HOMOLOGICAL MULTIPLICITIES IN REPRESENTATION THEORY OF p-ADIC GROUPS

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ABSTRACT. We study homological multiplicities of spherical varieties of reductive group G over a p-adic field F. Based on Bernstein's decomposition of the category of smooth representations of a p-adic group, we introduce a sheaf that measures these multiplicities.

We show that these multiplicities are finite whenever the usual mutliplicities are finite, in particular this holds for symmetric varieties, conjectured for all spherical varieties and known for a large class of spherical varieties. Furthermore, we show that the Euler-Poincaré characteristic is constant in families induced from admissible representations of a Levi M. In the case when $M = G$ we compute these multiplicities more explicitly.

CONTENTS

1. INTRODUCTION

Let $\mathbf{G}$ be a connected reductive group defined over a non-Archimedean local field F. Let X be an F-spherical transitive $G = \mathbf{G}(F)$-space. That is, a G-transitive space which has finitely many P_0 orbits, where P_0 is a minimal parabolic subgroup of G. Let $\mathcal{S}(X)$ be the space of Schwartz (i.e. locally constant compactly supported) functions on X. Harmonic analysis on X requires a deep understanding of the multiplicity spaces $\mathrm{Hom}_G(\mathcal{S}(X), \pi)$, where π is an admissible representation of G. A considerable amount of research was dedicated to the determination of the dimension of these multiplicity spaces (e.g. [GK75, Sha74, JR96, Pra90, vD08, AGRS10, AGS08, HM08, AG09a, AG09b, SZ12, GGP12, OS08b]) and to the construction of bases to these spaces (e.g. [vB88, OS08a, GSS15]).

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Assume that for any admissible representation π of G, the multiplicity

$$m_X(\pi) := \dim \mathrm{Hom}_G(\mathcal{S}(X), \pi)$$

is finite. This is known to be true in a number of cases (see [Del10, SV]), including the case of symmetric spaces, and conjectured to be true for any F-spherical space.

In this paper we study the homological version of those multiplicities:

$$m_X^i(\pi) = \dim \mathrm{Ext}_G^i(\mathcal{S}(X), \pi), \text{ for } i \geq 0.$$

The first result of this paper is the following:

Theorem A (See Theorem 3.1 for a more general result)**.** *Let π be an admissible representation of G. Under the assumption above, the homological multiplicities $m_X^i(\pi)$ are finite for every $i \geq 0$.*

Special cases of this result were studied in [Pra, §5] using a different method.

The category $\mathcal{M}(G)$ of smooth representations of G is of finite homological dimension ([BR, Theorem 29]). Thus, we can define

$$e_X(\pi) := EP_G(\mathcal{S}(X), \pi) := \sum (-1)^i \dim \mathrm{Ext}_G^i(\mathcal{S}(X), \pi).$$

More generally we will consider

$$p_X(\pi) := P_G(\mathcal{S}(X), \pi) := \sum \dim \mathrm{Ext}_G^i(\mathcal{S}(X), \pi) t^i \in \mathbb{Z}[t].$$

The second result of this paper is the following:

Theorem B (See Theorem 5.3 for more general result)**.** *Let $\mathbf{M} \subset \mathbf{G}$ be a Levi subgroup of $\mathbf{G}$. Let ρ be an irreducible cuspidal representation of $M = \mathbf{M}(F)$. For an unramified character χ of M, let $\pi_\chi := i_M^G(\rho \cdot \chi)$ be the normalized parabolic induction. Then $e_X(\pi_\chi)$ is independent of χ.*

In the case where $\mathbf{M} = \mathbf{G}$ we also prove:

Theorem C (See §6)**.** *Let ρ be an irreducible cuspidal representation of G. Let $\mathfrak{X}_G$ be the torus of unramified characters of G. Then there is a smooth subvariety $\mathfrak{X}' \subset \mathfrak{X}_G$ such that*

$$p_X(\chi\rho) = \begin{cases} 0 & \text{if } \chi \notin \mathfrak{X}' \\ (1+t)^{\dim \mathfrak{X}_G - \dim \mathfrak{X}'} & \text{if } \chi \in \mathfrak{X}' \end{cases}$$

In order to prove Theorems B and C above we introduce, for any cuspidal data (i.e. a Levi subgroup $\mathbf{M} < \mathbf{G}$ and an irreducible cuspidal representation ρ of $M = \mathbf{M}(F)$) a coherent sheaf $\mathcal{F}_X((M, \rho))$ over the torus $\mathfrak{X}_M$ of unramified characters of M. We prove:

Theorem D (See Theorem 4.3 and Theorem 6.1 for more general results)**.**

(1) In the notations of Theorem B we have $m_X^i(\pi_\chi) = \dim \mathrm{Ext}^i_{O_{\mathfrak{X}_M}}(\mathcal{F}_X((M,\rho)), \delta_\chi)$, where δ_χ is the skyscraper sheaf at $\chi \in \mathfrak{X}_M$ and $O_{\mathfrak{X}_M}$ is the sheaf of regular functions on the torus of unramified characters $\mathfrak{X}_M$

(2) In the notations of Theorem C we have $\mathcal{F}_X((G, \rho)) = i_*(\mathcal{L})$, where X' is as above and $i : \mathfrak{X}' \to \mathfrak{X}_G$ is the embedding and $\mathcal{L}$ is a locally free sheaf over $\mathfrak{X}'$

Infact, for every $V \in \mathcal{M}(G)$ we introduce a quasi-coherent sheaf $\mathcal{F}_V((M, \rho))$ which coincide with $\mathcal{F}_X((M, \rho))$ when $V = \mathcal{S}(X)$. We prove analogous results for this sheaf.

In Appendix B we prove a generalization of theorem B where we replace the cuspidal representation ρ with a general admissible representation τ. For this we need to generalize the construction of the sheaf $\mathcal{F}_X$ on $\mathfrak{X}(M)$ to the case when the representation of M is not cuspidal. We are thus led to introduce an object of the derived category of sheaves on $\mathfrak{X}(M)$, denoted $\mathcal{G}_V(M, \tau)$, that plays a similar role to the that of $\mathcal{F}_V((M, \rho))$.

Theorem E (See Appendix B for a more general result)**.** *Let $\mathbf{M} \subset \mathbf{G}$ be a Levi subgroup of $\mathbf{G}$. Let τ be an admissible representation of $M = \mathbf{M}(F)$, and let χ be an unramified character of M. Let $V \in \mathcal{M}(G)$. Then the following holds:*

(1) $\mathrm{RHom}_G(V, i_M^G(\chi\tau)) \cong \mathrm{RHom}_{O(\mathfrak{X}_M)}(\mathcal{G}_V(M,\tau), \delta_\chi)$.
(2) If τ is cuspidal then $\mathcal{G}_V(M,\tau) \cong \mathcal{F}_V((M,\tau))$.
(3) If $V = \mathcal{S}(X)$ then $\mathcal{G}_V(M,\tau)$ is perfect.
(4) $e_X(i_M^G(\chi\tau))$ is independent of χ.

Our approach to Theorem E is based on a slight modification of the the notion of multiplicities. Namely we replace Hom by the tensor product. The relation between the two notions of multiplicity is given in Corollary A.2

1.1. **Structure of the paper.** In §2 we give a brief overview on homological multiplicities and recall few properties of the Bernstein decomposition and Bernstein center. In §3 we prove Theorem A. In §4 we prove Theorem D(1). In §5 we prove Theorem B. In §6 we prove theorem D(2) and deduce Theorem C. In Appendix §A we study the relation between tensor multiplicity and usual multiplicity. In Appendix §B we introduce the object $\mathcal{G}_V(M,\tau)$, and prove Theorem E.

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2. Preliminaries

2.1. **Homological multiplicities.** Fix $H \subset G$ a closed subgroup. As in the case of multiplicity spaces, one can define homological multiplicity in several ways. The relation between them is given by the following:

Proposition 2.1 ([Cas],[Pra, Proposition 2.5])**.** *Let π be an admissible representation of G and η a character of H. Then*

$$\mathrm{Ext}^i_H(\pi|_H, \eta) = \mathrm{Ext}^i_G(\pi, Ind_H^G(\eta)) = \mathrm{Ext}^i_G(ind_H^G(\eta'), \tilde{\pi}),$$

where $\eta' = \eta^{-1}\Delta_H^{-1}$ and Δ_H is the modulus character of H.

Following this proposition we introduce some notation:

Definition 2.2 (Homological Multiplicities)**.** *In the setting of the proposition above we define*

$$\widetilde{m}^i_{(H,\eta)}(\pi) := \dim \mathrm{Ext}^i_H(\pi,\eta) \text{ and } m^i_{(H,\eta)}(\pi) := \dim \mathrm{Ext}^i_G(ind_H^G(\eta),\pi).$$

If η is trivial, we will omit it from the notation.

To relate those notions to the ones introduced in [Pra] we need:

Proposition 2.3. *Let π_1 be a smooth representation of G and π_2 be an admissible representation of H. Then $\mathrm{Ext}^i_H(\pi_1,\pi_2) \cong \mathrm{Ext}^i_H(\pi_1 \otimes \tilde{\pi}_2, \mathbb{C})$. In particular we have*

$$\dim \mathrm{Ext}^i_H(\pi_1,\pi_2) = \tilde{m}^i_H(\pi_1 \otimes \tilde{\pi}_2).$$

Proof.

Step 1 Proof for $i = 0$.

We have $\mathrm{Hom}_{\mathbb{C}}(\pi_1, \tilde{\pi}_2^*) \cong Bil(\pi_1, \tilde{\pi}_2; \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(\pi_1 \otimes \tilde{\pi}_2, \mathbb{C})$. We get

$$\mathrm{Hom}_H(\pi_1,\pi_2) \cong \mathrm{Hom}_H(\pi_1, \tilde{\tilde{\pi}}_2) = \mathrm{Hom}_H(\pi_1, \tilde{\pi}_2^*) \cong \mathrm{Hom}_H(\pi_1 \otimes \tilde{\pi}_2, \mathbb{C}).$$

Step 2 Proof for the general case.
To deduce the general case from the previous step we need to show that there are enough projective representations of G that stay projective after restrictions to H. This follows from the fact that $\mathcal{S}(G) \otimes V$ is a projective representation of G for any vector space V (See [Cas], [Pra, Propositions 2.5,2.6] for more details).

□

2.2. **Bernstein center.** We summarize the parts of the theory of Bernstein center and Bernstein decomposition that are used in this paper. We need few notations.

Notation 2.4. *Let* $\mathbf{P}$ *be a parabolic subgroup of* $\mathbf{G}$ *and* $\mathbf{M}$ *be its Levi factor. Let* ρ *be a cuspidal representation of* $M = \mathbf{M}(F)$*. Denote*

- $i_M^G : \mathcal{M}(M) \to \mathcal{M}(G)$ *the (normalized) parabolic induction of* M *w.r.t.* P
- $\bar{i}_M^G(\rho) : \mathcal{M}(M) \to \mathcal{M}(G)$ *the (normalized) parabolic induction from* M *w.r.t. an opposite parabolic subgroup* $\bar{\mathbf{P}}$*.*
- $r_M^G : \mathcal{M}(G) \to \mathcal{M}(M)$ *the (normalized) Jacquet functor.*

Given $M \subset G$, the objects defined above depend on the choice of a parabolic $P \subset G$. However, unless stated otherwise the result will not depend on this choice and we will ignore it.

Theorem 2.5 (Bernstein)**.**

(1) *There exists a local base* $\mathcal{B}$ *of the topology at the unit* $e \in G$*, such that every* $K \in \mathcal{B}$ *is a compact (open) subgroup satisfying the following:*
   (a) *The category* $\mathcal{M}(G, K)$ *of representations generated by their* K*-fixed vectors is a direct summand of* $\mathcal{M}(G)$*.*
   (b) *The functor* $V \to V^K$ *is an equivalence of categories from* $\mathcal{M}(G, K)$ *to the category of modules over the Hecke algebra* $\mathcal{H}(G, K)$*.*
   (c) *The algebra* $\mathcal{H}(G, K)$ *is Noetherian.*
   (d) *For any Levi subgroup* $M \subset G$ *and for any* $V \in \mathcal{M}(G)$ *the map*
   $$V^K \to r_M^G(V)^{K \cap M}$$
   is onto.

   We will call an open compact subgroup K *satisfing these properties, a splitting subgroup.*
(2) *The functor* r_M^G *is right adjoint to* $\bar{i}_M^G$*, that is*
$$\mathrm{Hom}_G(\bar{i}_M^G(V), W) \cong \mathrm{Hom}_M(V, r_M^G(W)).$$
(3) *For a cuspidal data* (M, ρ) *let* $\Psi_G(M, \rho) = \bar{i}_M^G(\rho \otimes O(\mathfrak{X}_M))$ *be the normalized parabolic induction of* $\rho \otimes O(\mathfrak{X}_M)$ *where the action of* M *is diagonal. Then* $\Psi_G(M, \rho) \in \mathcal{M}(G)$ *is a projective generator of a direct summand of the category* $\mathcal{M}(G)$*.*
(4) *Let* ρ *be an irreducible cuspidal representation of* G*, and set* $\mathfrak{I}_\rho := \{\psi \in \mathfrak{X}_G | \psi\rho \simeq \rho\}$ *and embed* $\mathcal{O}(\mathfrak{X}_G)$ *in* $\mathcal{R}_{(G,\rho)} := \mathrm{End}(\Psi_G(G, \rho))$*. Then there exists a decomposition*
$$\mathcal{R}_{(G,\rho)} = \bigoplus_{\psi \in \mathfrak{I}_\rho} O(\mathfrak{X}_G)\nu_\psi,$$
such that $\nu_\psi f = f_\psi \nu_\psi$ *and* $\nu_\psi \nu_{\psi'} = c_{\psi,\psi'} \nu_{\psi\psi'}$ *where* f_ψ *is the translation of* $f \in O(\mathfrak{X}_G)$ *by* ψ *and* $c_{\psi,\psi'}$ *are scalars.*

For completeness we supply exact references:

- For statement(1) See [BD84, Corollary 3.9, Proposition 2.10 and its proof, Corollary 3.4, the proof of Theorem 2.13 and Proposition 3.5.2].
- For statement (2) see [Ber87].

- Statement (3) follows from [BR, Propositions 34,35] and [BD84, Proposition 2.10].
- For statement (4) see [BR, Proposition 28].

3. Finiteness of Homological multiplicities

In this section we prove a generalization of Theorem A. Namely,

Theorem 3.1. *Assume that G/H is an F-spherical G-variety. Let η be a character of H. Assume that for any irreducible representation (π, V) we have $m^0_{(H,\eta)}(\pi) < \infty$. Then for any admissible representation π and any natural j we have $m^j_{(H,\eta)}(\pi) < \infty$.*

We will say the the pair (G, H) and the G-space G/H is of **finite type**, if the conditions of Theorem 3.1 are satisfied for any η. As mentioned earlier, it is conjectured that any F-spherical pair is of finite type and it is known in many cases.

For the proof we recall the following facts:

Theorem 3.2 ([AAG11], [AGS15, Appendix B])**.** *Assume that G/H is an F-spherical G-variety and that for any irreducible representation (π, V) we have $m^0_{(H,\eta)}(\pi) < \infty$. Let K be an open subgroup of G. Then the module $ind^G_H(\eta)^K$ is finitely generated over $\mathcal{H}(G, K)$.*

Proof of Theorem 3.1. Let K be such that V^K generates V. By Theorem 2.5(1) we can assume that K is a splitting subgroup.

Let W be the sub-representation of $ind^G_H(\eta)$ generated by K fixed vectors. We have

$$m^i_{(H,\eta)}(\pi) = \dim \mathrm{Ext}^i_G(ind^G_H(\eta), \pi) = \dim \mathrm{Ext}^i_G(W, \pi) = \dim \mathrm{Ext}^i_{\mathcal{H}(G,K)}(ind^G_H(\eta)^K, \pi^K).$$

The fact that $\mathcal{H}(G, K)$ is Noetherian and Theorem 3.2 imply that the module $ind^G_H(\eta)^K$ has a resolution by finitely generated free $\mathcal{H}(G, K)$-module. Thus

$$\dim \mathrm{Ext}^i_{\mathcal{H}(G,K)}(ind^G_H(\eta)^K, \pi^K) < \infty.$$

□

4. Homological multiplicities and the multiplicity sheaf

In this section we study the homological multiplicities of cuspidally induced representations.

Definition 4.1. *Let M, ρ as Theorem 2.5(3). Let $V \in \mathcal{M}(G)$. We define*

$$\mathcal{F}_V((M, \rho)) := \mathrm{Hom}_G(\Psi_G(M, \rho), V),$$

This is a module over $\mathrm{End}_G(\Psi_G(M, \rho))$ and thus a module over $\mathcal{R}_{(M,\rho)} = \mathrm{End}_M(\Psi_M(M, \rho))$. In particular, it is a module over $\mathcal{O}(\mathfrak{X}_M)$. We shall view it as a quasi-coherent sheaf over the space $\mathfrak{X}_M$ of unramified characters of M.

Notation 4.2. *Let P, M, ρ as above.*

- *When $V = ind^G_H(\eta)$ for a character η of a subgroup $H \subset G$ we denote $\mathcal{F}_{H,\eta}((M, \rho)) := \mathcal{F}_V((M, \rho))$.*
- *When $V = \mathcal{S}(X, \mathcal{L})$ for a G-equivariant sheaf $\mathcal{L}$ on X we will denote $\mathcal{F}_{X,\mathcal{L}}((M, \rho)) := \mathcal{F}_V((M, \rho))$.*
- *If η or $\mathcal{L}$ are trivial we will omit them from the notation.*

The following generalizes Theorem D(1) of the introduction.

Theorem 4.3. *Let $V \in \mathcal{M}(G)$ Then,*

$$\mathrm{Ext}^i_G(V, i^G_M(\chi\rho)) \cong \mathrm{Ext}^i_{O(\mathfrak{X}_M)}(\mathcal{F}_V((M, \rho)), \delta_\chi),$$

where δ_χ is the skyscraper sheaf over $\chi \in \mathfrak{X}_M$.

For the proof we need the following standard lemma:

Lemma 4.4. *Let Z be a commutative finitely generated algebra over $\mathbb{C}$ without nilpotent elements. Let A be an algebra over Z and let M be an A-module. Let $\operatorname{Spec}(C) \to \operatorname{Spec}(Z)$ be an étale map. Let Δ be an irreducible A-module with annihilator $\mathfrak{m}$. Let $\mathfrak{m}'$ be a maximal ideal of C lying over $\mathfrak{m}$. Let $\Delta' := (\Delta \otimes_Z C)/\mathfrak{m}'$ considered as a $A \otimes_Z C$-module. Then*

$$\operatorname{Ext}^*_A(M, \Delta) \cong \operatorname{Ext}^*_{A\otimes_Z C}(M \otimes_Z C, \Delta').$$

Proof. Since C is flat over Z we have isomorphism of C modules

$$\operatorname{Ext}^*_{A\otimes_Z C}(M \otimes_Z C, \Delta \otimes_Z C) \cong \operatorname{Ext}^*_A(M, \Delta) \otimes_Z C.$$

Since $\mathfrak{m}$ annihilates Δ and $\operatorname{Ext}^*_A(M, \Delta)$, we have

$$\Delta \otimes_Z C \cong \Delta \otimes_{Z/\mathfrak{m}} C/\mathfrak{m} \text{ and } \operatorname{Ext}^*_A(M, \Delta) \otimes_Z C \cong \operatorname{Ext}^*_A(M, \Delta) \otimes_{Z/\mathfrak{m}} C/\mathfrak{m}.$$

Thus, we obtain

$$\operatorname{Ext}^*_{A\otimes_Z C}(M \otimes_Z C, \Delta \otimes_{Z/\mathfrak{m}} C/\mathfrak{m}) \cong \operatorname{Ext}^*_A(M, \Delta) \otimes_{Z/\mathfrak{m}} C/\mathfrak{m}.$$

Since $C/\mathfrak{m}'$ is a direct summand of $C/\mathfrak{m}$ we obtain

$$\operatorname{Ext}^*_{A\otimes_Z C}(M \otimes_Z C, \Delta \otimes_{Z/\mathfrak{m}} C/\mathfrak{m}') \cong \operatorname{Ext}^*_A(M, \Delta) \otimes_{Z/\mathfrak{m}} C/\mathfrak{m}'.$$

This implies the assertion. □

We will also need the following proposition:

Proposition 4.5. *Let ρ be a an irreducible cuspidal representation of G, recall that*

$$\mathfrak{I}_\rho := \{\chi \in \mathfrak{X}_G | \chi\rho \simeq \rho\}.$$

Then there exists an onto étale map $\operatorname{Spec}(C) \to \operatorname{Spec}(Z(\mathcal{R}_{(G,\rho)}))$ such that the triple

$$C \subset C \otimes_{Z(\mathcal{R}_{(G,\rho)})} O(\mathfrak{X}_G) \subset C \otimes_{Z(\mathcal{R}_{(G,\rho)})} \mathcal{R}_{(G,\rho)}$$

is isomorphic to

$$C \subset \left\{ \left. \begin{pmatrix} f_1 & & \\ & \ddots & \\ & & f_n \end{pmatrix} \right| f_i \in C \right\} \subset Mat_n(C).$$

Proof. By Theorem 2.5(4) we have

$$\mathcal{R}_{(G,\rho)} = \bigoplus_{\psi \in \mathfrak{I}_\rho} O(\mathfrak{X}_G)\nu_\psi,$$

Thus the center of $\mathcal{R}_{(G,\rho)}$ is $\mathcal{O}(\mathfrak{X}_G/\mathfrak{I}_\rho) \subset \mathcal{O}(\mathfrak{X}_G)$. Let $C = O(\mathfrak{X}_G)$. Let

$$\phi : \ O(\mathfrak{X}_G) \otimes_{O(\mathfrak{X}_G/\mathfrak{I}_\rho)} O(\mathfrak{X}_G) \to O(\mathfrak{X}_G \times \mathfrak{I}_\rho)$$

be the map given by $\phi(f_1 \otimes f_2)(x, j) = f_1(x)f_2(jx)$. It is easy to see that ϕ is an isomorphism. This yields an identification

$$C \otimes_{Z(\mathcal{R}_{(G,\rho)})} \mathcal{R}_{(G,\rho)} \cong \bigoplus_{\psi \in \mathfrak{I}_\rho} O(\mathfrak{X}_G \times \mathfrak{I}_\rho)\nu_\psi.$$

Define a map $\mu : \bigoplus_{\psi \in \mathfrak{I}_\rho} O(\mathfrak{X}_G \times \mathfrak{I}_\rho)\nu_\psi \to \operatorname{End}_{O(\mathfrak{X}_G)}(O(\mathfrak{X}_G \times \mathfrak{I}_\rho))$ by

$$\mu(\nu_\psi)(f)(\chi, \psi') = c_{\psi,\psi'} f(\chi, \psi^{-1}\psi') \tag{1}$$

and for $g \in O(\mathfrak{X}_G \times \mathfrak{I}_\rho)$

$$\mu(g)(f)(\chi, \psi) = g(\chi, \psi)f(\chi, \psi). \tag{2}$$

It remains to show that μ is an isomorphism. Fixing a point $\chi \in \mathfrak{X}_G$, Burnside's theorem on matrix algebras implies that the specialization

$$\mu|_\chi : \left. \left(C \otimes_{Z(\mathcal{R}_{(G,\rho)})} \mathcal{R}_{(G,\rho)} \right) \right|_\chi \to \operatorname{End}(\mathbb{C}[\mathfrak{I}_\rho])$$

is onto, and hence an isomorphism. This implies that μ is isomorphism.

□

Proof of Theorem 4.3. We will compute the left and the right hand side of the desired isomorphism and see that they are the same. We start with the left hand side.

Deriving the Frobenius reciprocity we have

$$\mathrm{Ext}^i_G(V, i^G_M(\chi\rho)) = \mathrm{Ext}^i_M(r^G_M(V), \chi\rho).$$

By Theorem 2.5 (3)

$$\mathrm{Ext}^i_M(r^G_M(V), \chi\rho) = \mathrm{Ext}^i_{\mathcal{R}_{(M,\rho)}}(\mathrm{Hom}_M(\Psi_M(M,\rho), r^G_M(V)), \mathrm{Hom}_M(\Psi_M(M,\rho), \chi\rho)),$$

Note that by the second adjointness theorem (Theorem 2.5 (2)) we have

$$\mathrm{Hom}_M(\Psi_M(M,\rho), r^G_M(V)) = \mathcal{F}_V((M,\rho)).$$

Furthermore, $\Delta := \mathrm{Hom}_M(\Psi_M(M,\rho), \chi\rho)$ is an irreducible module over $\mathcal{R}_{(M,\rho)}$. Let C be as in Proposition 4.5 (applied for the group M). Let Δ' be a $\mathcal{R}_{(M,\rho)}{}' := \mathcal{R}_{(M,\rho)} \otimes_{Z(\mathcal{R}_{(M,\rho)})} C$-module which coincides with Δ as a $Z(\mathcal{R}_{(M,\rho)})$-module. By Lemma 4.4 we have

$$\mathrm{Ext}^i_{\mathcal{R}_{(M,\rho)}}(\mathcal{F}_V((M,\rho)), \Delta) \cong \mathrm{Ext}^i_{\mathcal{R}_{(M,\rho)}{}'}(\mathcal{F}_V((M,\rho)) \otimes_{Z(\mathcal{R}_{(G,\rho)})} C, \Delta').$$

Identify $\mathcal{R}_{(M,\rho)}{}' \cong Mat_n(C)$ as in Proposition 4.5. We get that there is a C-module T such that $\mathcal{F}_V((M,\rho)) \otimes_{Z(\mathcal{R}_{(M,\rho)})} C = T^n$. This implies that

$$\mathrm{Ext}^i_{\mathcal{R}_{(M,\rho)}{}'}(\mathcal{F}_V((M,\rho)) \otimes_{Z(\mathcal{R}_{(M,\rho)})} C, \Delta') \cong \mathrm{Ext}^i_C(T, \delta_{\chi'}),$$

where χ' is the character with which C acts on Δ'.

To compute the right hand side we apply Lemma 4.4 again and obtain

$$\mathrm{Ext}^i_{O(\mathfrak{X}_M)}(\mathcal{F}_V((M,\rho)), \delta_\chi) \cong \mathrm{Ext}^i_{O(\mathfrak{X}_M)\otimes_{Z(\mathcal{R}_{(M,\rho)})}C}(\mathcal{F}_V((M,\rho)) \otimes_{Z(\mathcal{R}_{(M,\rho)})} C, \delta_{\chi''}),$$

where χ'' is a character of $O(\mathfrak{X}_M) \otimes_{Z(\mathcal{R}_{(M,\rho)})} C$ whose restriction to C is χ'. Since $\mathcal{F}_V((M,\rho)) \otimes_{Z(\mathcal{R}_{(G,\rho)})} C = T^n$ we get

$$\mathrm{Ext}^i_{O(\mathfrak{X}_M)\otimes_{Z(\mathcal{R}_{(M,\rho)})}C}(\mathcal{F}_V((M,\rho)) \otimes_{Z(\mathcal{R}_{(M,\rho)})} C, \delta_{\chi''}) \cong \mathrm{Ext}^i_C(T, \delta_{\chi'}).$$

□

5. The Euler Characteristics

Recall that the homological dimension of the abelian category $\mathcal{M}(G)$ is finite (See [BR, Theorem 29]). This allows us to give the following notation:

Notation 5.1 (cf. [Pra])**.** *Let $\pi, \tau \in \mathcal{M}(G)$. Define*

$$EP_G(\pi, \tau) = \sum (-1)^i \dim \mathrm{Ext}^i_G(\pi, \tau).$$

In order to formulate the main result of this section we need:

Definition 5.2. *We will call $V \in \mathcal{M}(G)$ locally finitely generated if for any compact open $K < G$ the module V^K is finitely generated over $\mathcal{H}(G, K)$.*

The following is a generalization of Theorem B.

Theorem 5.3. *Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ and $\mathbf{M}$ be its Levi factor. Let ρ be an irreducible cuspidal representation of $M = \mathbf{M}(F)$ and $\chi \in \mathfrak{X}_M$ be an unramified character of M. Let $V \in \mathcal{M}(G)$ locally finitely generated representation.*

Then $EP_G(V, ind^G_M(\chi \cdot \rho))$ is constant as a function of χ.

This theorem follows from Theorem 4.3 and the following propositions:

Proposition 5.4. *Let $\mathcal{F}$ be a coherent sheaf over a (complex) smooth algebraic variety X. Then the function $x \mapsto \sum (-1)^i \dim \operatorname{Ext}^i_{O_X}(\mathcal{F}, \delta_x)$ is a locally constant function on X.*

Proof. Since X have finite homological dimension, it is enough to prove the proposition for localy free $\mathcal{F}$. In this case the proposition is trivial. □

Proposition 5.5. *Let $V \in \mathcal{M}(G)$ be a locally finitely generated representation and let (M, ρ) be a cuspidal data. Then $\mathcal{F}_V((M, \rho))$ is coherent.*

Proof. Denote the category generated by $\Psi_G(M, \rho)$ by $\mathcal{M}_{(M,\rho)}(G)$. Since $\Psi_G(M, \rho)$ is finitely generated there exist an open compact subgroup $K < G$ such that $\mathcal{M}_{(M,\rho)}(G) \subset \mathcal{M}_K(G)$. By Theorem 2.5(1) we can assume that K is splitting.

Let $V_{(M,\rho)}$ be the maximal subrepresentation of V which is contained in $\mathcal{M}_{(M,\rho)}(G)$ and let $\mathcal{S}(G/K)_{(M,\rho)}$ be the maximal subrepresentation of $\mathcal{S}(G/K)$ which is contained in $\mathcal{M}_{(M,\rho)}(G)$. By Theorem 2.5(1a, 3), the object $\mathcal{S}(G/K)_{(M,\rho)}$ is a projective genarator of $\mathcal{M}_{(M,\rho)}(G)$. Since V is locally finitely generated, $V^K_{(M,\rho)} = \operatorname{Hom}_G(\mathcal{S}(G/K), V_{(M,\rho)})$ is finitely generated over $\mathcal{H}_K(G) = \operatorname{End}(\mathcal{S}(G/K))$. Thus it is finitely generated over $\operatorname{End}(\mathcal{S}(G/K)_{(M,\rho)})$. Thus, by Theorem 2.5(1c) it is finitely presented over $\operatorname{End}(\mathcal{S}(G/K)_{(M,\rho)})$. Therefore by [Len, Satz 3], $\operatorname{Hom}_G(\Psi_G(M, \rho), V_{(M,\rho)}) = \mathcal{F}_V((M, \rho))$ it is finitely presented over $\operatorname{End}(\Psi_G(M, \rho))$ and thus over $\mathfrak{X}(M)$. □

Following a suggestion of D. Prasad we provide a class of representations that are of locally finitely generated. They are obtained as restrictions of an admissible representation of a group G to a subgroup $H \subset G$.

Lemma 5.6. *Let $H \subset G$ be a pair of p-adic groups such that $\mathcal{S}(G)$ is locally finitely generated as a representation of $G \times H$. Then for any admissible representation π of G, the representation $\pi|_H$ is locally finitely generated as an H-representation.*

Proof. Let K_G be an open compact subgroup of G such that π^{K_G} generates π as a representation of G. We clearly have

$$\pi \cong \mathcal{S}(G) \otimes_{\mathcal{H}(G,K_G)} \pi^{K_G}.$$

Now given $K_H \subset H$ an open compact subgroup, we obtain

$$\pi^{K_H} = \mathcal{S}(G)^{K_H \times K_G} \otimes_{\mathcal{H}(G,K_G)} \pi^{K_G}$$

By our assumpsion $\mathcal{S}(G)^{K_H \times K_G}$ is finitely generated as a representation of $\mathcal{H}(G, K_G) \otimes \mathcal{H}(H, K_H)$, and as π is admissible, the space π^{K_G} is finite dimensional. Thus, the last equality implies that π^{K_H} is finitely generated over the algebra $\mathcal{H}(H, K_H)$. □

As an example we note that since $(GL_{n+1}(F) \times GL_n(F), \Delta GL_n(F))$ is of finite type we obtain:

Corollary 5.7. *Let π be an admissible representation of $GL_{n+1}(F)$. Then the restriction $\pi|_{GL_n(F)}$ is of locally finite type.*

6. Cuspidal case

The following theorem generalizes theorem D(2).

Theorem 6.1. *Let ρ be an irreducible cuspidal representation of G. Let $\mathbf{H} < \mathbf{G}$ be a Zariski closed subgroup and $H = \mathbf{H}(F)$. Let $X = G/H$ and let $\mathcal{L}$ be a G-equivariant line bundle over X. Assume that X is of finite type.*

Then, the sheaf $\mathcal{F}_{X,\mathcal{L}}(G, \rho)$ is a direct image of a locally free sheaf on a smooth subvariety of $\mathfrak{X}_G$.

For the proof we will need the following lemmas

Lemma 6.2. *Let $\phi : S \to S'$ be finite étale map of algebraic varieties. Let $\mathcal{F}$ be a coherent sheaf over S. Suppose that $\phi_*(\mathcal{F})$ is a direct image of a locally free sheaf on a smooth subvariety of S'. Then, $\mathcal{F}$ is a direct image of a locally free sheaf on a smooth subvariety of S.*

Proof. Without loss of generality we can assume that $\phi_*(\mathcal{F})$ is a locally free sheaf on S'. Now recall that a sheaf is locally free if and only if it is locally free in the étale topology. So we can assume that the map ϕ is a projection from a product of S' by a reduced zero dimensional scheme. In this case the assertion follows from the fact that a direct summand of a locally free sheaf is locally free. $\square$

Lemma 6.3. *Let $\Lambda_{\mathcal{Z}} = \mathcal{X}_*(\mathcal{Z}(\mathbf{G}))$ be the lattice of co-characters of the center of G. Consider the map $\Lambda_{\mathcal{Z}} \to G$ given by evaluation at the uniformizer ϖ, and consider $\Lambda_{\mathcal{Z}}$ as a subset of G. Let X, H be as in Theorem 6.1. Then $\Lambda_{\mathcal{Z}} \cap H$ has finite index in $\Lambda_{\mathcal{Z}} \cap (H \cdot G^0)$, where G^0 is the subgroup generated by compact subgroups of G.*

Proof. Let H^0 be the subgroup generated by compact subgroups of H. We also denote $H^1 := H^0\mathcal{Z}(H)$ and $G^1 = G^0\mathcal{Z}(G)$. Let $\Lambda_{\mathcal{Z}(H)} = \mathcal{X}_*(\mathcal{Z}(\mathbf{H}))$. We have the following commutative diagram:

$$\begin{tikzcd}
\Lambda_{\mathcal{Z}} \cap (H \cdot G^0) \arrow[r] & \Lambda_{\mathcal{Z}} \arrow[r, "i_G"] & G \arrow[r, "p_G"] & G/G^0 \\
 & & H \arrow[u, "i"] \arrow[r, "p_H"] & H/H^0 \arrow[u, "i_0"] \\
\Lambda_{\mathcal{Z}} \cap H \arrow[uu] \arrow[r] & \Lambda_{\mathcal{Z}(H)} \arrow[r, "i_H"] & H^1 \arrow[u, "j"] \arrow[r, "p_{H^1}"] & H^1/H^0 \arrow[u, "j_0"]
\end{tikzcd}$$

We have $\Lambda_{\mathcal{Z}} \cap (H \cdot G^0) = (p_G \circ i_G)^{-1}(Im(i_0))$. Note that $\mathcal{Z}(G) \cap G^0$ is compact. Hence $\Lambda_{\mathcal{Z}} \cap G^0$ is trivial. Thus we have $\Lambda_{\mathcal{Z}} \cap H = (p_G \circ i_G)^{-1}(Im(i_0 \circ j_0 \circ p_{H^1} \circ i_H))$. Since $p_{H^1} \circ i_H$ is onto, we have $\Lambda_{\mathcal{Z}} \cap H = (p_G \circ i_G)^{-1}(Im(i_0 \circ j_0))$. The assertion follows now from the fact that $Im(j_0)$ has finite index in H/H^0. $\square$

Proof of Theorem 6.1.

Step 1. Proof for the case when $G = G^1 := G^0\mathcal{Z}(G)$

Let $\Lambda_{\mathcal{Z}}$ be as in the above lemma (Lemma 6.3). Decompose $G = G^0 \times \Lambda_{\mathcal{Z}}$. Let $\Lambda^0_{\mathcal{Z}} := \Lambda_{\mathcal{Z}} \cap H$ and $\Lambda^1_{\mathcal{Z}} := \Lambda_{\mathcal{Z}} \cap (H \cdot G^0)$. We can find a decomposition $\Lambda_{\mathcal{Z}} := \Lambda^2_{\mathcal{Z}} \oplus \Lambda^3_{\mathcal{Z}}$, such that $\Lambda^1_{\mathcal{Z}}$ is a subgroup of finite index in $\Lambda^2_{\mathcal{Z}}$. We define $X^0 := \Lambda^2_{\mathcal{Z}} \cdot G^0 \cdot [e]$, where $[e] \in X$ is the class of the unit element in G.

Using the fact that G^0 is normal in G, we get

$$X \cong X^0 \times \Lambda^3_{\mathcal{Z}}$$

as $G^0 \cdot \Lambda^3_{\mathcal{Z}} \cdot \Lambda^0_{\mathcal{Z}} \cong G^0 \times \Lambda^3_{\mathcal{Z}} \times \Lambda^0_{\mathcal{Z}}$-spaces. Here the action of $\Lambda^0_{\mathcal{Z}}$ on $X^0 \times \Lambda^3_{\mathcal{Z}}$ is trivial, the action of $\Lambda^3_{\mathcal{Z}}$ is on the second component and the action of G^0 is on the first.

Now consider the fiber $\mathcal{L}|_{[e]}$ as a character of $G^0_{[e]} \times \Lambda^0_{\mathcal{Z}}$ and decompose it into a product $\chi_1 \otimes \chi_2$.

Thus we have isomorphisms of $G^0 \times \Lambda^3_{\mathcal{Z}} \times \Lambda^0_{\mathcal{Z}}$ representations:

$$\mathcal{F}_{X,\mathcal{L}}((G,\rho)) = \mathrm{Hom}_{G^0}(\rho, \mathcal{S}(X,\mathcal{L})) \cong \mathrm{Hom}_{G^0}(\rho, \mathcal{S}(X^0, \mathcal{L}|_{X^0})) \otimes \mathbb{C}[\Lambda^3_{\mathcal{Z}}] \otimes \chi_2.$$

Let $L := \mathrm{Spec}(\mathbb{C}[\Lambda^3_{\mathcal{Z}} \times \Lambda^0_{\mathcal{Z}}])$. By Lemma 6.3 the map

$$\pi : \mathfrak{X}_G = \mathrm{Spec}(\mathbb{C}[G/G^0]) \to L$$

is a finite étale morphism. We see that $\pi_*(\mathcal{F}_{X,\mathcal{L}}((G,\rho)))$ is a direct image of a free sheaf on $\mathrm{Spec}(\mathbb{C}[\Lambda^3_{\mathcal{Z}}])\times\{\chi_2\}$ which is a smooth subvariety of L. Thus Lemma 6.2 implies the assertion.

Step 2. Proof for the general case.
As in the previous step consider $\Lambda_{\mathcal{Z}} \subset G/G^0$. This gives a finite étale map

$$\mathfrak{X}(G) = \mathrm{Spec}(\mathbb{C}[G/G^0]) \xrightarrow{p} \mathrm{Spec}(\mathbb{C}[\Lambda_{\mathcal{Z}}]) \cong \mathbb{C}[G^1/G^0].$$

Similarly to the previous step, the sheaf $p_*(\mathcal{F}_{X,\mathcal{L}}((G,\rho)))$ is a direct image of a locally free sheaf on a smooth subvariety of $\mathrm{Spec}(\mathbb{C}[\Lambda_{\mathcal{Z}}])$. Again Lemma 6.2 implies the assertion.

□

Theorem 6.1 together with Theorem 4.3, imply Theorem C using the following standard fact:

Lemma 6.4. *Let X be a smooth variety and Y be its closed subvariety. Consider O_Y as a coherent sheaf over X. Then for any $y \in Y$ we have*

$$\dim \mathrm{Ext}^i_{O_X}(O_Y, \delta_y) = \binom{\dim X - \dim Y}{i}.$$

Appendix A. Tensor Multipicities

Let $V, W \in \mathcal{M}(G)$ be smooth representations of G. Consider V, W as left $\mathcal{H}(G)$-modules. Since $\mathcal{H}(G)$ is equipped with an anti-involution induced by $g \mapsto g^{-1}$, we can consider V as a right $\mathcal{H}(G)$-module. Thus we can define $V \otimes_{\mathcal{H}(G)} W$.

Lemma A.1. *Let $V, W \in \mathcal{M}(G)$ be smooth representations of G. Then we have:*

$$(V \otimes_{\mathcal{H}(G)} W)^* \cong \mathrm{Hom}_G(V, \tilde{W}).$$

Proof. Define $\phi : (V \otimes W)^* \to \mathrm{Hom}_{\mathbb{C}}(V, W^*)$ by

$$\phi(\ell)(v)(w) = \ell(v \otimes w).$$

Since $V \otimes_{\mathcal{H}(G)} W$ is a quotient of $V \otimes W$ we can consider $(V \otimes_{\mathcal{H}(G)} W)^*$ as a subset of $(V \otimes W)^*$ spesificaly $(V \otimes_{\mathcal{H}(G)} W)^* \cong \{\ell \in (V \otimes_{\mathcal{H}(G)} W)^* | \ell(g^{-1}v \otimes w) = \ell(v \otimes gw)\}$

It is easy to see that $\phi((V \otimes_{\mathcal{H}(G)} W)^*) \subset \mathrm{Hom}_G(V, W^*)$ and that $\mathrm{Hom}_G(V, W^*) \cong \mathrm{Hom}_G(V, \tilde{W})$. So we can consider ϕ as a map $(V \otimes_{\mathcal{H}(G)} W)^* \to \mathrm{Hom}_G(V, \tilde{W})$.

Now define $\psi : \mathrm{Hom}_{\mathbb{C}}(V, \tilde{W}) \to (V \otimes W)^*$ by:

$$\psi(T)(v \otimes w) = \langle T(v), w\rangle.$$

Again $\psi(\mathrm{Hom}_G(V, \tilde{W})) \subset (V \otimes_{\mathcal{H}(G)} W)^*$.

Finally we notice that $\psi \circ \phi = \mathrm{Id}$ and $\phi \circ \psi = \mathrm{Id}$. □

As an immidiate consequence we get:

Corollary A.2.

$$(V \otimes^L_{\mathcal{H}(G)} W)^* \cong \mathrm{RHom}_G(V, \tilde{W}).$$

Appendix B. Induction of general families

In this Appendix we introduce the object $\mathcal{G}_V(M, \tau)$ and prove Theorem E.

Definition B.1. *Let $\mathbf{P} \subset \mathbf{G}$ be a parabolic subgroup and $\mathbf{M}$ be its Levi factor. Let τ be an admissible representation of $M = \mathbf{M}(F)$ Let $V \in \mathcal{M}(G)$. We define*

$$\mathcal{G}_V(M, \tau) := ind^M_{M_0} \tilde{\tau}|_{M_0} \otimes^L_{\mathcal{H}(M)} r^G_M(V),$$

and consider it as an object in the derived category of the category of M/M_0-modules. Equivalently, we will consider it as an object in the derived category of the category of quasi-coherent sheaves over $\mathfrak{X}_M$.

It is enough to prove Theorem E(1-3) since Theorem E(4) follows from Theorem E(1,3) as in §5.

To compare $\mathcal{G}_V(M,\tau)$ with $\mathcal{F}_V((M,\rho))$ we need the following standard lemma:

Lemma B.2. *Let A and B be an associative algebras (not necessarily unital). Let M be a left A-module, K be a left B-module, and M be an (A,B)-bi-module*

Then

$$\operatorname{Hom}_A(N,M)\otimes_B K=\operatorname{Hom}_A(N,M\otimes_B K).$$

Corollary B.3. *For any $V,W\in\mathcal{M}(G)$ we have*

$$\operatorname{Hom}_G(V,W)\cong\operatorname{Hom}_G(V,\mathcal{S}(G))\otimes_{\mathcal{H}(G)}W.$$

We will also need:

Lemma B.4. *Let ρ be an irreducible cuspidal representation of G and recall that $\Psi_G(\rho,G)=ind_{G^0}^G(\rho|_{G^0})$. Consider $\operatorname{Hom}_G(\Psi_G(\rho,G),\mathcal{S}(G))$ as G-representation by letting G act on $S(G)$. We have an isomorphism of G-representations:*

$$\operatorname{Hom}_G(\Psi_G(\rho,G),\mathcal{S}(G))\cong\Psi_G(\tilde\rho,G).$$

Proof.

Step 1 Proof that $\tilde\rho\cong\operatorname{Hom}_{G^0}(\rho,\mathcal{S}(G^0))$ as G^0 representations.

This follow immidately from [BR, §§5.3 Theorem 8]

Step 2 Proof of the lemma.

$$\operatorname{Hom}_G(\Psi_G(\rho,G),\mathcal{S}(G))=\operatorname{Hom}_{G^0}(\rho,\mathcal{S}(G))=\operatorname{Hom}_{\mathcal{H}(G^0)}(\rho,\mathcal{S}(G^0)\otimes_{\mathcal{H}(G^0)}\mathcal{S}(G))$$

By Lemma B.2

$$\operatorname{Hom}_{\mathcal{H}(G^0)}(\rho,\mathcal{S}(G^0)\otimes_{\mathcal{H}(G^0)}\mathcal{S}(G))=\operatorname{Hom}_{\mathcal{H}(G^0)}(\rho,\mathcal{S}(G^0))\otimes_{\mathcal{H}(G^0)}\mathcal{S}(G)=\operatorname{Hom}_{G^0}(\rho,\mathcal{S}(G^0))\otimes_{\mathcal{H}(G^0)}\mathcal{S}(G)$$

Finaly, by the previous step

$$\operatorname{Hom}_{G^0}(\rho,\mathcal{S}(G^0))\otimes_{\mathcal{H}(G^0)}\mathcal{S}(G)=\tilde\rho\otimes_{\mathcal{H}(G^0)}\mathcal{S}(G)=\Psi_G(\rho,G).$$

□

Proposition B.5. *Theorem E(2) holds. Namely, if ρ is cuspidal then $\mathcal{G}_V(M,\rho)\cong\mathcal{F}_V((M,\rho))$.*

Proof. By definition we have

$$\mathcal{F}_V((M,\rho))=\operatorname{Hom}_G(\Psi_G(M,\rho),V)=\operatorname{Hom}_G(\bar i_M^G(ind_{M_0}^M(\rho|_{M_0})),V).$$

Applying the second adjointness theorem we get

$$\operatorname{Hom}_G(\bar i_M^G(ind_{M_0}^M(\rho|_{M_0})),V)\cong\operatorname{Hom}_M(ind_{M_0}^M(\rho|_{M_0}),r_M^G V).$$

By Corollary B.3 we get

$$\mathcal{F}_V((M,\rho))\cong\operatorname{Hom}_M(ind_{M_0}^M(\rho|_{M_0}),\mathcal{S}(M))\otimes_{\mathcal{H}(M)}r_M^G(V).$$

By Lemma B.4 We obtain

$$\mathcal{F}_V((M,\rho))\cong ind_{M_0}^M(\tilde\rho|_{M_0})\otimes_{\mathcal{H}(M)}r_M^G(V)$$

Finally, since $\Psi_M(M,\rho)$ is a projective object, we obtain

$$\mathcal{G}_V(M,\rho)\cong\Psi_M(M,\tilde\rho)\otimes^L_{\mathcal{H}(M)}r_M^G(V)\cong\Psi_M(M,\tilde\rho)\otimes_{\mathcal{H}(M)}r_M^G(V)=\mathcal{F}_V((M,\rho))$$

□

Proposition B.6. *Theorem E(1) holds. Namely, let $V\in\mathcal{M}(G)$ Then,*

$$\operatorname{RHom}_G(V,i_M^G(\chi\tau))\cong\operatorname{RHom}_{O(\mathfrak{X}_M)}(\mathcal{G}_V(M,\tau),\delta_\chi),$$

where δ_χ is the skyscraper sheaf over $\chi\in\mathfrak{X}_M$.

Proof.

Step 1 Proof for the case $M = G$.
By Corollary A.2 it is enough to prove that

$$V \otimes^L_{\mathcal{H}(G)} i^G_M(\chi\tau)) \cong \mathcal{G}_V(M,\tau) \otimes_{O(\mathfrak{X}_M)} \delta_\chi$$

This follows from associativity of tensor product.

Step 2 Proof the general case.
By the previous step we have

$$\mathrm{RHom}_{O(\mathfrak{X}_M)}(\mathcal{G}_V(M,\tau), \delta_\chi)) \cong \mathrm{RHom}_M(r^G_M(V), \chi\tau)$$

By Frobenius reciprocity we get

$$\mathrm{RHom}_M(r^G_M(V), \chi\tau) \cong \mathrm{RHom}_G(V, i^G_M(\chi\tau))$$

$\square$

Proposition B.7. *The following generalization of Theorem E(3) holds. Namely, let $V \in \mathcal{M}(G)$ be such that for any compact open $K < G$ the module V^K is finitely generated over $\mathcal{H}(G,K)$. Then, $\mathcal{G}_V(M,\tau)$ is perfect.*

For the proposition we will need the following lemmas

Lemma B.8. *Let $V \in \mathcal{M}(G)$ be a locally finitely generated module. Then for any Levi group M, the representation $r^G_M(V)$ is locally finitely generated.*

Proof. First note that by Iwasawa decomposition (see e.g. page 40 [BR]) if V is finitely generated over G then it is finitely generated over P. Thus if V is finitely generated over G then $r^G_M(V)$ is finitely generated over M.

Now, let $V \in \mathcal{M}(G)$ be a locally finitely generated module, and let $K' \subset M$ be an open compact subgroup. For any open compact subgroup $K' < M$, one can find open compact subgroup $K'' < G$ s.t. $K'' \cap M \subset K'$. By Bruhat theorem (page 41 [BR]) one can find open compact subgroup $K < K''$ satisfying the conditions of Jacquet's lemma (page 65 in loc. cit.). Thus the map

$$V^K \to r^G_M(V)^{K\cap M}$$

is onto.

This implies that

$$r^G_M(G \cdot V^K) \supset r^G_M(M \cdot V^K) = M \cdot r^G_M(V)^{K\cap M} \supset M \cdot r^G_M(V)^{K'}.$$

We know that $G \cdot V^K$ is finitely generated over G. Thus $r^G_M(G \cdot V^K)$ finitely generated over M. By Noetherity, this implies that $M \cdot r^G_M(V)^{K'}$ is finitely generated over M, in other words $r^G_M(V)^{K'}$ is finitely generated over $\mathcal{H}_{K'}(M)$.

$\square$

Lemma B.9. *Let B a unital associative algebra and C be a commutative algebra. Assume that $B \otimes C$ have finite homological dimension. Let V be finitely generated module over $B \otimes C$ and T be finite dimensional right module over $B \otimes C$. Then $\mathrm{Tor}^*_B(T,V)$ is finitely generated over C with respect to the diagonal action.*

Proof.

Case 1 $V = B \otimes C$ and the action of C on T is given by a character.
This is Obvious.

Case 2 V is free module
Follows from the previoues step.

Case 3 V is projective
Follows from the previous step.

Case 4 The general case
Follows from the previous step, by induction on the homological dimension of V.

□

The following lemma is straightforward.

Lemma B.10. *Let $\tau, V \in \mathcal{M}(G)$ and let $K < G$ be an open compact subgroup. Assume that τ is generated by τ^K. Then*

$$\tau \otimes_{\mathcal{H}(G)} V \cong \tau^K \otimes_{\mathcal{H}_K(G)} V^K$$

We thus obtain:

Corollary B.11. *Let $\tau, V \in \mathcal{M}(G)$ and let $K < G$ be an splitting open compact subgroup. Assume that τ is generated by τ^K. Then*

$$\tau \otimes^L_{\mathcal{H}(G)} V \cong \tau^K \otimes^L_{\mathcal{H}_K(G)} V^K$$

Lemma B.12. *Let $G_1 = G_0 \cdot Z(G)$ where G_0 is the subgroup of G generated by compact subgroups, $Z(G)$ the center of G. Let $\Lambda_{Z(G)} = \mathfrak{X}_*(Z(G))$. Our choise of a uniformizer allows us to identify $\Lambda_{Z(G)}$ with a subgroup of G. Then $\nu : G_0 \times \Lambda_{Z(G)} \to G_1$ is an isomorphism. In particular $\mathcal{H}(G_1) = \mathcal{H}(G_0) \otimes \mathbb{C}[\Lambda_{Z(G)}]$*

Proof. By [BR, page 86] the map $Z(G)/Z(G)^0 \to G/G^0$ is injective and hence $Z(G)^0 = Z(G) \cap G^0$. Thus $Ker(\nu) \cong G_0 \cap \Lambda = Z_0 \cap \Lambda = \{1\}$ and $Im(\nu) = G_0 \cdot \Lambda = G_0 \cdot Z(G)_0 \cdot \Lambda = G_0 \cdot Z(G) = G_1$. □

Proof of Proposition B.7. Since $\mathcal{M}(G)$ has finit homological dimension it is enough to prove that $H^*(\mathcal{G}_V(M, \tau))$ is finitly generated.

Step 1 Proof for the case $G = M$

Since G_1 is of finite index in G we get that V is localy finitely generated over G_1. Chose splitting open compact $K < G$ such that τ is generated by τ^K.

By Corollary B.11 we have:

$$\mathcal{G}_V(G, \tau) := ind^G_{G_0}(\tau|_{G_0}) \otimes^L_{\mathcal{H}(G)} V = (ind^G_{G_0}(\tau|_{G_0}))^K \otimes^L_{\mathcal{H}_K(G)} V^K.$$

Now we have:

$$(ind^G_{G_0}(\tau|_{G_0}))^K \otimes^L_{\mathcal{H}_K(G)} V^K = (\tau^K \otimes_{\mathcal{H}_K(G_0)} \mathcal{H}_K(G)) \otimes^L_{\mathcal{H}_K(G)} V^K =$$
$$= \tau^K \otimes^L_{\mathcal{H}_K(G_0)} \mathcal{H}_K(G) \otimes^L_{\mathcal{H}_K(G)} V^K = \tau^K \otimes^L_{\mathcal{H}_K(G_0)} V^K$$

This implies (by lemmas B.9 and B.12) that $H^*(\mathcal{G}_V(M, \tau))$ is finitely generated over $\mathbb{C}[G^1/G^0]$, and hence over G/G^0

Step 2 Proof for the general case

Follows from the previous step and Lemma B.8.

□

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