Frobenius formula and representation count

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- *Let $x \in G$. Then*
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- $\chi_\pi*\chi_\tau=\begin{cases}0 & \pi\not\cong\tau\\ \frac{\chi_\pi}{\dim\pi} & \pi=\tau\end{cases}$

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Proposition (A.- Avni)

$\forall d, k, n \exists p_0$ *s.t.* $\forall p > p_0$.

$$\zeta_{GL_d(\mathbb{Z}/p^k\mathbb{Z})}(2n) = \zeta_{GL_d(\mathbb{F}_p[t]/t^k\mathbb{F}_p[t])}(2n)$$

Pro-finite case

Theorem (A.- Avni)

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- This result with 22 replaced by $\dim G$ is due to Lubotzky–Martin.

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$$\lim_{\varepsilon \to 0} \frac{Vol\{(g_1,h_1,\ldots g_{12},h_{12}) \in SL_d(\mathbb{Z}_p)^{24} : \|[g_1,h_1]\cdots[g_{12},h_{12}]-1\| < \varepsilon\}}{Vol\{g \in SL_d(\mathbb{Z}_p) : \|g-1\| < \epsilon\}} < \infty$$

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Let ϕ be an algebraic map of p-adic smooth algebraic varieties. When is the pushforward ϕ_* of a smooth compactly supported measure on X a measure on Y with continuous density ?

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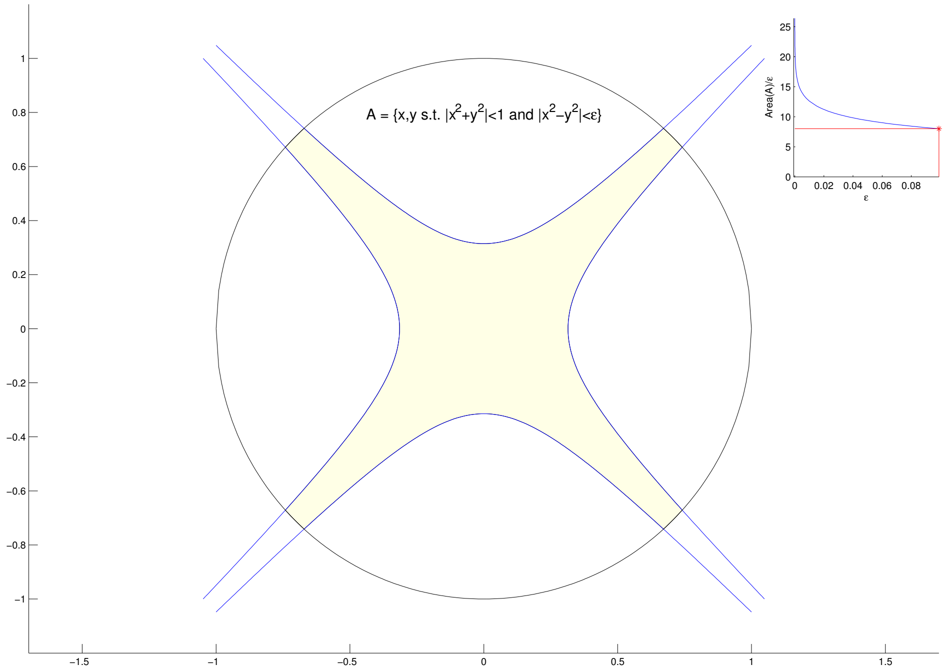
- ϕ is submersive (a.k.a. smooth) $\Rightarrow \phi_*(m)$ is smooth
- ϕ is (locally) dominant $\Rightarrow \phi_*(m)$ has L^1 density (Radon–Nikodym)


A = {x,y s.t. |x²+y²|<1 and |x²−y²|<ε}
Area(A)/ε
ε
0
0.02
0.04
0.06
0.08
5
10
15
20
25
−1.5
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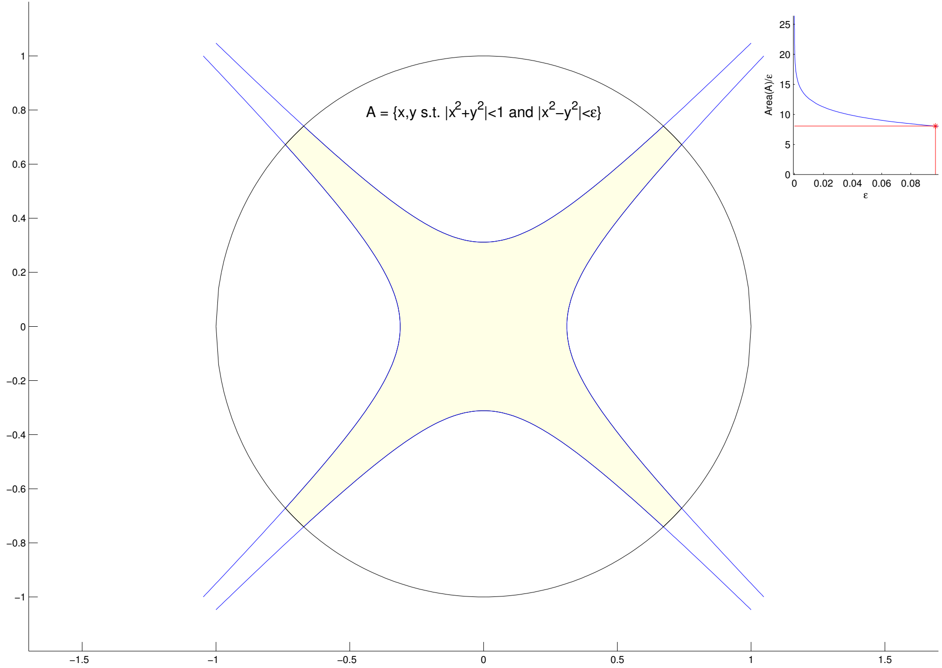

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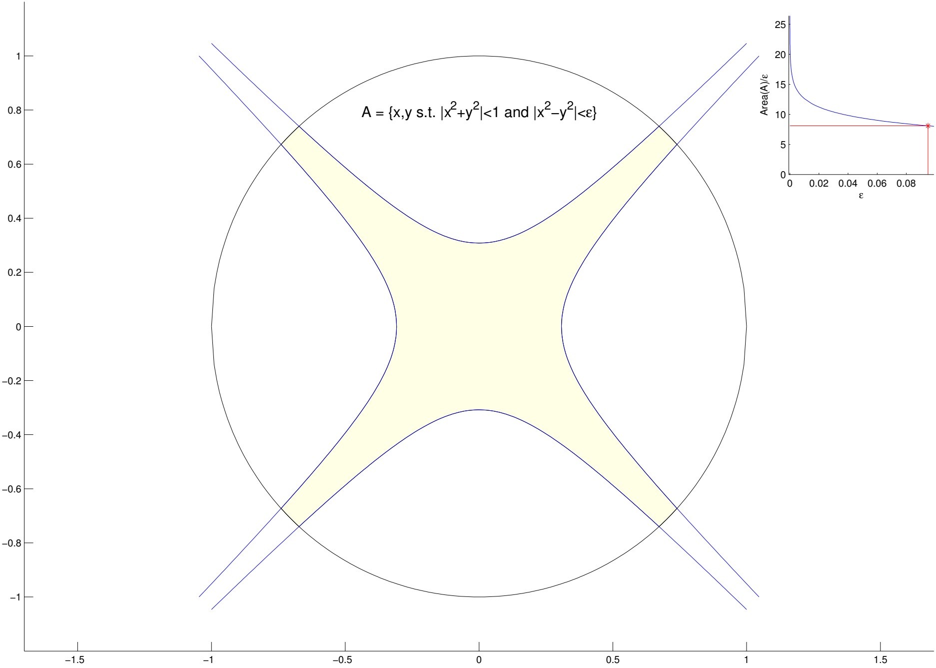
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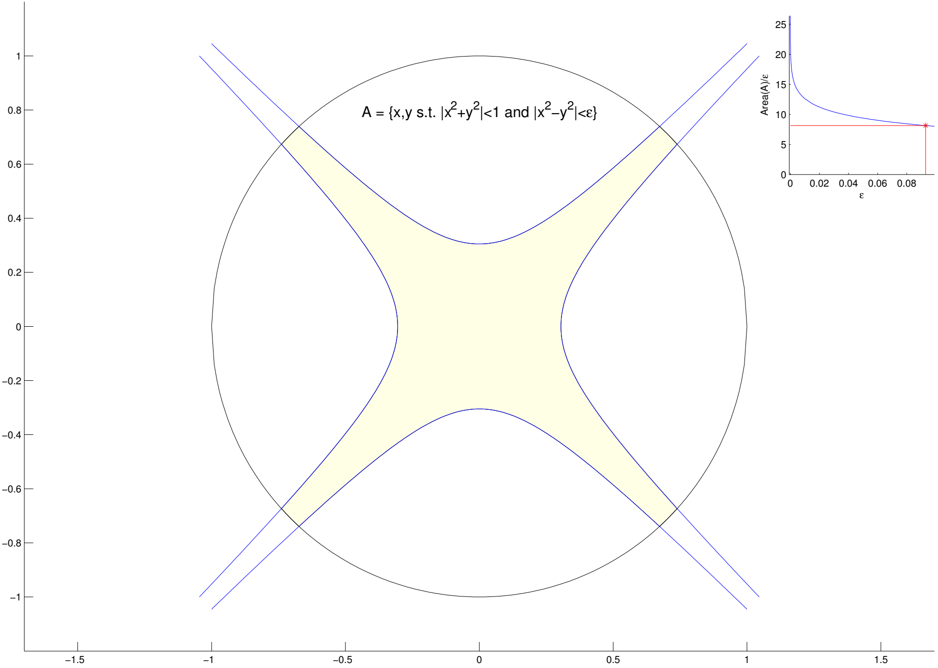
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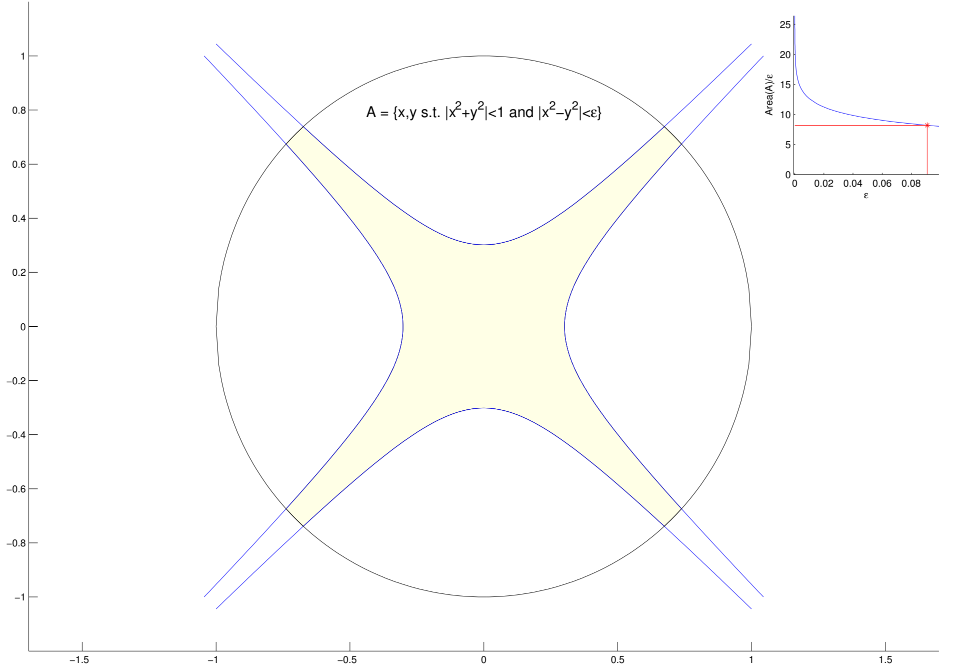

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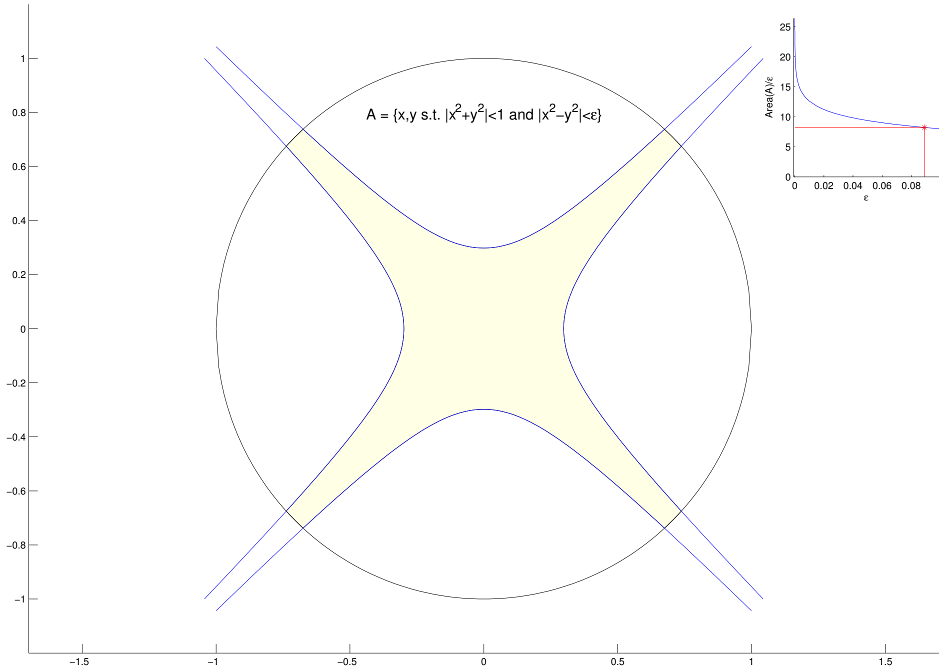

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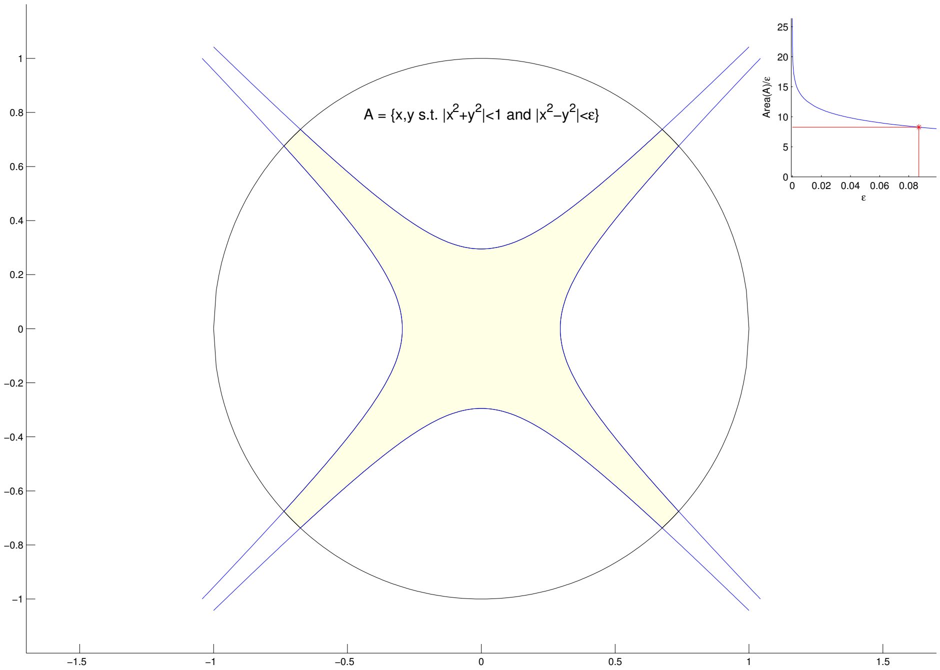
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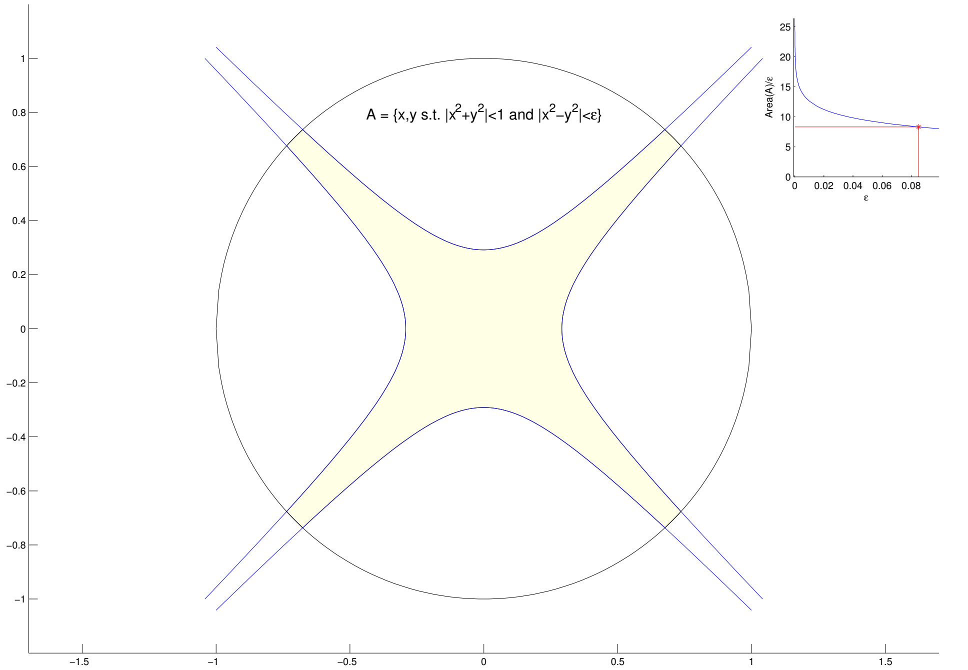
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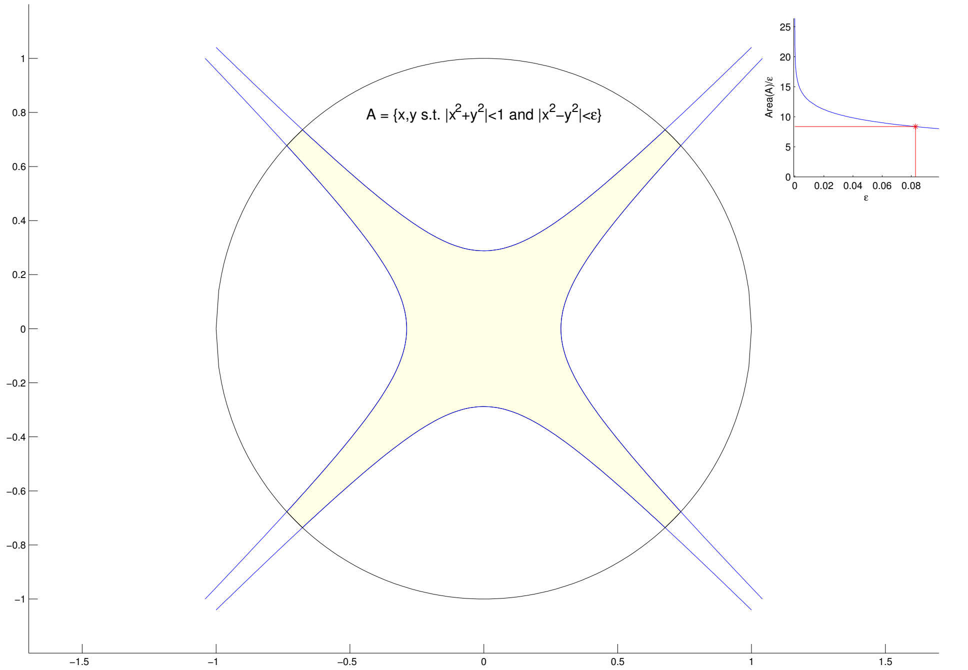

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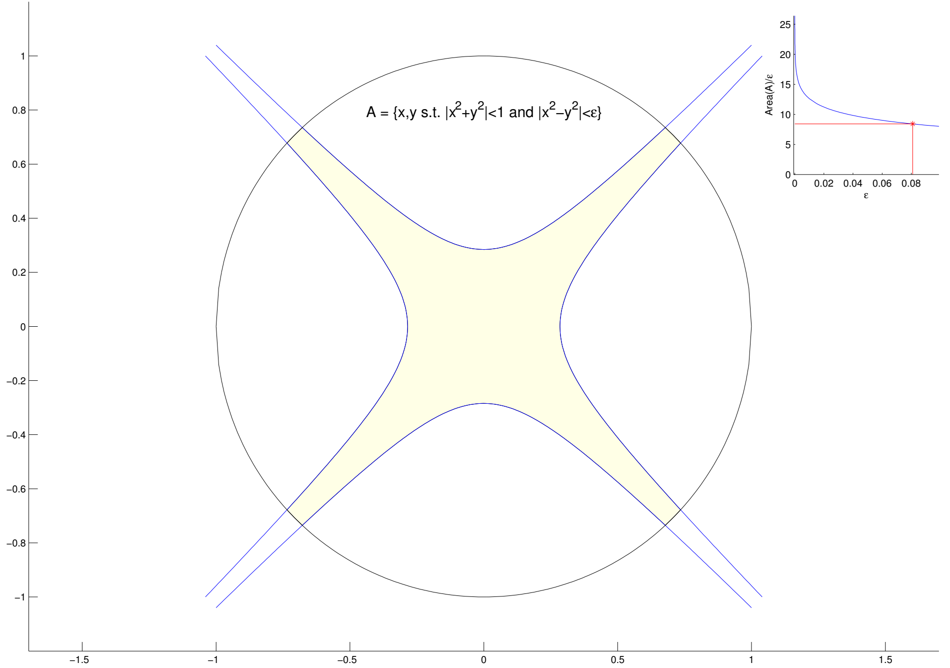
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−0.5
0.5
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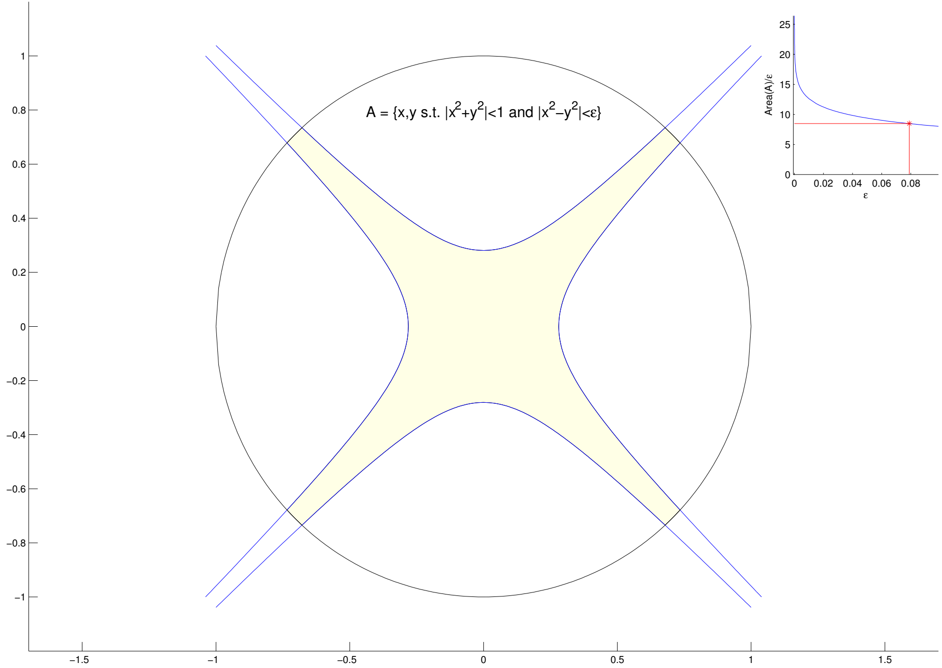
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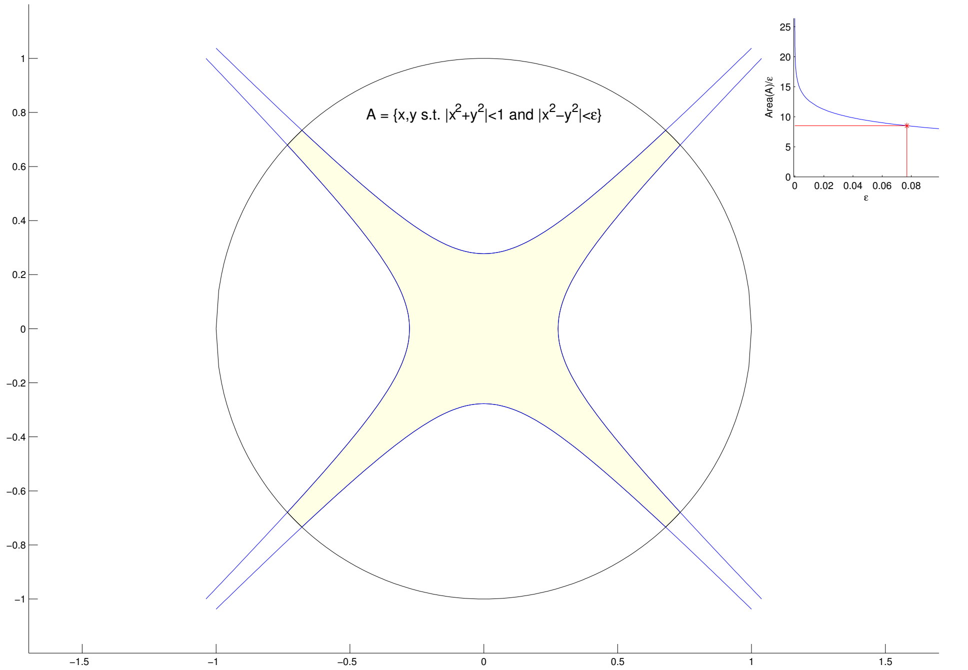
Area(A)/ε
ε



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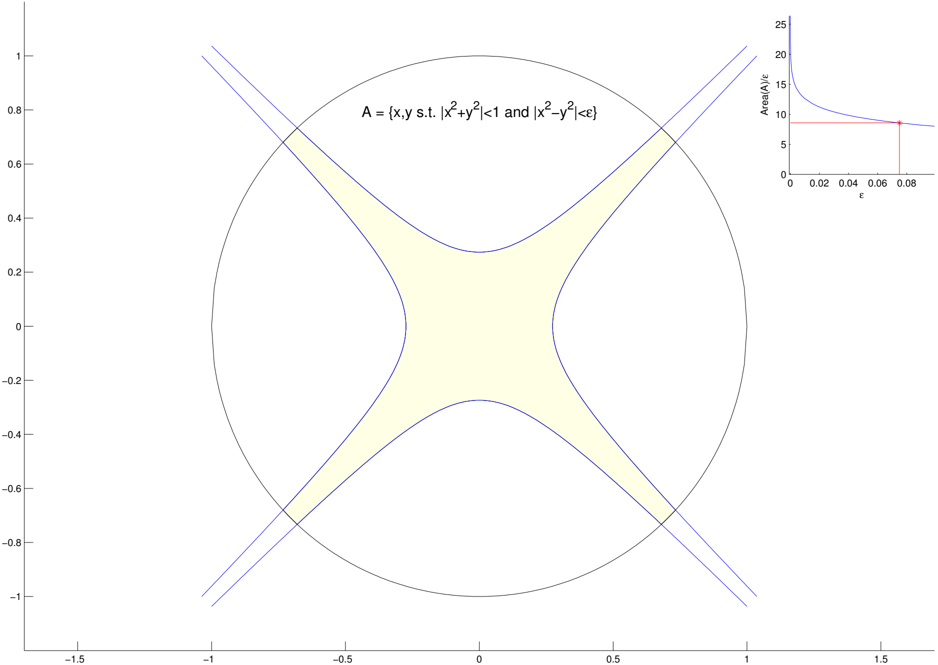
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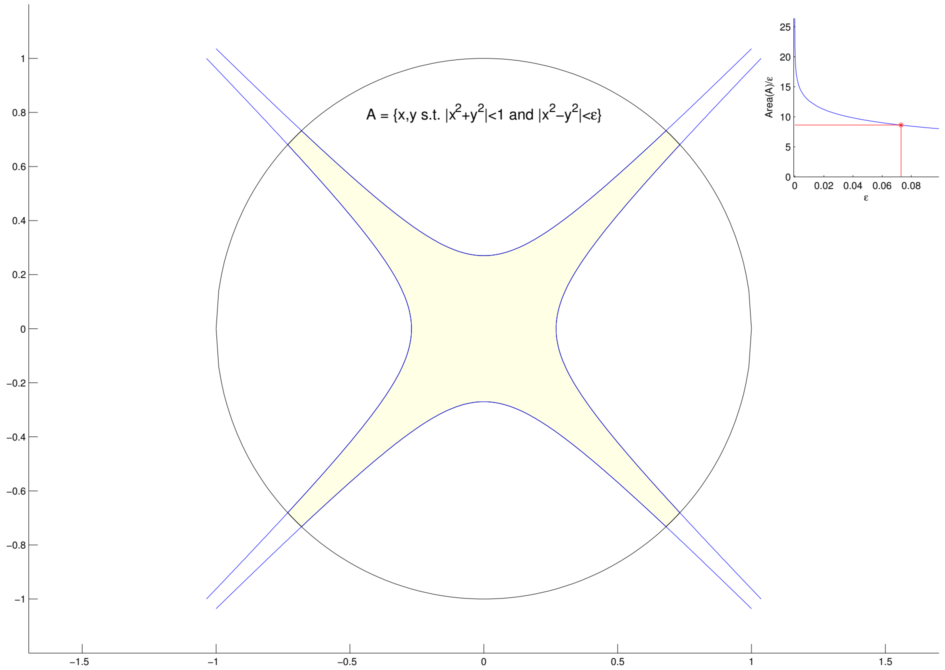
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Area(A)/ε
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0.08
5
10
15
20
25
1
0.8
0.6
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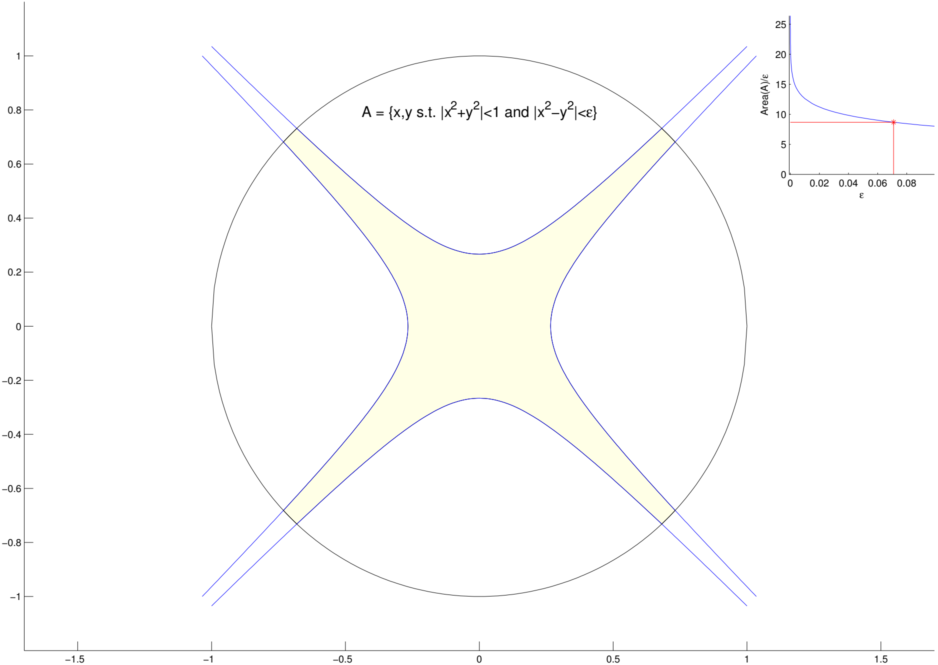
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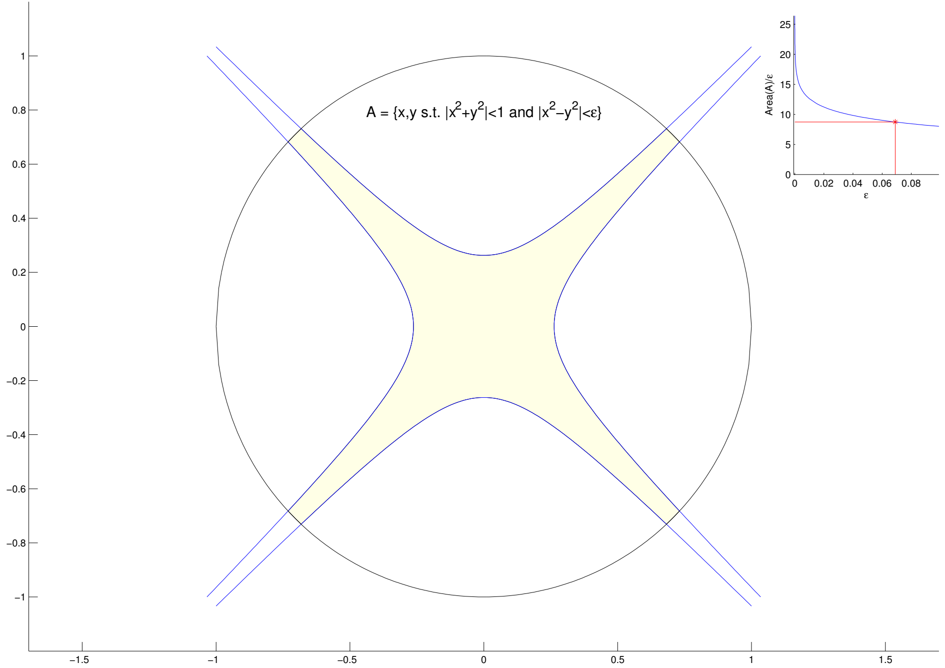
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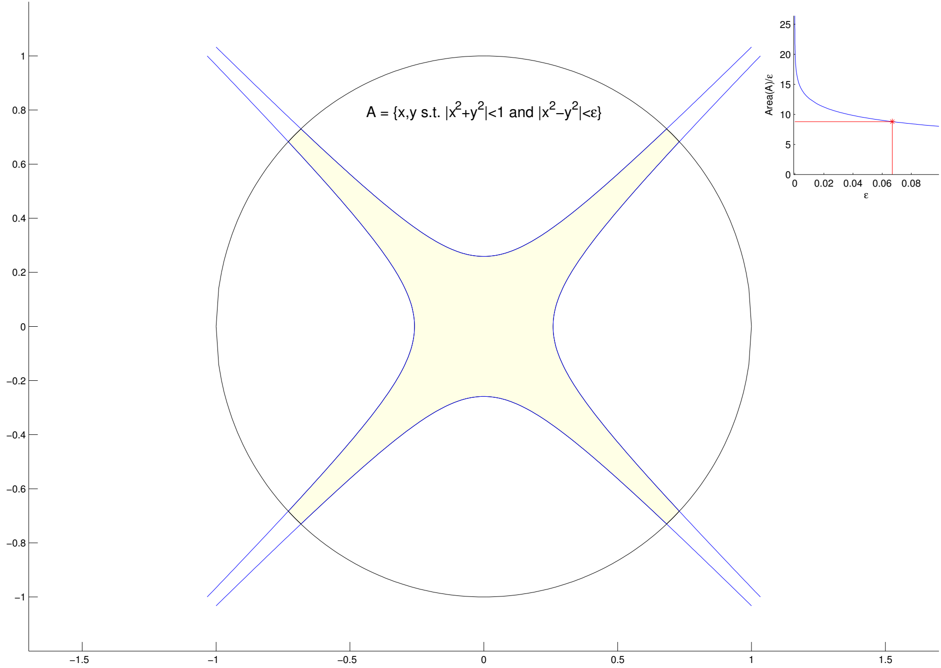
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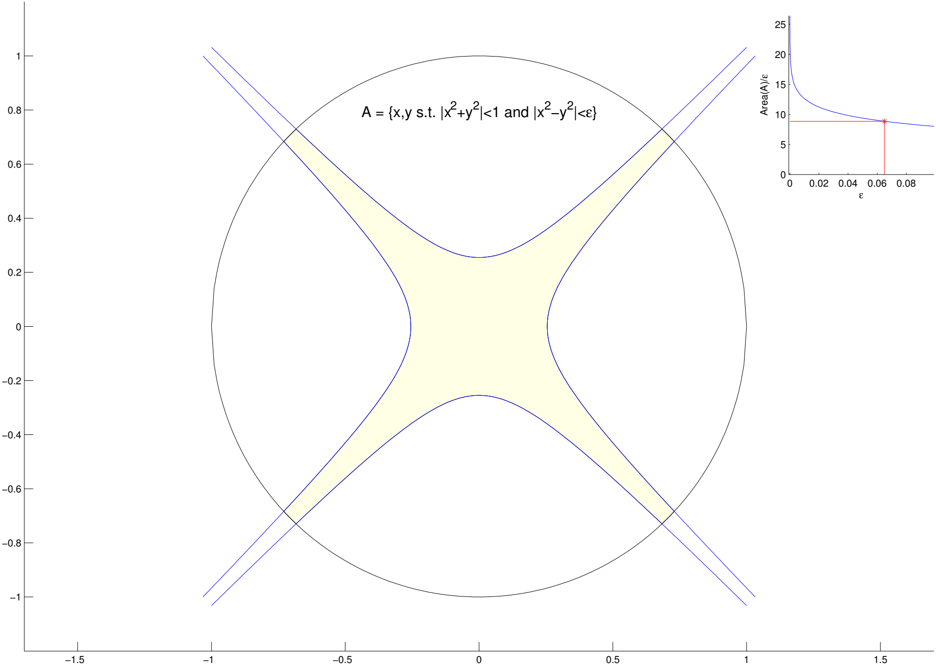
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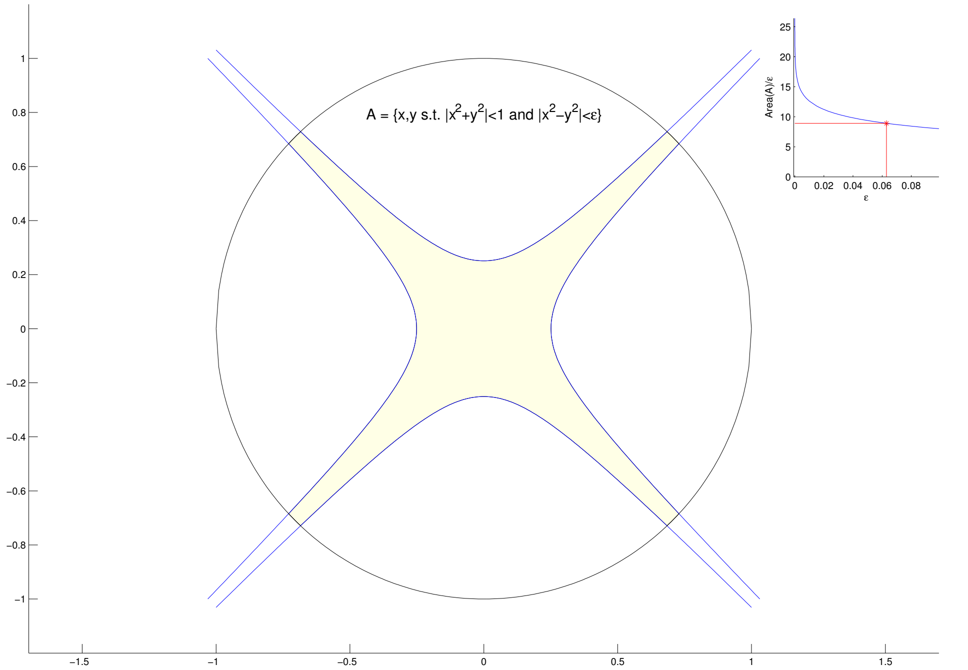
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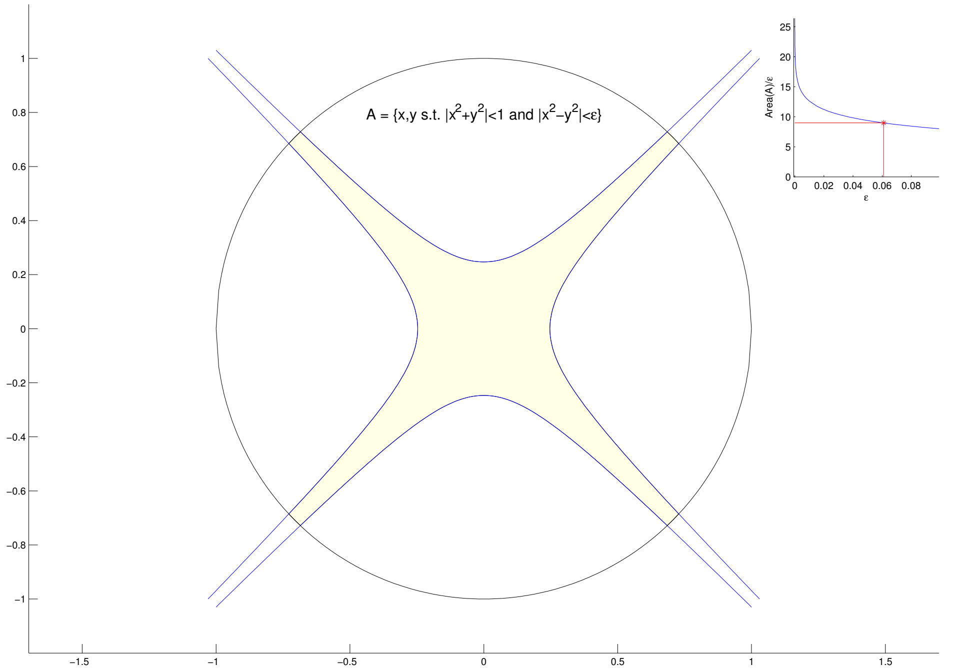
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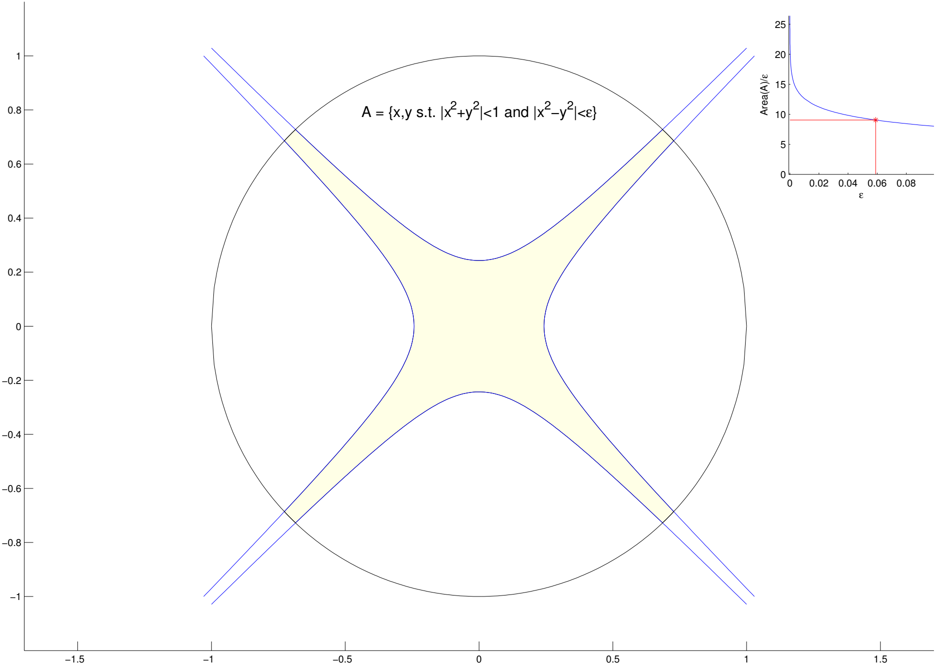

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Area(A)/ε
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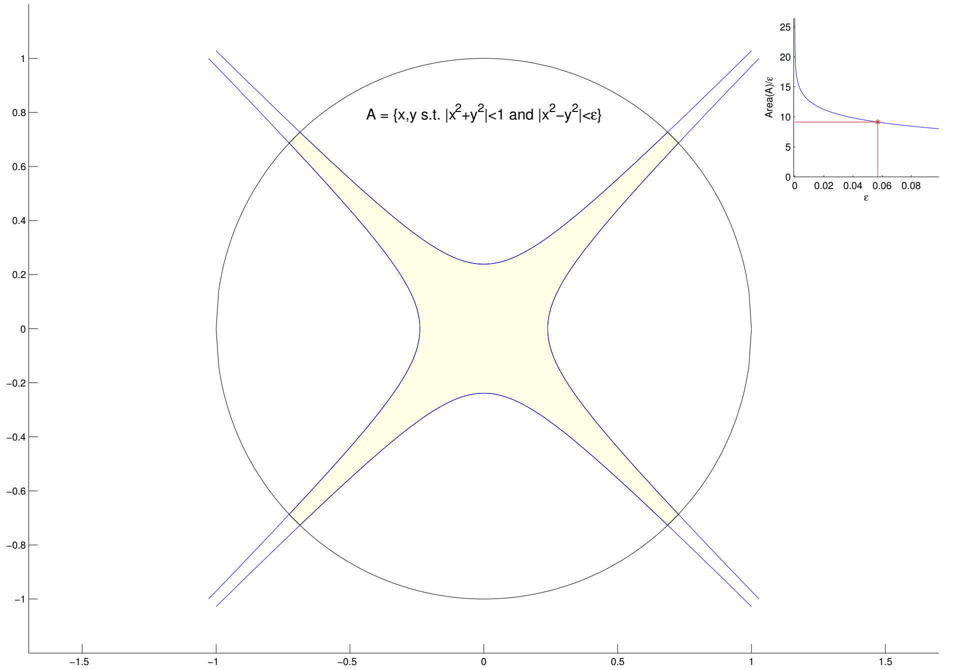
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1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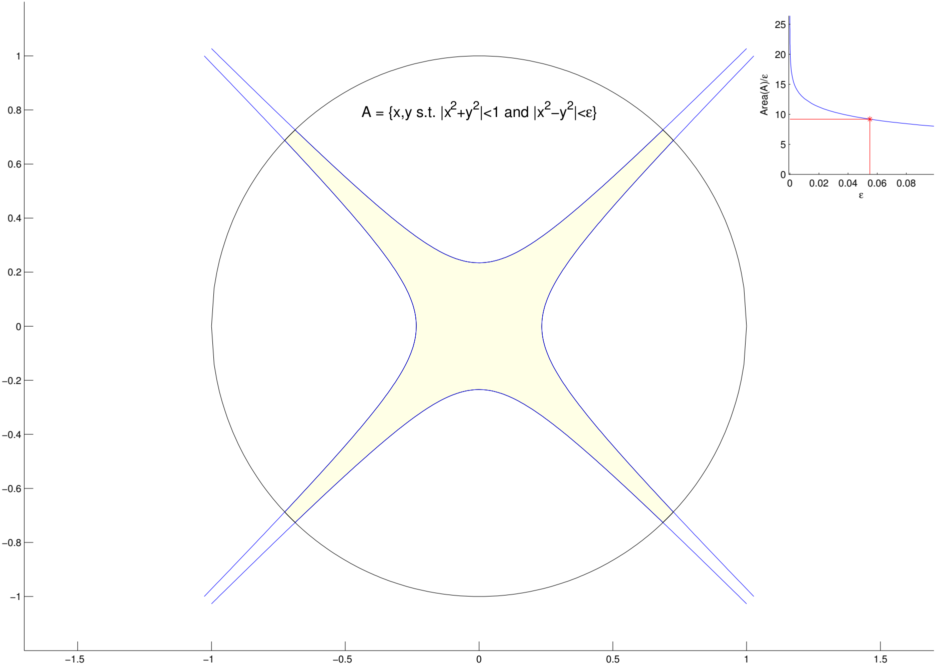
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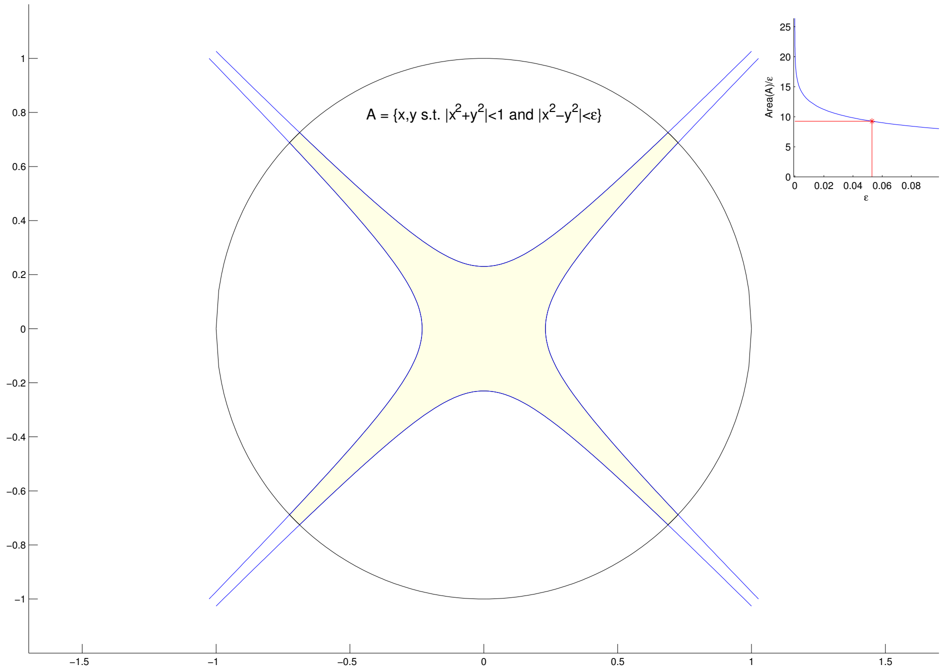
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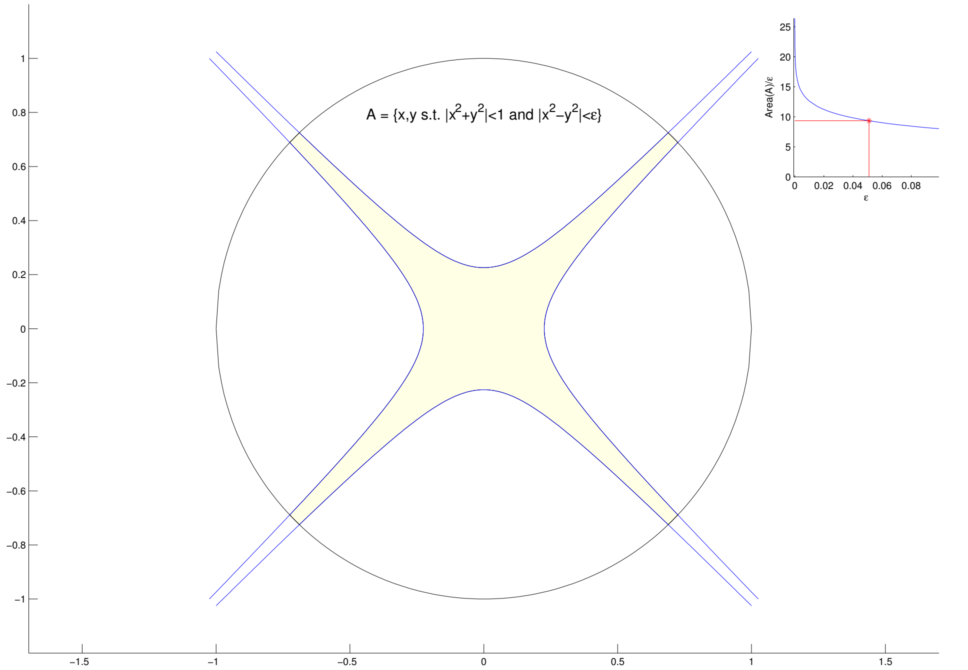
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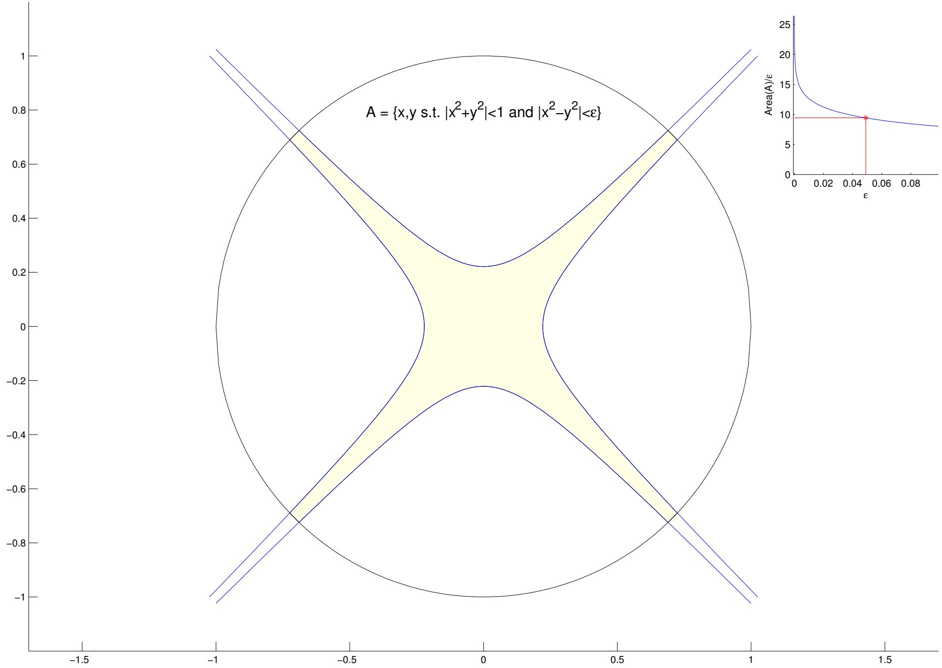
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
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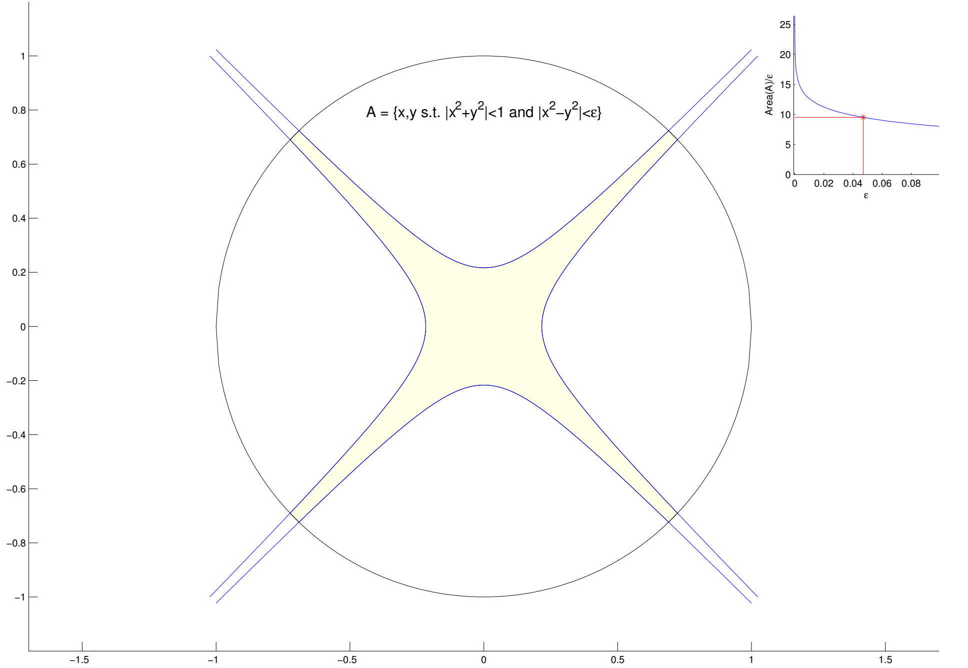
A = {x,y s.t. |x2+y2|<1 and |x2−y2|<ε}
Area(A)/ε
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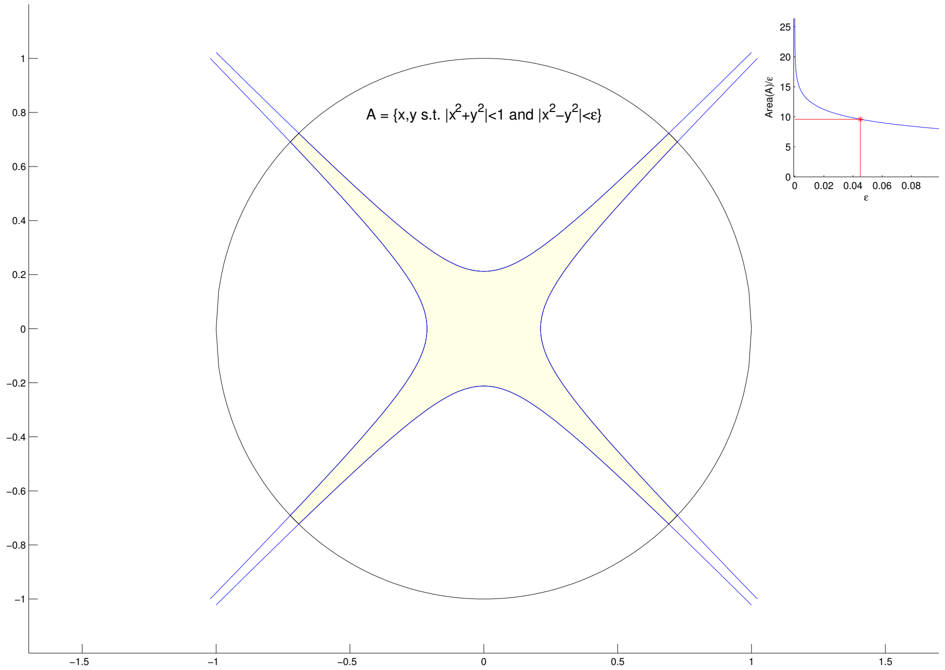

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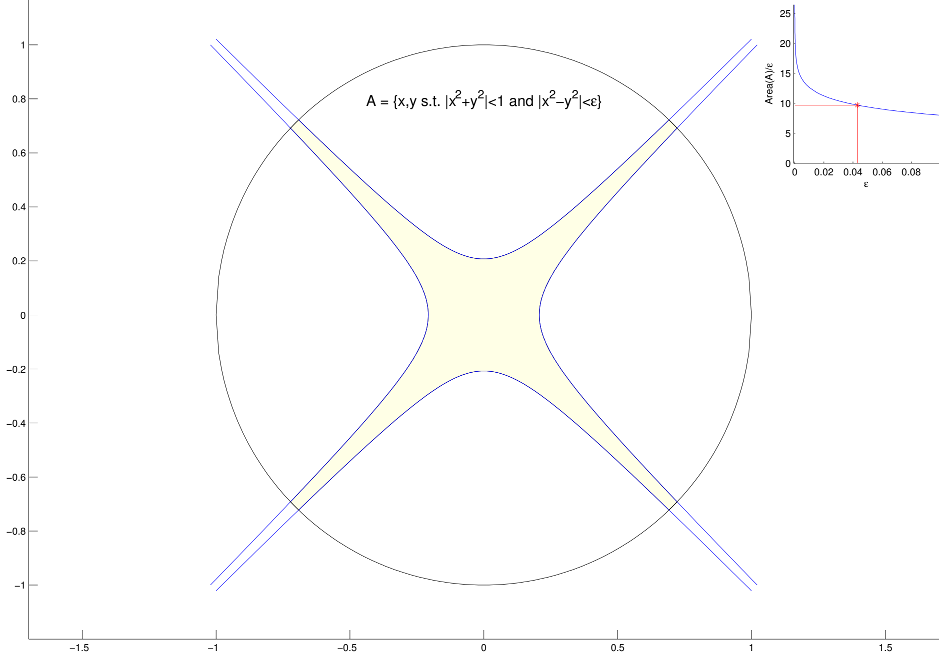
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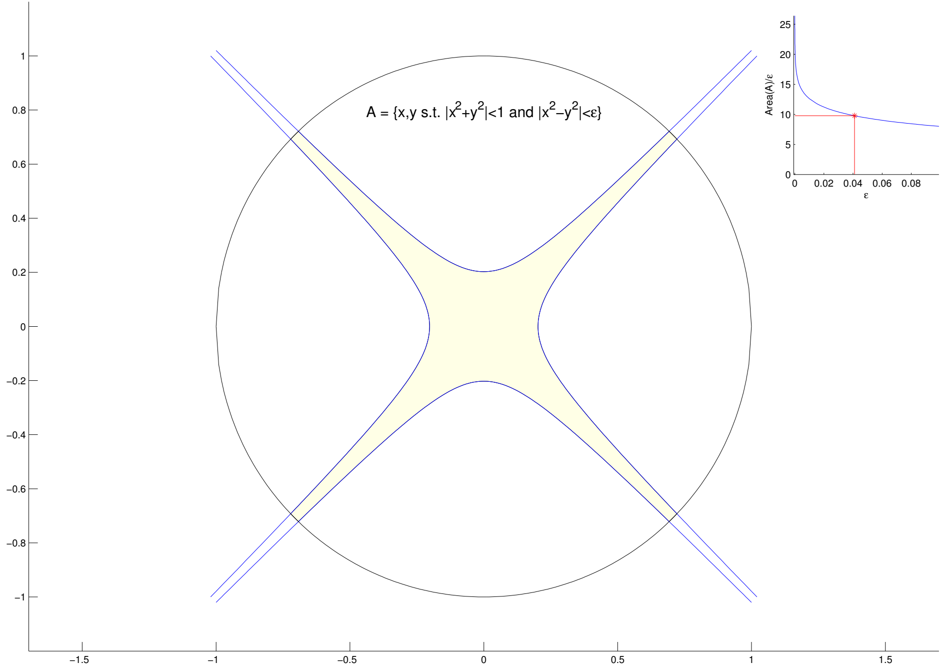
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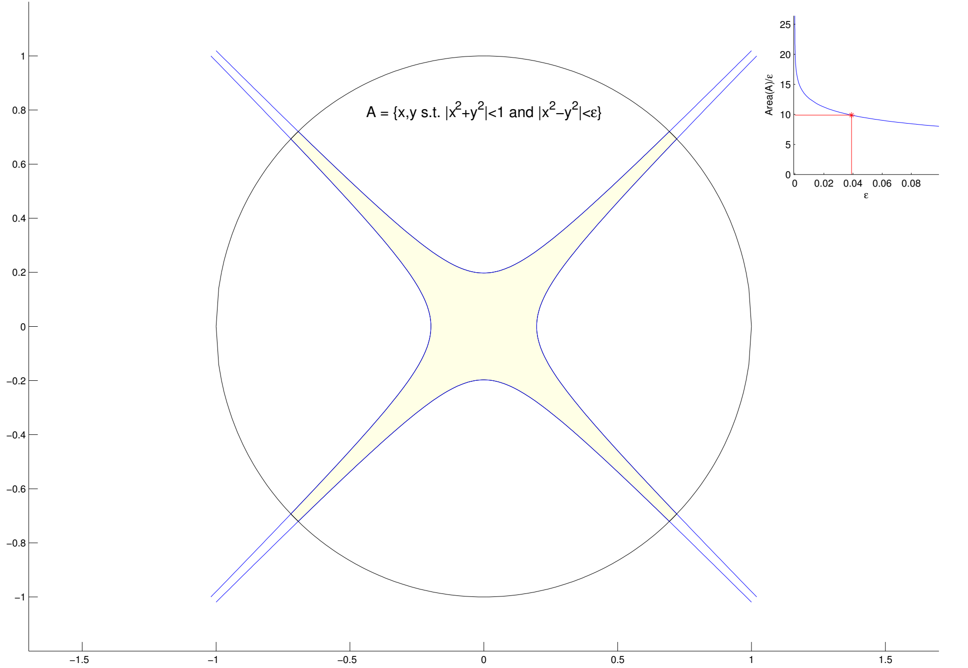

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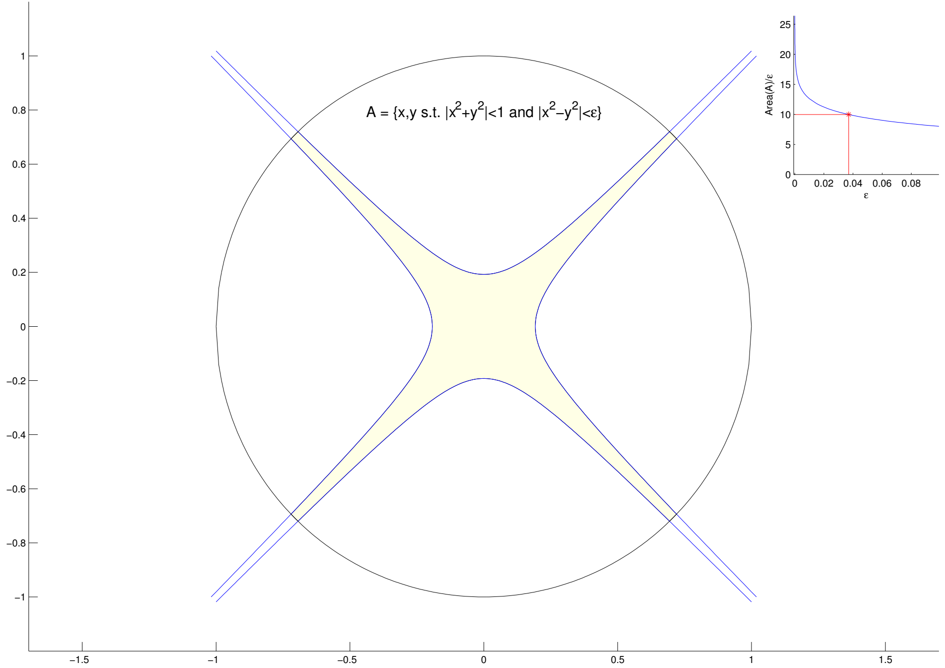
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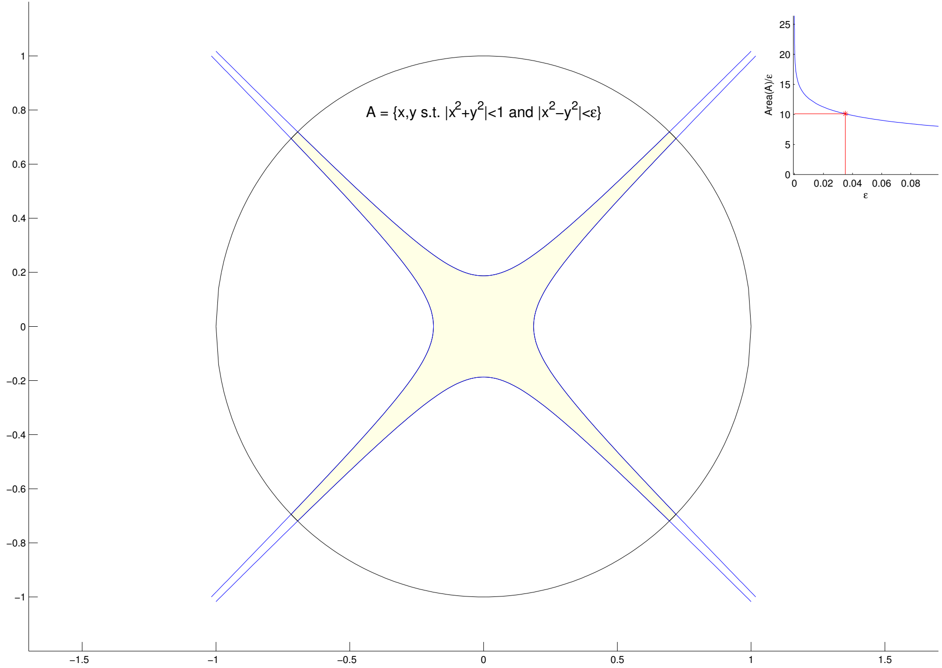
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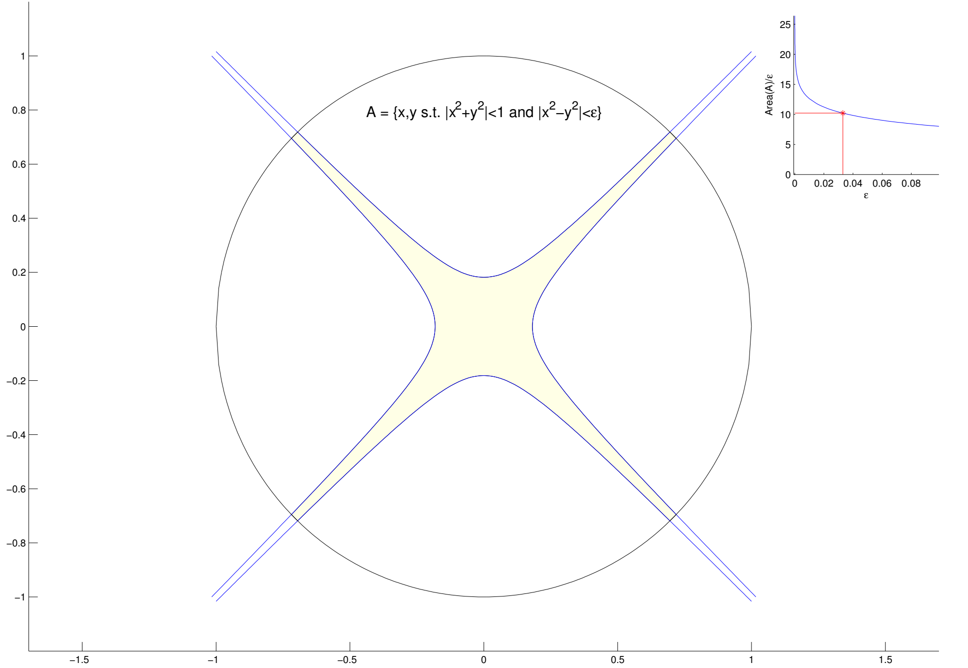

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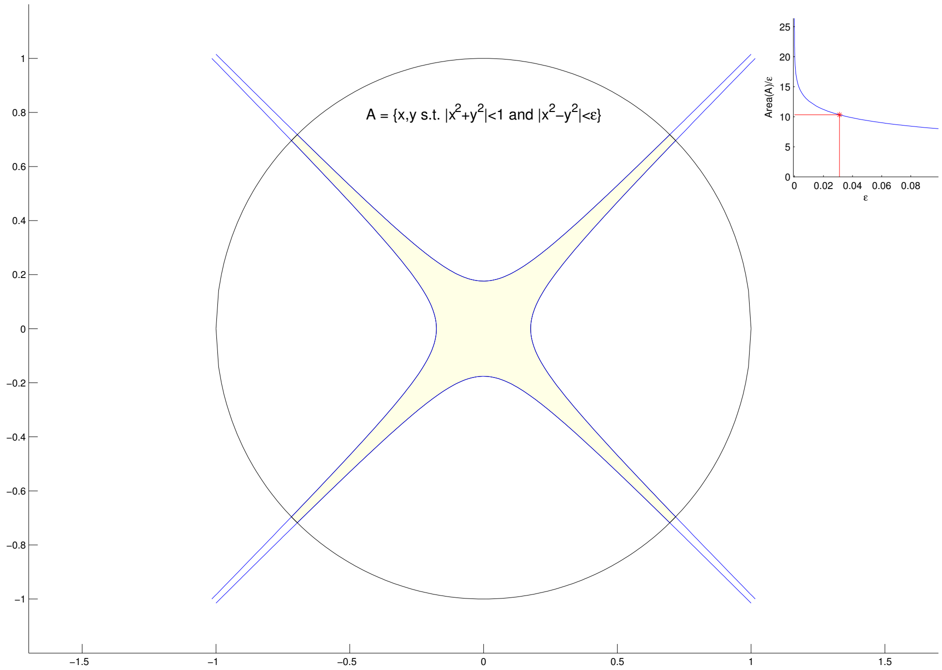
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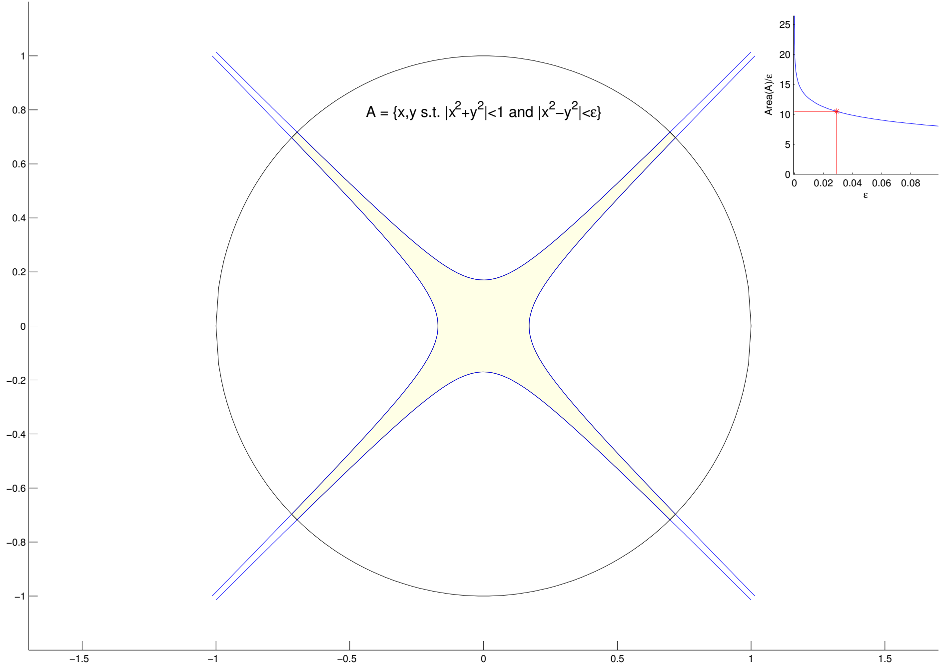

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
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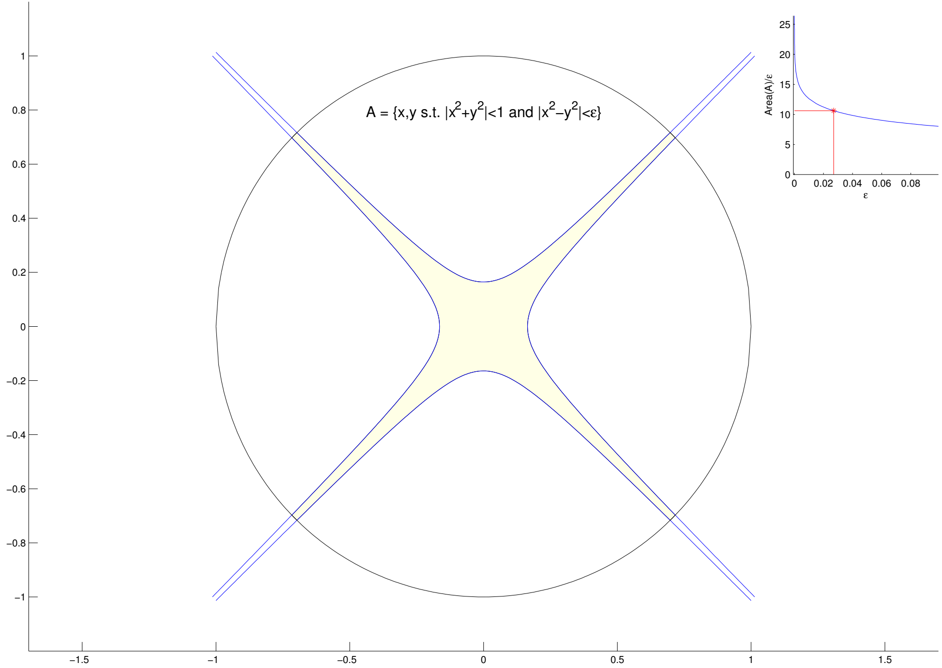
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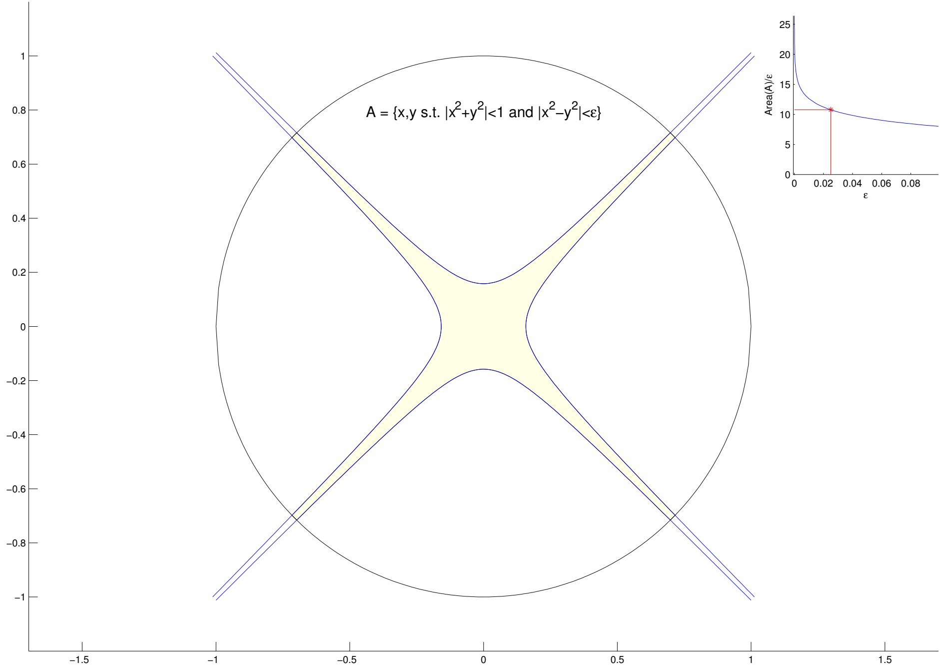
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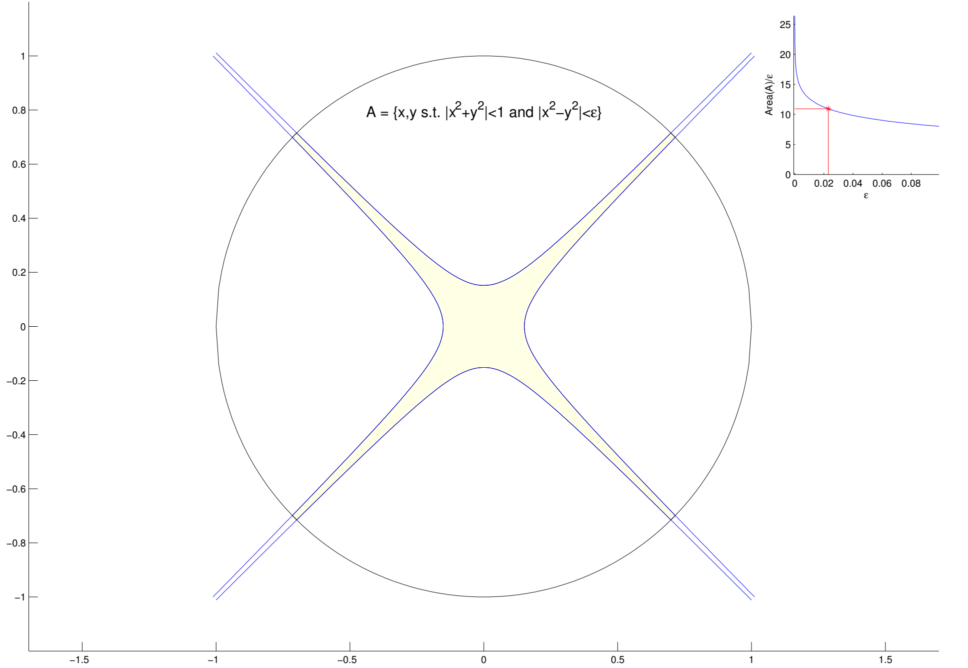
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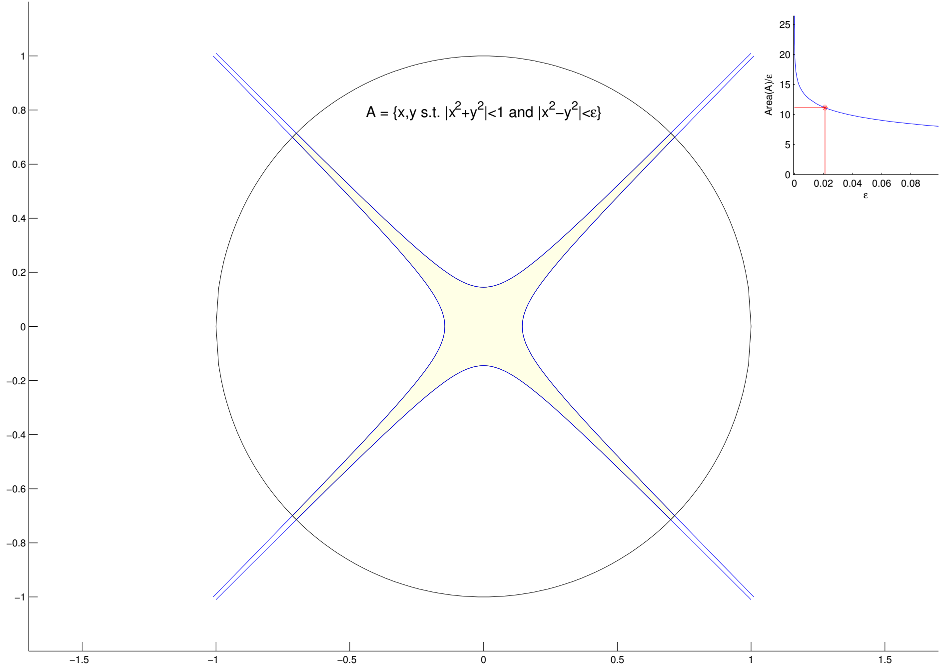

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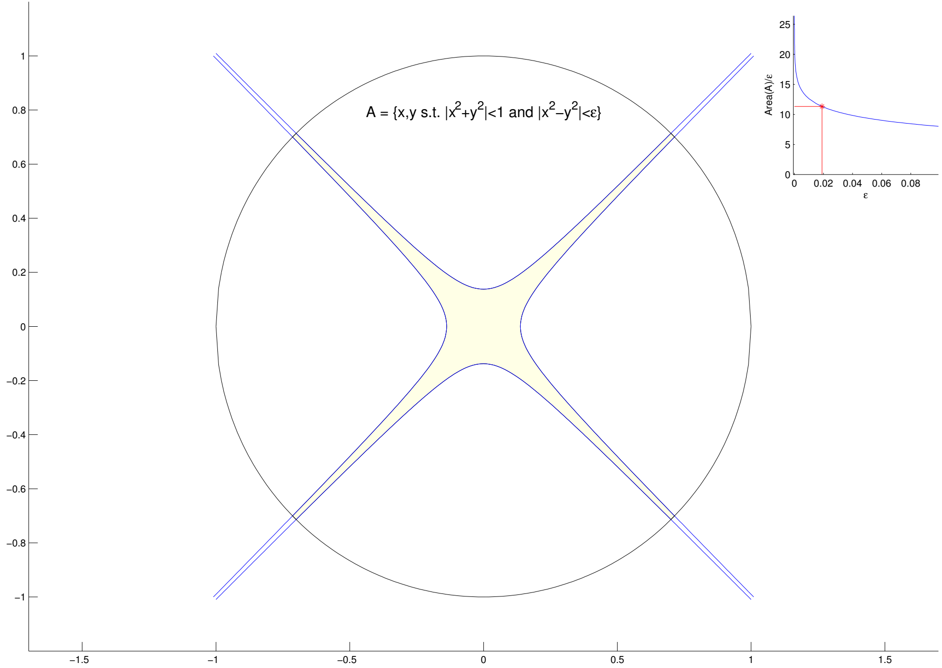

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
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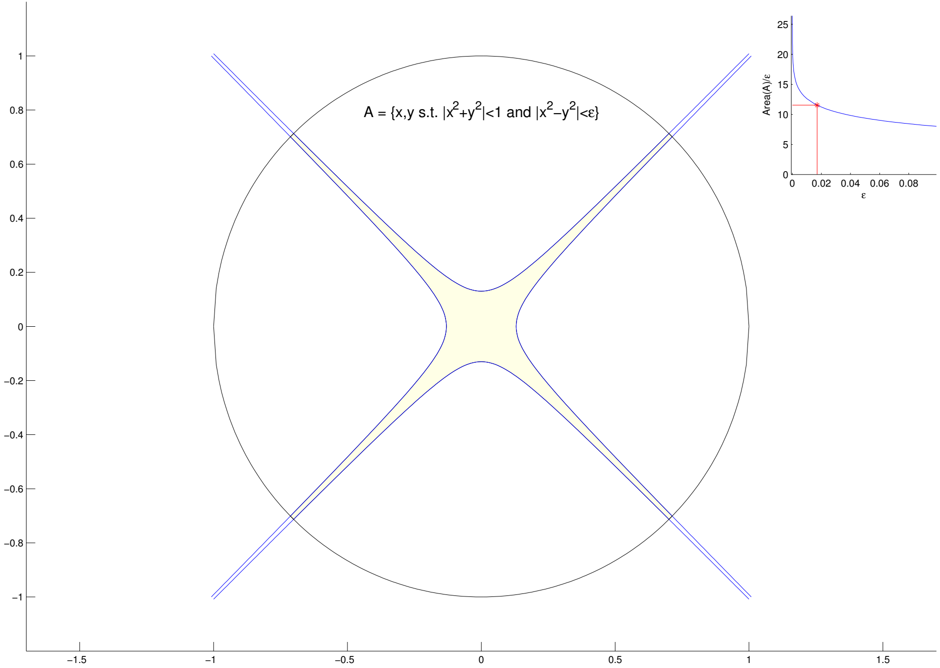

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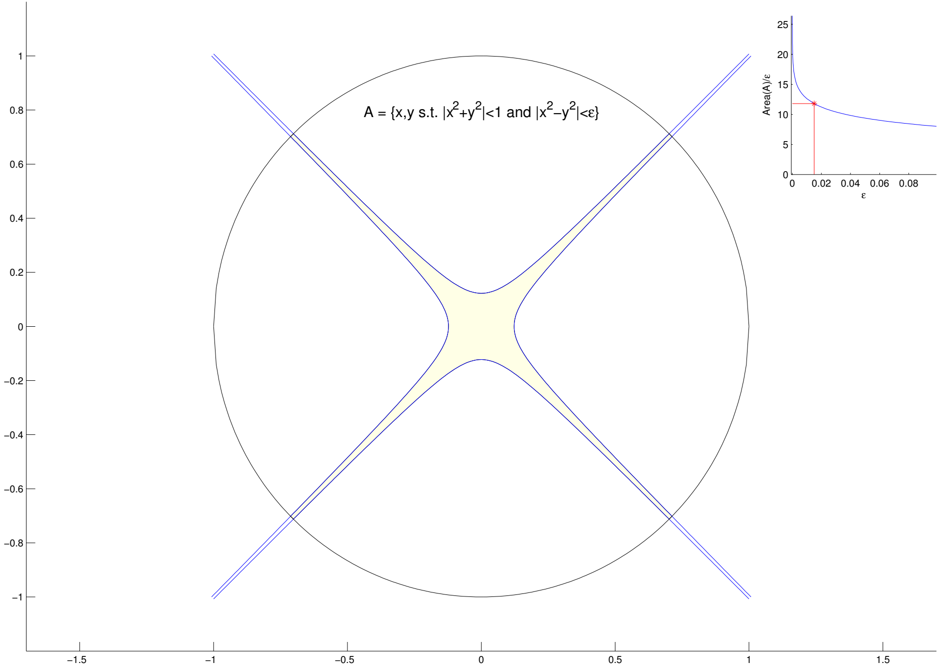
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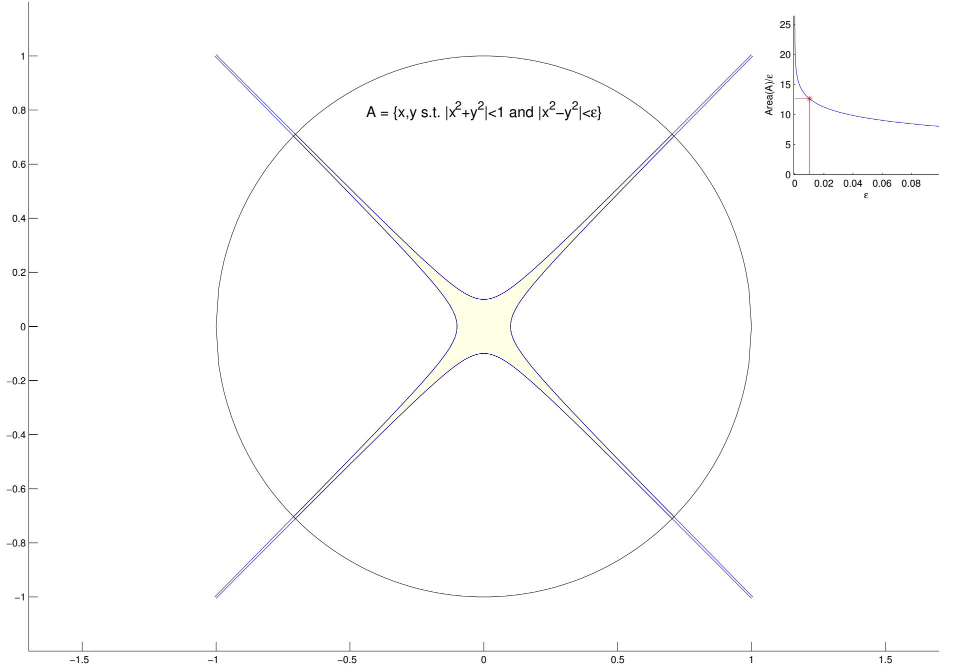

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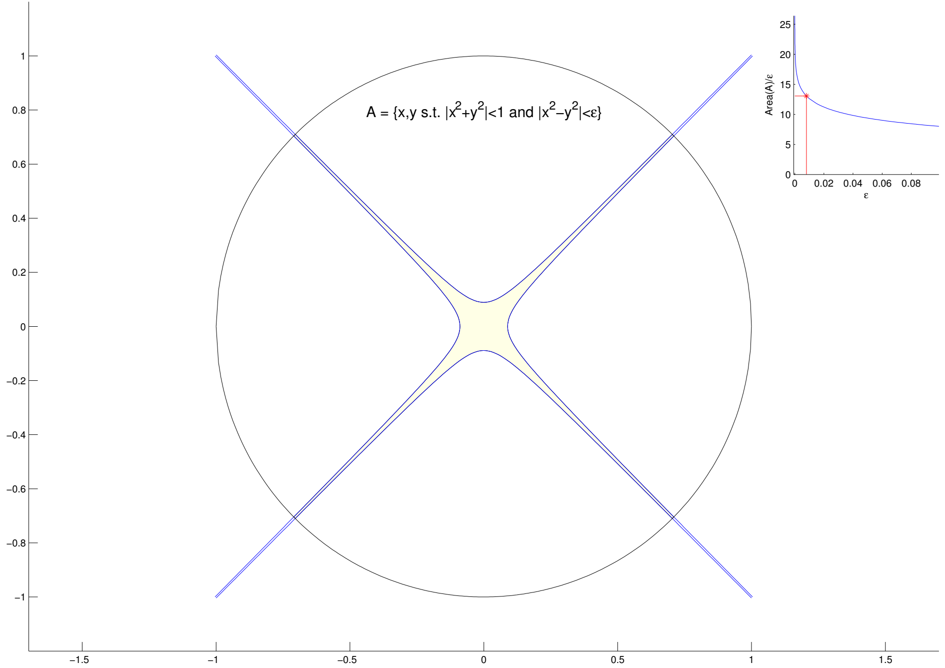

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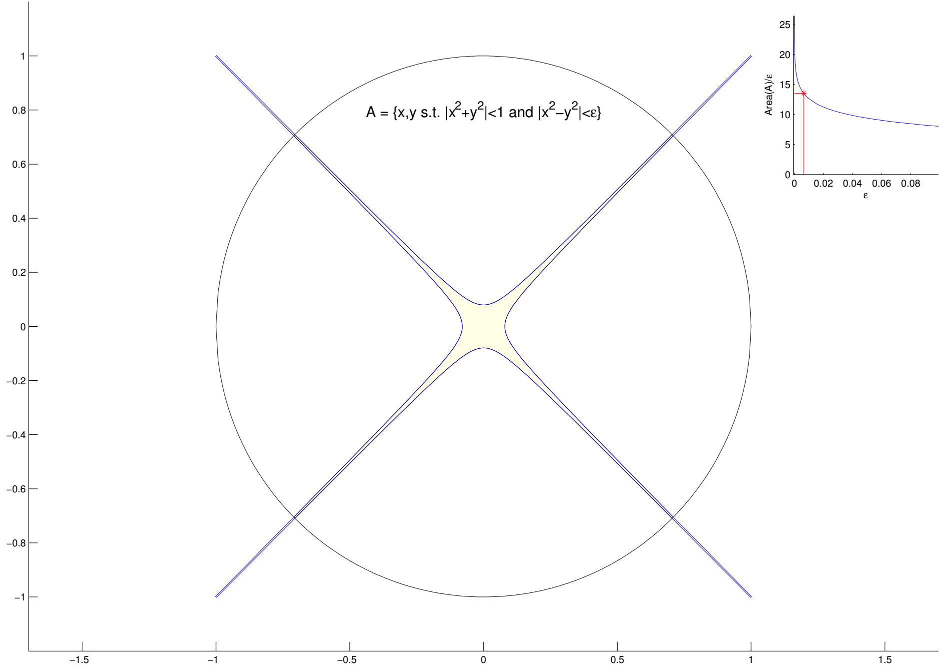
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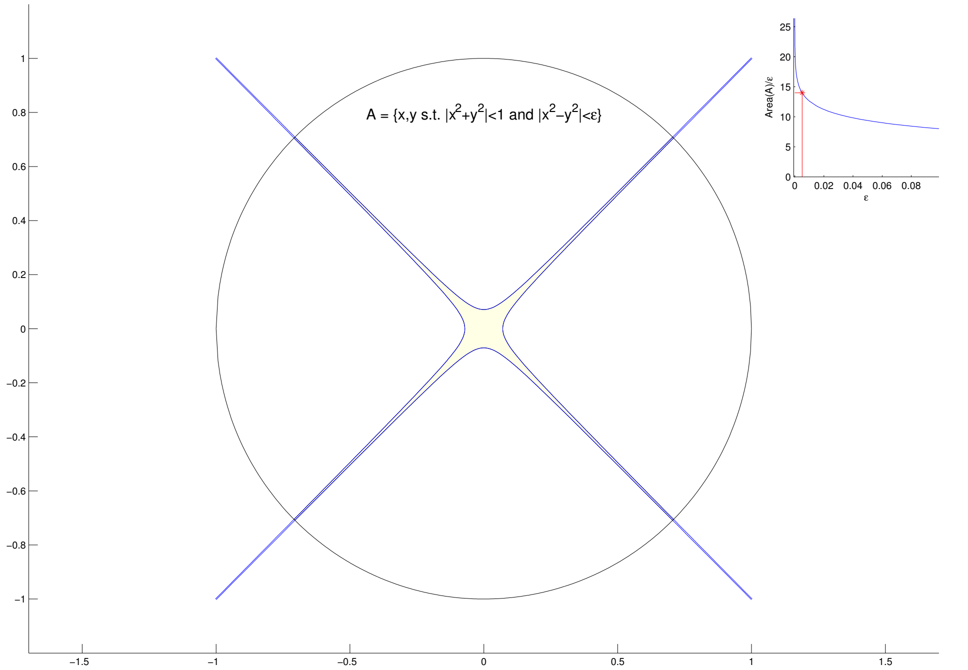
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
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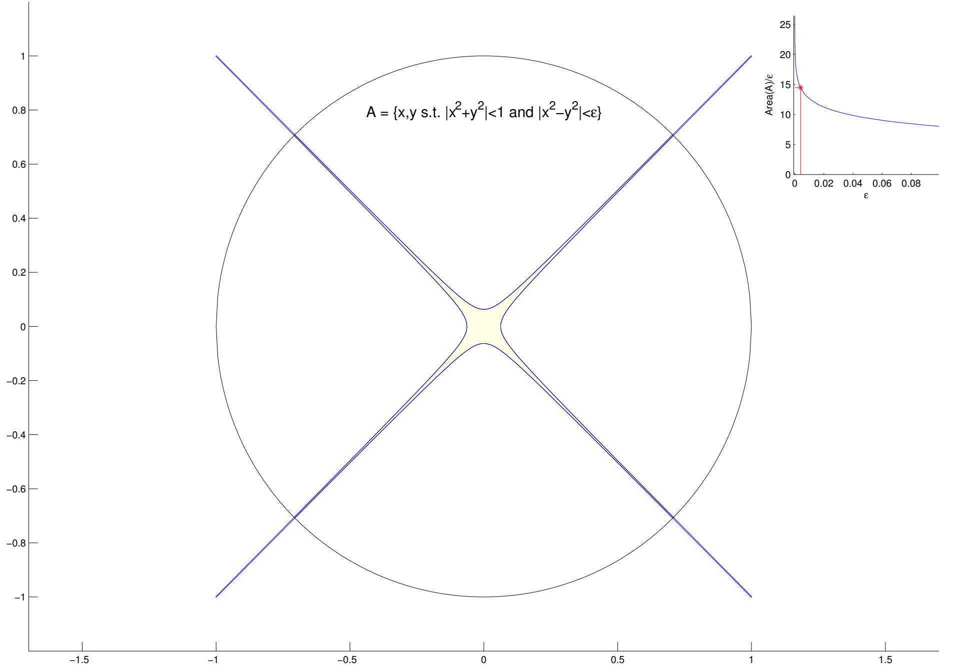

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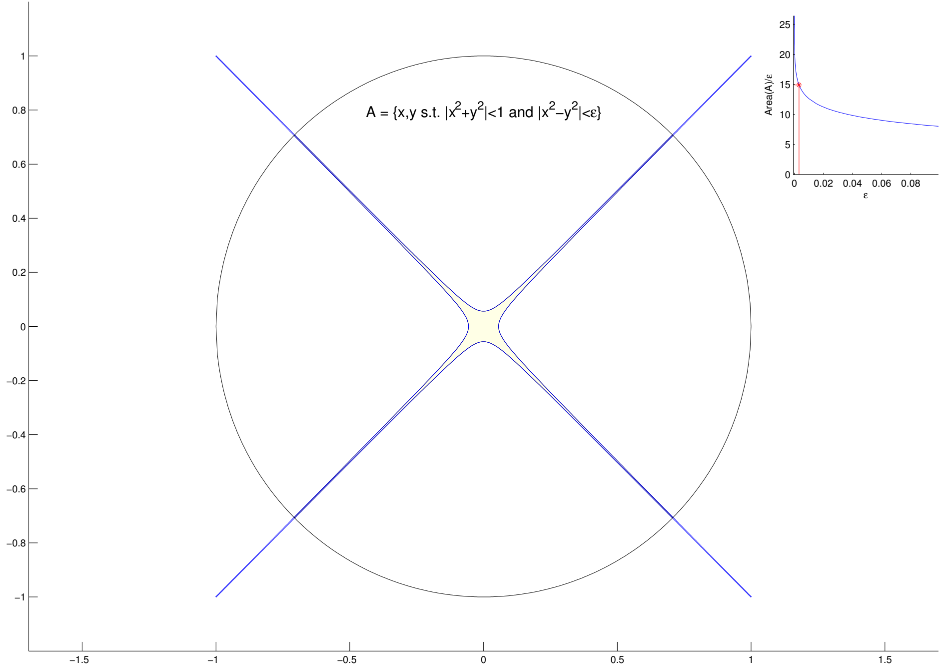
ε


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0
0.02
0.04
0.06
0.08
5
10
15
20
25
−1.5
−1
−0.5
0.5
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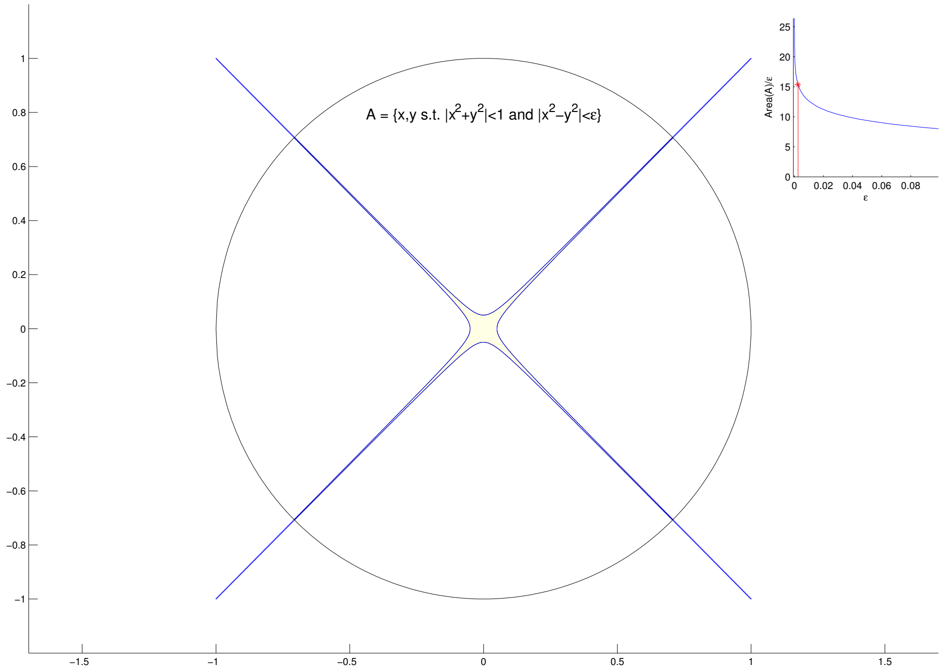
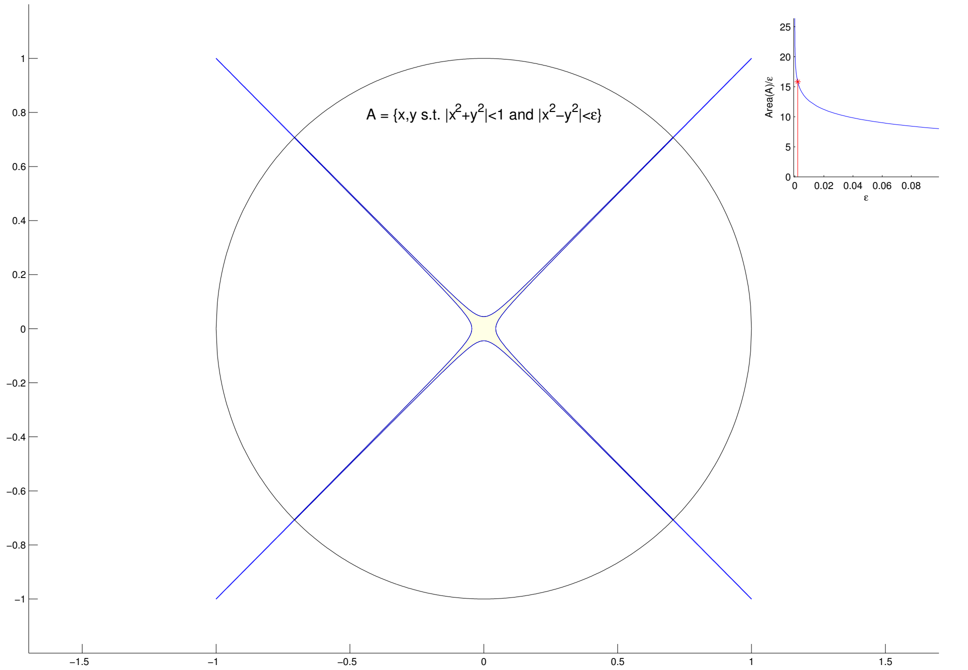
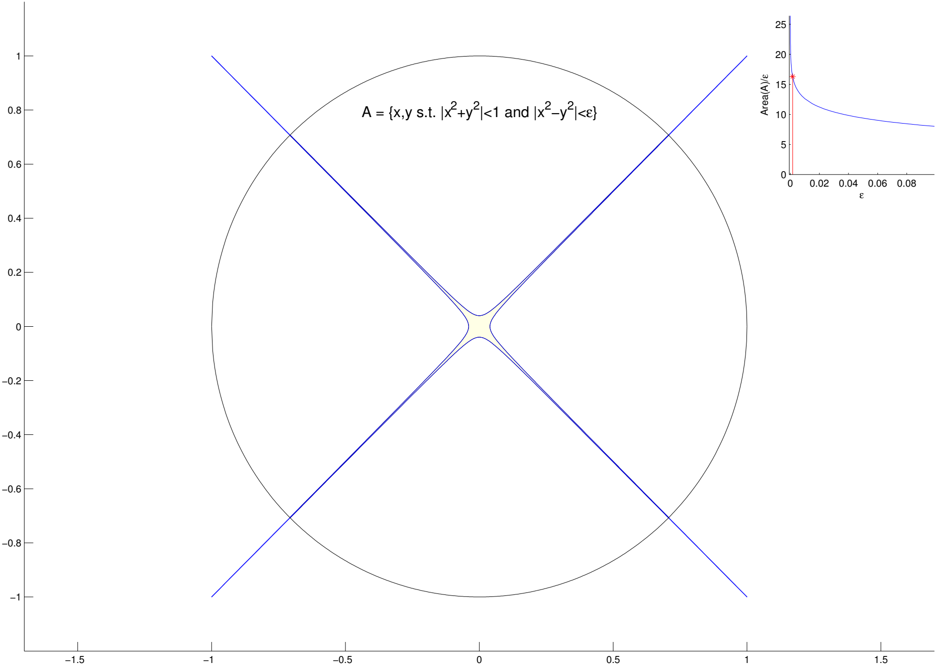
1.5
0.2
0.4
0.6
0.8
−0.2
−0.4
−0.6
−0.8



A = {x,y s.t. |x2+y2|<1 and |x2−y2|<ε}
Area(A)/ε
ε



A = {x,y s.t. |x²+y²|<1 and |x²−y²|<ε}
Area(A)/ε
ε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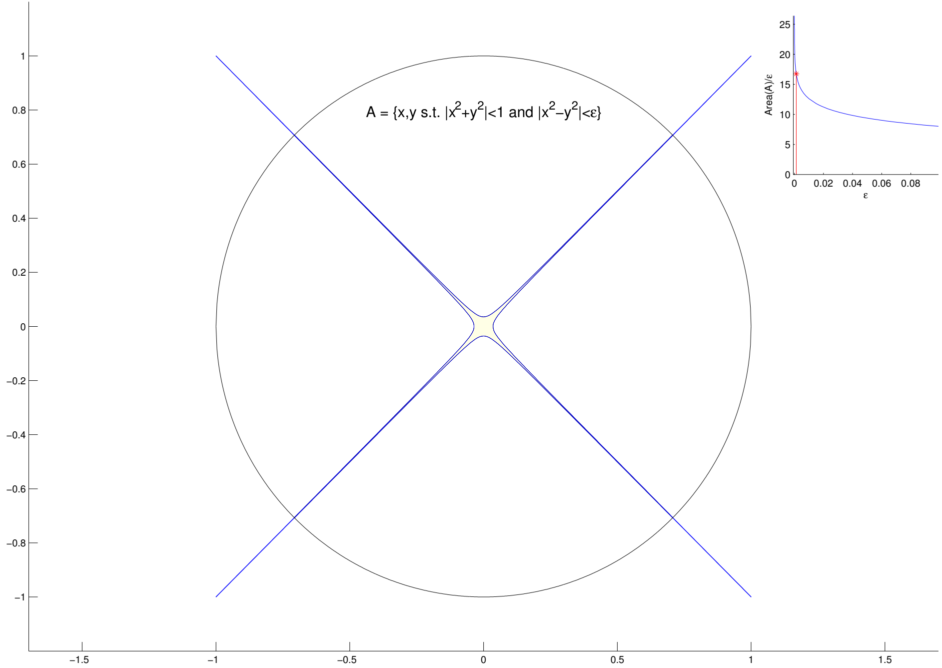
Area(A)/ε
ε


A = {x,y s.t. $|x^2+y^2|<1$ and $|x^2-y^2|<\varepsilon$}
Area(A)/ε
ε
0
0.02
0.04
0.06
0.08
0
5
10
15
20
25
−1.5
−1
−0.5
0
0.5
1
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0.4
0.2
0
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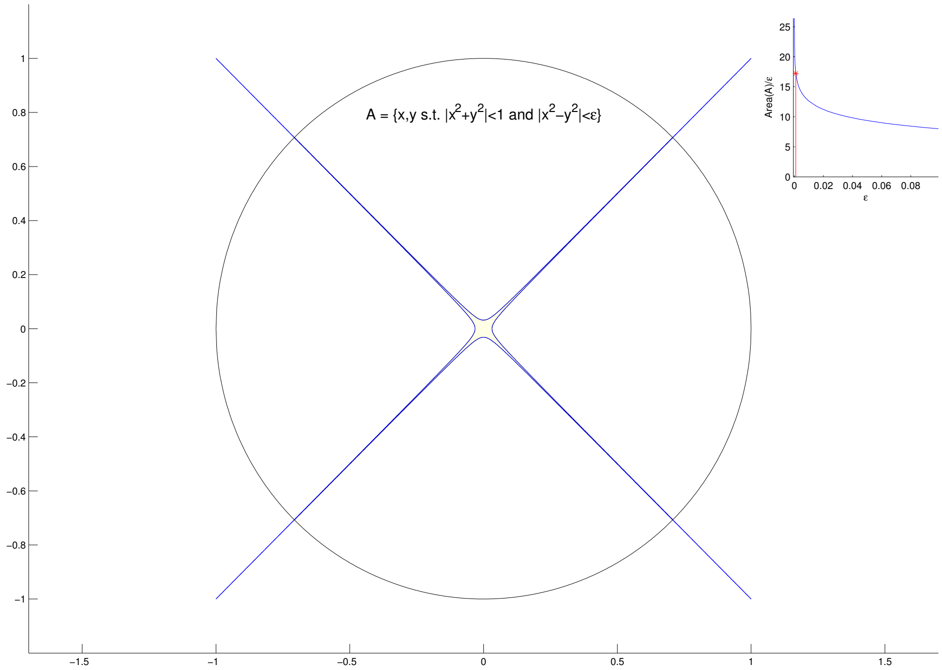
−0.4
−0.6
−0.8
−1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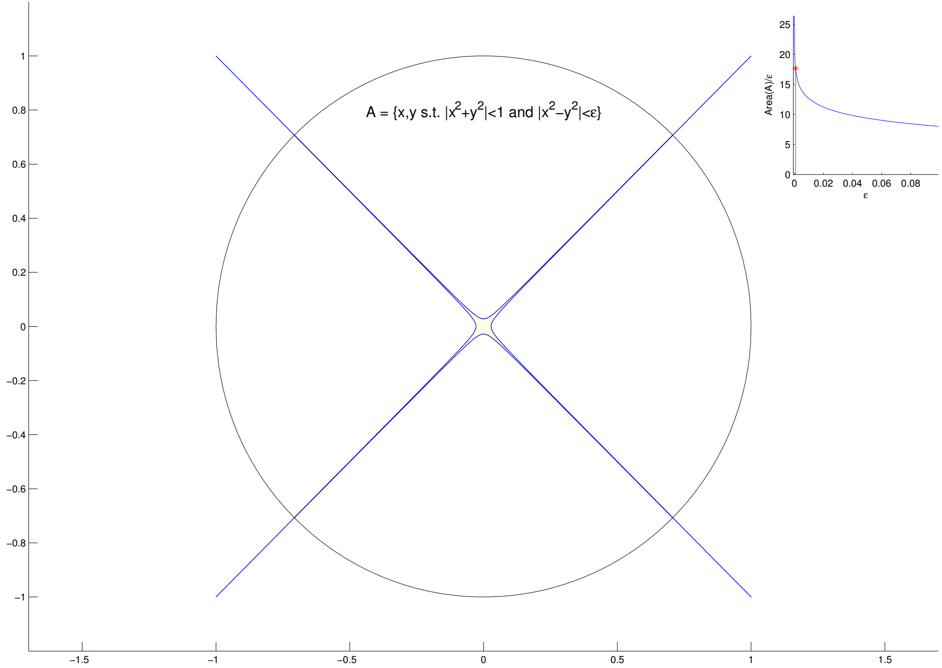
Area(A)/ε
ε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Area(A)/ε
ε
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20
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10
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0
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0.04
0.06
0.08
1
0.8
0.6
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−0.4
−0.6
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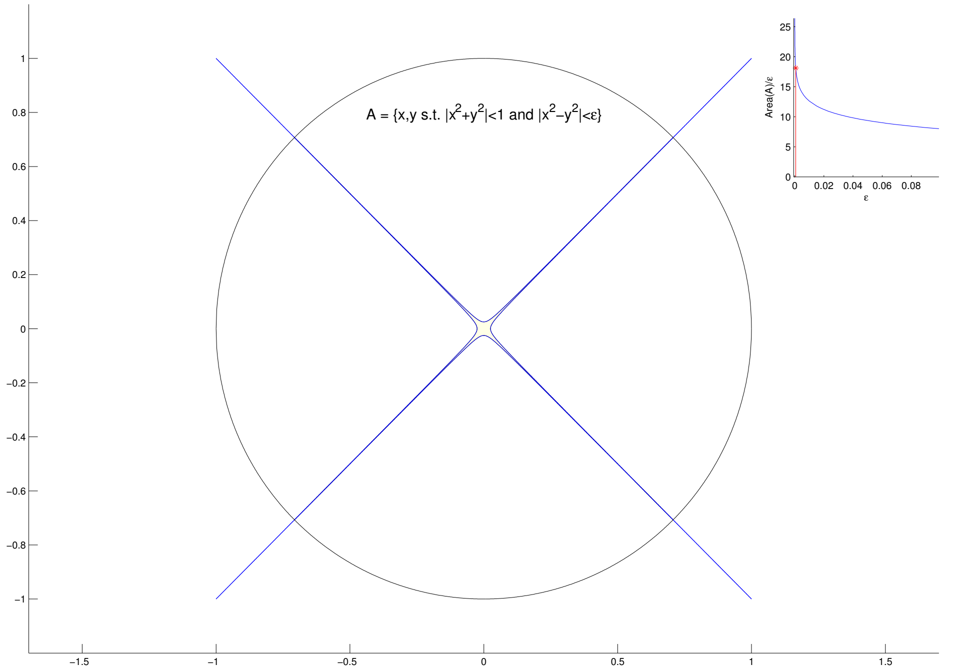
−1
−1.5
−0.5
0.5
1.5



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Area(A)/ε
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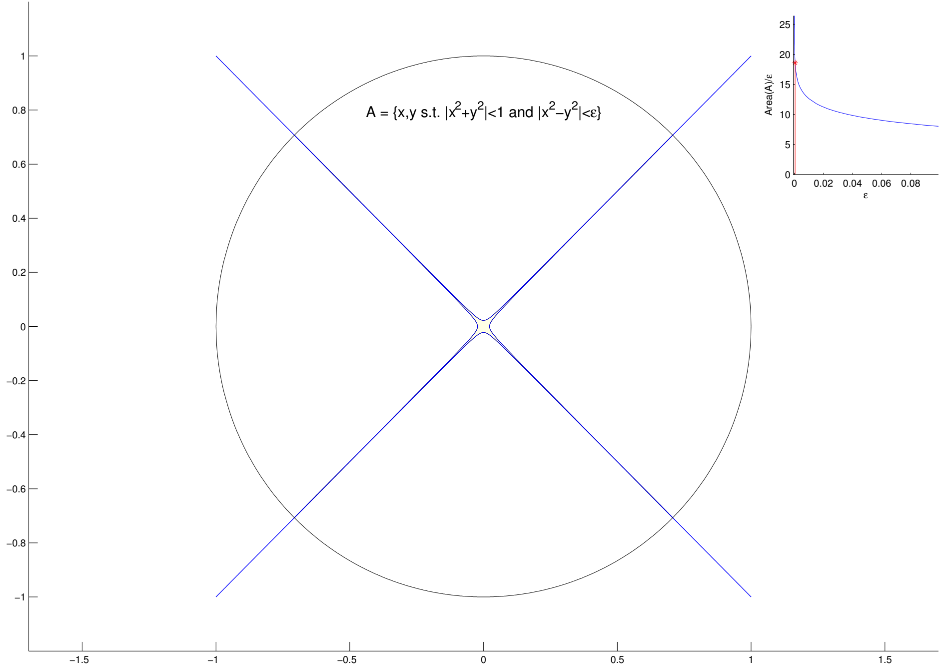

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Area(A)/ε
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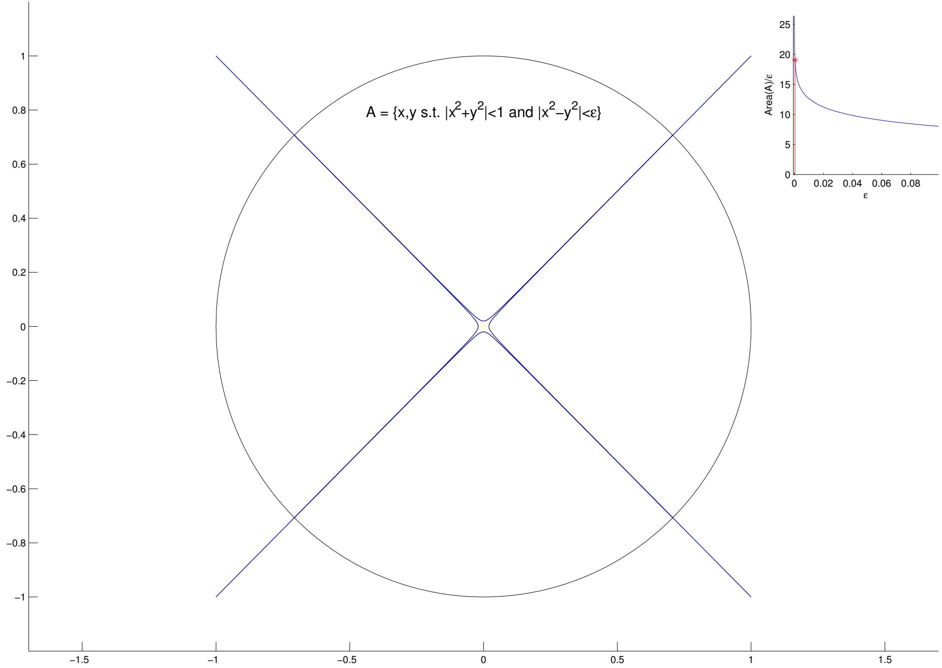
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0
5
10
15
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25
0.02
0.04
0.06
0.08
1
0.8
0.6
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−0.4
−0.6
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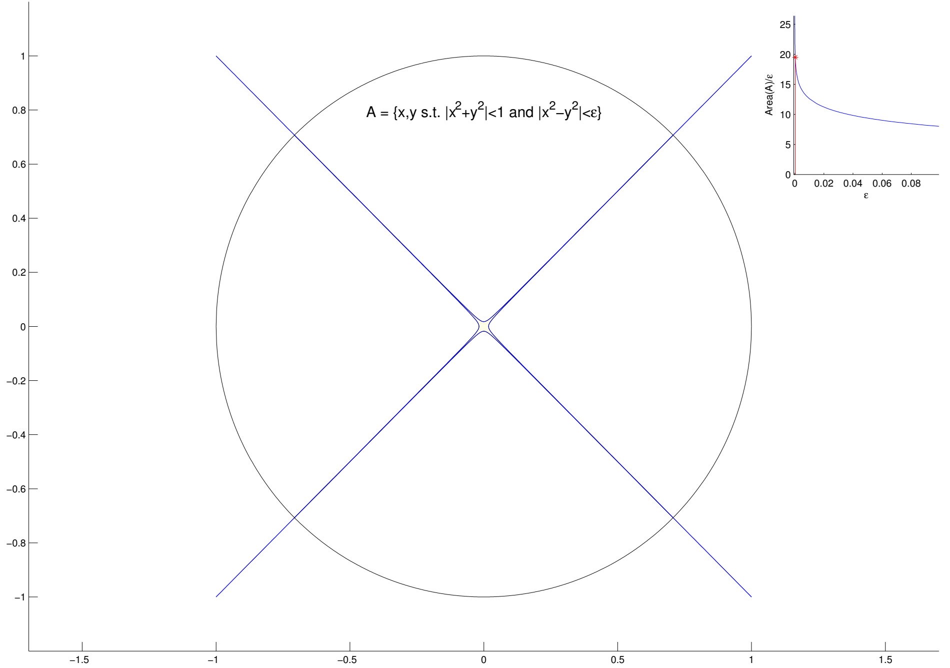
−1
−1.5
−0.5
0.5
1.5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Area(A)/ε
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20
15
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0.02
0.04
0.06
0.08
1
0.8
0.6
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−0.8
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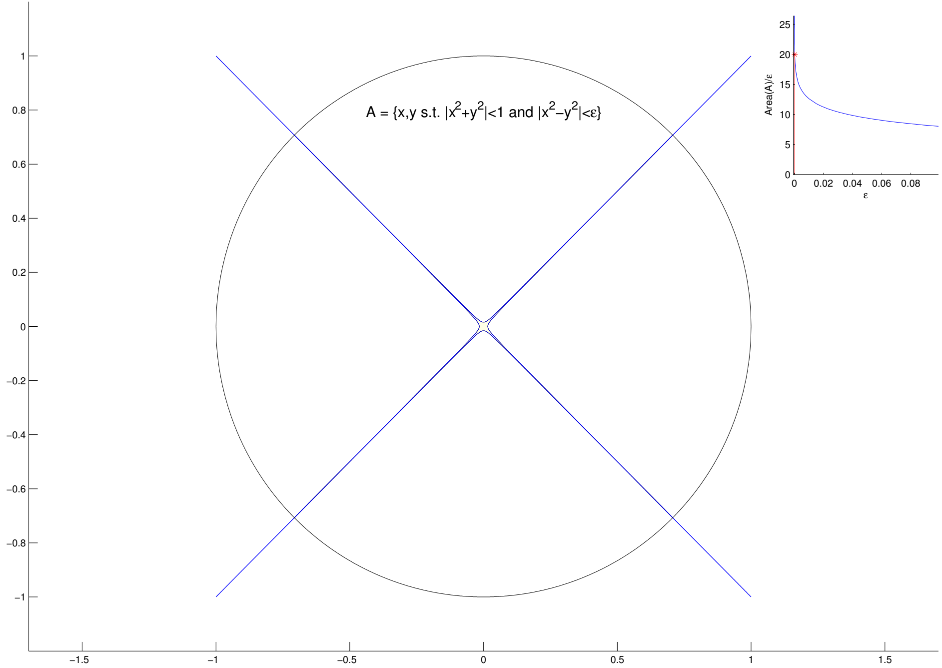
−1.5
−0.5
0.5
1.5


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
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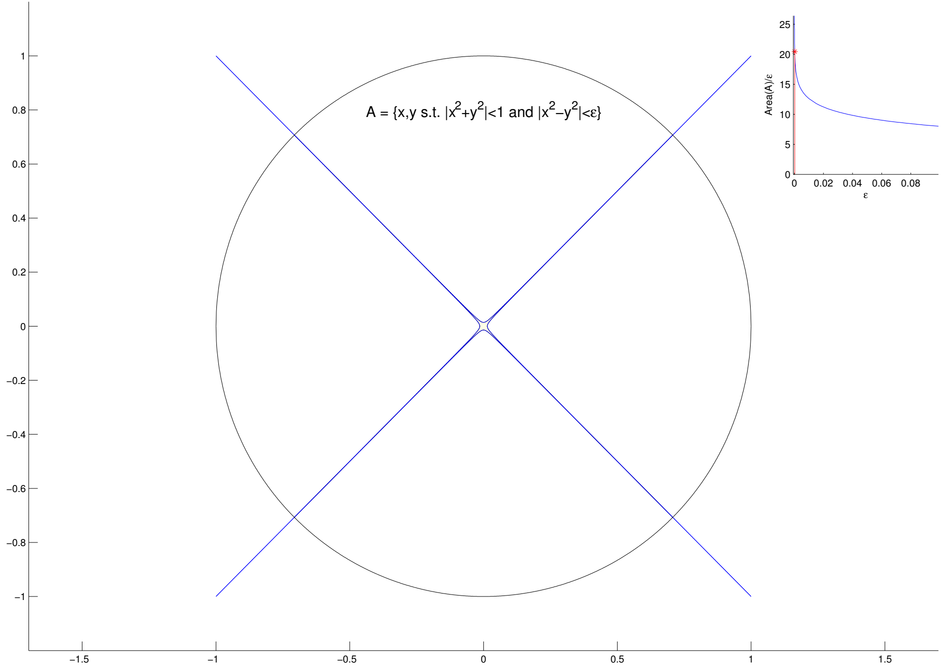
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5
Area(A)/ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
ε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−0.6
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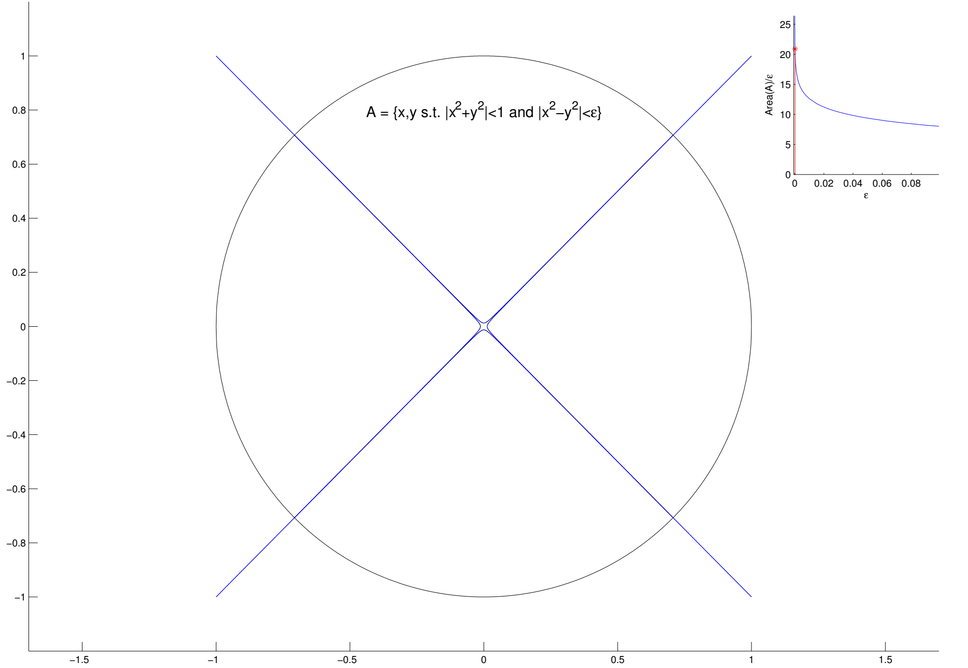
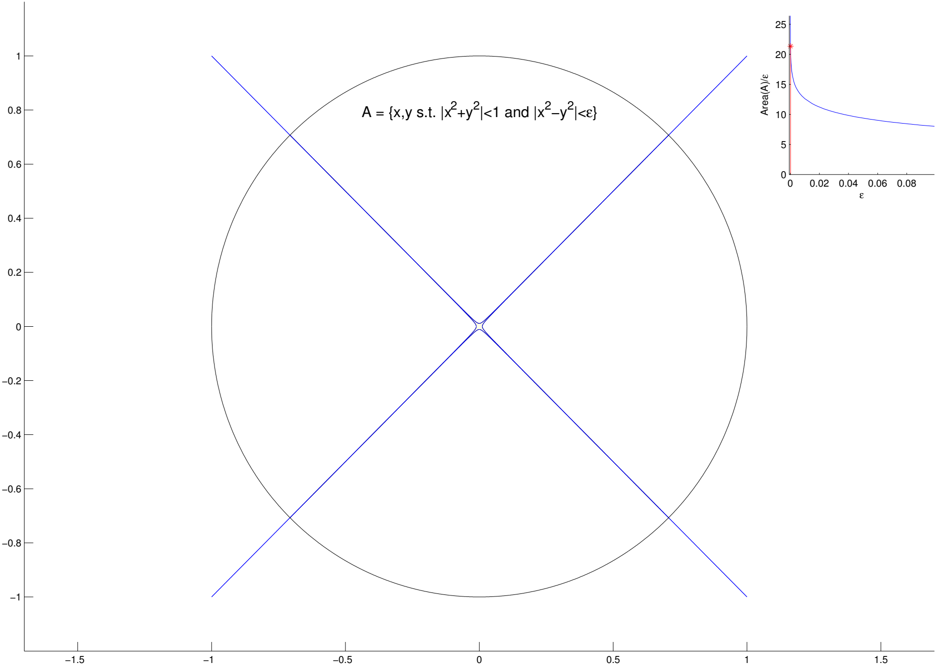
−0.5
0.5
1.5


A = {x,y s.t. |x²+y²|<1 and |x²−y²|<ε}
Area(A)/ε
ε


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
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0.08
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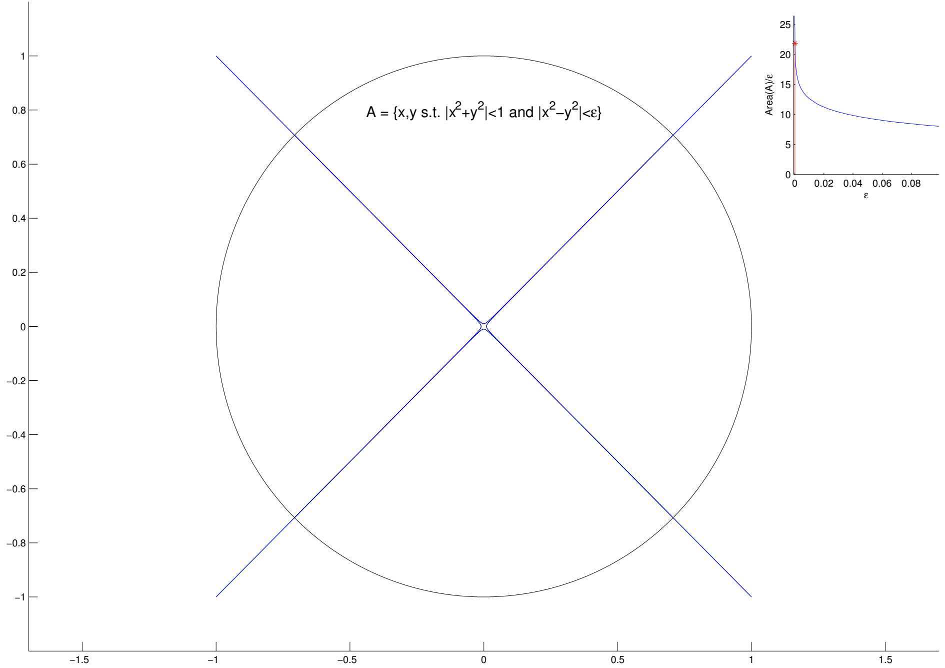
1
1.5


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
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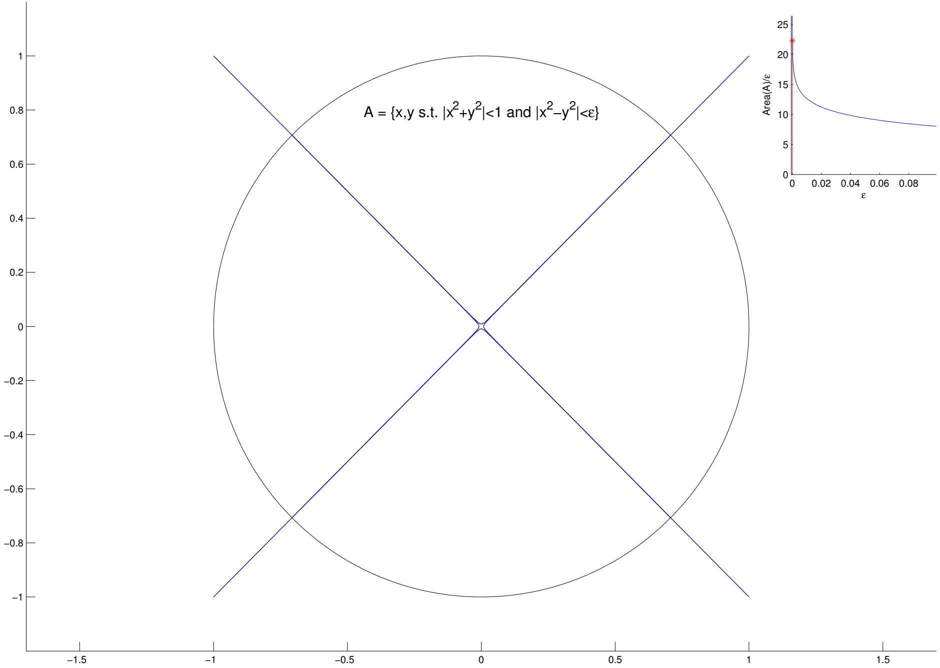
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0
5
10
15
20
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0.02
0.04
0.06
0.08
1
0.8
0.6
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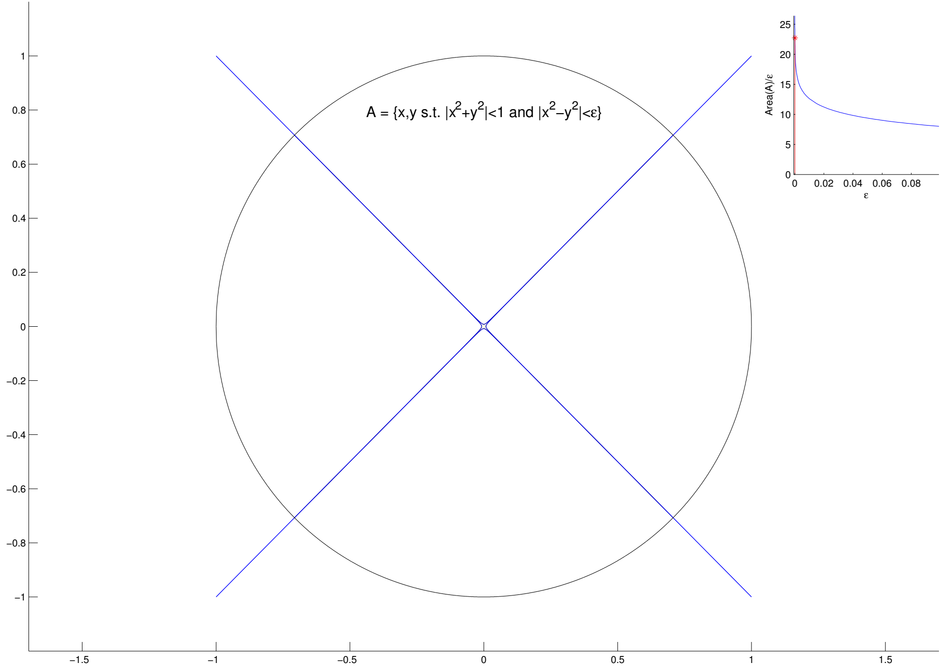
0.5
1.5



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ε
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15
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0.08
1
0.8
0.6
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0.2
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−0.5
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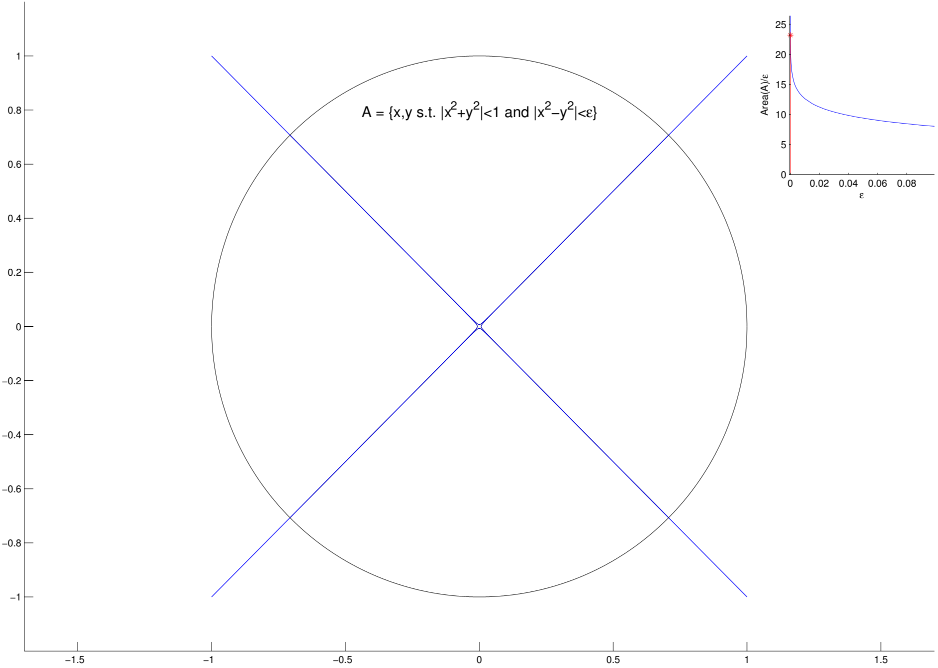
1
1.5


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
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0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
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5
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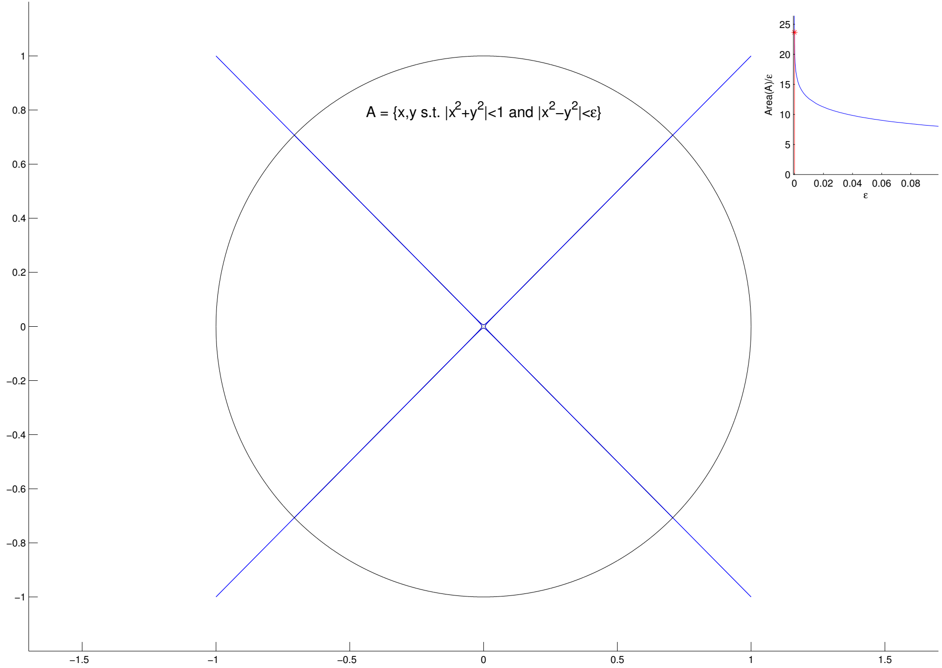
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
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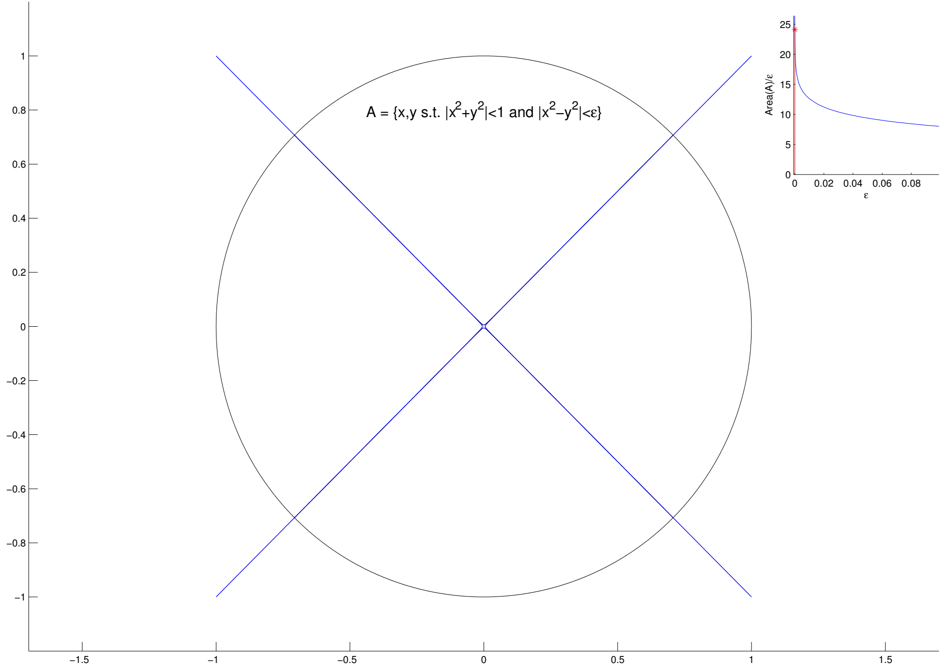
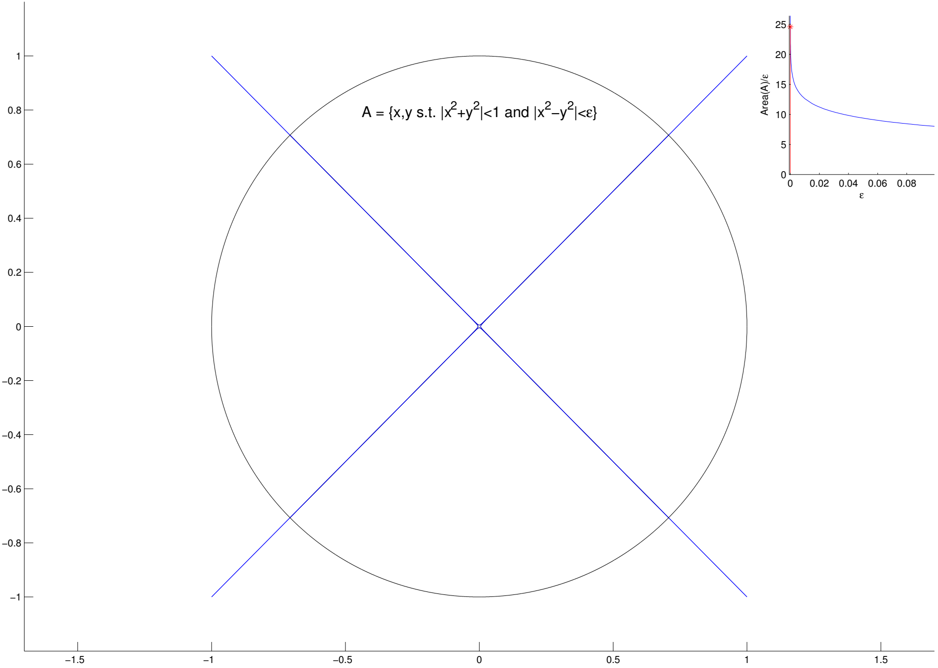
1.5


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε



A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
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20
15
10
5
0
0.02
0.04
0.06
0.08
1
0.8
0.6
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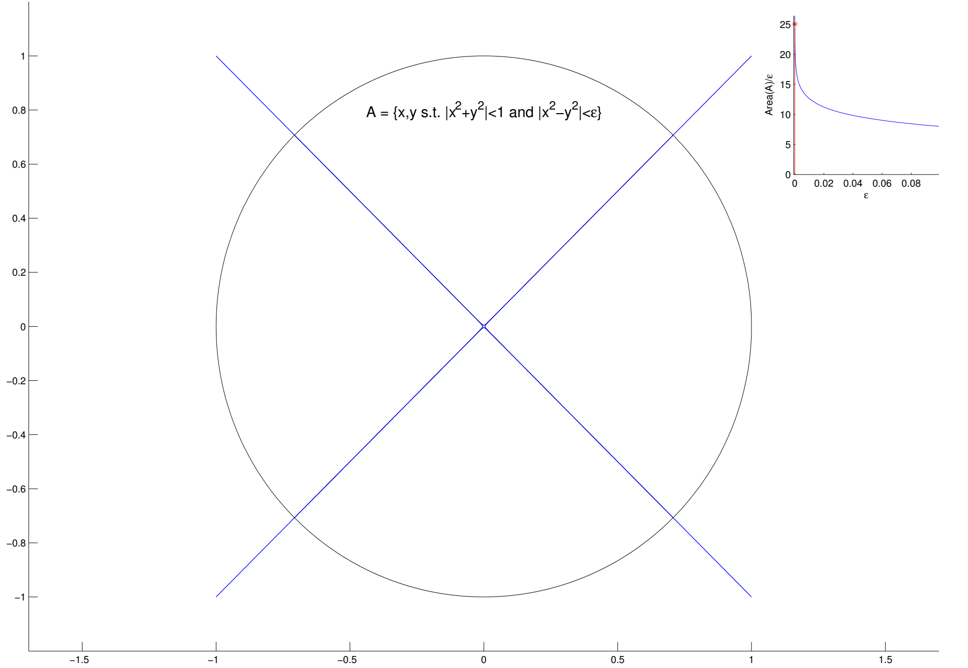
−0.8
−1
−1.5
−0.5
0.5
1.5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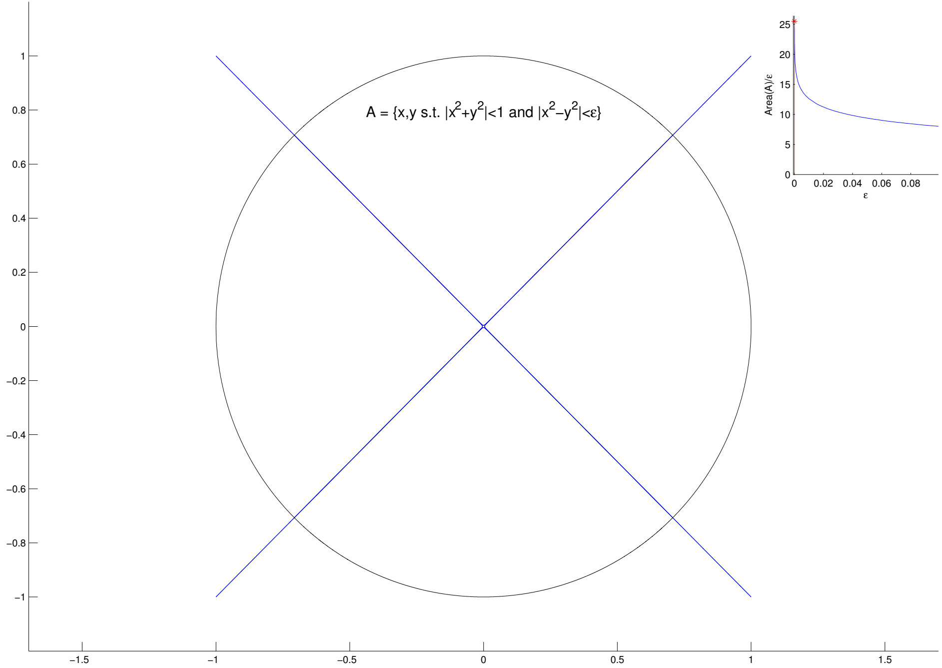
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
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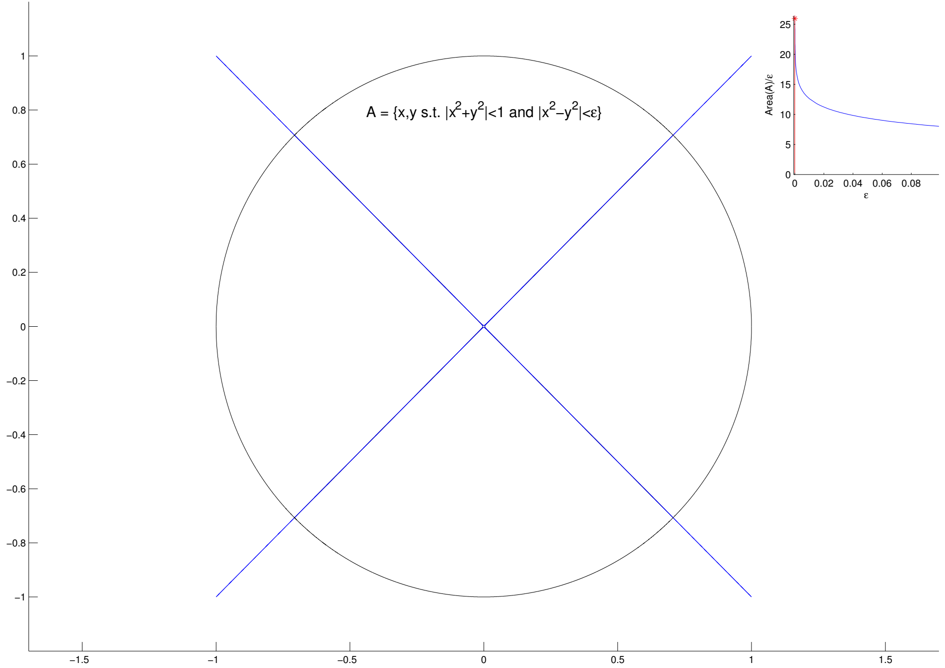
ε



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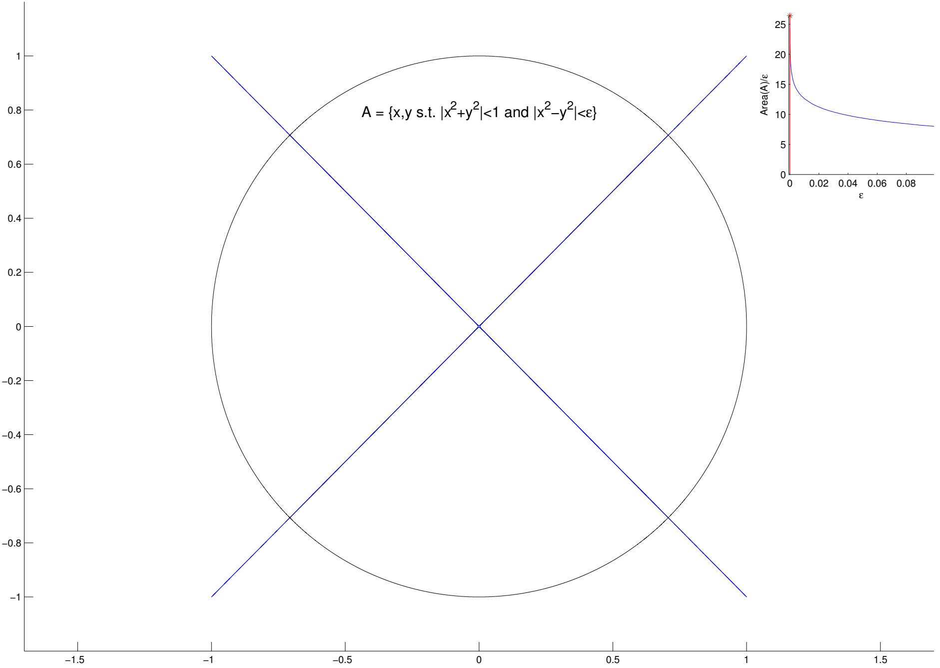
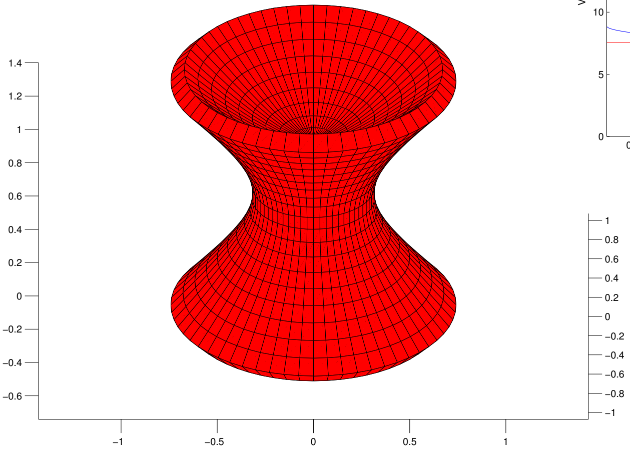
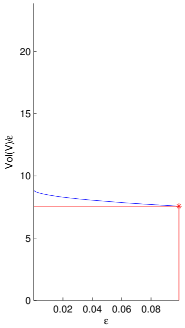


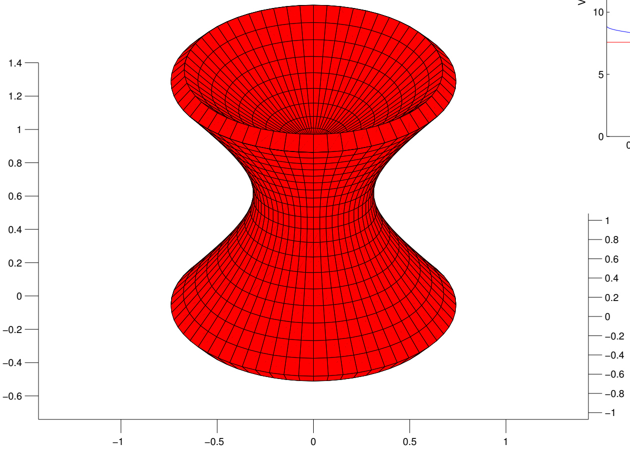
A = {x,y s.t. $|x^2+y^2|<1$ and $|x^2-y^2|<\varepsilon$}
Area(A)/ε
ε

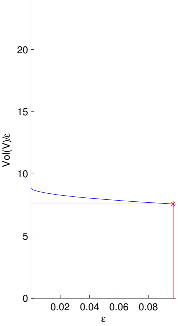
V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}



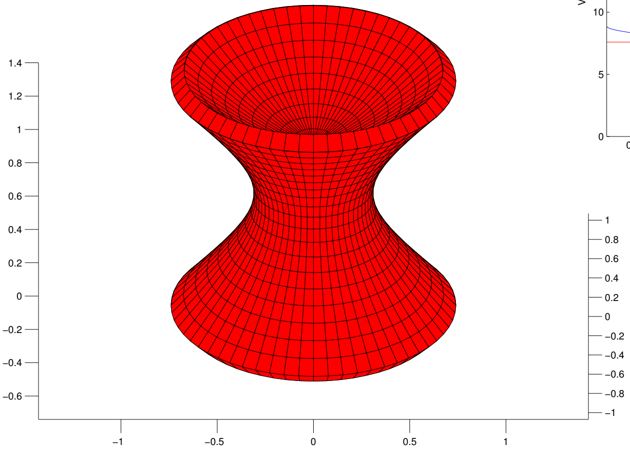


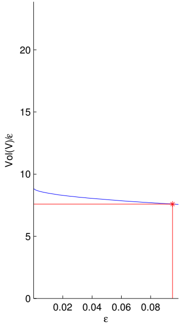
V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}

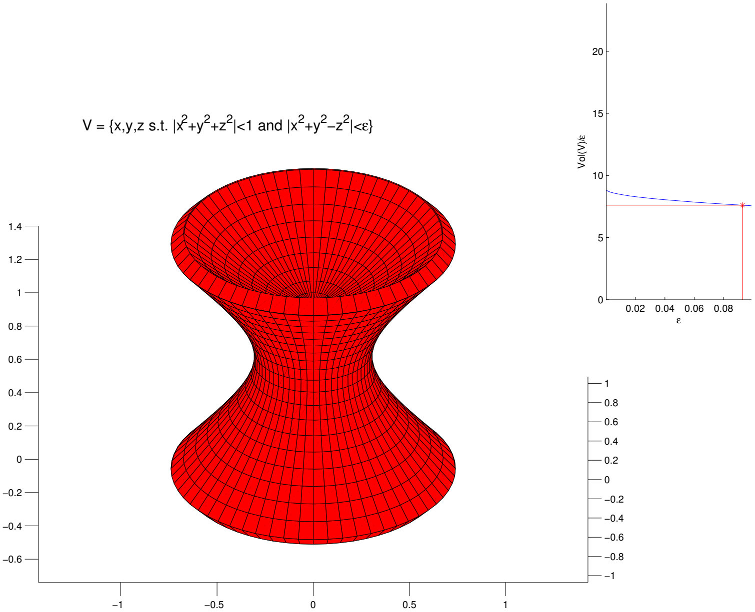


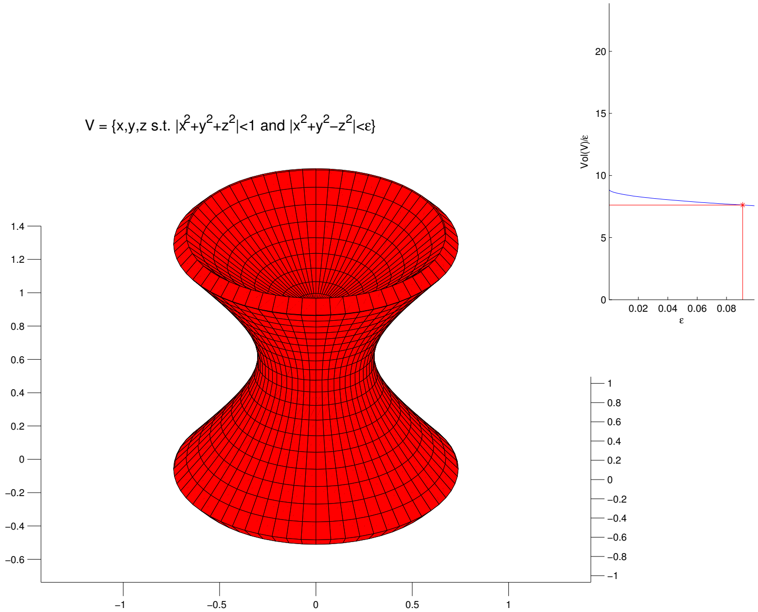


$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$






V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1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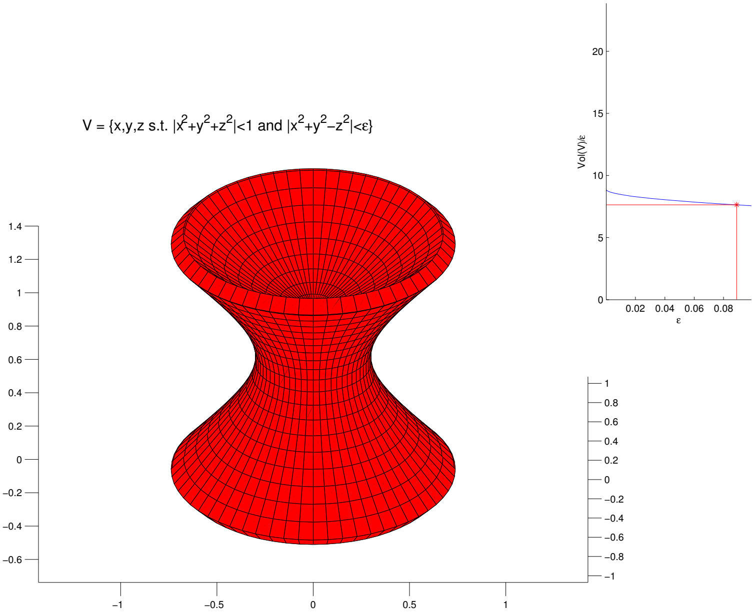
Vol(V)/ε
ε


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0
5
10
15
20
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
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−0.4
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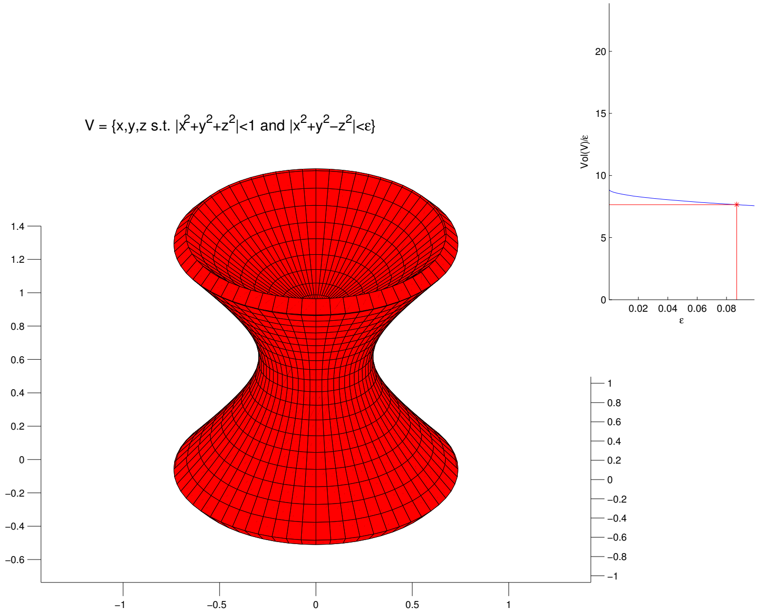
−0.8
−1


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
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0.08
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−0.4
−0.6
−1
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−0.4
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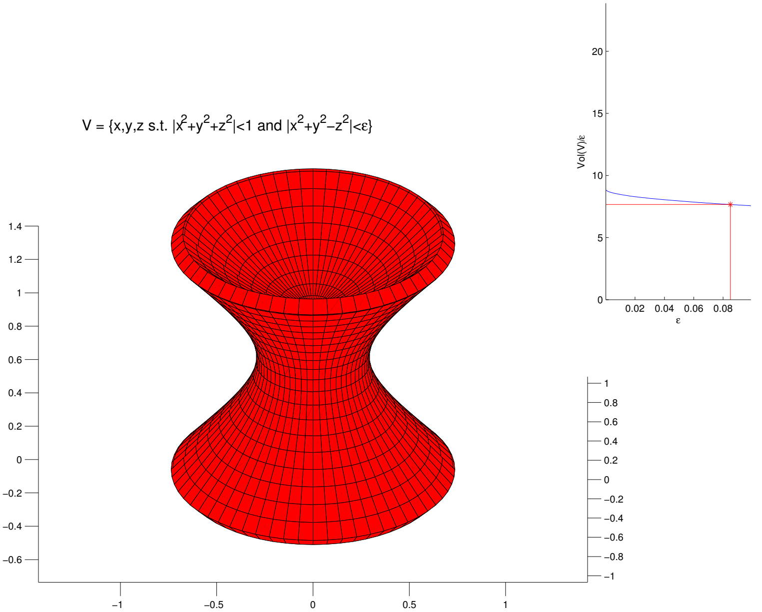
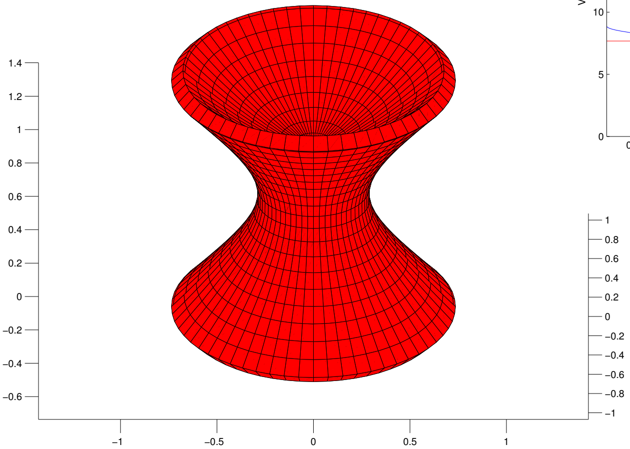
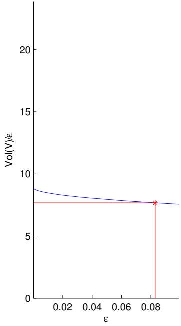
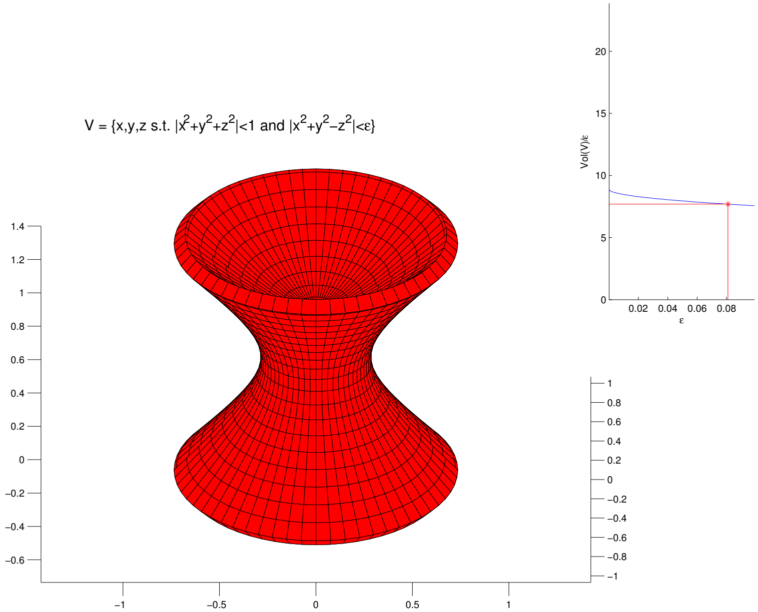
−1


V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
0 5 10 15 20
0.02 0.04 0.06 0.08
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

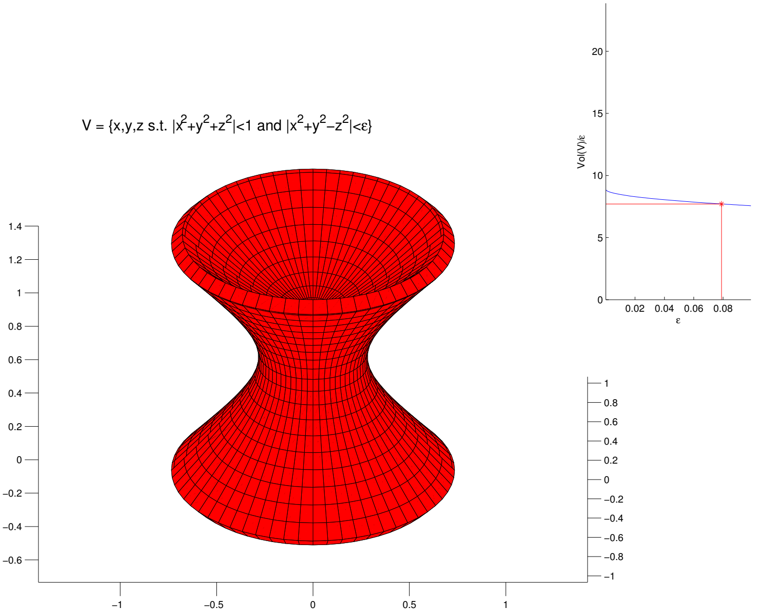
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



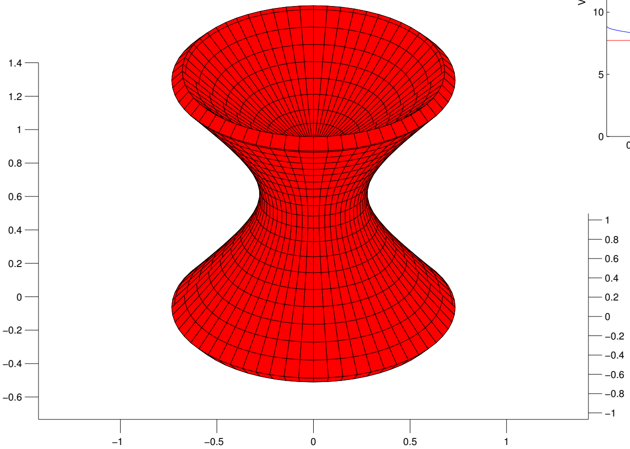


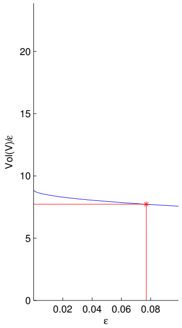


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

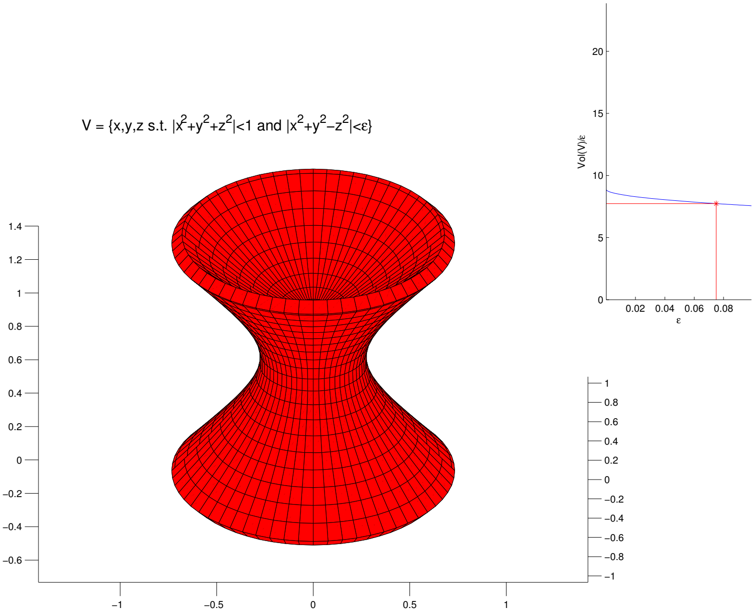

V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
−1 −0.5 0 0.5 1
−0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8 1 1.2 1.4
−1 −0.8 −0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8 1

$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

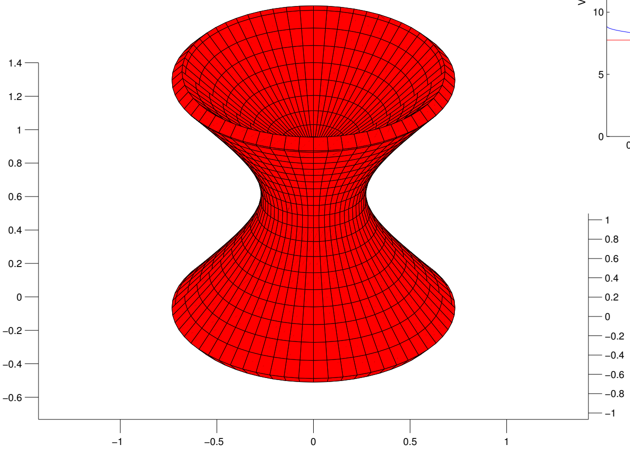


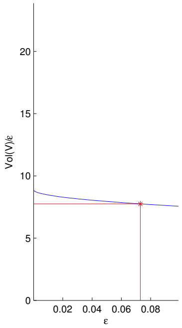


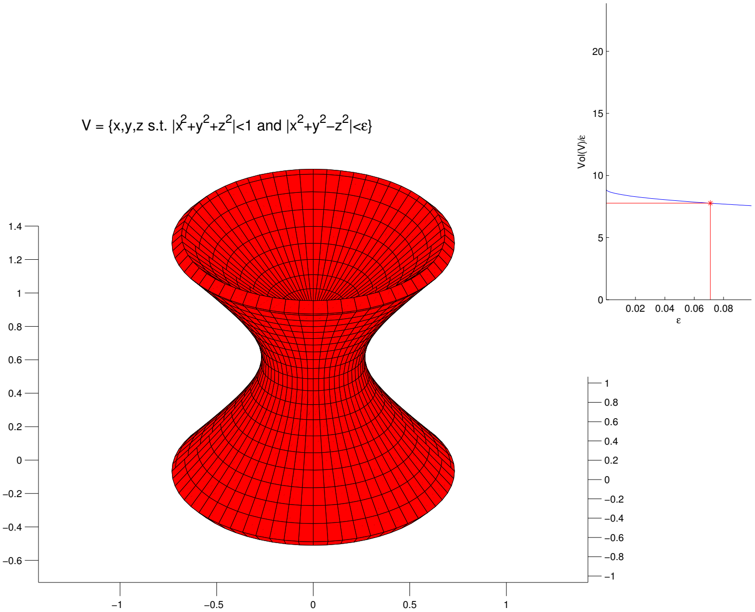
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



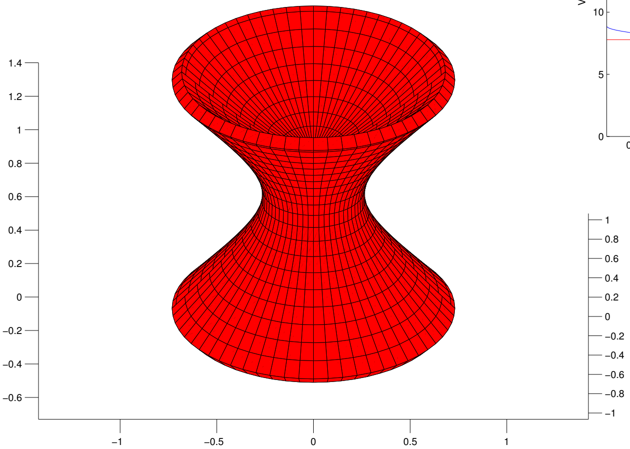
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

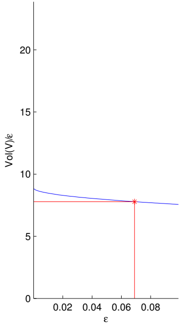




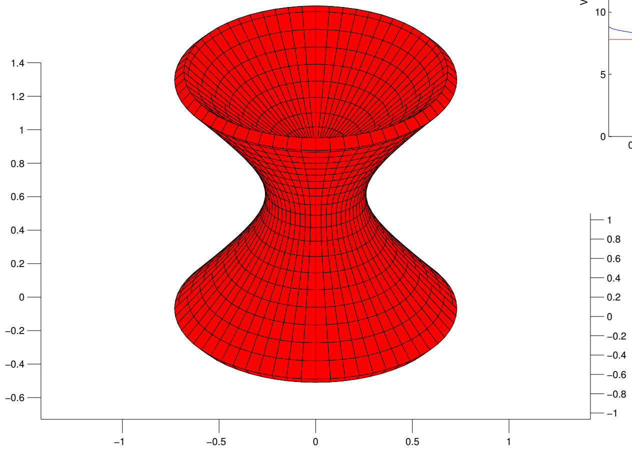

V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε

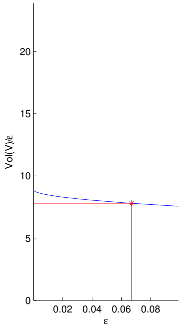
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



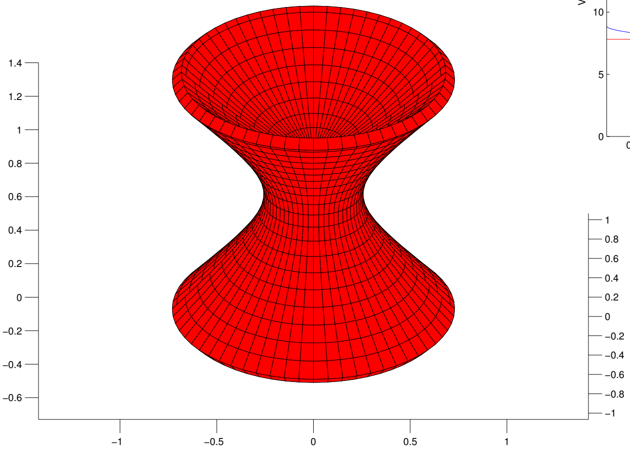


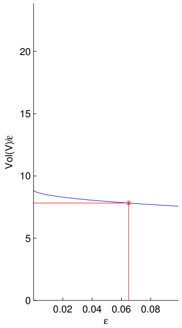
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



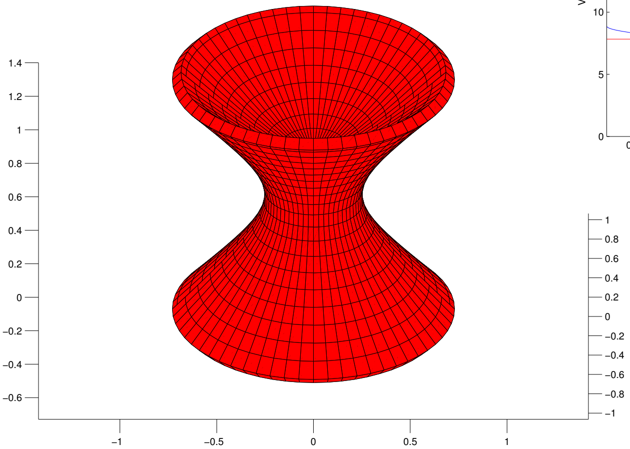


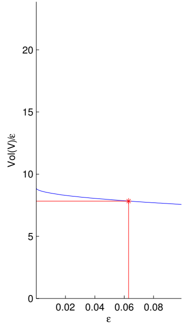
V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}



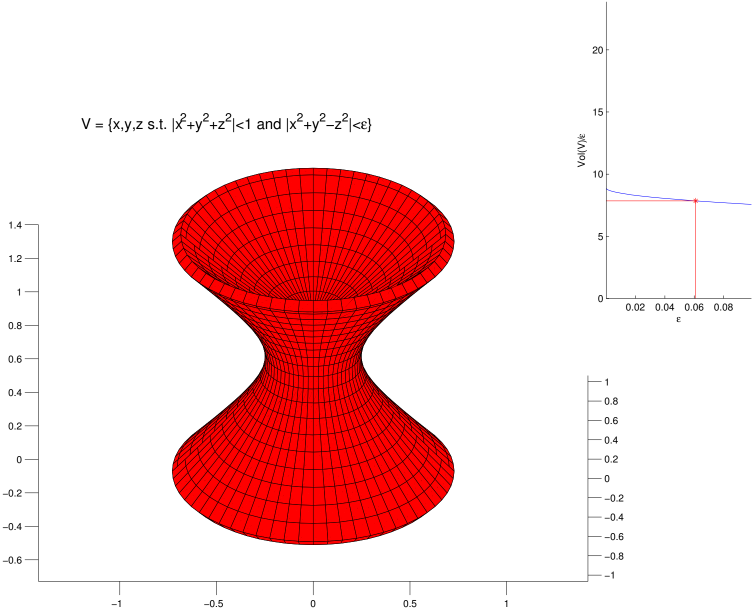


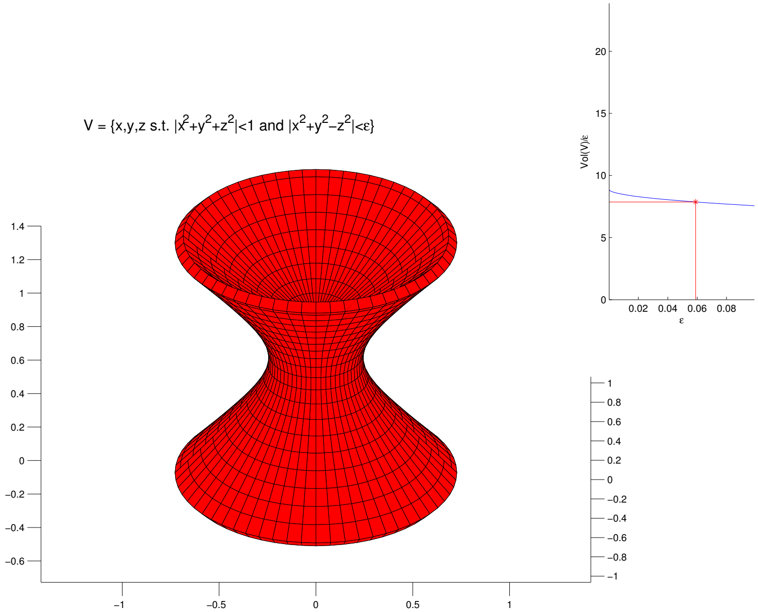
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



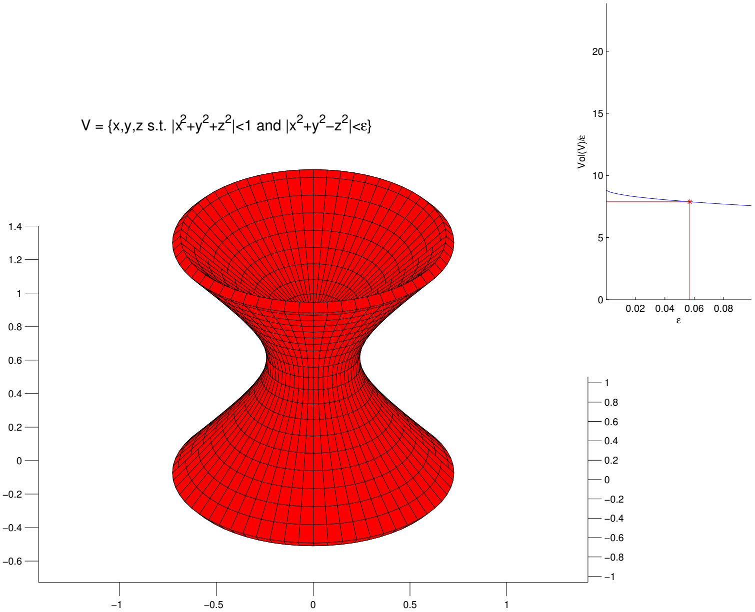


$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



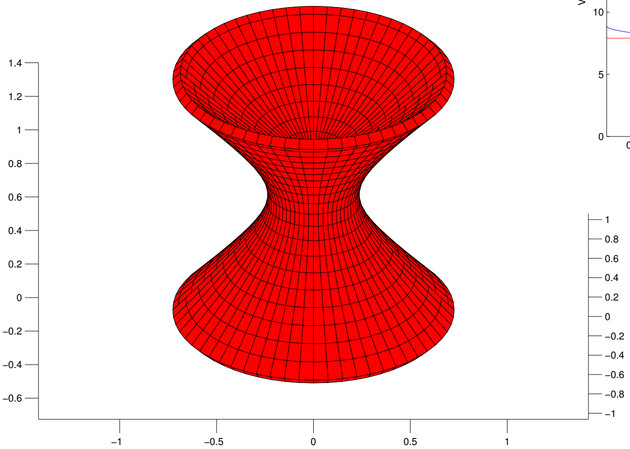


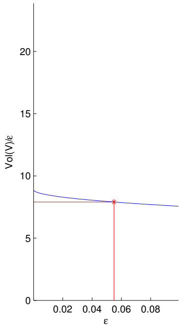
V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
20
15
10
5
0
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1



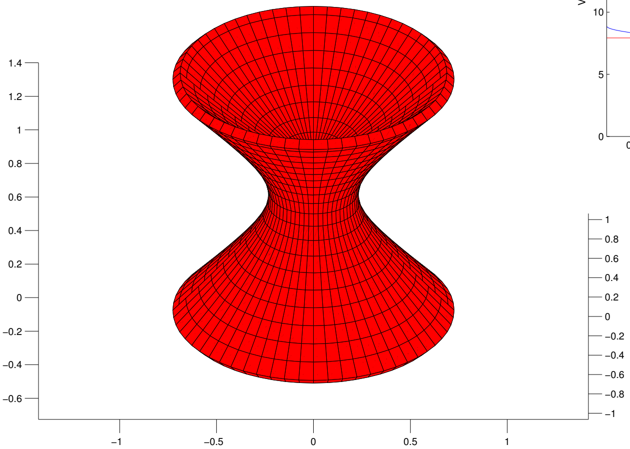
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε

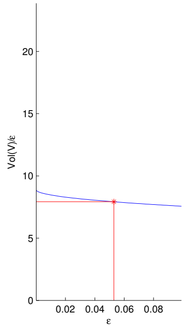
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

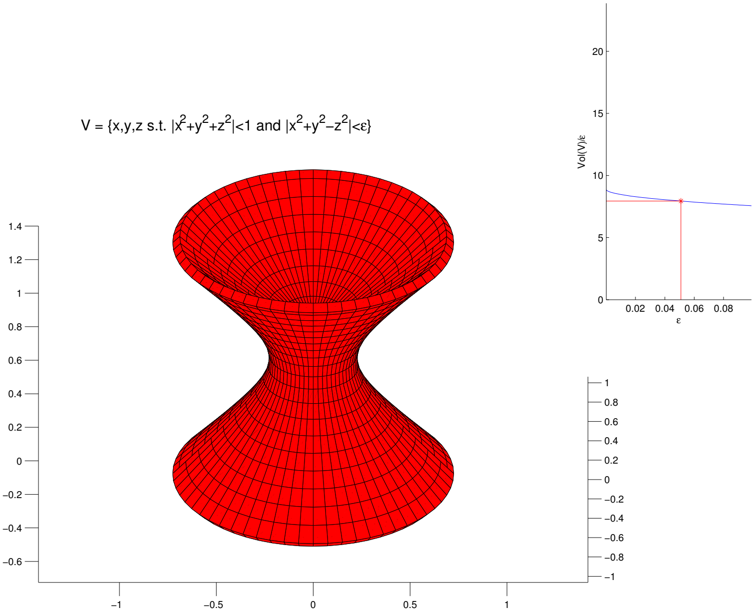


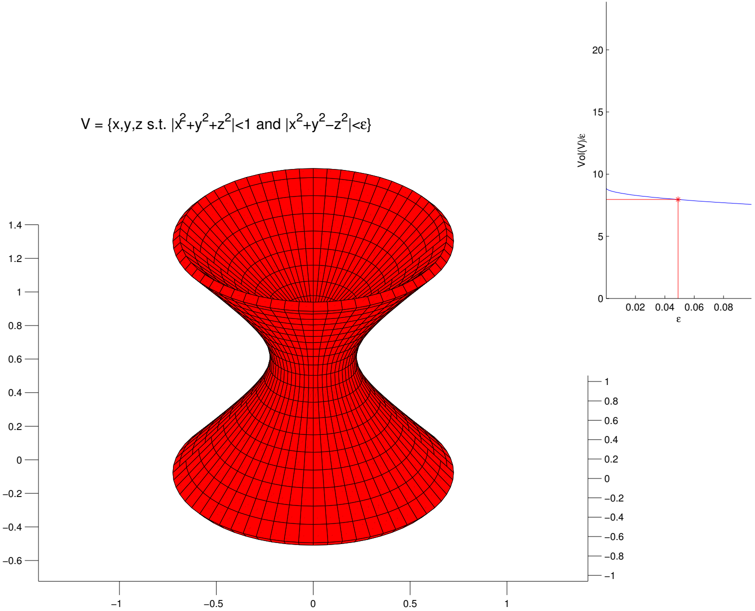


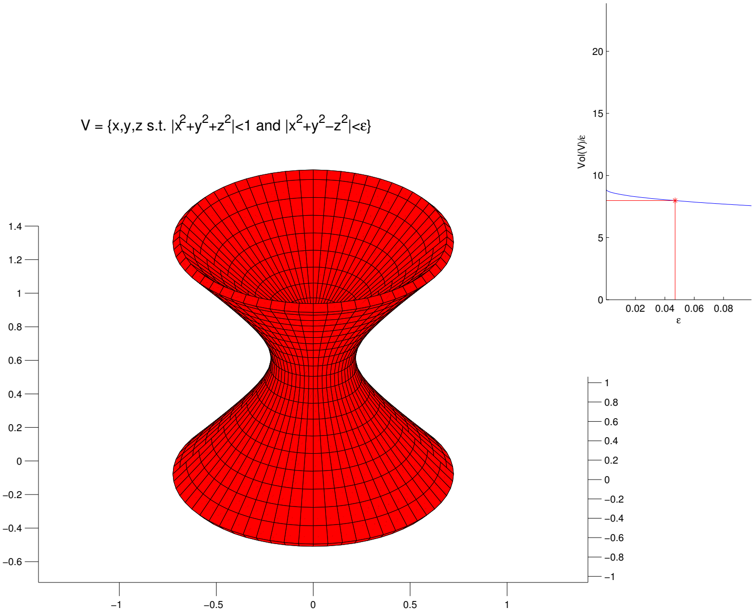
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

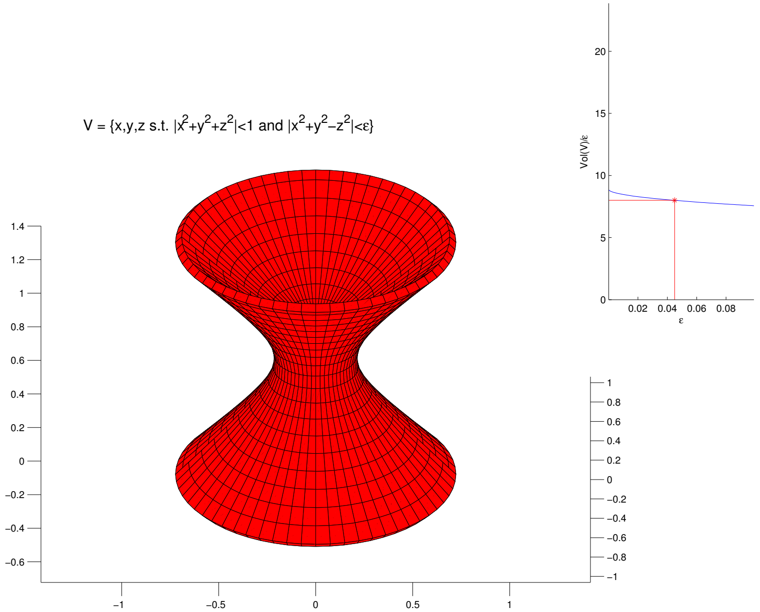


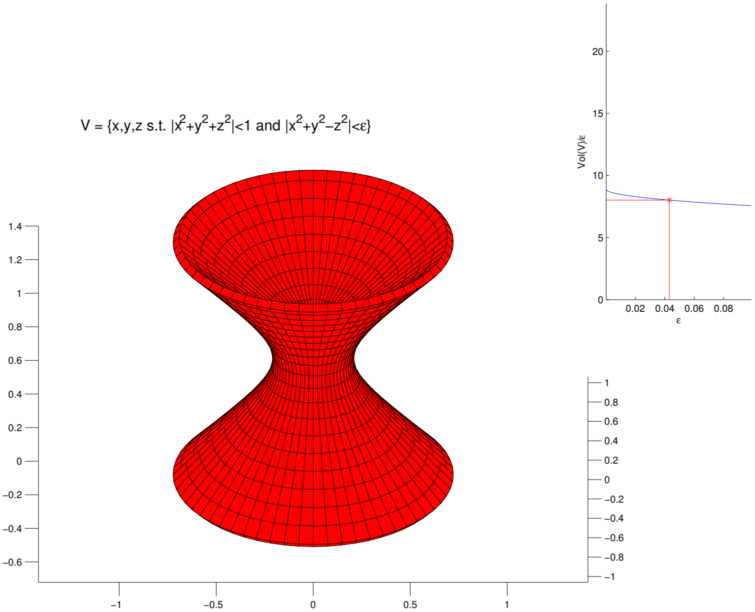



V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0.02
0.04
0.06
0.08
0
5
10
15
20
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

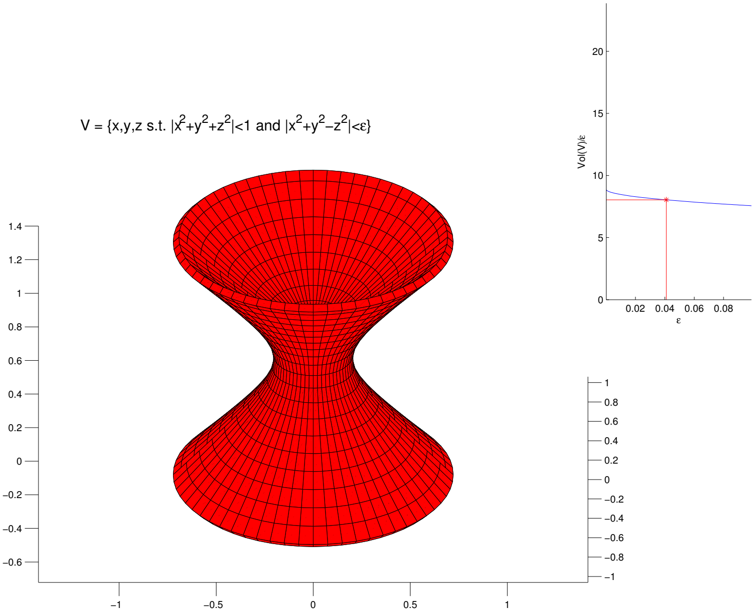

V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε

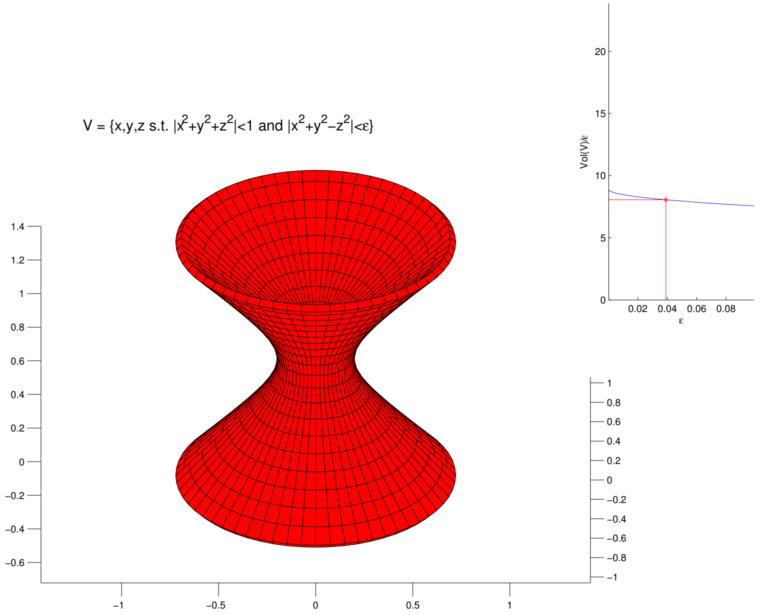

V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε

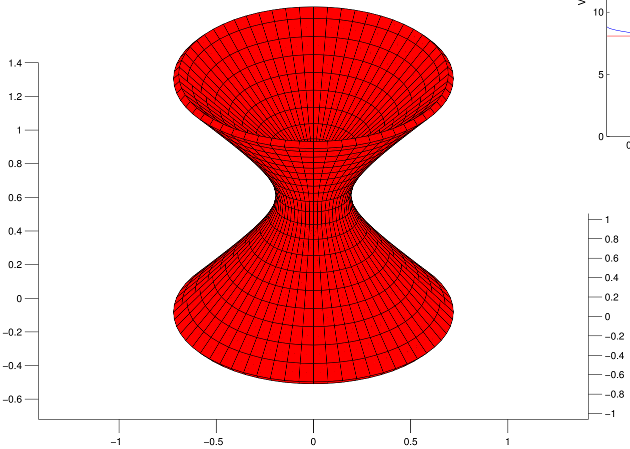

V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

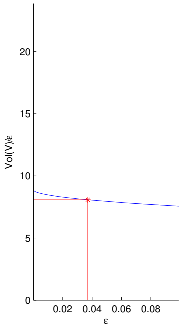
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



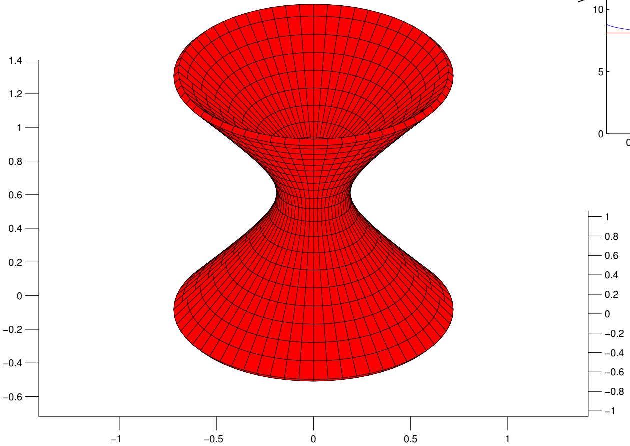

V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε

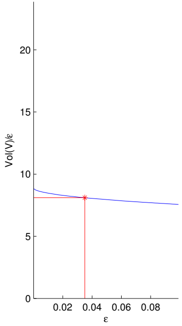
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



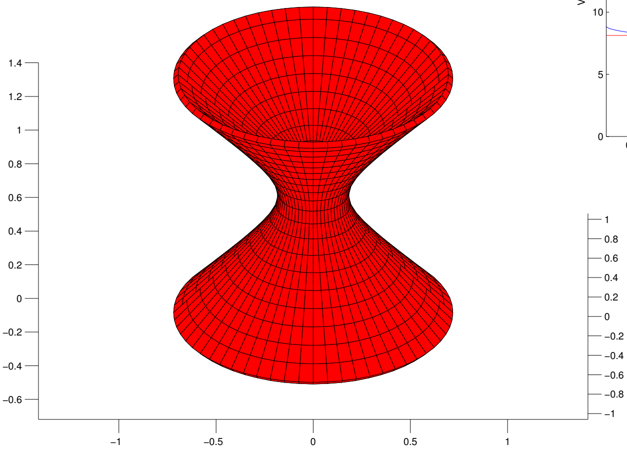


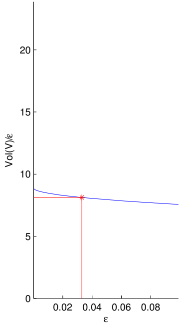
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



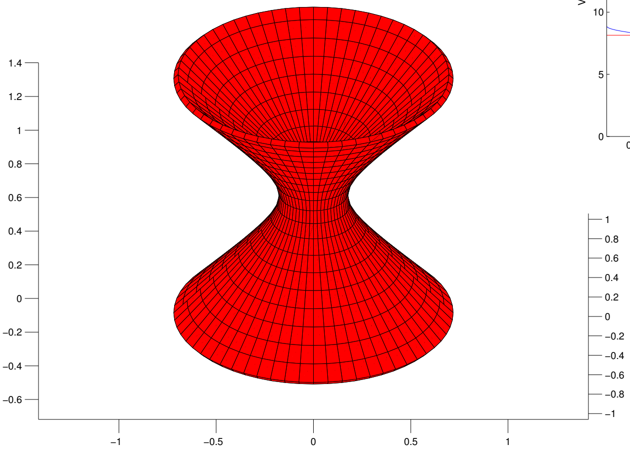


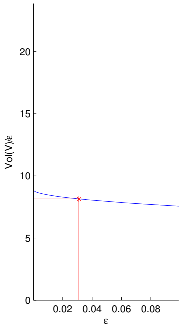
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



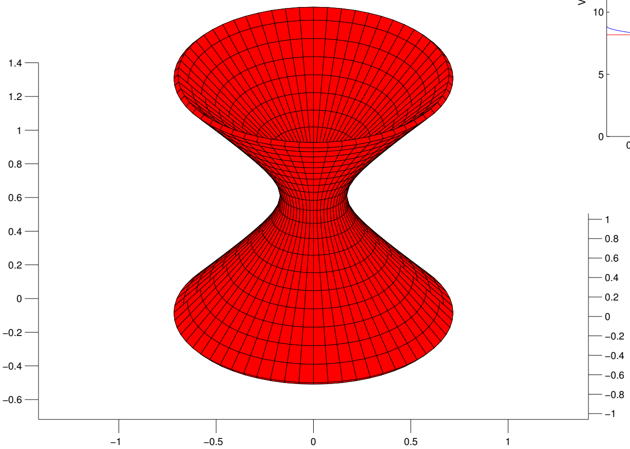


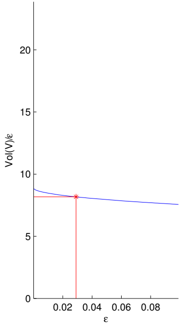
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



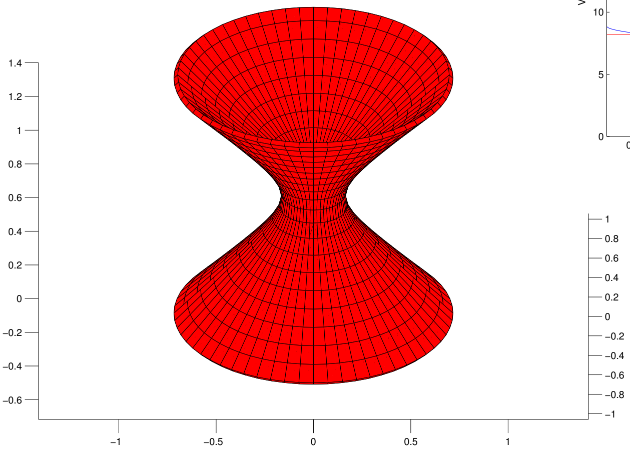


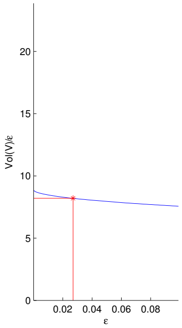
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



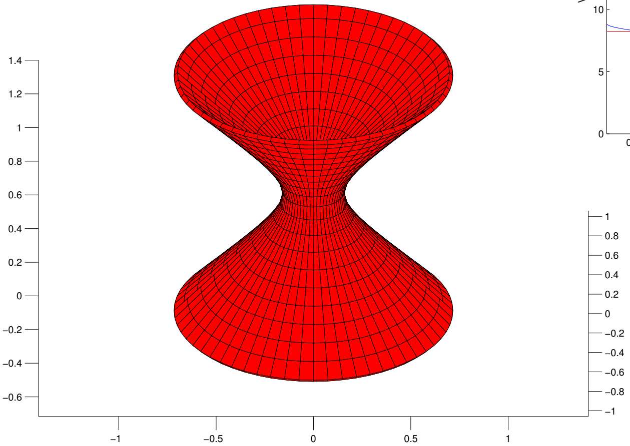


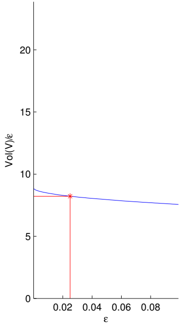
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

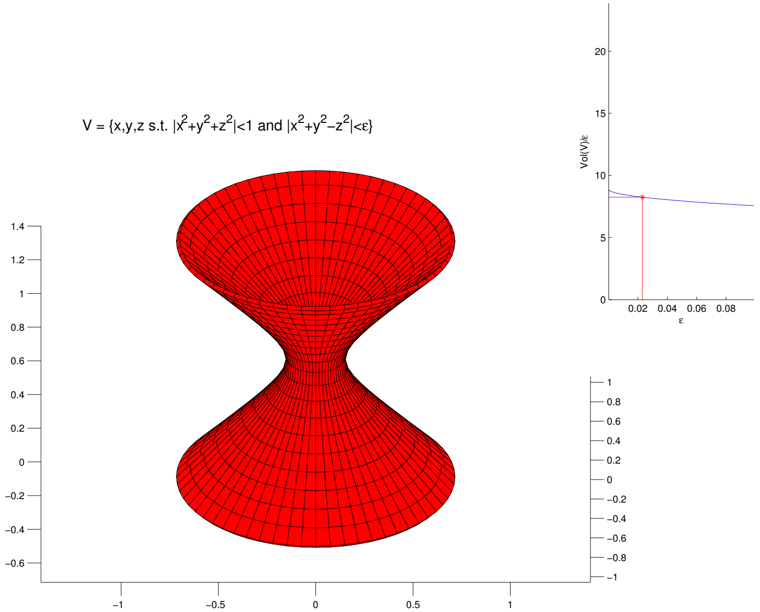


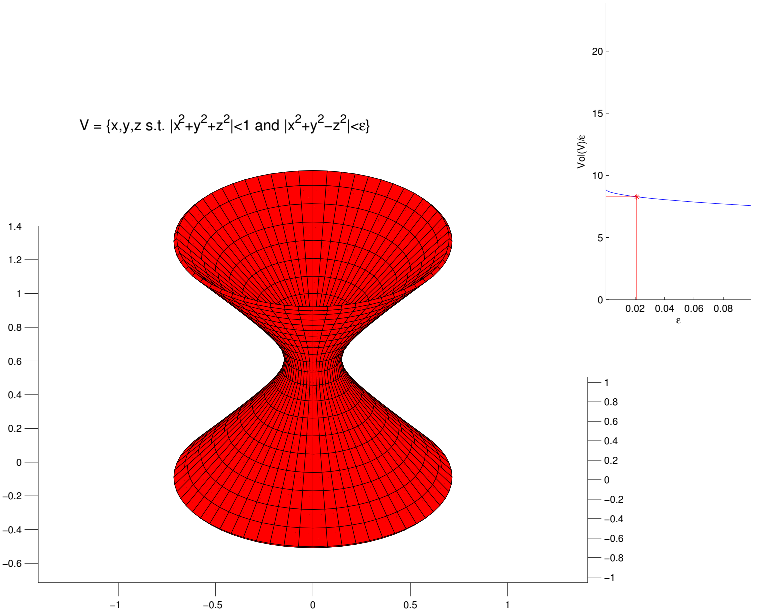


$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

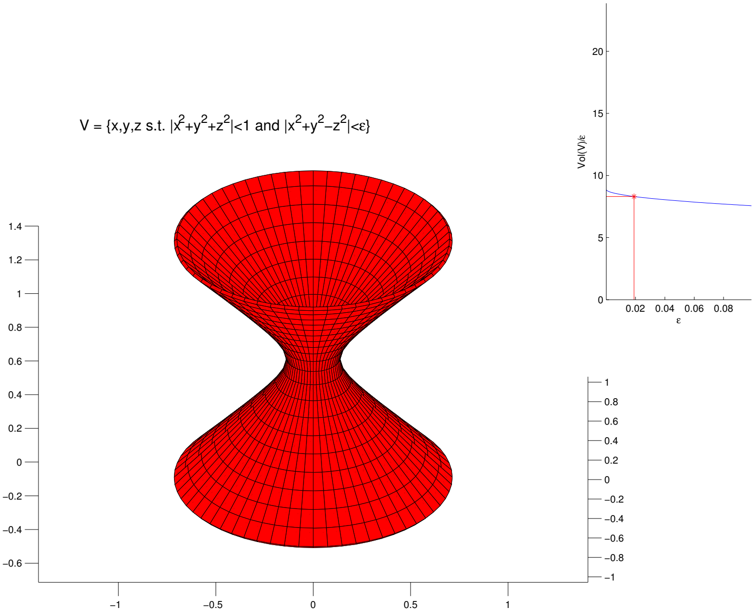





V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
20
15
10
5
0
0.02 0.04 0.06 0.08
ε
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1 −0.5 0 0.5 1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

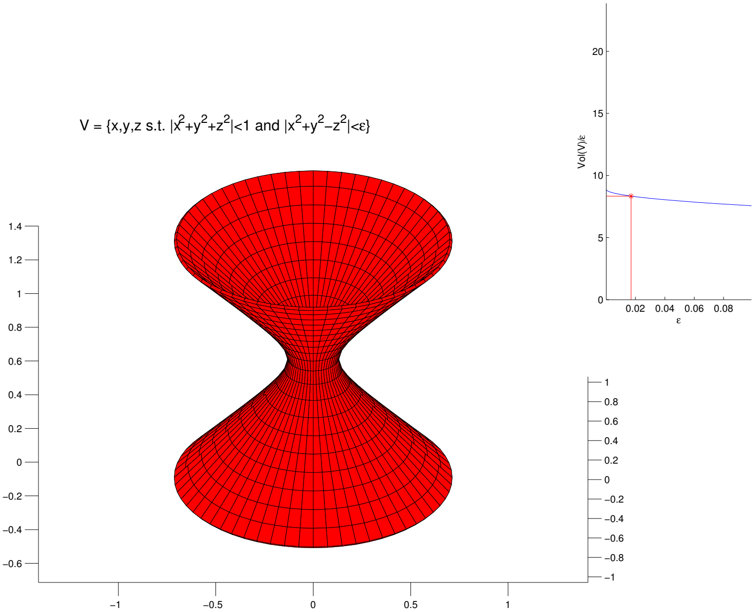


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0.02
0.04
0.06
0.08
0
5
10
15
20
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

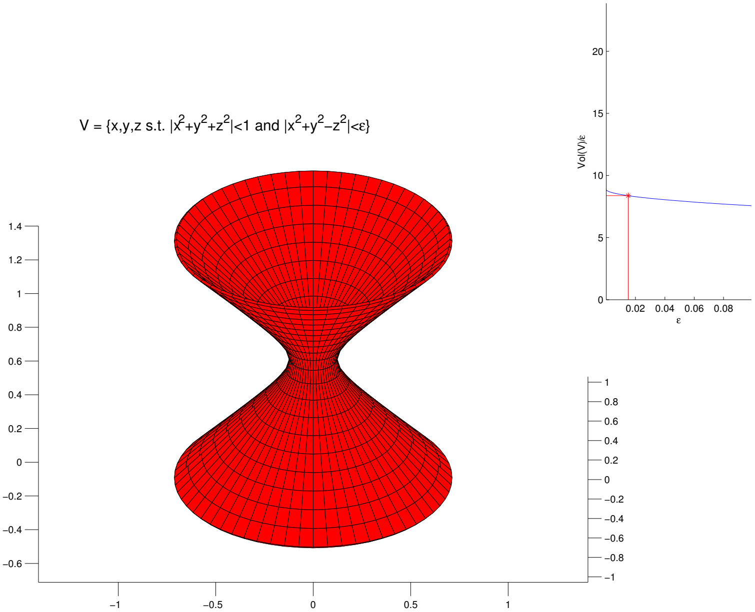


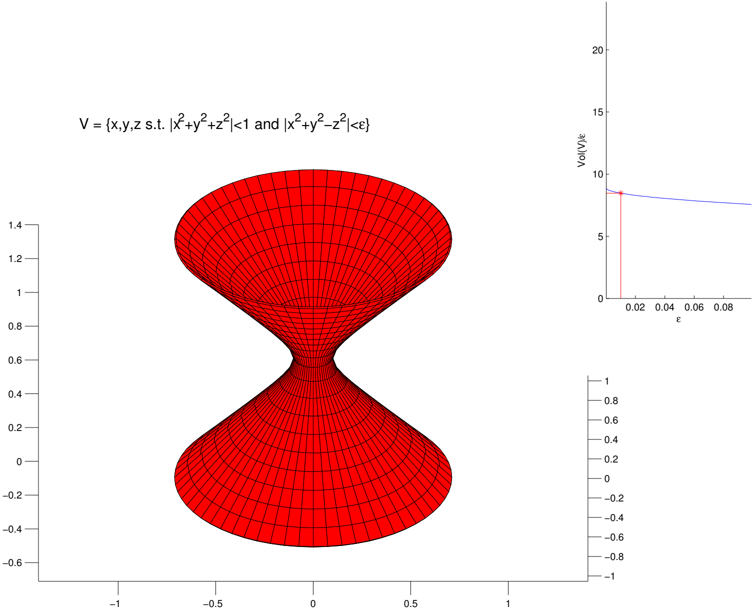
V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε

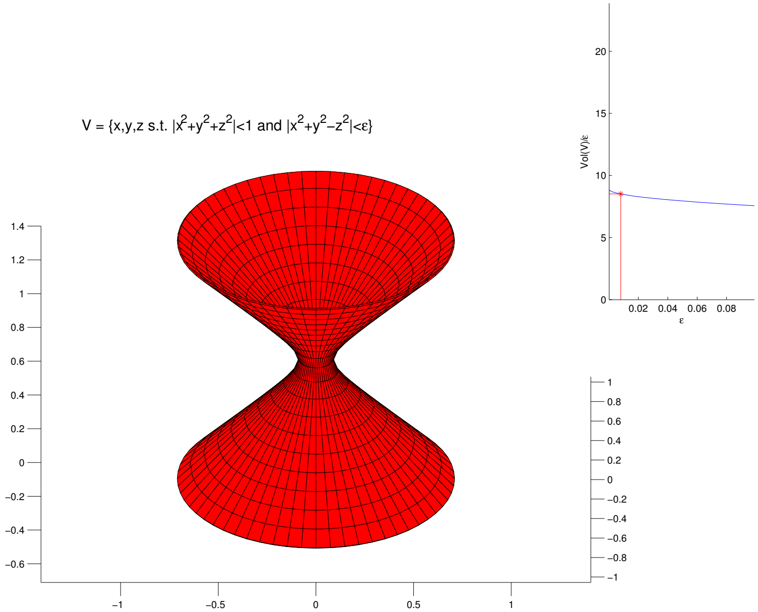
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}




V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0
5
10
15
20
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
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0.4
0.2
0
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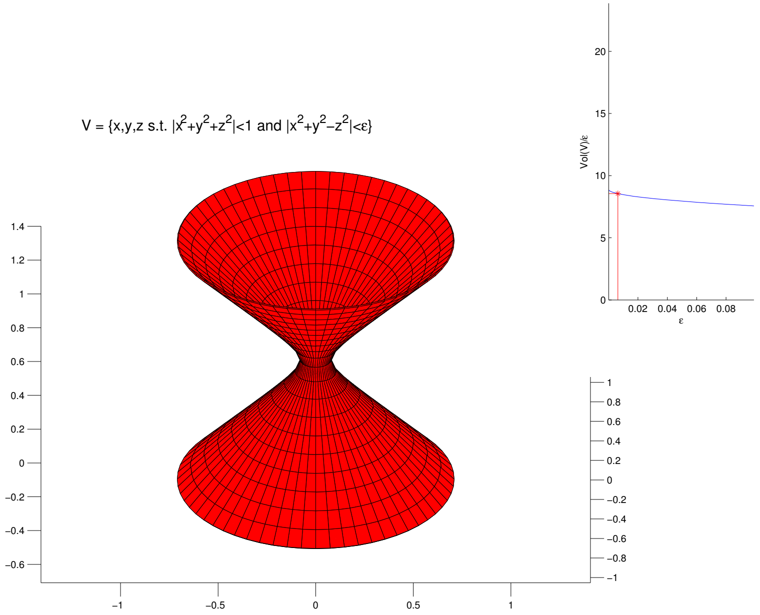
−0.4
−0.6
−0.8
−1


V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
0.02
0.04
0.06
0.08
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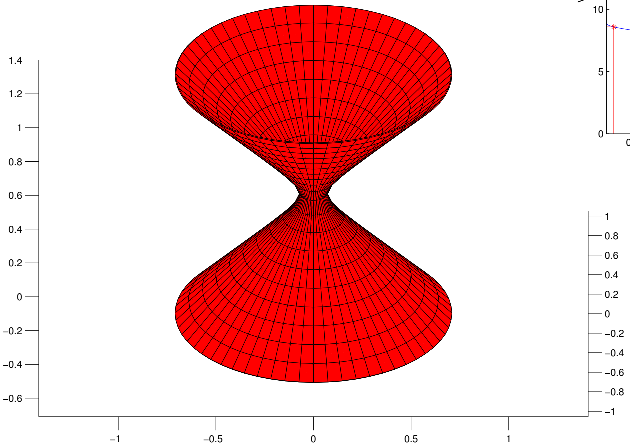
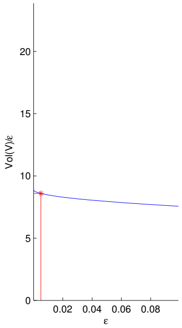
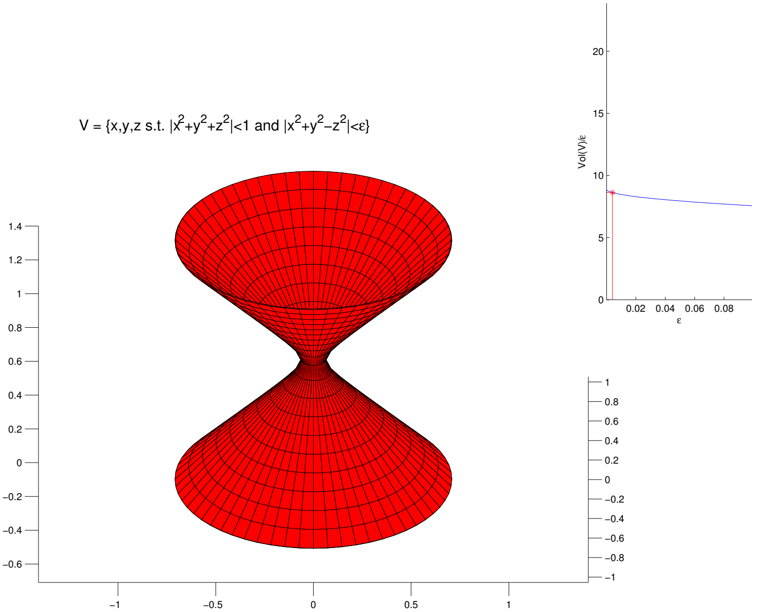
5
10
15
20
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$






V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
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5
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0.02
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1.4
1.2
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0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
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0.6
0.4
0.2
0
−0.2
−0.4
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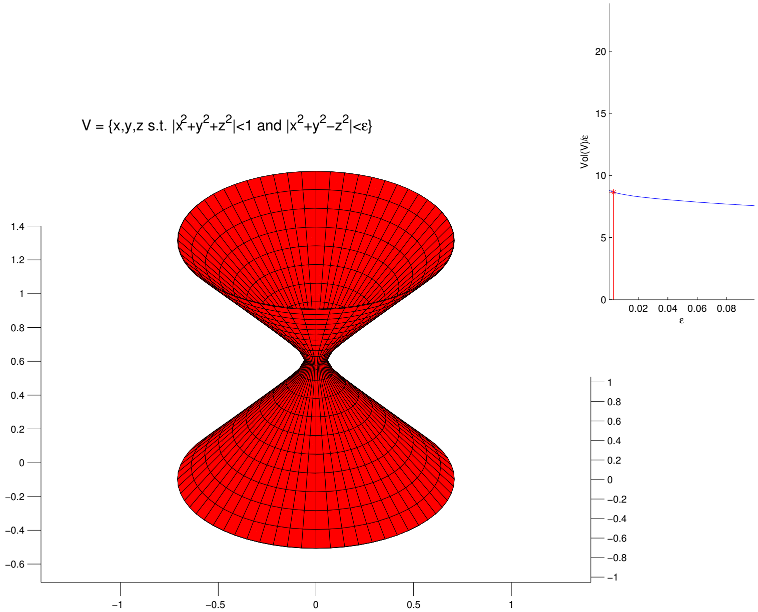
−1


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
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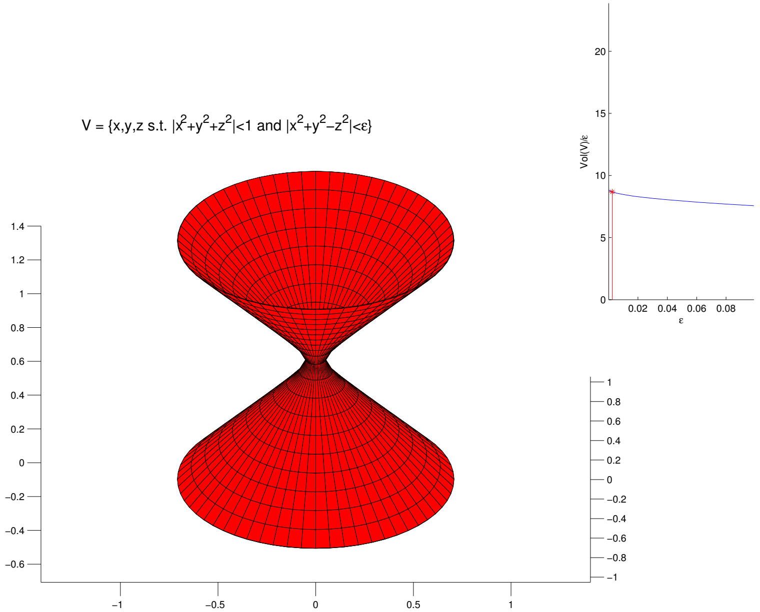
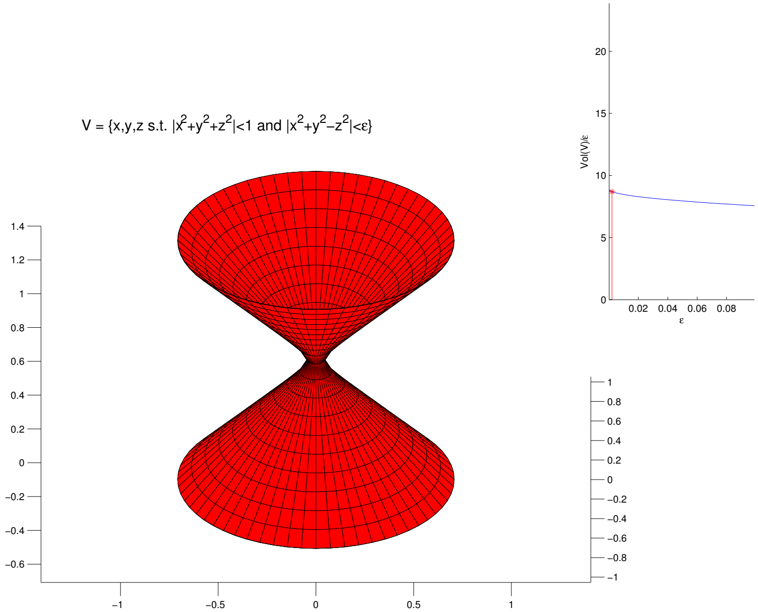
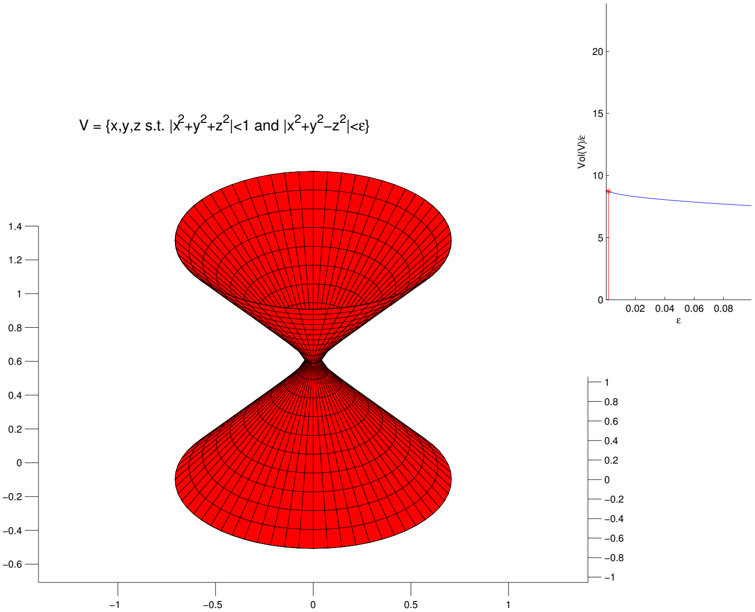


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0
5
10
15
20
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1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15
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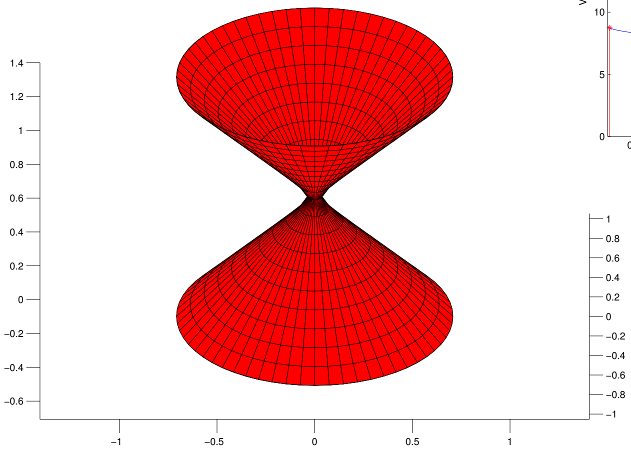
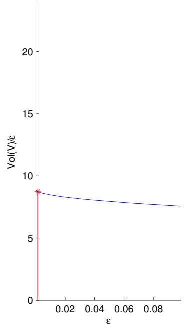
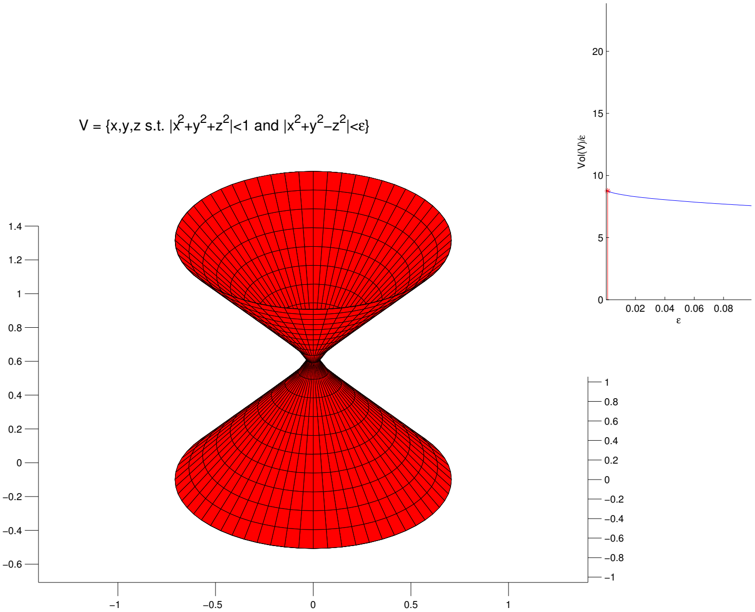
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$






V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
0.02
0.04
0.06
0.08
0
5
10
15
20
1.4
1.2
1
0.8
0.6
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0.2
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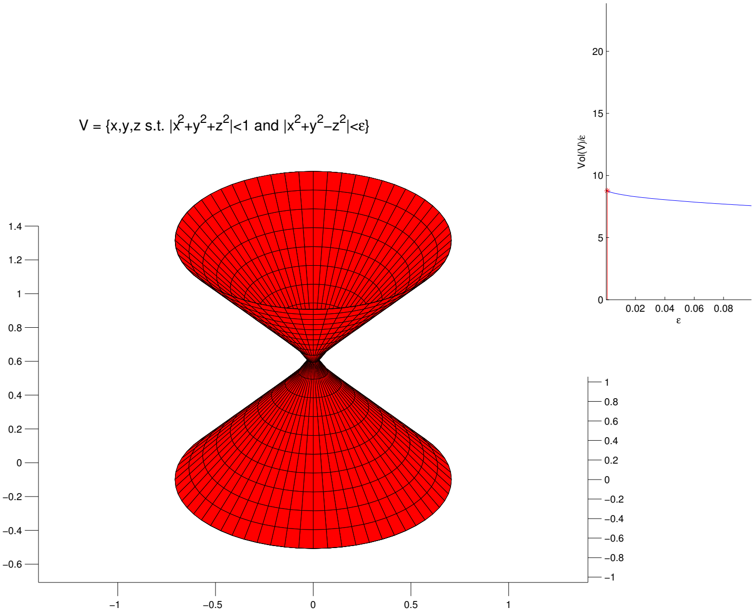
−0.2
−0.4
−0.6
−1
−0.5
0.5
−0.8


V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
20
15
10
5
0
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
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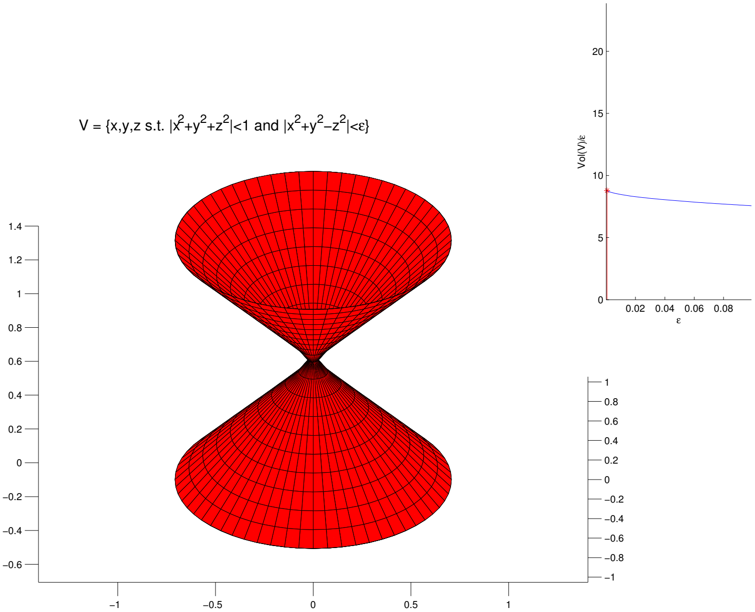
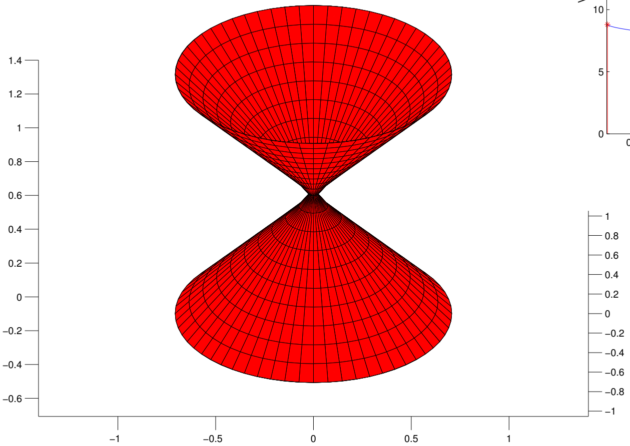
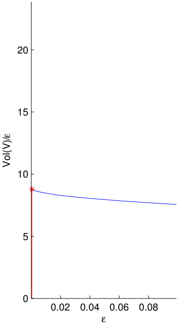
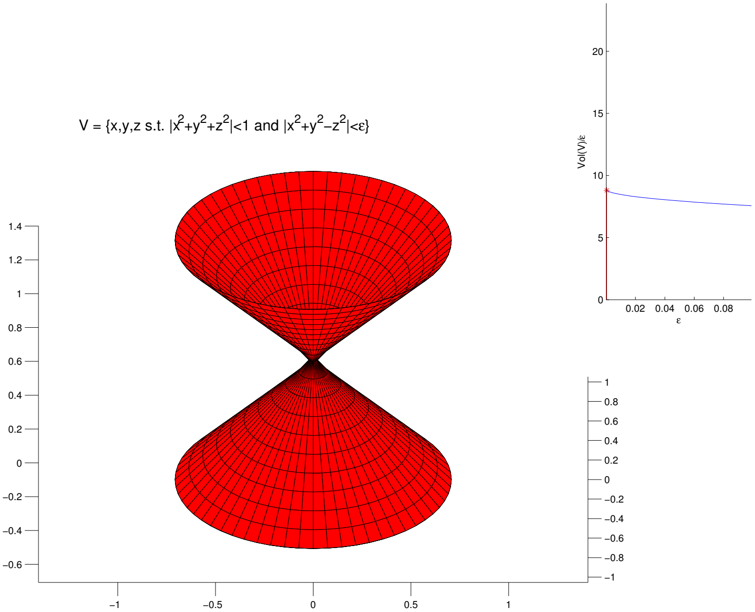
−0.8
−1

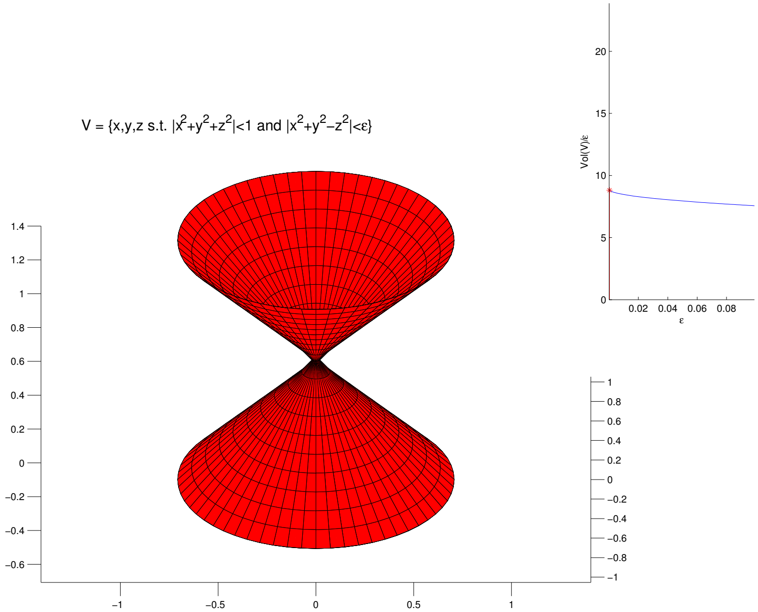

V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε

$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$






V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
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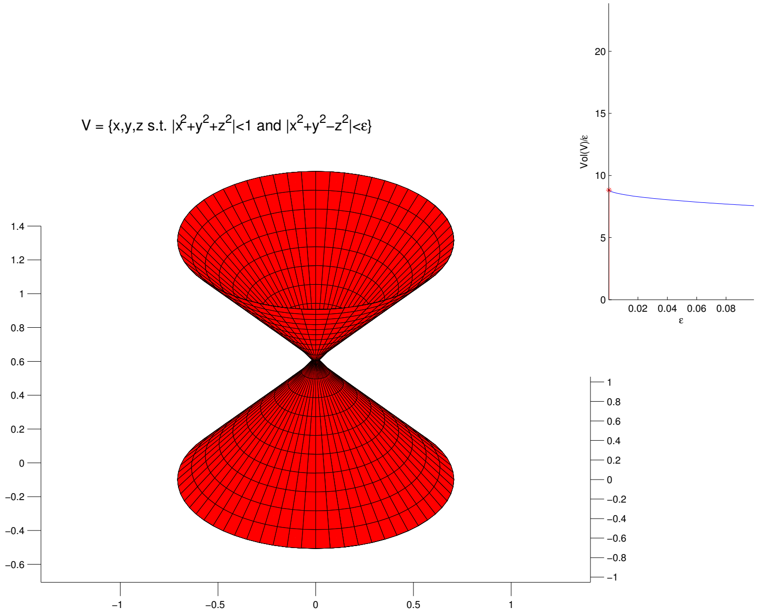
V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
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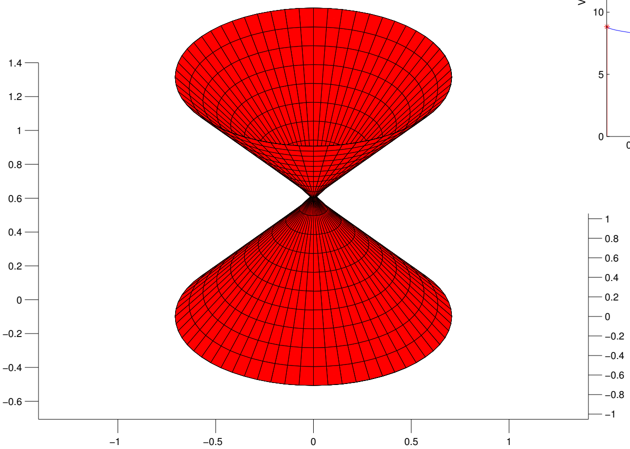
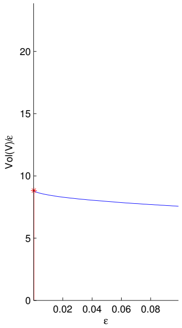
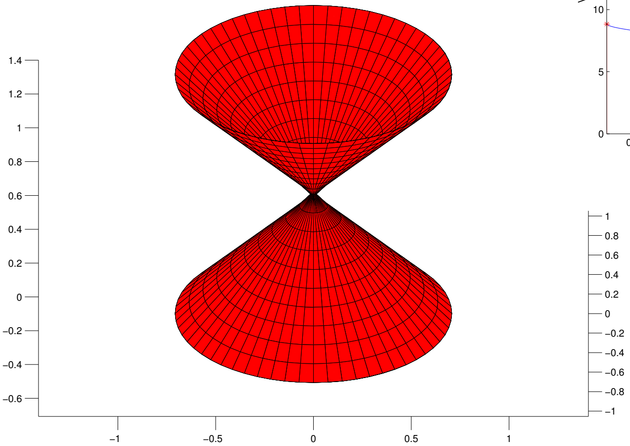
0 5 10 15 20
0.02 0.04 0.06 0.08
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

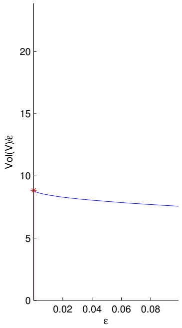
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



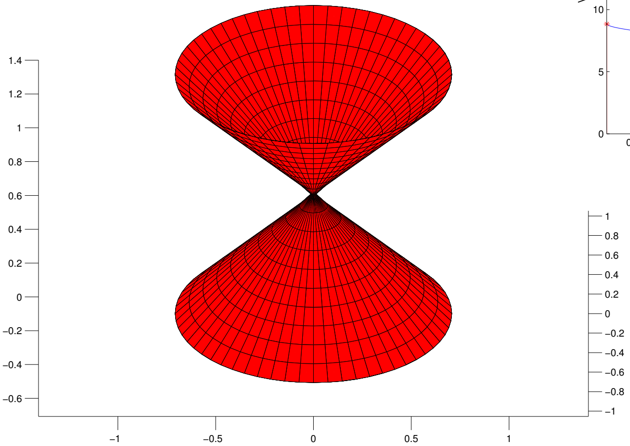


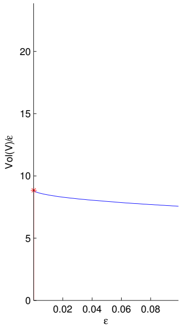
V = $\{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

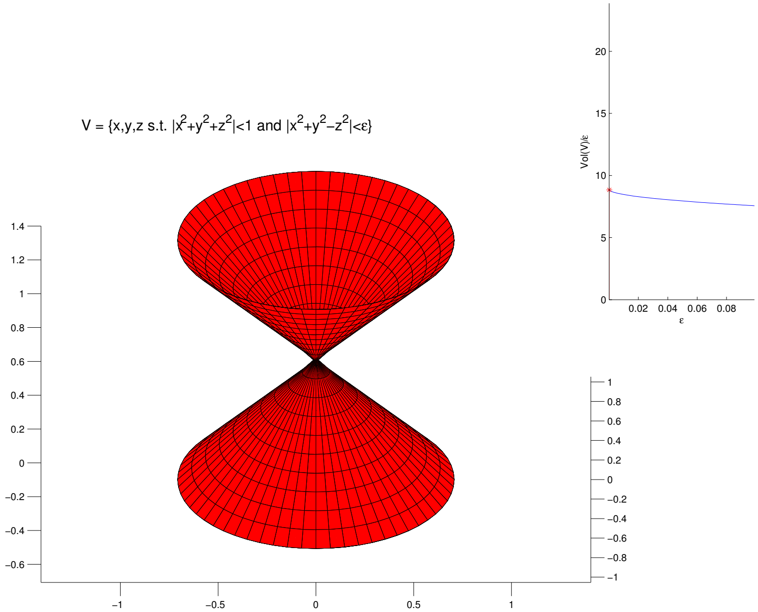


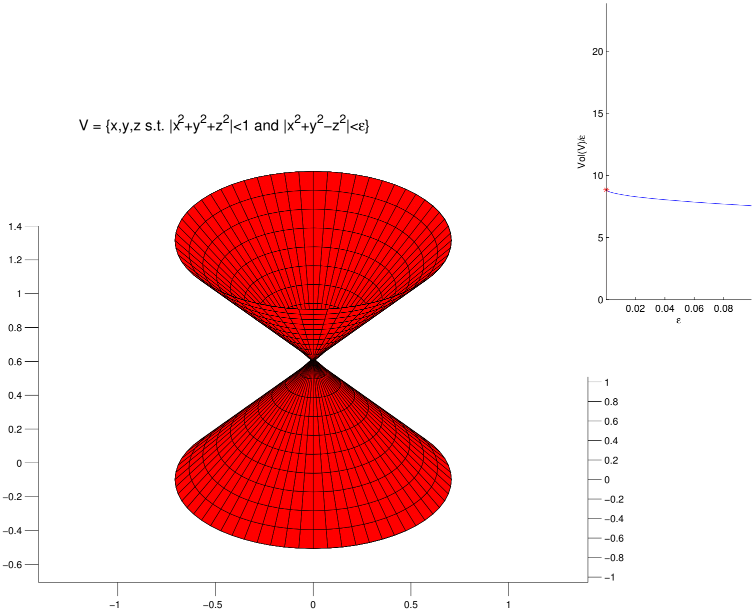


V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}

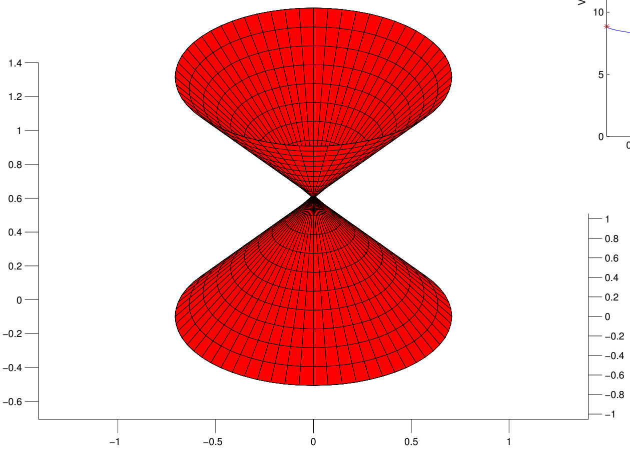


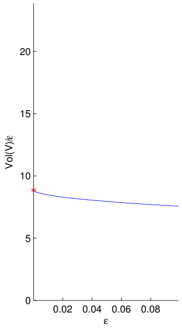



V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε

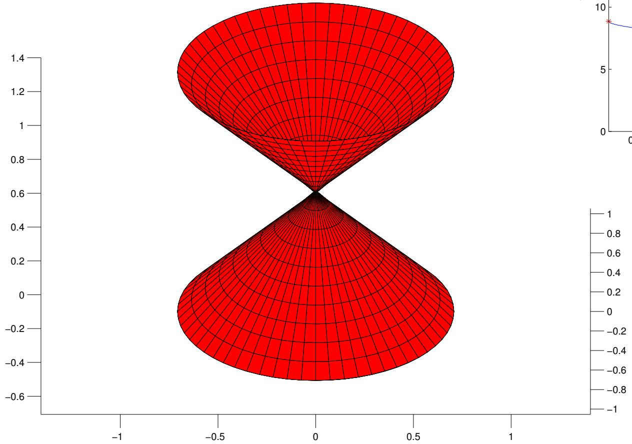

V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0.02
0.04
0.06
0.08
0
5
10
15
20
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

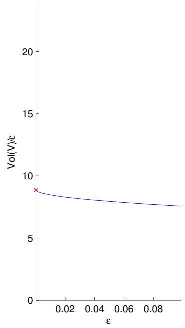
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

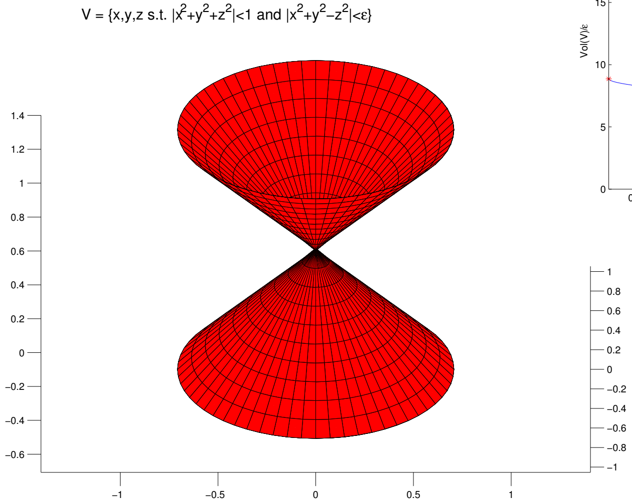




$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



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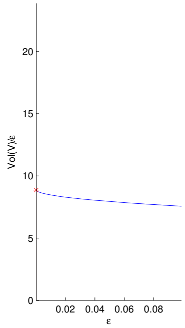
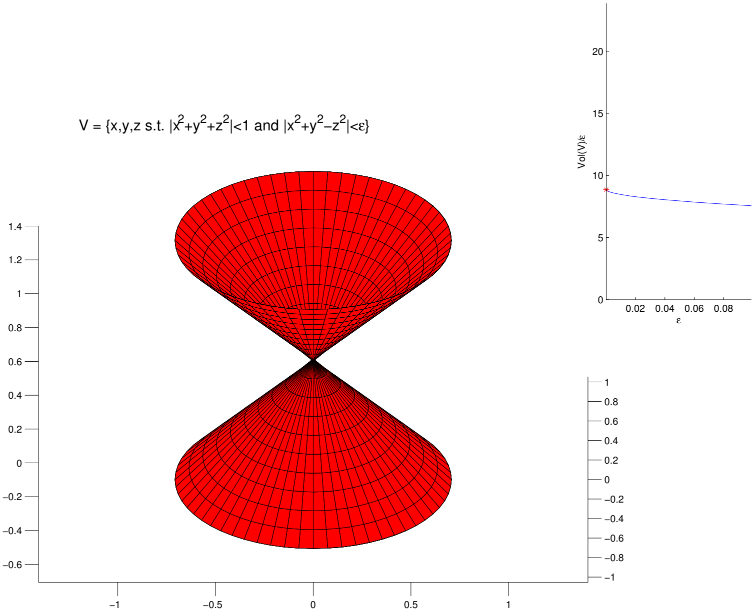
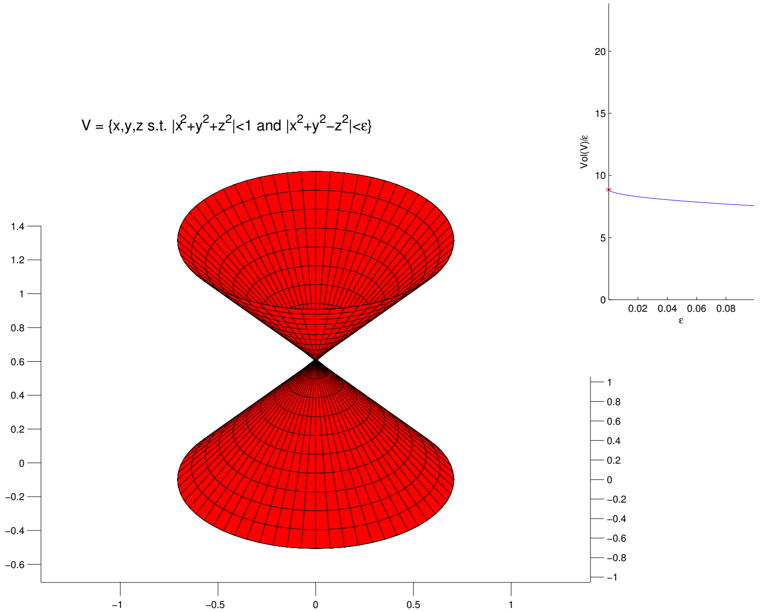
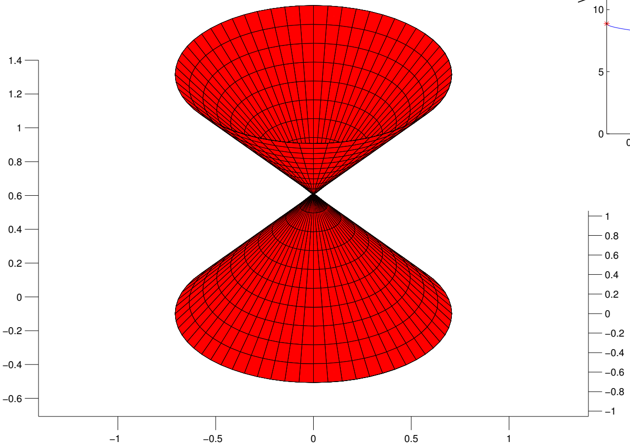

Vol(V)/ε
ε

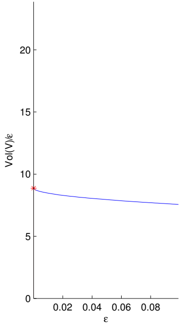
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$




V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$





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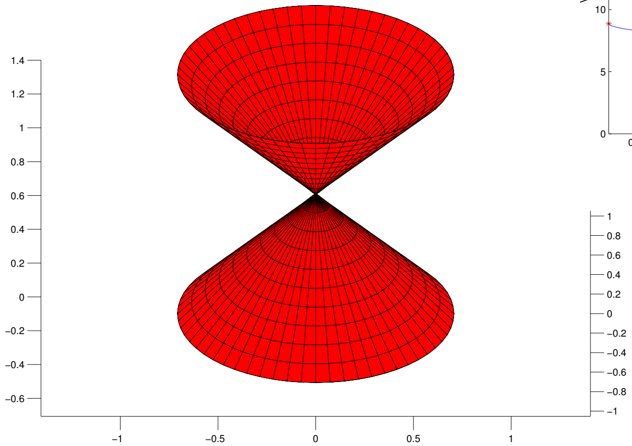

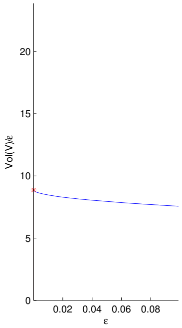

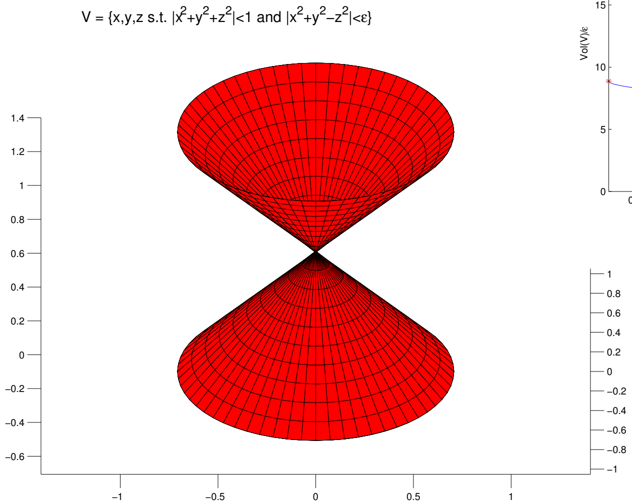
V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
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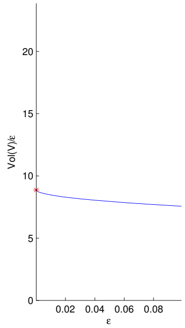
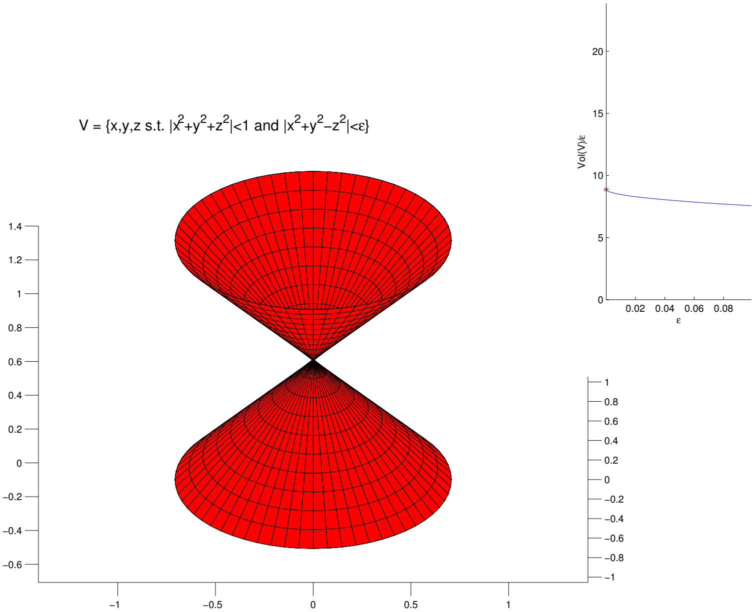
−0.4
−0.6
−0.8
−1


Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1
−1 −0.5 0 0.5 1

Theorem (A.- Avni)

Let $n \geq 12$. The map is FRS .

$$(g_1, h_1, \dots g_n, h_n) \mapsto [g_1, h_1] \cdots [g_n, h_n].$$

Rationality of the singularities of deformation varieties

Theorem (A.- Avni)

Let $n \geq 12$. The map is FRS .

$$(g_1, h_1, \ldots g_n, h_n) \mapsto [g_1, h_1] \cdots [g_n, h_n].$$

In particular, the singularities of the deformation variety

$$\{(g_1, h_1, \ldots g_n, h_n) \in SL_d^{2n} : [g_1, h_1] \cdots [g_n, h_n] = 1\} = \\ = Hom(\pi_1(\Sigma_n), SL_d),$$

are rational and complete intersection.

Theorem (A.- Avni)

Let $n \geq 12$. The map is FRS .

$$(g_1, h_1, \dots g_n, h_n) \mapsto [g_1, h_1] \cdots [g_n, h_n].$$

In particular, the singularities of the deformation variety

$$\{(g_1, h_1, \dots g_n, h_n) \in SL_d^{2n} : [g_1, h_1] \cdots [g_n, h_n] = 1\} = \\ = Hom(\pi_1(\Sigma_n), SL_d),$$

are rational and complete intersection.

- In our case we do not have an explicit resolution of singularities, unlike other cases of proofs of rationality of singularity of varieties from algebraic group theory.

Sum up

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Φ is FRS

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continuity criterion $\Downarrow$

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$\Phi_*(\mu)$ has continuous density

Sum up

Φ is FRS

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$\Phi_*(\mu)$ has continuous density

Frobenius Formula $\Downarrow$

Sum up

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$\Phi_*(\mu)$ has continuous density

Frobenius Formula $\Downarrow$

$$\zeta_{SL_d(\mathbb{Z}_p)}(2n) < \infty$$