Push-Forward of Smooth Measures and Strong Thom Stratifications

Joint work of
A. Aizenbud, N. Avni, and S. Carmeli

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- By [BZ] we can consider it as a sheaf over $Y(F)$
- We can look at its stalk $\varphi_*(\mathcal{M}_c^\infty(X(F)))_y$ at $y \in Y(F)$.

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We will concentrate on the second one:

Question

When are the stalks

$$\left(\varphi_*\mathcal{M}_c^\infty(X(F))\right)_y$$

finite-dimensional (over $\mathbb{C}$) for all $y \in Y(F)$?

Example: Shalika Germs

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- We will generalize this phenomenon to the infinitesimal setting.

The Scheme $\mathfrak{B}_\varphi$ and Quasitransitivity

Definition (The scheme $\mathfrak{B}_\varphi$)

Let $\varphi : X \to Y$ be as above. Let

$$D\varphi : \mathcal{T}_X \to \varphi^*(\mathcal{T}_Y)$$

be the differential, and let $\mathrm{Im}(D\varphi)$ be its image sheaf. Define

$$\mathfrak{B}_\varphi := \mathrm{Spec}_X\big(\mathrm{Sym}(\mathrm{Im}(D\varphi))\big).$$

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First main result

Theorem (A., Avni, Carmeli)

Let $\varphi : X \to Y$ be a quasi-transitive morphism of smooth algebraic varieties over a p-adic field F. Then there exists an integer $N(\varphi)$ such that for every $y \in Y(F)$,

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Step 3 does not work. Non-archimedean exponentiation is too local.

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"many" vertically extendable vectors $\iff$ quasi-transitivity

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- *φ admits a strong Thom stratification.*

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Remark

Quasi-transitivity is not stable under flat deformations of morphisms.

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- *Is $\mathcal{M}$-finiteness equivalent to having a theory of nearby cycles?*

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- If φ is quasi-transitive, dimension counts imply that this stratification is also vertically extendable.

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  - The sheaf $\mathrm{Im}(D\varphi)$ has constant rank along strata.
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- If φ is quasi-transitive, dimension counts imply that this stratification is also vertically extendable.
- We deduce the strong Thom condition.

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Let $\varphi : X \to Y$ be strongly Thom-stratified, and let $S \subset X$ be a closed stratum with

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Thank you very much!