 \le k < n,\ \langle v^*, X^k v\rangle = 0\}.$$
□

End of the proof of Lemma 4.1 For a principal X, we proceed by induction on n. Keep the above notation. The centralizer C of X is the space of polynomials (of degree at most $n-1$) in X with non-zero constant term. In particular the orbit Ω of e_n is the open subset $x_n \neq 0$. We shall prove that the restriction of T to $\Omega \times V^*$ is 0. Note that the centralizer of

e_n in C is trivial. By Frobenius descent (Theorem 2.2), to the restriction of T corresponds a distribution R on V^* with support in the set of v^* such that $(e_n, v^*) \in Q(X)$. By the last Lemma this means that R is a multiple $a\delta$ of the Dirac measure at the origin. The distribution T satisfies the two conditions

$$\langle T, f(v, v^*)\rangle = \langle T, f(tv, t^{-1}v^*)\rangle = |t|^n \langle T, f(tv, tv^*)\rangle.$$

therefore

$$\langle T, f(v, t^2v^*) = |t|^{-n} \langle T, f(v, v^*)\rangle.$$

Now T is recovered from R by the formula

$$\langle T, f(v, v^*) = \int_C \langle R, f(ce_n, {}^t c^{-1}v^*\rangle dc = a \int_C f(ce_n, 0)dc, \;\; f \in \mathcal{S}(\Omega \times V^*).$$

Unless $a = 0$ this is not compatible with this last homogeneity condition.

Exactly in the same way one proves that T is 0 on $V \times \Omega^*$ where Ω^* is the open orbit $x_1^* \neq 0$ of C in V^*. The same argument is valid for $\widehat{T}$ (which is even equal to T ...).

If $n = 1$ then T is obviously 0. If $n \geq 2$ then there exists a distribution T' on

$$\bigoplus_{1<j<n} \mathbb{F}e_j \oplus \mathbb{F}e_j^*$$

such that,

$$T = T' \otimes \delta_{x_n=0} \otimes dx_1 \otimes \delta_{x_1^*=0} \otimes dx_n^*.$$

Let u be the isomorphism of V onto V^* given by $u(e_j) = e_{n+1-j}^*$. Recall that it acts on $\mathfrak{g} \times V \times V^*$ by $(X, v, v^*) \mapsto ({}^t(uXu^{-1}), {}^t u^{-1}v^*, uv)$. It belongs to $\widetilde{C}$ but not to C so it must transform T into $-T$.

The case $n = 1$ has just been settled. If $n = 2$ in the above formula T' should be replaced by a constant. The constant must be 0 if we want $u(T) = -T$. If $n > 2$ let

$$V' = \left(\oplus_1^{n-1}\mathbb{F}e_i\right)/\mathbb{F}e_1$$

and let X' be the nilpotent endomorphism of V' defined by X. We may consider T' as a distribution on $V' \oplus V'^*$ and one easily checks that, with obvious notation, it transforms according to the character χ of the the centralizer $\widetilde{C}'$ of X' in $\widetilde{G}'$. By induction $T' = 0$, hence $T = 0$.

□

5. Reduction to the singular set: the orthogonal and unitary cases

We now turn our attention to the unitary case. We keep the notation of the introduction. In particular $W = V \oplus \mathbb{D}e$ is a vector space over $\mathbb{D}$ of dimension $n+1$ with a non-degenerate hermitian form $\langle ., .\rangle$ such that e is orthogonal to V. The unitary group G of V is embedded into the unitary group M of W.

Let A be the set of all bijective maps u from V to V such that

$$u(v_1 + v_2) = u(v_1) + u(v_2), \;\; u(\lambda v) = \overline{\lambda}u(v), \;\; \langle u(v_1), u(v_2)\rangle = \overline{\langle v_1, v_2\rangle}.$$

An example of such a map is obtained by choosing a basis $e_1, \dots, e_n$ of V such that $\langle e_i, e_j\rangle \in \mathbb{F}$ and defining

$$u(\sum x_i e_i) = \sum \overline{x}_i e_i.$$

Any $u \in A$ is extended to W by the rule $u(v + \lambda e) = u(v) + \overline{\lambda} e$ and we define an action on $\mathrm{GL}(W)$ by $m \mapsto u m^{-1} u^{-1}$. The group G acts on $\mathrm{GL}(W)$ by conjugation.

Let $\widetilde{G}$ be the group of bijections of $\mathrm{GL}(W)$ onto itself generated by the actions of G and A. It is a semi direct product of G and S_2. We identify G with a subgroup of $\widetilde{G}$ and A with $\widetilde{G}\backslash G$. Note that $\widetilde{G}$ preserves M. When a confusion is possible we denote the product in $\widetilde{G}$ by $\times$.

We define a character χ of $\widetilde{G}$ by $\chi(g) = 1$ for $g \in G$ and $\chi(u) = -1$ for $u \in \widetilde{G} \setminus G$. Our overall goal is to prove that $\mathcal{S}'(M)^{\widetilde{G},\chi} = (0)$.

Let $\widetilde{G}$ act on $G \times V$ as follows:

$$g(x, v) = (gxg^{-1}, g(v)),\ u(x, v) = (ux^{-1}u^{-1}, -u(v)),\quad g \in G, u \in A, x \in G, v \in V$$

Our first step is to replace M by $G \times V$.

Proposition 5.1. *Suppose that for any V and any hermitian form $\mathcal{S}'(G \times V)^{\widetilde{G},\chi} = (0)$, then $\mathcal{S}'(M)^{\widetilde{G},\chi} = (0)$.*

Proof. We have in particular $\mathcal{S}'(M \times W)^{\widetilde{M},\chi} = (0)$. Let Y be the set of all (m, w) such that $\langle w, w\rangle = \langle e, e\rangle$; it is a closed subset, invariant under $\widetilde{M}$, hence $\mathcal{S}'(Y)^{\widetilde{M},\chi} = (0)$. By Witt's theorem M acts transitively on $\Gamma = \{w | \langle w, w\rangle = \langle e, e\rangle\}$. We can apply Frobenius descent (Theorem 2.2) to the map $(m, w) \mapsto w$ of Y onto Γ. The centralizer of e in $\widetilde{M}$ is isomorphic to $\widetilde{G}$ acting as before on the fiber $M \times \{e\}$. We have a linear bijection between $\mathcal{S}'(M)^{\widetilde{G},\chi}$ and $\mathcal{S}'(Y)^{\widetilde{M},\chi}$; therefore $\mathcal{S}'(M)^{\widetilde{G},\chi} = (0)$. □

The proof that $\mathcal{S}'(G \times V)^{\widetilde{G},\chi} = (0)$ is by induction on n. If $\mathfrak{g}$ is the Lie algebra of G we shall prove simultaneously that $\mathcal{S}'(\mathfrak{g} \times V)^{\widetilde{G},\chi} = (0)$. In this case G acts on its Lie algebra by the adjoint action and for $u \in \widetilde{G} \setminus G$ one puts, for $X \in \mathfrak{g}$, $u(X) = -uXu^{-1}$.

The case $n = 0$ is trivial so we may assume that $n \geq 1$. If $T \in \mathcal{S}'(G \times V)^{\widetilde{G},\chi}$ in this section we will prove that the support of T must be contained in the "singular set".

Let Z (resp. $\mathfrak{z}$) be the center of G (resp. $\mathfrak{g}$) and $\mathcal{U}$ (resp. $\mathcal{N}$) the (closed) set of all unipotent (resp. nilpotent) elements of G (resp. $\mathfrak{g}$).

Lemma 5.1. *If $T \in \mathcal{S}'(G \times V)^{\widetilde{G},\chi}$ (resp. $T \in \mathcal{S}'(\mathfrak{g} \times V)^{\widetilde{G},\chi}$) then the support of T is contained in $Z\mathcal{U} \times V$ (resp. $(\mathfrak{z} + \mathcal{N}) \times V$).*

Proof. This is Harish-Chandra's descent. We first review some facts about the centralizers of semi-simple elements, following [SS70].

Let $a \in G$, semi-simple; we want to describe its centralizer G_a (resp. $\widetilde{G}_a$) in G (resp. in $\widetilde{G}$) and to show that $\mathcal{S}'(G_a \times V)^{\widetilde{G}_a,\chi} = (0)$.

View a as a $\mathbb{D}$-linear endomorphism of V and call P its minimal polynomial. Then, as a is semi-simple, P decomposes into distinct irreducible factors $P = P_1 \dots P_r$. Let $V_i = \operatorname{Ker} P_i(a)$ so that $V = \oplus V_i$. Any element x which commutes with a will satisfy $xV_i \subseteq V_i$ for each i. For

$$R(\xi) = d_0 + \cdots + d_m \xi^m, \quad d_0 d_m \neq 0$$

let

$$R^*(\xi) = \overline{d_0}\xi^m + \cdots + \overline{d_m}.$$

Then, from $aa^* = 1$ we obtain, if m is the degree of P

$$\langle P(a)v, v' \rangle = \langle v, a^{-m} P^*(a) v' \rangle$$

(note that the constant term of P can not be 0 because a is invertible). It follows that $P^*(a) = 0$ so that P^* is proportional to P. Now $P^* = P_1^* \dots P_r^*$; hence there exists a bijection τ from $\{1, 2, \dots, r\}$ onto itself such that P_i^* is proportional to $P_{\tau(i)}$. Let m_i be the degree of P_i. Then, for some non-zero constant c

$$0 = \langle P_i(a) v_i, v_j \rangle = \langle v_i, a^{-m_i} P_i^*(a) v_j \rangle = c \langle v_i, a^{-m_i} P_{\tau(i)}(a) v_j \rangle, \quad v_i \in V_i, \ v_j \in V_j.$$

We have two possibilities.

Case 1: $\tau(i) = i$. The space V_i is orthogonal to V_j for $j \neq i$; the restriction of the hermitian form to V_i is non-degenerate. Let $\mathbb{D}_i = \mathbb{D}[\xi]/(P_i)$ and consider V_i as a vector space over $\mathbb{D}_i$ through the action $(R(\xi), v) \mapsto R(a)v$. As $a_{|V_i}$ is invertible, ξ is invertible modulo (P_i); choose η such that $\xi\eta = 1$ modulo (P_i). Let σ_i be the semi-linear involution of $\mathbb{D}_i$, as an algebra over $\mathbb{D}$:

$$\sum d_j \xi^j \mapsto \sum \overline{d_j} \eta^j \pmod{P}_i$$

Let $\mathbb{F}_i$ be the subfield of fixed points for σ_i. It is a finite extension of $\mathbb{F}$, and $\mathbb{D}_i$ is either a quadratic extension of $\mathbb{F}_i$ or equal to $\mathbb{F}_i$. There exists a $\mathbb{D}$-linear form $\ell \neq 0$ on $\mathbb{D}_i$ such that $\ell(\sigma_i(d)) = \overline{\ell(d)}$ for all $d \in \mathbb{D}_i$. Then any $\mathbb{D}$-linear form L on $\mathbb{D}_i$ may be written as $d \mapsto \ell(\lambda d)$ for some unique $\lambda \in \mathbb{D}_i$.

If $v, v' \in V_i$ then $d \mapsto \langle d(a)v, v' \rangle$ is $\mathbb{D}$-linear map on $\mathbb{D}_i$; hence there exists $S(v, v') \in \mathbb{D}_i$ such that

$$\langle d(a)v, v' \rangle = \ell(dS(v, v')).$$

One checks that S is a non-degenerate hermitian form on V_i as a vector space over $\mathbb{D}_i$. Also a $\mathbb{D}$-linear map x_i from V_i into itself commutes with a_i if and only if it is $\mathbb{D}_i$-linear and it is unitary with respect to our original hermitian form if and only if it is unitary with respect to S. So in this case we call G_i the unitary group of S. It does not depend upon the choice of ℓ. As no confusion may arise, for $\lambda \in \mathbb{D}_i$ we define $\overline{\lambda} = \sigma_i(\lambda)$.

We choose an $\mathbb{F}_i$-linear map u_i from V_i onto itself, such that $u_i(\lambda v) = \overline{\lambda} u(v)$ and $S(u_i(v), u_i(v')) = \overline{S(v, v')}$. Then because of our original choice of ℓ we also have $\langle u_i(v), u_i(v') \rangle = \overline{\langle v, v' \rangle}$. Note that $u(a_{|V_i})^{-1} u^{-1} = a_{|V_i}$.

Case 2. Suppose now that $j = \tau(i) \neq i$. Then $V_i \oplus V_j$ is orthogonal to V_k for $k \neq i, j$ and the restriction of the hermitian form to $V_i \oplus V_j$ is non-degenerate, both V_i and V_j being totally isotropic subspaces. Choose an inverse η of ξ modulo P_j. Then for any $P \in \mathbb{D}[\xi]$

$$\langle P(a)v_i, v_j\rangle = \langle v_i, \overline{P}(\eta(a))v_j\rangle, \quad v_i \in V_i,\ v_j \in V_j$$

where $\overline{P}$ is the polynomial obtained from P by conjugating its coefficients. This defines a map, which we call σ_i from $\mathbb{D}_i$ onto $\mathbb{D}_j$. In a similar way we have a map σ_j which is the inverse of σ_i. Then, for $\lambda \in \mathbb{D}_i$ we have $\langle \lambda v_i, v_j\rangle = \langle v_i, \sigma_i(\lambda)v_j\rangle$.

View V_i as a vector space over $\mathbb{D}_i$. The action

$$(\lambda, v_j) \mapsto \sigma_i(\lambda)v_j$$

defines a structure of $\mathbb{D}_i$ vector space on V_j. However note that for $\lambda \in \mathbb{D}$ we have $\sigma_i(\lambda) = \overline{\lambda}$ so that $\sigma_i(\lambda)v_j$ may be different from λv_j. To avoid confusion we shall write, for $\lambda \in \mathbb{D}_i$

$$\lambda v_i = \lambda * v_i \quad \text{and} \quad \sigma_i(\lambda)v_j = \lambda * v_j.$$

As in the first case choose a non-zero $\mathbb{D}$-linear form ℓ on $\mathbb{D}_i$. For $v_i \in V_i$ and $v_j \in V_j$ the map $\lambda \mapsto \langle \lambda * v_i, v_j\rangle$ is a $\mathbb{D}$-linear form on $\mathbb{D}_i$; hence there exists a unique element $S(v_i, v_j) \in \mathbb{D}_i$ such that, for all λ

$$\langle \lambda * v_i, v_j\rangle = \ell(\lambda S(v_i, v_j)).$$

The form S is $\mathbb{D}_i$- bilinear and non-degenerate so that we can view V_j as the dual space over $\mathbb{D}_i$ of the $\mathbb{D}_i$ vector space V_i.

Let $(x_i, x_j) \in \mathrm{End}_{\mathbb{D}}(V_i) \times \mathrm{End}_{\mathbb{D}}(V_j)$. They commute with (a_i, a_j) if and only if they are $\mathbb{D}_i$-linear. The original hermitian form will be preserved, if and only if $S(x_i v_i, x_j v_j) = S(v_i, v_j)$ for all v_i, v_j. This means that x_j is the inverse of the transpose of x_i. In this situation we define G_i as the linear group of the $\mathbb{D}_i$-vector space V_i.

Let u_i be a $\mathbb{D}_i$-linear bijection of V_i onto V_j. Then $u_i(av_i) = a^{-1}u_i(v_i)$ and $u_i^{-1}(av_j) = a^{-1}u_i^{-1}(v_j)$.

Recall that G_a is the centralizer of a in G. Then (G_a, V) decomposes as a "product", each "factor" being either of type (G_i, V_i) with G_i a unitary group (case 1) or $(G_i, V_i \times V_j)$ with G_i a general linear group (case 2). Gluing together the u_i (case 1) and the (u_i, u_i^{-1}) (case 2) we get an element $u \in \widetilde{G} \setminus G$ such that $ua^{-1}u^{-1} = a$ which means that it belongs to the centralizer of a in $\widetilde{G}$. Finally if $\widetilde{G}_a$ is the centralizer of a in $\widetilde{G}$ then $(\widetilde{G}_a, V)$ is imbedded into a product each "factor" being either of type $(\widetilde{G}_i, V_i)$ with G_i a unitary group (case 1) or $(\widetilde{G}_i, V_i \times V_j)$ with G_i a general linear group (case 2).

If a is not central then for each i the dimension of V_i is strictly smaller than n and from the result for the general linear group and the inductive assumption in the orthogonal or unitary case we conclude that $\mathcal{S}'(G_a \times V)^{\widetilde{G}_a, \chi} = (0)$.

Proof of Lemma 5.1 in the group case. Consider the map $g \mapsto P_g$ where P_g is the characteristic polynomial of g. It is a continuous map from G into the set of polynomials of degree at most n. Each non-empty fiber $\mathcal{F}$ is stable under G but also under $\widetilde{G} \setminus G$. Bernstein's localization principle tells us that it is enough to prove that $\mathcal{S}'(\mathcal{F} \times V)^{\widetilde{G}, \chi} = (0)$.

Now it follows from [SS70, chapter IV] that $\mathcal{F}$ contains only a finite number of semi-simple orbits; in particular the set of semi-simple elements $\mathcal{F}_s$ in $\mathcal{F}$ is closed. Let us use the multiplicative Jordan decomposition into a product of a semi-simple and a unipotent element. Consider the map θ from $\mathcal{F} \times V$ onto $\mathcal{F}_s$ which associates to (g, v) the semi-simple part g_s of g. This map is continuous (see the corresponding proof for GL) and commutes with the action of $\widetilde{G}$. In $\mathcal{F}_s$ each orbit γ is both open and closed therefore $\theta^{-1}(\gamma)$ is open and closed and invariant under $\widetilde{G}$. It is enough to prove that for each such orbit $\mathcal{S}'(\theta^{-1}(\gamma))^{\widetilde{G},\chi} = (0)$. By Frobenius descent (Theorem 2.2), if $a \in \gamma$ and is not central, this follows from the above considerations on the centralizer of such an a and the fact that $\theta^{-1}(a)$ is a closed subset of the centralizer of a in $\widetilde{G}$, the product of the set of unipotent element commuting with a by V. Now g_s is central if and only if g belongs to $Z\mathcal{U}$, hence the Lemma. For the Lie algebra the proof is similar, using the additive Jordan decomposition. □

Going back to the group if a is central we see that it suffices to prove that $\mathcal{S}'(\mathcal{U} \times V)^{\widetilde{G},\chi} = (0)$ and similarly for the Lie algebra it is enough to prove that $\mathcal{S}'(\mathcal{N} \times V)^{\widetilde{G},\chi} = (0)$.

Now the exponential map (or the Cayley transform) is a homeomorphism of $\mathcal{N}$ onto $\mathcal{U}$ commuting with the action of $\widetilde{G}$. Therefore it is enough to consider the Lie algebra case.

We now turn our attention to V. Let

$$\Gamma = \{v \in V | \langle v, v \rangle = 0\}$$

Proposition 5.2. *If $T \in \mathcal{S}'(\mathcal{N} \times V)^{\widetilde{G},\chi}$ then the support of T is contained in $\mathcal{N} \times \Gamma$.*

Proof. Let

$$\Gamma_t = \{v \in V \mid \langle v, v \rangle = t\}$$

Each Γ_t is stable by $\widetilde{G}$, hence, by Bernstein's localization principle, to prove that the support of T is contained in $\mathcal{N} \times \Gamma_0$ it is enough to prove that, for $t \neq 0$, $\mathcal{S}'(\mathcal{N} \times \Gamma_t)^{\widetilde{G},\chi} = (0)$.

By Witt's theorem the group G acts transitively on Γ_t. We can apply Frobenius descent to the projection from $\mathcal{N} \times \Gamma_t$ onto Γ_t. Fix a point $v_0 \in \Gamma_t$. The fiber is $\mathcal{N} \times \{v_0\}$. Let $\widetilde{G}_1$ be the centralizer of v_0 in $\widetilde{G}$. We have to show that $\mathcal{S}'(\mathcal{N})^{\widetilde{G}_1,\chi} = (0)$ and it is enough to prove that $\mathcal{S}'(\mathfrak{g})^{\widetilde{G}_1,\chi} = (0)$.

The vector v_0 is not isotropic so we have an orthogonal decomposition

$$V = \mathbb{D}v_0 \oplus V_1$$

with V_1 orthogonal to v_0. The restriction of the hermitian form to V_1 is non-degenerate and G_1 is identified with the unitary group of this restriction, and $\widetilde{G}_1$ is the expected semi-direct product with S_2. As a $\widetilde{G}_1$-module the Lie algebra $\mathfrak{g}$ is isomorphic to a direct sum

$$\mathfrak{g} \approx \mathfrak{g}_1 \oplus V_1 \oplus W$$

where $\mathfrak{g}_1$ is the Lie algebra of G_1 and W a vector space over $\mathbb{F}$ of dimension 0 or 1 and on which the action of $\widetilde{G}_1$ is trivial. The action on $\mathfrak{g}_1 \oplus V_1$ is the usual one so that, by induction, we know that $\mathcal{S}'(\mathfrak{g}_1 \oplus V_1)^{\widetilde{G}_1,\chi} = (0)$. This readily implies that $\mathcal{S}'(\mathfrak{g})^{\widetilde{G}_1,\chi} = (0)$. □

Summarizing: it remains to prove that $\mathcal{S}'(\mathcal{N} \times \Gamma)^{\widetilde{G},\chi} = (0)$.

6. End of the proof in the orthogonal and unitary cases

We keep our general notation. We have to show that a distribution on $\mathcal{N} \times \Gamma$ which is invariant under G is invariant under $\widetilde{G}$. To some extent the proof will be similar to the one we gave for the general linear group.

In particular we will use the fact that if T is such a distribution then its partial Fourier transform on V is also invariant under G. The Fourier transform on V is defined using the bilinear form

$$(v_1, v_2) \mapsto \langle v_1, v_2 \rangle + \langle v_2, v_1 \rangle$$

which is invariant under $\widetilde{G}$.

For $v \in V$ put

$$\varphi_v(x) = \langle x, v \rangle v, \quad x \in V.$$

It is a rank one endomorphism of V and $\langle \varphi_v(x), y \rangle = \langle x, \varphi_v(y) \rangle$.

Lemma 6.1.

(1) *In the unitary case, for $\lambda \in \mathbb{D}$ such that $\lambda = -\overline{\lambda}$ the map*

$$\nu_\lambda : \quad (X, v) \mapsto (X + \lambda\varphi_v, v)$$

is a homeomorphism of $[\mathfrak{g}, \mathfrak{g}] \times \Gamma$ onto itself which commutes with $\widetilde{G}$.

(2) *In the orthogonal case, for $\lambda \in \mathbb{F}$ the map*

$$\mu_\lambda : \quad (X, v) \mapsto (X + \lambda X\varphi_v + \lambda\varphi_v X, v)$$

is a homeomorphism of $[\mathfrak{g}, \mathfrak{g}] \times \Gamma$ onto itself which commutes with $\widetilde{G}$.

The proof is a trivial verification.

We now use the stratification of $\mathcal{N}$. Let us first check that a G-orbit is stable by $\widetilde{G}$. [1]

Choose a basis $e_1, \ldots, e_n$ of V such that $\langle e_i, e_j \rangle \in \mathbb{F}$; this gives a conjugation $u : v = \sum x_i e_i \mapsto \overline{v} = \sum \overline{x_i} e_i$ on V. If A is any endomorphism of V then $\overline{A}$ is the endomorphism $v \mapsto \overline{A(\overline{v})}$. The conjugation u is an element of $\widetilde{G} \setminus G$ and, as such, it acts on $\mathfrak{g} \times V$ by $(X, v) \mapsto (-uXu^{-1}, -u(v)) = (-\overline{X}, -\overline{v})$. In [MVW87, Chapter 4, Proposition 1-2] it is shown that for $X \in \mathfrak{g}$ there exists an $\mathbb{F}$-linear automorphism a of V such that $\langle a(x), a(y) \rangle = \overline{\langle x, y \rangle}$ (this implies that $a(\lambda x) = \overline{\lambda} x$) and such that $aXa^{-1} = -X$. Then $g = ua \in G$ and $gXg^{-1} = -\overline{X}$ so that $-\overline{X}$ belongs to the G-orbit of X. Note that $a \in \widetilde{G} \setminus G$ and as such acts as $a(X, v) = (X, -a(v))$; it is an element of the centralizer of X in $\widetilde{G} \setminus G$.

Let $\mathcal{N}_i$ be the union of all nilpotent orbits of dimension at most i. We shall prove, by descending induction on i, that the support of a distribution $T \in \mathcal{S}'(\mathcal{N} \times \Gamma)^{\widetilde{G},\chi}$ must be contained in $\mathcal{N}_i \times \Gamma$.

[1] In fact, we only need this for nilpotent orbits and this will be done later in an explicit way, using the canonical form of nilpotent matrices.

So now assume that $i \geq 0$ and that we already know that the support of any $T \in \mathcal{S}'(\mathcal{N} \times \Gamma)^{\widetilde{G},\chi}$ must be contained in $\mathcal{N}_i \times \Gamma$. Let $\mathcal{O}$ be a nilpotent orbit of dimension i; we have to show that the restriction of T to $\mathcal{O}$ is 0.

In the unitary case fix $\lambda \in \mathbb{D}$ such that $\lambda = -\overline{\lambda}$ and consider, for every $t \in \mathbb{F}$ the homeomorphism $\nu_{t\lambda}$; the image of T belongs to $\mathcal{S}'(\mathcal{N} \times \Gamma)^{\widetilde{G},\chi}$ so that the image of the support of T must be contained in $\mathcal{N}_i \times \Gamma$. If (X, v) belongs to this support this means that $X + t\lambda\varphi_v \in \mathcal{N}_i$.

If $i = 0$ so that $\mathcal{N}_i = \{0\}$ this implies that $v = 0$ so that T must be a multiple of the Dirac measure at the point $(0, 0)$ and hence is invariant under $\widetilde{G}$ so must be 0.

If $i > 0$ and $X \in \mathcal{O}$ then as $\mathcal{O}$ is open in $\mathcal{N}_i$, we get that, at least for $|t|$ small enough, $X + t\lambda\varphi_v \in \mathcal{O}$ and therefore $\lambda\varphi_v$ belongs to the tangent space $\mathrm{Im}\ \mathrm{ad}(X)$ of $\mathcal{O}$ at the point X. Define

$$Q(X) = \{v \in V | \varphi_v \in \mathrm{Im}\ \mathrm{ad}(X)\}, \quad X \in \mathcal{N}, \text{ (unitary case)}.$$

Then we know that the support of the restriction of T to $\mathcal{O}$ is contained in

$$\{(X, v) | X \in \mathcal{O}, v \in Q(X)\}$$

and the same is true for the partial Fourier transform of T on V.

In the orthogonal case for $i = 0$, the distribution T is the product of the Dirac measure at the origin of $\mathfrak{g}$ by a distribution T' on V. The distribution T' is invariant under G but the image of $\widetilde{G}$ in $\mathrm{End}(V)$ is the same as the image of G so that T' is invariant under $\widetilde{G}$ hence must be 0.

If $i > 0$ we proceed as in the unitary case, using μ_λ. We define

$$Q(X) = \{v \in V | X\varphi_v + \varphi_v X \in \mathrm{Im}\ \mathrm{ad}(X)\}, \quad X \in \mathcal{N}, \text{ (orthogonal case)}$$

and we have the same conclusion.

In both cases, for $i > 0$, fix $X \in \mathcal{O}$. We use Frobenius descent for the projection map $(Y, v) \mapsto Y$ of $\mathcal{O} \times V$ onto $\mathcal{O}$. Let C (resp. $\widetilde{C}$) be the stabilizer of X in G (resp. $\widetilde{G}$). We have a linear bijection of $\mathcal{S}'(\mathcal{O} \times \Gamma)^{\widetilde{G},\chi}$ onto $\mathcal{S}'(V)^{\widetilde{C},\chi}$.

Lemma 6.2. *Let $T \in \mathcal{S}'(V)^{\widetilde{C},\chi}$. If T and its Fourier transform are supported in $Q(X)$ then $T = 0$.*

Let us say that a nilpotent element X is nice if the above Lemma is true.

Suppose that we have a direct sum decomposition $V = V_1 \oplus V_2$ such that V_1 and V_2 are orthogonal. By restriction we get non-degenerate hermitian forms $\langle ., . \rangle_i$ on V_i. We call G_i the unitary group of $\langle ., . \rangle_i$, $\mathfrak{g}_i$ its Lie algebra and so on. Suppose that $X(V_i) \subseteq V_i$ so that $X_i = X_{|V_i}$ is a nilpotent element of $\mathfrak{g}_i$.

Lemma 6.3. *If X_1 and X_2 are nice so is X.*

Proof. We claim that $Q(X) \subseteq Q(X_1) \times Q(X_2)$. Indeed if

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \in \mathfrak{g}$$

then from

$$\langle A\begin{pmatrix}x_1\\x_2\end{pmatrix},\begin{pmatrix}y_1\\y_2\end{pmatrix}\rangle+\langle\begin{pmatrix}x_1\\x_2\end{pmatrix},A\begin{pmatrix}y_1\\y_2\end{pmatrix}\rangle=0$$

we get in particular

$$\langle A_{i,i}x_i,y_i\rangle+\langle x_i,A_{i,i}y_i\rangle=0$$

so that $A_{i,i}\in\mathfrak{g}_i$. Note that

$$[X,A]=\begin{pmatrix}[X_1,A_{1,1}] & *\\ * & [X_2,A_{2,2}]\end{pmatrix}.$$

If $v_i\in V_i$ and $v_j\in V_j$ we define $\varphi_{v_i,v_j}:V_i\mapsto V_j$ by $\varphi_{v_i,v_j}(x_i)=\langle x_i,v_i\rangle v_j$. Then, for $v=v_1+v_2$

$$\varphi_v=\begin{pmatrix}\varphi_{v_1,v_1} & \varphi_{v_2,v_1}\\ \varphi_{v_1,v_2} & \varphi_{v_1,v_2}\end{pmatrix}.$$

Therefore if, for $A\in\mathfrak{g}$ we have $\varphi_v=[X,A]$ then $\varphi_{v_i,v_i}=[X_i,A_{i,i}]$. This proves the assertion for the unitary case. The orthogonal case is similar.

The end of the proof is the same as the end of the proof of Lemma 4.2. □

Now in both orthogonal and unitary cases nilpotent elements have normal forms which are orthogonal direct sums of "simple" nilpotent matrices. This is precisely described in [SS70] IV 2-19 page 259. By the above Lemma it is enough to prove that each "simple" matrix is nice.

Unitary case. There is only one type to consider. There exists a basis $e_1,\dots,e_n$ of V such that $Xe_1=0$ and $Xe_i=e_{i-1}$, $i\geq 2$. The hermitian form is given by

$$\langle e_i,e_j\rangle=0 \text{ if } i+j\neq n+1,\quad \langle e_i,e_{n+1-i}\rangle=(-1)^{n-i}\alpha$$

with $\alpha\neq 0$. Note that $\overline{\alpha}=(-1)^{n-1}\alpha$. Suppose that $v\in Q(X)$; for some $A\in\mathfrak{g}$ we have $\lambda\varphi_v=XA-AX$. For any integer $p\geq 0$

$$\mathrm{Tr}(\lambda\varphi_vX^p)=\mathrm{Tr}(XAX^p-AX^{p+1})=0.$$

Now $\mathrm{Tr}(\varphi_vX^p)=\langle X^pv,v\rangle$ Let $v=\sum x_ie_i$. Hence

$$\langle X^pv,v\rangle=\sum_1^{n-p}x_{i+p}\langle e_i,v\rangle=\sum_1^{n-p}(-1)^{n-i}\alpha x_{i+p}\overline{x}_{n+1-i}=0.$$

For $p=n-1$ this gives $x_n\overline{x}_n=0$. For $p=n-2$ we get nothing new but for $p=n-3$ we obtain $x_{n-1}=0$. Going on, by an easy induction, we conclude that $x_i=0$ if $i\geq(n+1)/2$.

If $n=2p+1$ is odd put $V_1=\oplus_1^p\mathbb{D}e_i$, $V_0=\mathbb{D}e_{p+1}$ and $V_2=\oplus_{p+2}^{2p+1}\mathbb{D}e_i$. If $n=2p$ is even put $V_1=\oplus_1^p\mathbb{D}e_i$, $V_0=(0)$ and $V_2=\oplus_{p+1}^{2p}\mathbb{D}e_i$. In both cases we have $V=V_1\oplus V_0\oplus V_2$. We use the notation $v=v_2+v_0+v_1$

The distribution T is supported by V_1. Call δ_i the Dirac measure at 0 on V_i. Then we may write $T=U\otimes\delta_0\otimes\delta_2$ with $U\in\mathcal{S}'(V_1)$. The same thing must be true of the Fourier transform of T. Note that $\widehat{U}$ is a distribution on V_2, that $\widehat{\delta_2}$ is a Haar measure dv_1 on V_1 and that, for n odd $\widehat{\delta_0}$ is a Haar measure dv_0 on V_0. So we have $\widehat{T}=dv_1\otimes\widehat{U}$ if n is even and $\widehat{T} = dv_1 \otimes dv_0 \otimes \widehat{U}$ if n is odd. In the odd case this forces $T = 0$. In the even case, up to a scalar multiple the only possibility is $T = dv_1 \otimes \delta_2$.

Let

$$a : \sum x_i e_i \mapsto \sum (-1)^i \overline{x}_i e_i.$$

Then $a \in \widetilde{G} \setminus G$. It acts on $\mathfrak{g}$ by $Y \mapsto -aYa^{-1}$ and in particular $-aXa^{-1} = X$ so that $a \in \widetilde{C} \setminus C$. The action on V is given by $v \mapsto -a(v)$. It is an involution. The subspace V_1 is invariant and so dv_1 is invariant. This implies that T is invariant under $\widetilde{C}$ so it must be 0.

Orthogonal case. There are two different types of "simple" nilpotent matrices.

The first type is the same as the unitary case, with $\alpha = 1$ and thus n odd but now our condition is that $X\varphi_v + \varphi_v X = [X, A]$ for some $A \in \mathfrak{g}$. As before this implies that $\mathrm{Tr}(\varphi_v X^q) = 0$ but only for $q \geq 1$. Put $n = 2p+1$; we get $x_j = 0$ for $j > p+1$. Decompose V as before: $V = V_1 \oplus V_0 \oplus V_2$. Our distribution T is supported by the subspace $v_2 = 0$ so we write it $T = U \otimes \delta_2$ with $U \in \mathcal{S}'(V_1 \oplus V_0)$. This is also true for the distribution $\widehat{T}$ so we must have $U = dv_1 \otimes R$ with R a distribution on V_0. Finally $T = dv_1 \otimes R \otimes \delta_2$. Now $-\mathrm{Id} \in C$ and T is invariant under C so that R must be an even distribution. On the other end the endomorphism a of V defined by $a(e_i) = (-1)^{i-p-1} e_i$ belongs to C and $aXa^{-1} = -X$ and $u : (X, v) \mapsto (-X, -v)$ belongs to $\widetilde{G} \setminus G$. The product $a \times u$ of a and u in $\widetilde{G}$ belongs to $\widetilde{C} \setminus C$. Clearly T is invariant under $a \times u$ so that T is invariant under $\widetilde{C}$ so it must be 0.

The second type is as follows. We have $n = 2m$, an even integer and a decomposition $V = E \oplus F$ with both E and F of dimension m. We have a basis $e_1, \ldots, e_m$ of E and a basis $f_1, \ldots, f_m$ of F such that

$$\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$$

and

$$\langle e_i, f_j \rangle = 0 \,\mathrm{if}\, i + j \neq m+1 \quad \text{and} \quad \langle e_i, f_{m+1-i} \rangle = (-1)^{m-i}.$$

Finally X is such that $Xe_i = e_{i-1},\ Xf_i = f_{i-1}$.

Let ξ be the matrix of the restriction of X to E or to F. Write an element $A \in \mathfrak{g}$ as 2×2 matrix $A = (a_{i,j})$. Then

$$[X, A] = \begin{pmatrix} [\xi, a_{1,1}] & [\xi, a_{1,2}] \\ [\xi, a_{2,1}] & [\xi, a_{2,2}] \end{pmatrix}.$$

Suppose that $v \in Q(X)$ and let

$$v = e + f \quad \text{with} \quad e = \sum x_i e_i, \;\; f = \sum y_i f_i.$$

We get

$$X\varphi_v + \varphi_v X = \begin{pmatrix} \xi\varphi_{f,e} + \varphi_{f,e}\xi & \xi\varphi_{f,f} + \varphi_{f,f}\xi \\ \xi\varphi_{e,e} + \varphi_{e,e}\xi & \xi\varphi_{e,f} + \varphi_{e,f}\xi \end{pmatrix}$$

where, for example $\varphi_{e,e}$ is the map $f' \mapsto \langle f', e \rangle e$ from F into E. Thus, for some A,

$$\xi\varphi_{e,e} + \varphi_{e,e}\xi = \xi a_{2,1} - a_{2,1}\xi$$

In this formula, using the basis (e_i), (f_i) replace all the maps by their matrices.

Then, as before, we have $\mathrm{Tr}(\varphi_{e,e}\xi^q) = 0$ for $1 \le q \le m-1$. If $e' = \sum x_i f_i$ (the x_i are the coordinates of e), then $\mathrm{Tr}(\xi^q \varphi_{e,e})$ is $\langle \xi^q e, e' \rangle$. Thus, as in the other cases, we have $x_j = 0$ for $j > m/2$ if m is even and $j > (m+1)/2$ if m is odd. The same thing is true for the y_i.

If $m = 2p$ is even, let $V_1 = \oplus_{i \le p}(\mathbb{F}e_i \oplus \mathbb{F}f_i)$ and $V_2 = \oplus_{i>p}(\mathbb{F}e_i \oplus \mathbb{F}f_i)$; write $v = v_1 + v_2$ the corresponding decomposition of an arbitrary element of V. Let δ_2 be the Dirac measure at the origin in V_2 and dv_1 a Haar measure on V_1. Then, as in the unitary case, using the Fourier transform, we see that the distribution T must be a multiple of $dv_1 \otimes \delta_2$.

The endomorphism a of V defined by $a(e_i) = (-1)^i e_i$ and $a(f_i) = (-1)^{i+1} f_i$ belongs to G and $aXa^{-1} = -X$. The map $u : (Y, v) \mapsto (-Y, -v)$ belongs to $\widetilde{G} \setminus G$ so that the product $a \times u$ in $\widetilde{G}$ belongs to $\widetilde{C} \setminus C$. It clearly leaves T invariant so that $T = 0$.

Finally if $m = 2p+1$ is odd we put $V_1 = \oplus_{i \le p}(\mathbb{F}e_i \oplus \mathbb{F}f_i)$, $V_0 = \mathbb{F}e_{p+1} \oplus \mathbb{F}f_{p+1}$, $V_2 = \oplus_{i \ge p+2}(\mathbb{F}e_i \oplus \mathbb{F}f_i)$. As in the unitary case we find that $T = dv_1 \otimes R \otimes \delta_2$ with R a distribution on V_0. As $-\mathrm{Id} \in C$ we see that R must be even. Then again, define $a \in G$ by $a(e_i) = (-1)^i e_i$ and $a(f_i) = (-1)^i f_i$ and consider $a \times u$ with $u(Y, v) = (-Y, -v)$. As before $a \times u \in \widetilde{C} \setminus C$ and leaves T invariant so we have to take $T = 0$. □

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