Representation count of arithmetic groups, moduli spaces of local systems, and pushforward of smooth measures

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Theorem (A.-Avni 2014)

Let G be a semi-simple group defined over $\mathbb{Z}$ whose $\mathbb{Q}$-split rank is > 1. Then, $\zeta_{G(\mathbb{Z})}(40)$ converges.

Application of CSP

Theorem (Lubotzky-Larsen 2007)

Let $d > 2$. Any irreducible representation π of $SL_d(\mathbb{Z})$ can be written as

$$\pi = \pi_{fin} \otimes \pi_{alg},$$

where π_{fin} factors through $SL_d(\mathbb{Z}|_n)$ and π_{alg} extends to an algebraic representation of $SL_d(\mathbb{C})$

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Lemma

$G_0 \rtimes SL_d(\mathbb{Z}|_n) = G_0 \times SL_d(\mathbb{Z}|_n)$

□

Local to Global

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$$\zeta_{SL_d(\mathbb{Z})} = \zeta_{SL_d(\mathbb{C})}\zeta_{SL_d(\hat{\mathbb{Z}})} = \zeta_{SL_d(\mathbb{C})} \prod_p \zeta_{SL_d(\mathbb{Z}_p)}$$

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Equivalently that $\zeta_{G(\mathbb{Z}/k\mathbb{Z})}(s)$ is bounded.

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- $\zeta_H(2n-2) = \frac{\#\{(g_1,h_1,\ldots g_n,h_n) \in H^{2n} | [g_1,h_1] \cdots [g_n,h_n]=1\}}{\# H^{2n-1}}$

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$$\mathrm{Prob}([g_1, h_1] \cdots [g_n, h_n] \in A) < c \cdot \mathrm{Prob}(g \in A),$$

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or equivalently:

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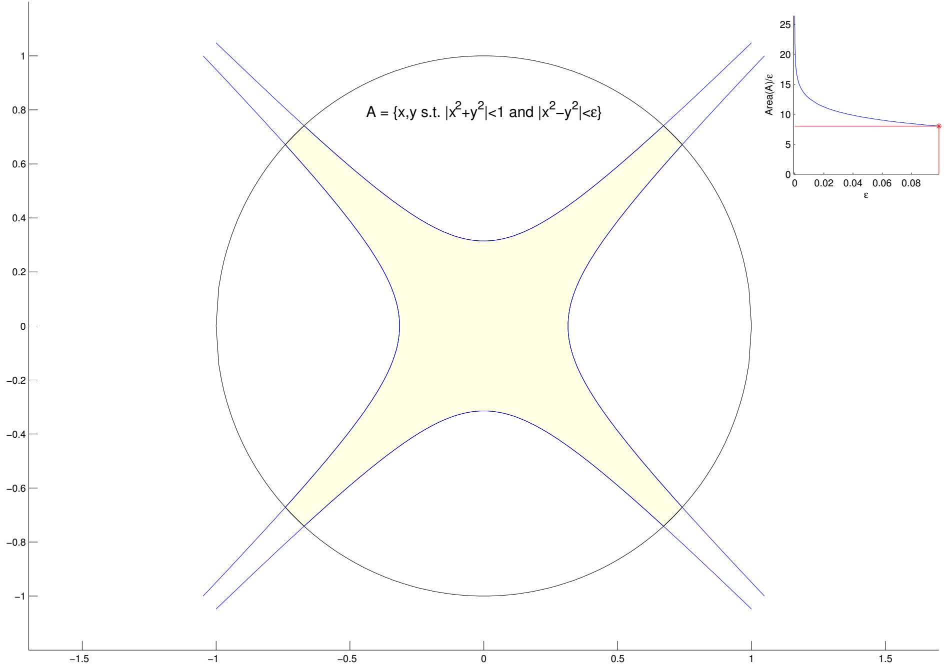
Then $\phi_*(m)$ has continuous density.



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Area(A)/ε
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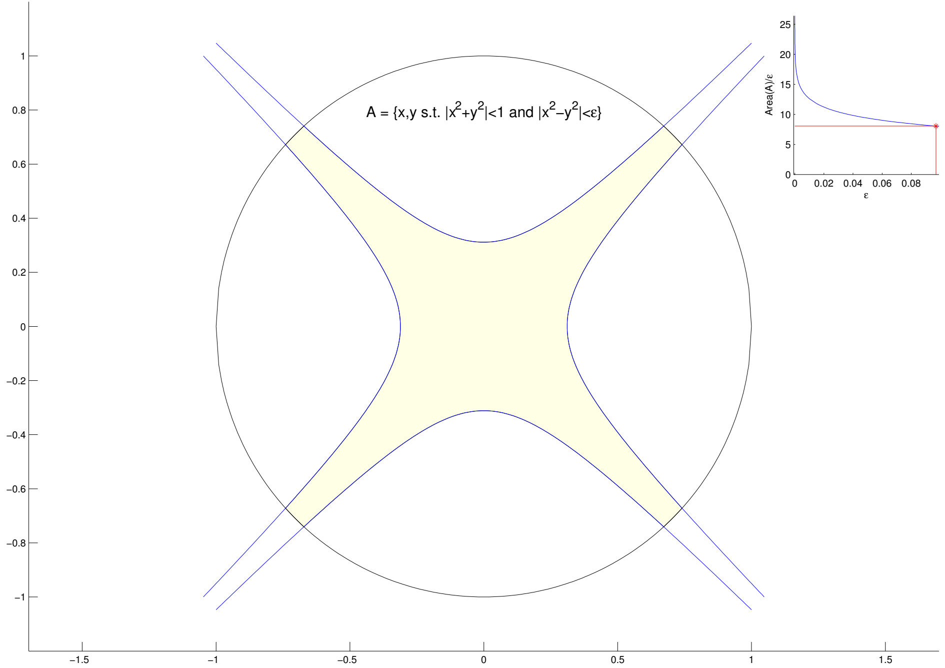

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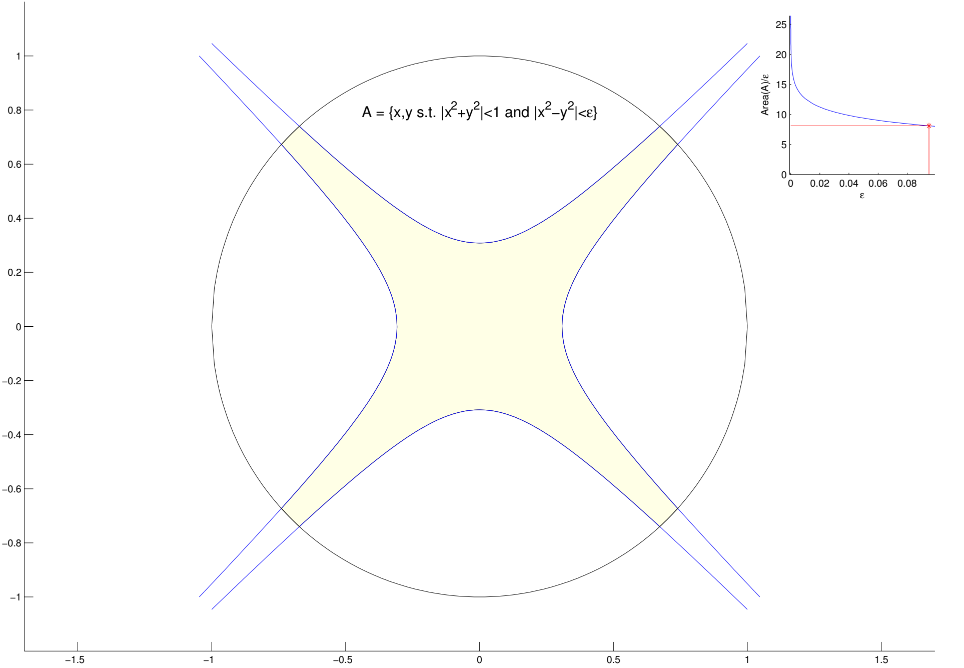
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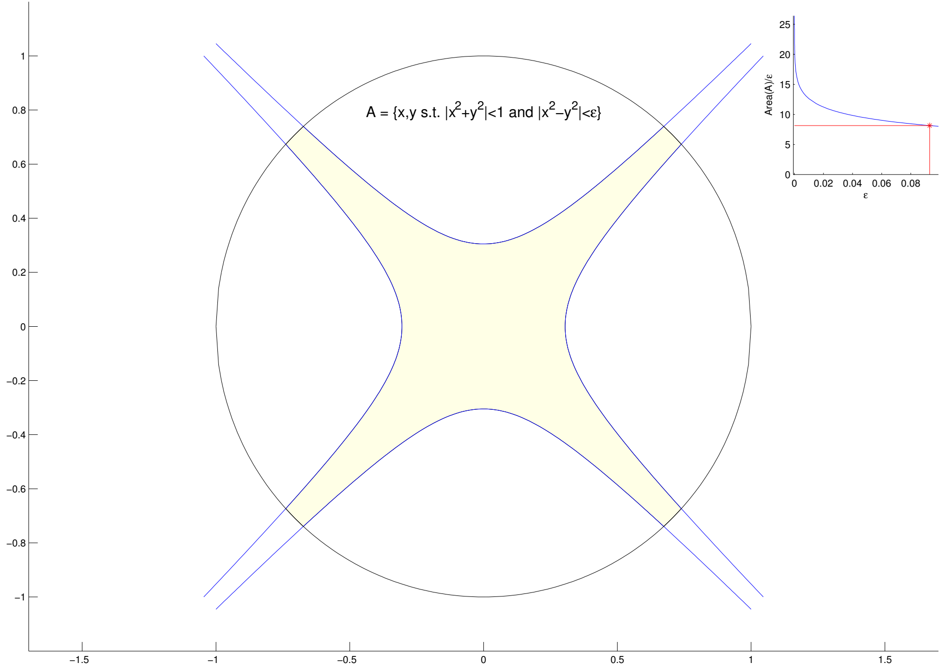

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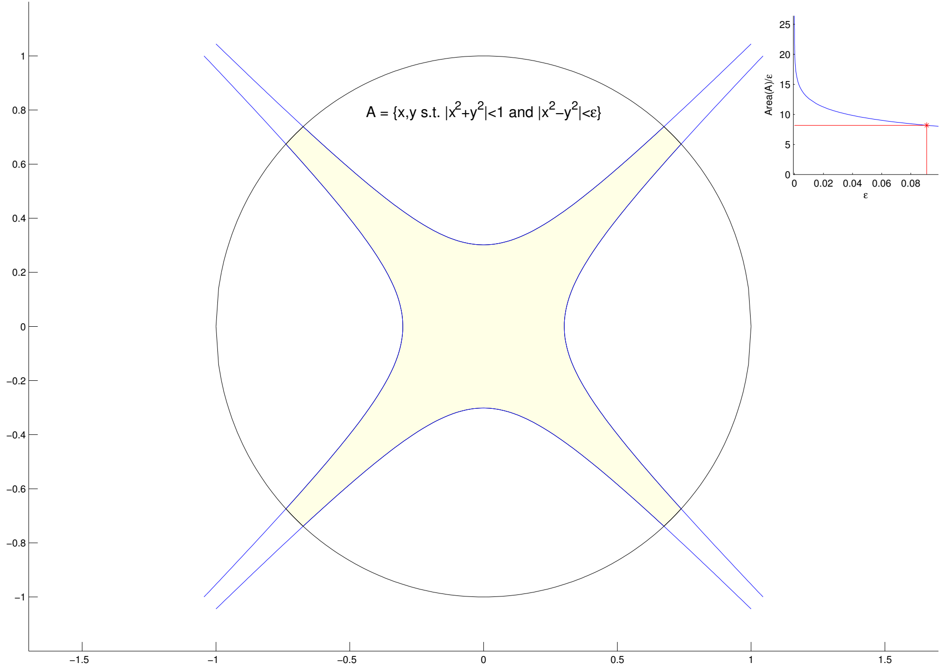
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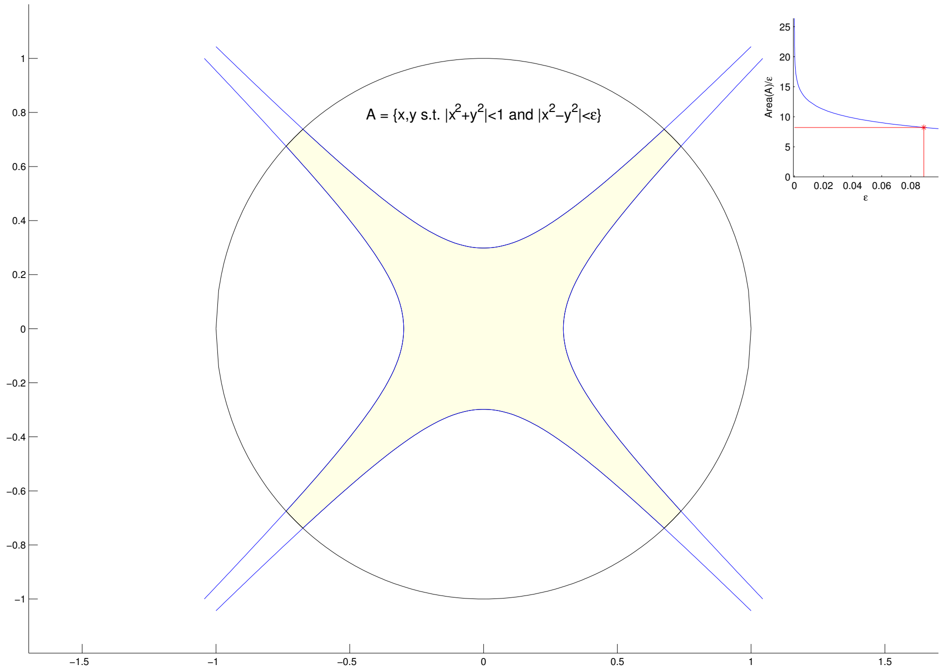

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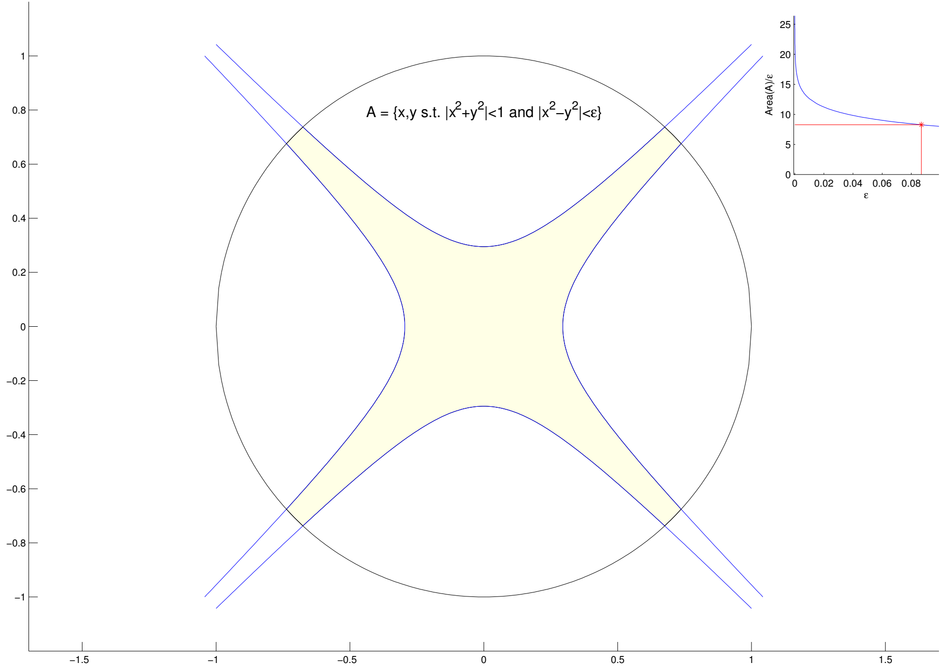
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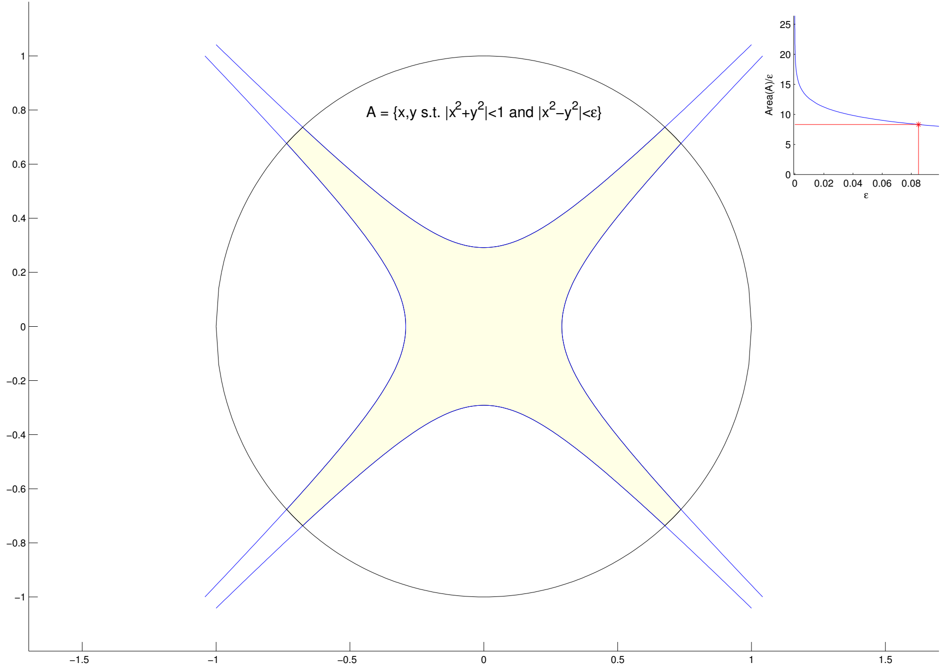
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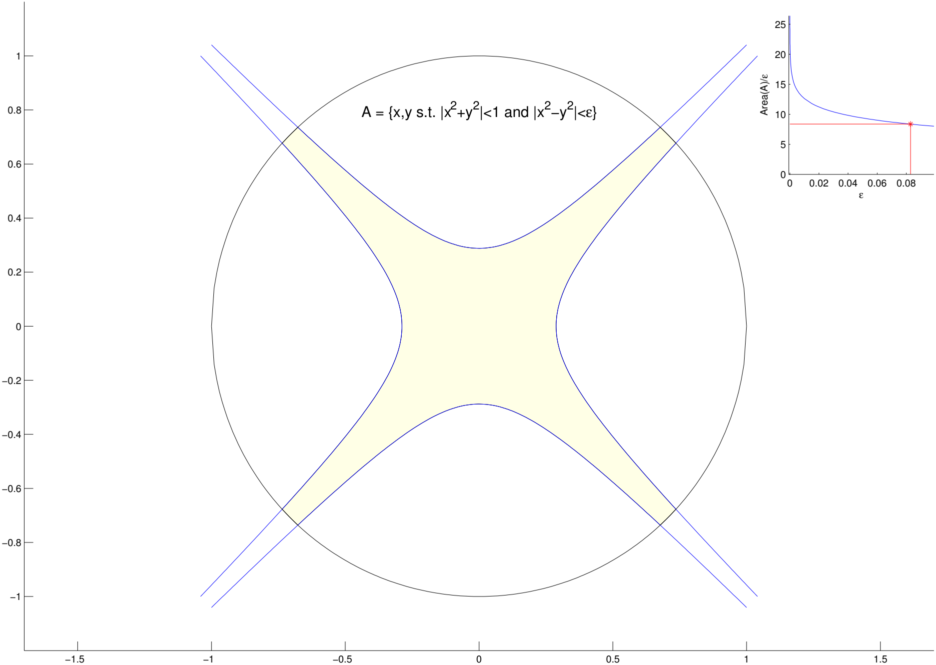
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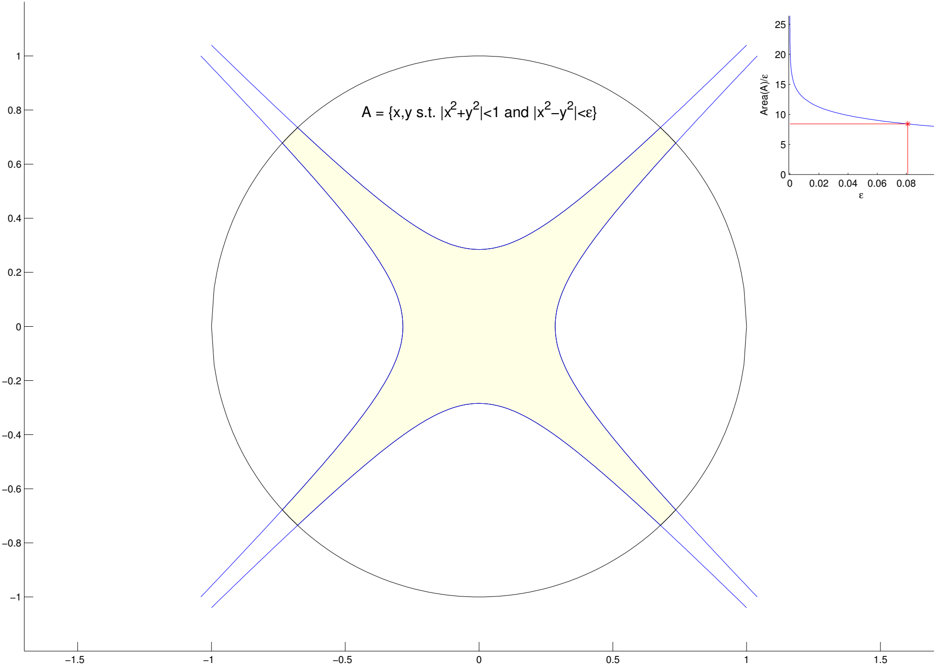
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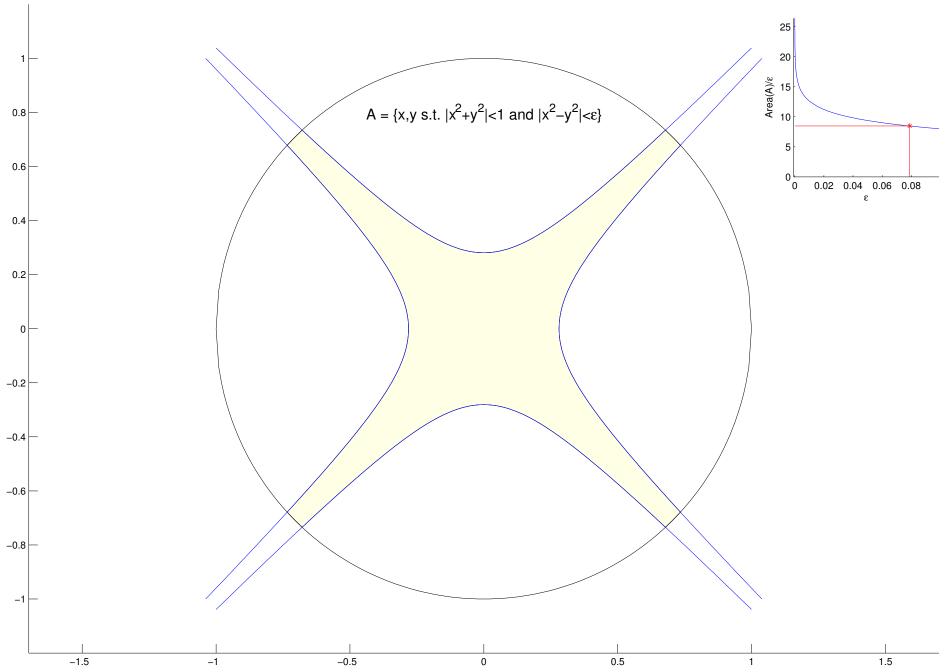
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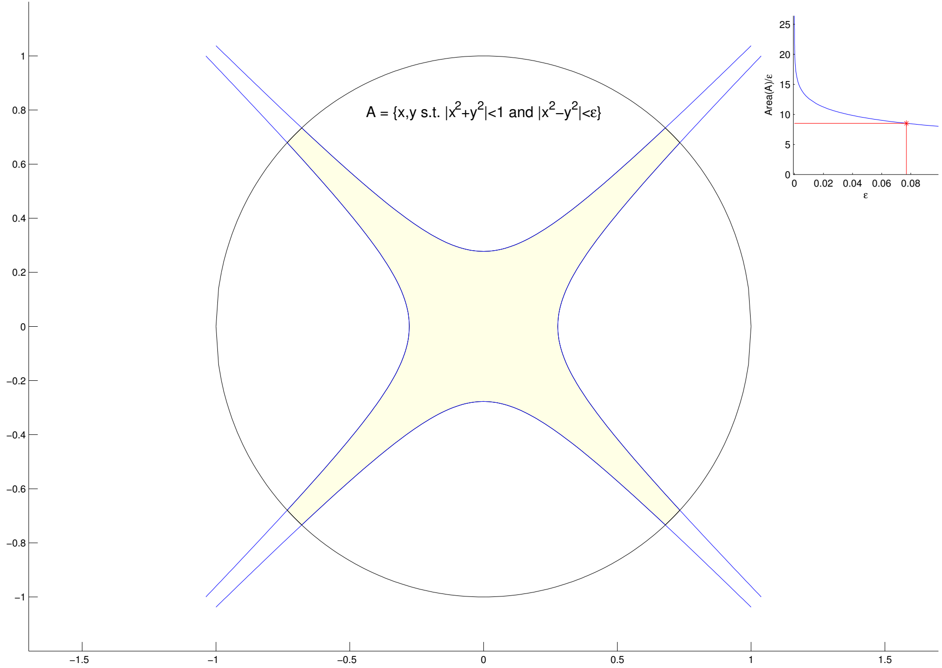
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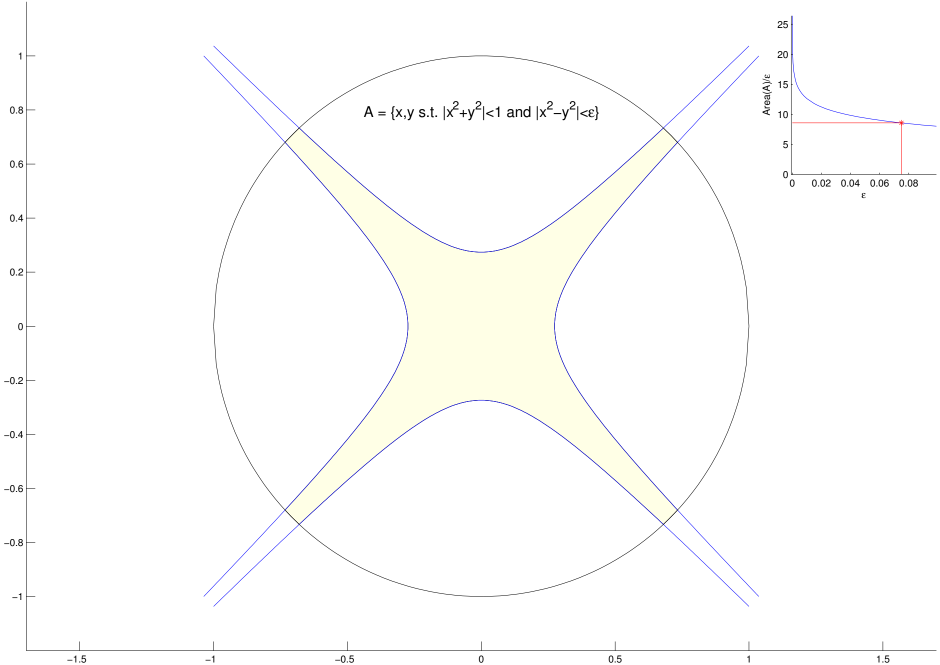
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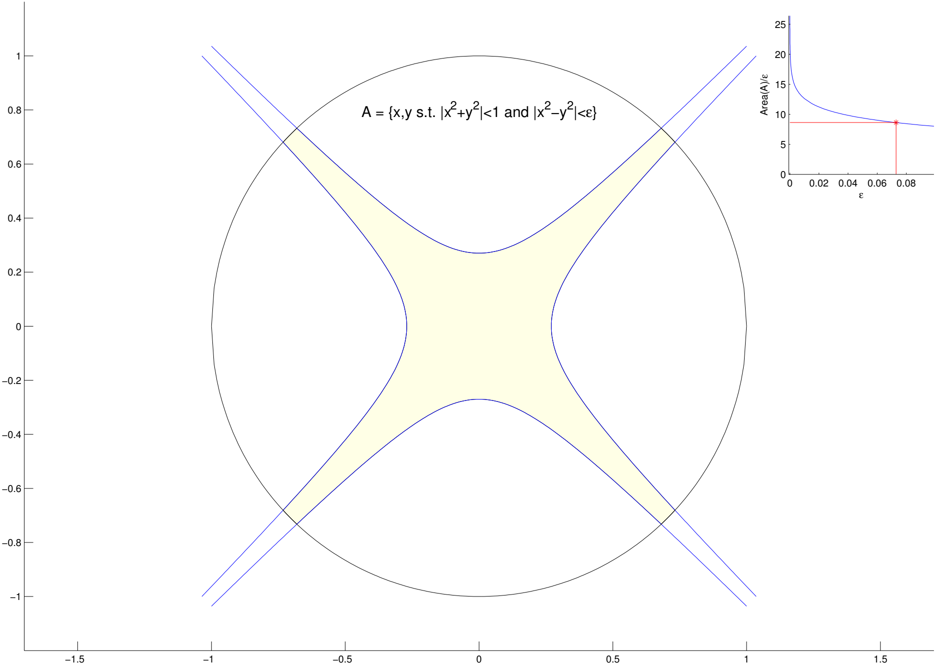
A = {x,y s.t. |x2+y2|<1 and |x2−y2|<ε}
Area(A)/ε
ε
0 0.02 0.04 0.06 0.08
0 5 10 15 20 25
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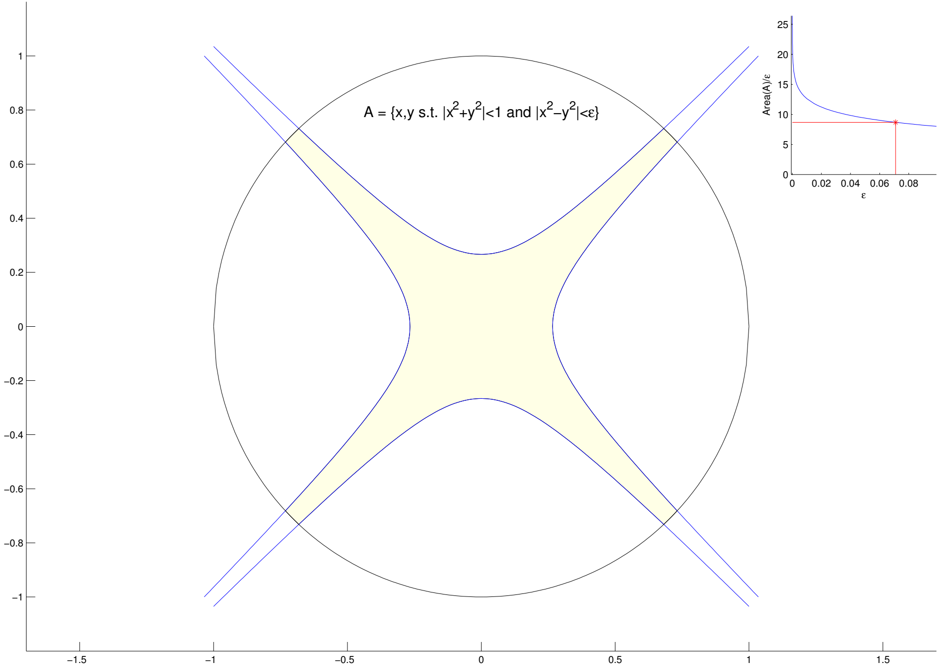
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Area(A)/ε
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20
15
10
5
0
0.02
0.04
0.06
0.08
1
0.8
0.6
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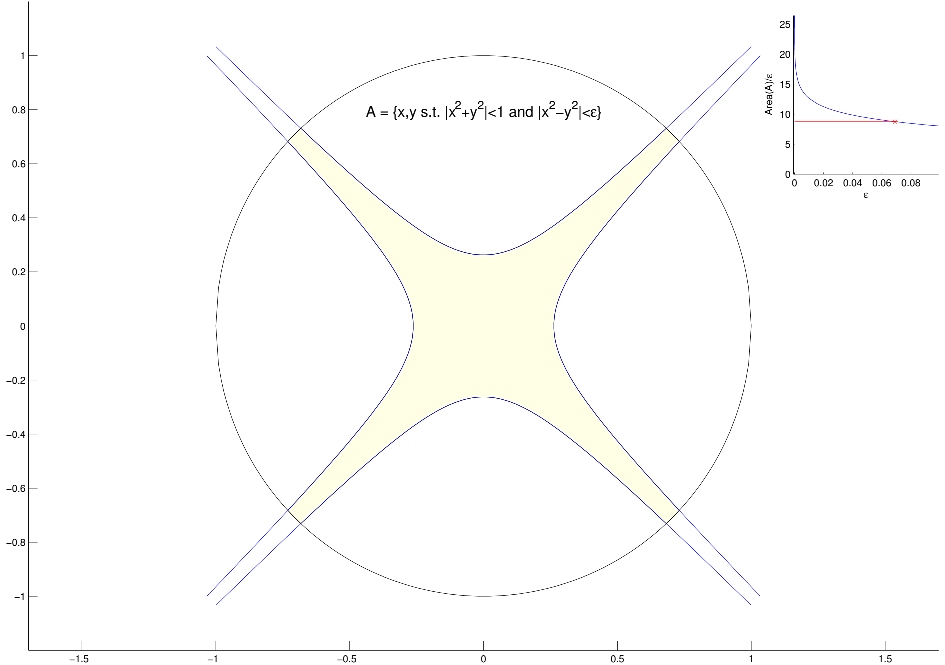
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0.02
0.04
0.06
0.08
1
0.8
0.6
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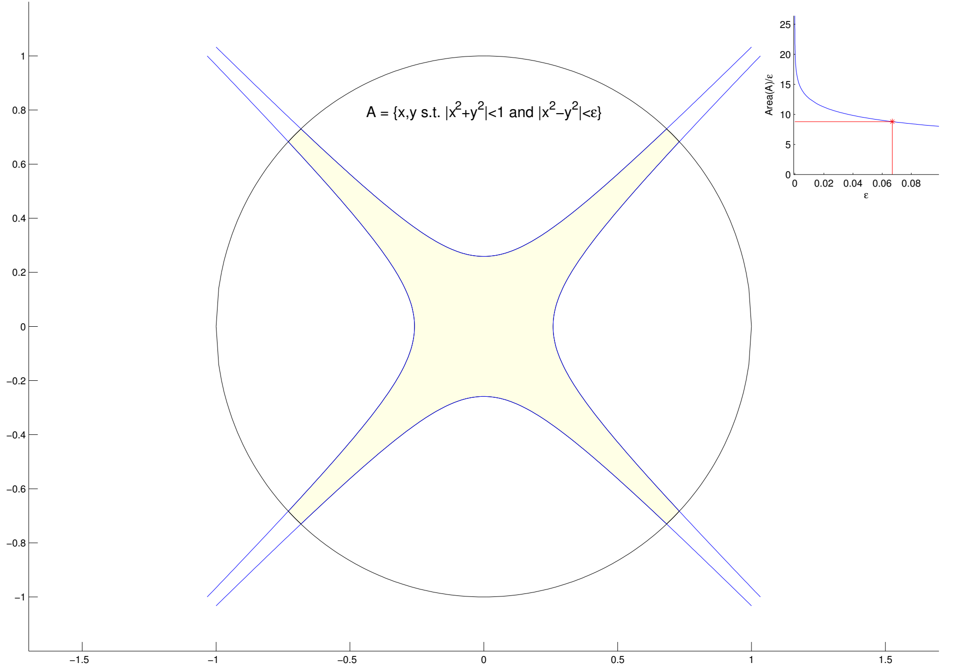
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Area(A)/ε
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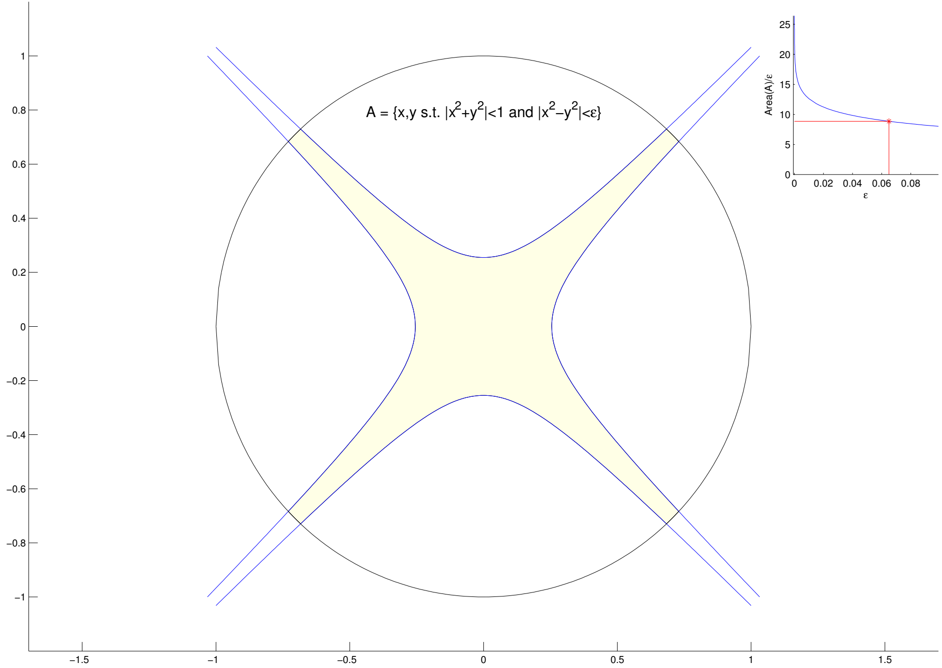

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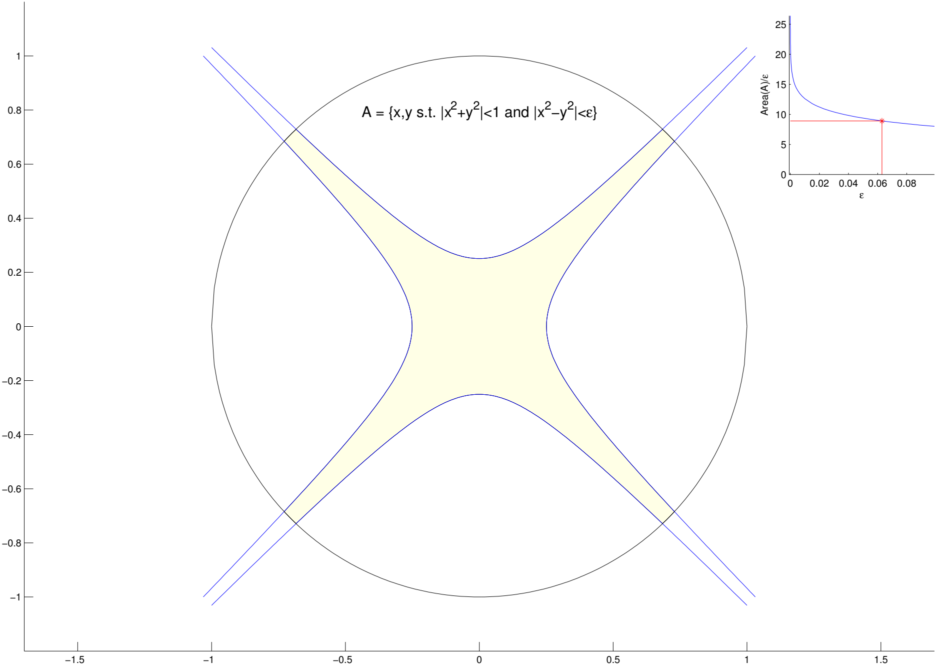
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5
Area(A)/ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
ε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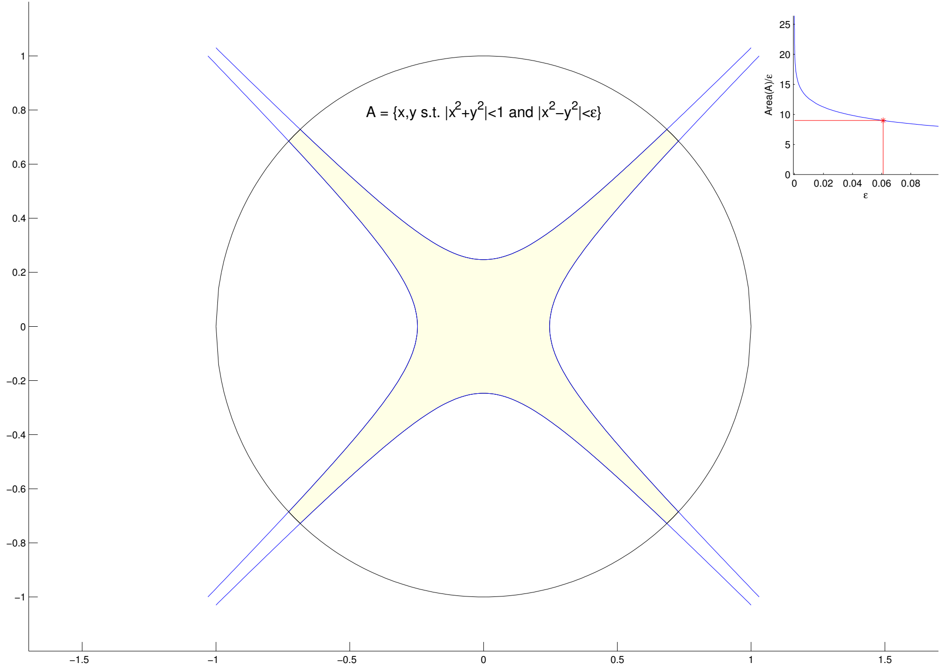

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ε
0 0.02 0.04 0.06 0.08
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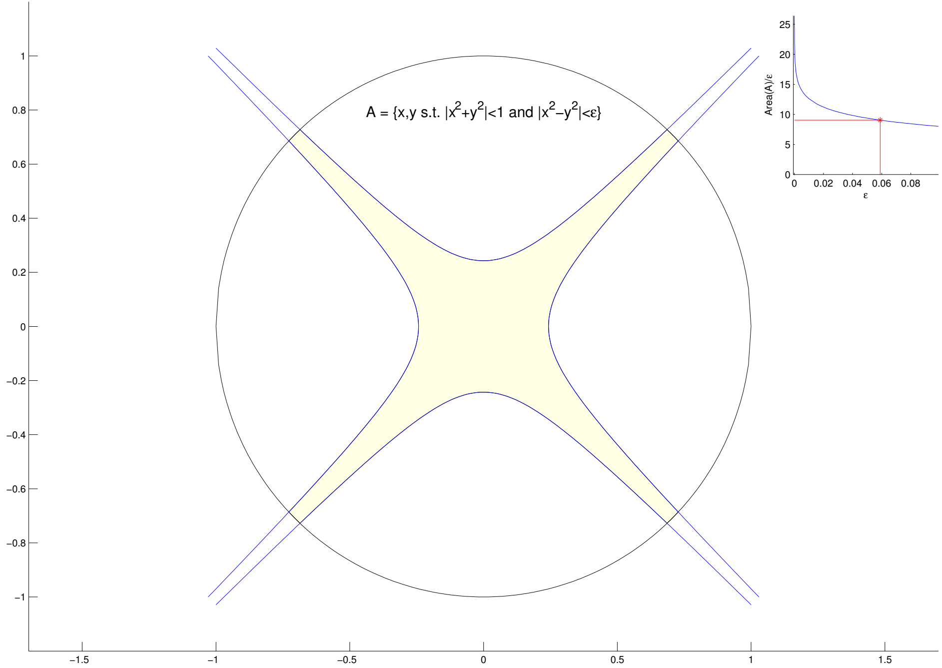
−1 −0.8 −0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8 1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0.08
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0.4
0.2
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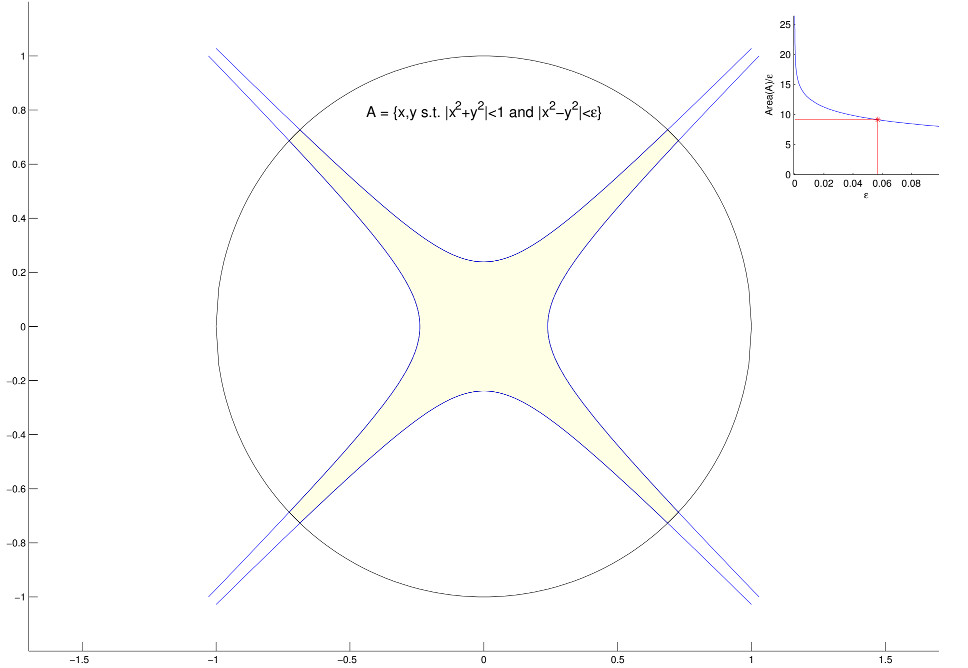
1.5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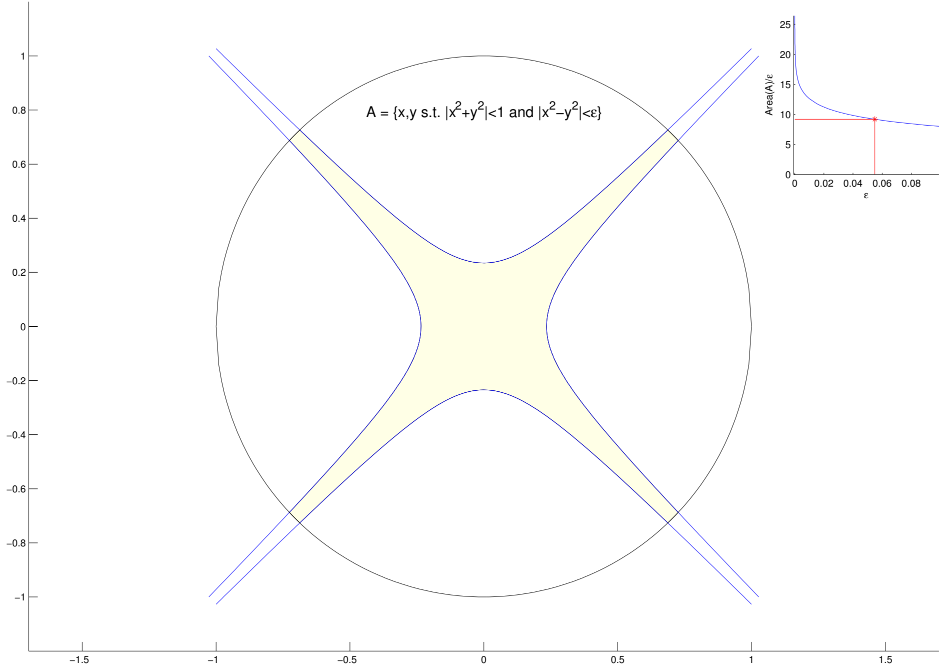
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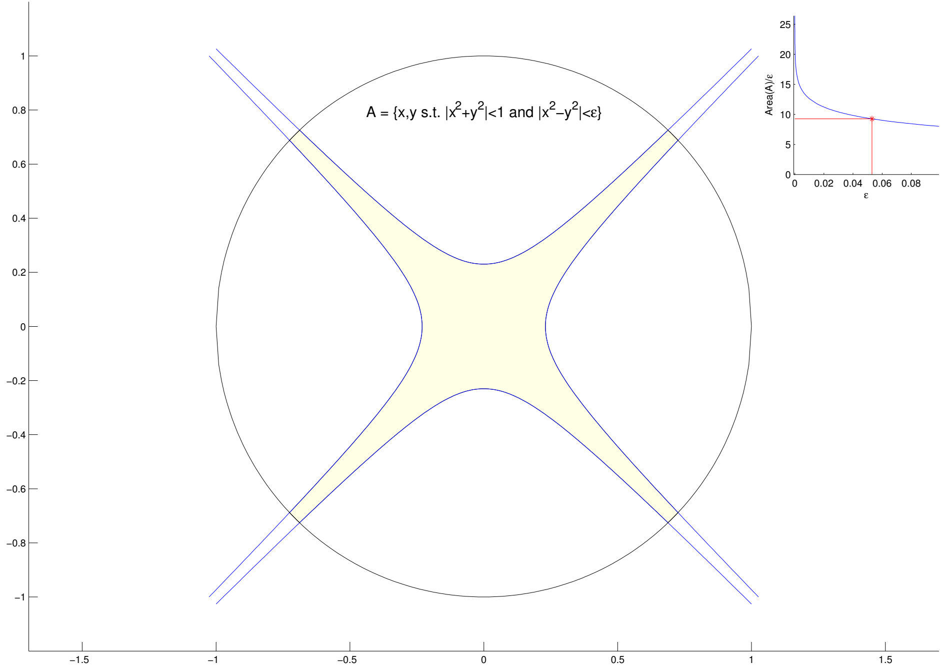
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−0.5
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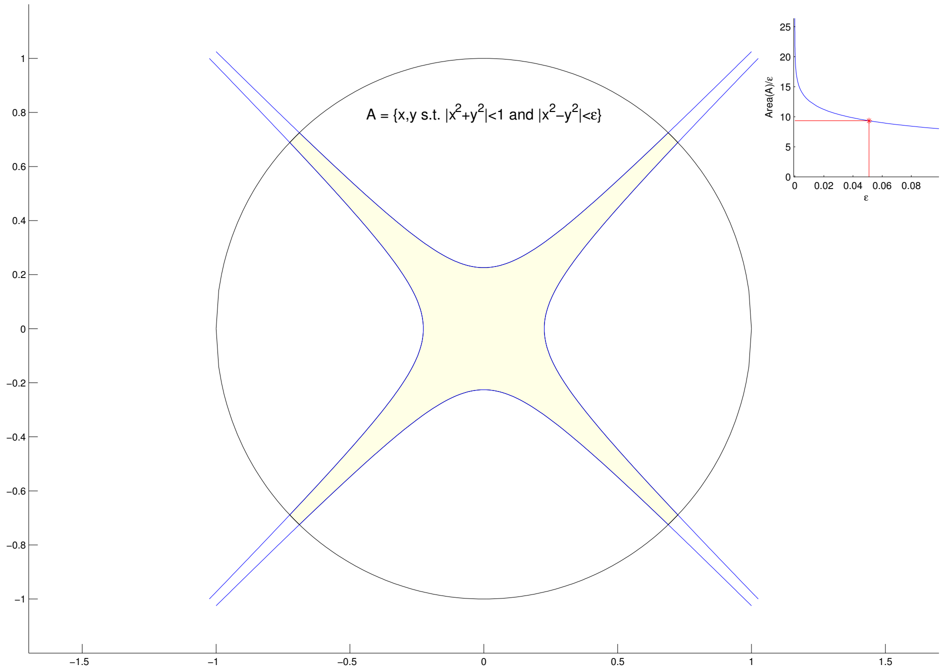
$A = \{x,y \text{ s.t. } |x^2+y^2|<1 \text{ and } |x^2-y^2|<\varepsilon\}$
Area(A)/ε
ε
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20
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0.08
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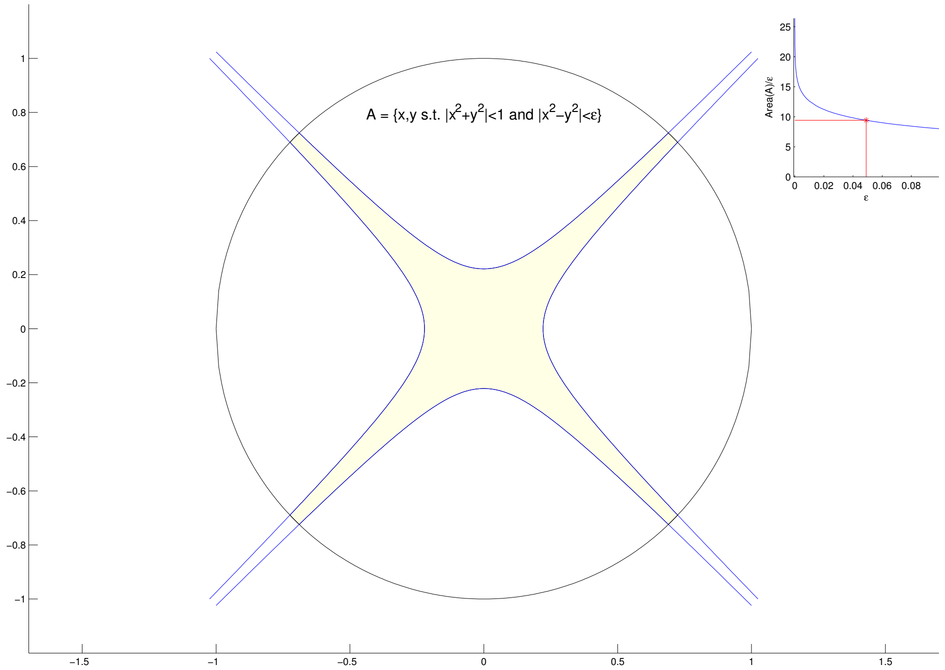
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
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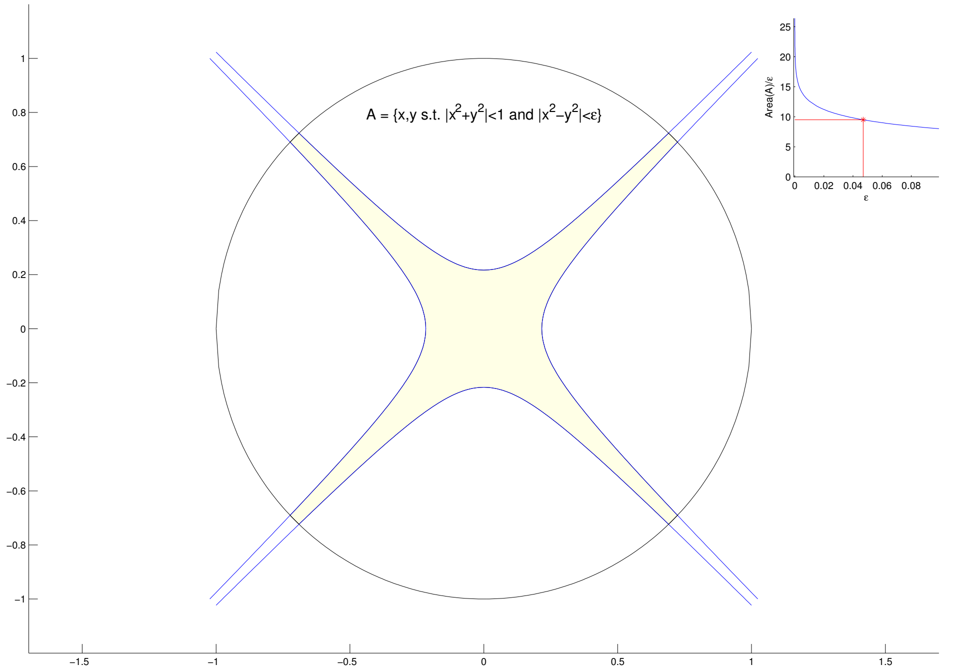

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Area(A)/ε
ε
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0.04
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15
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5
−1.5
−1
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−0.6
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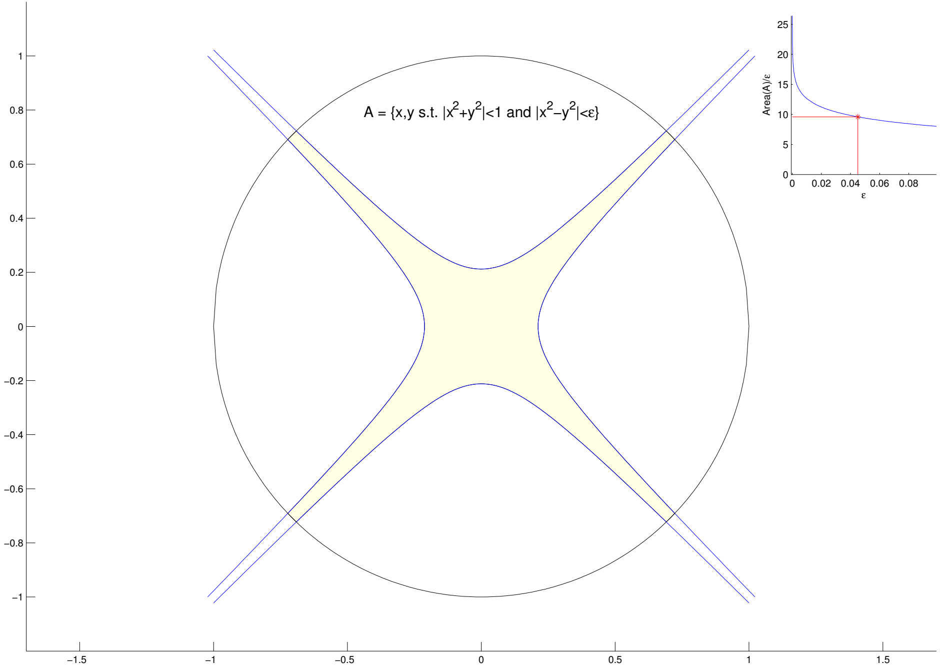

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Area(A)/ε
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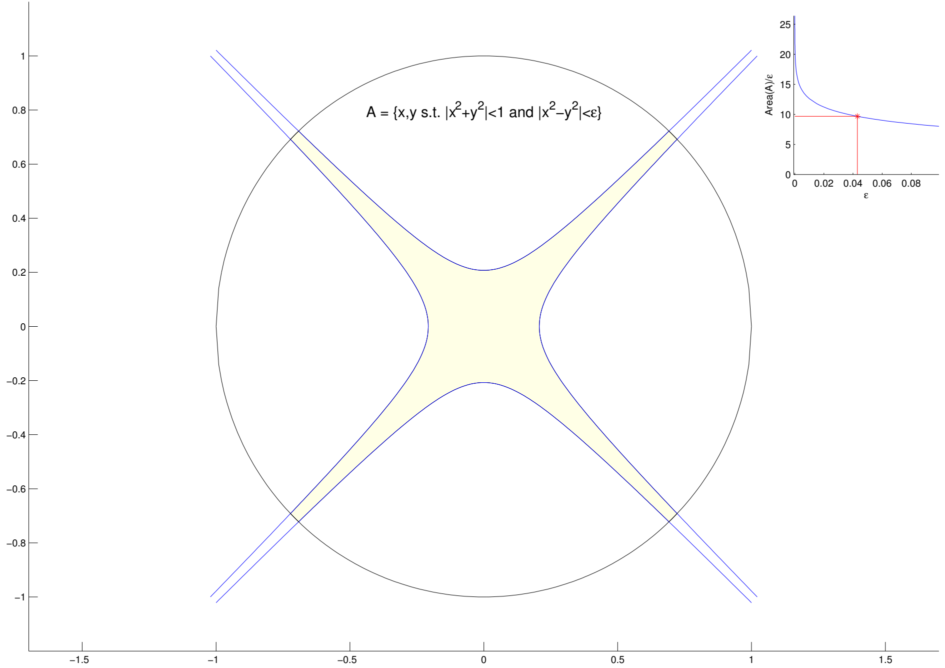

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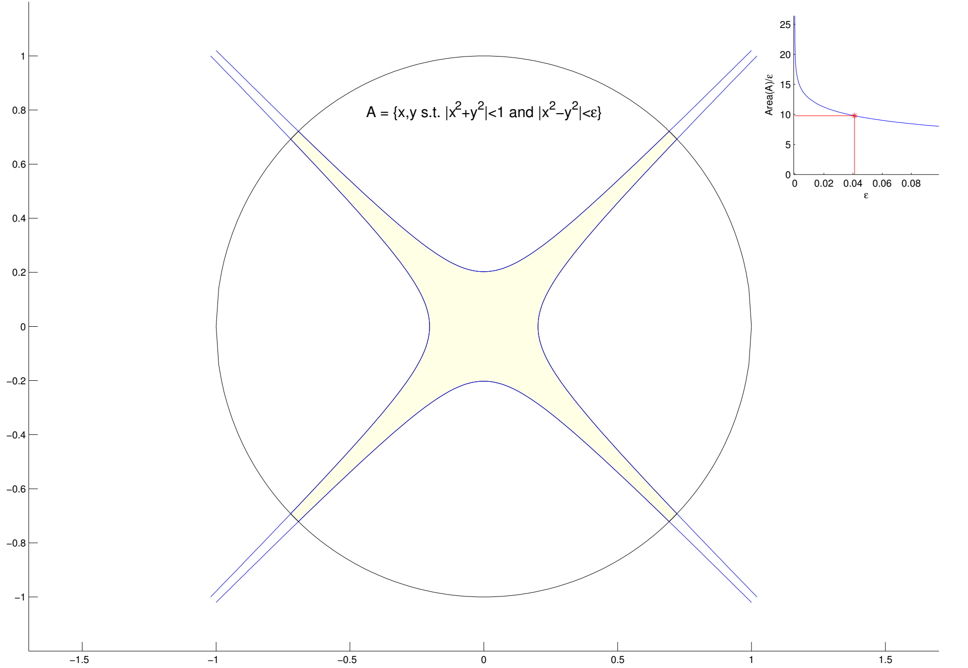

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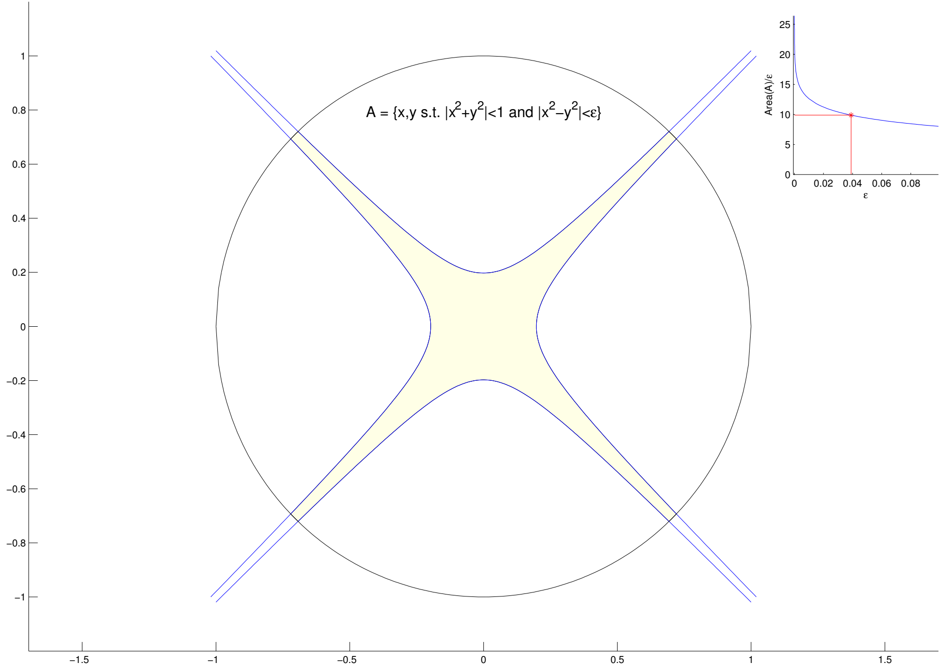
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0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
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Area(A)/ε
25
20
15
10
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0.04
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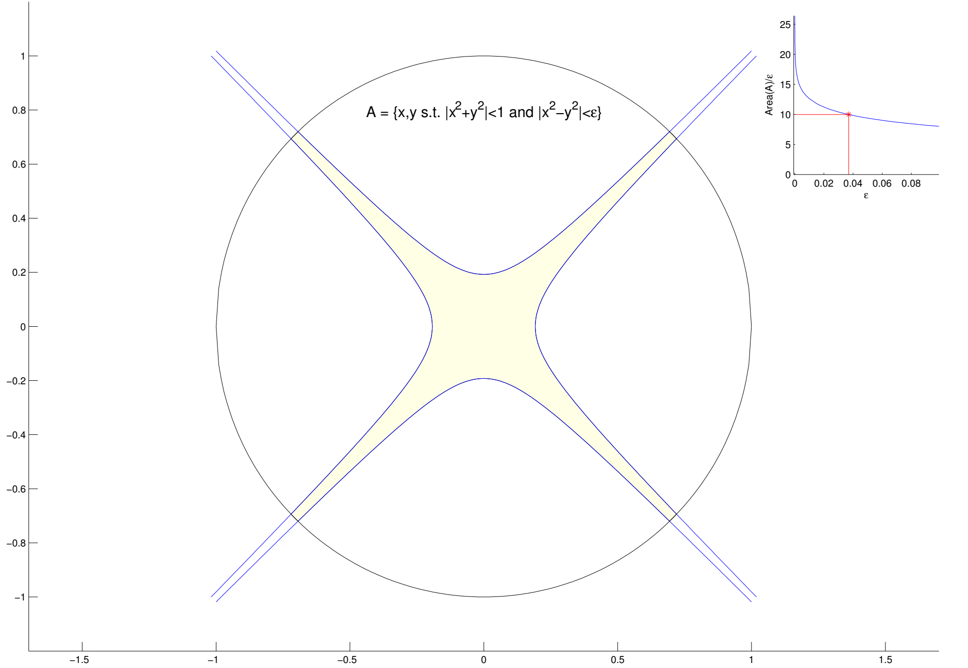
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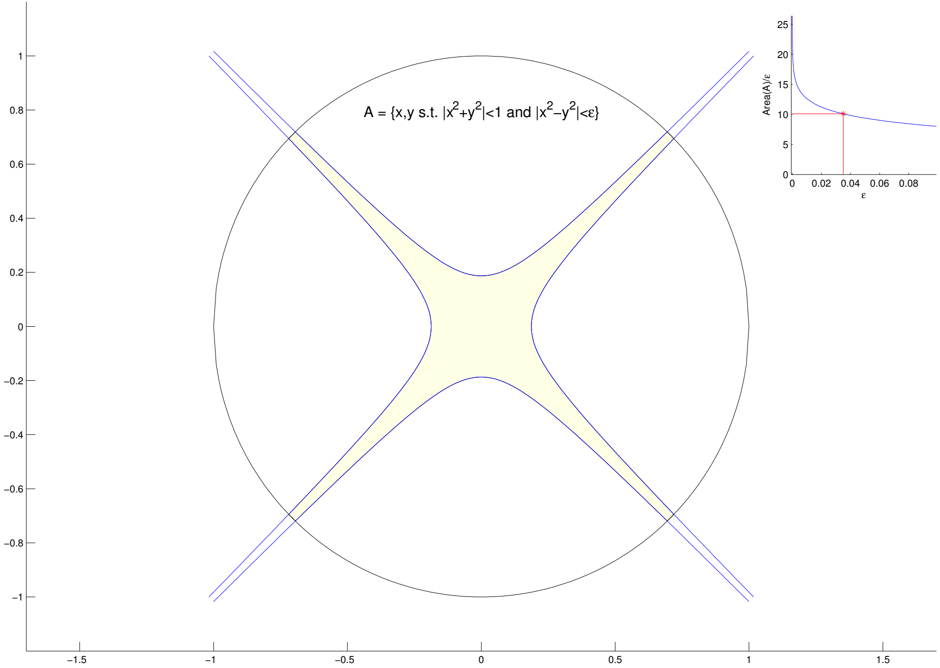
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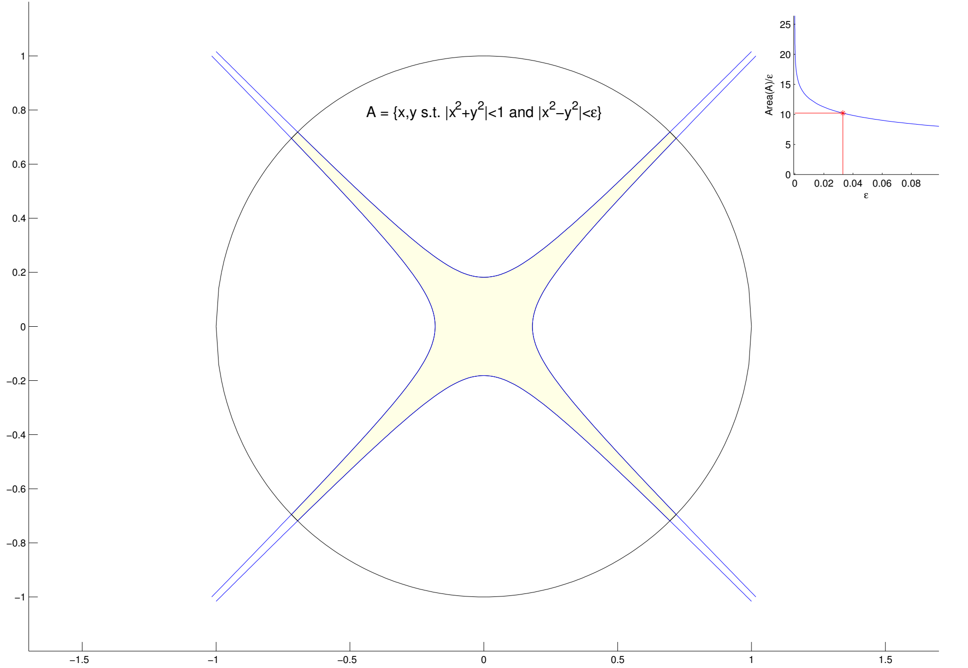
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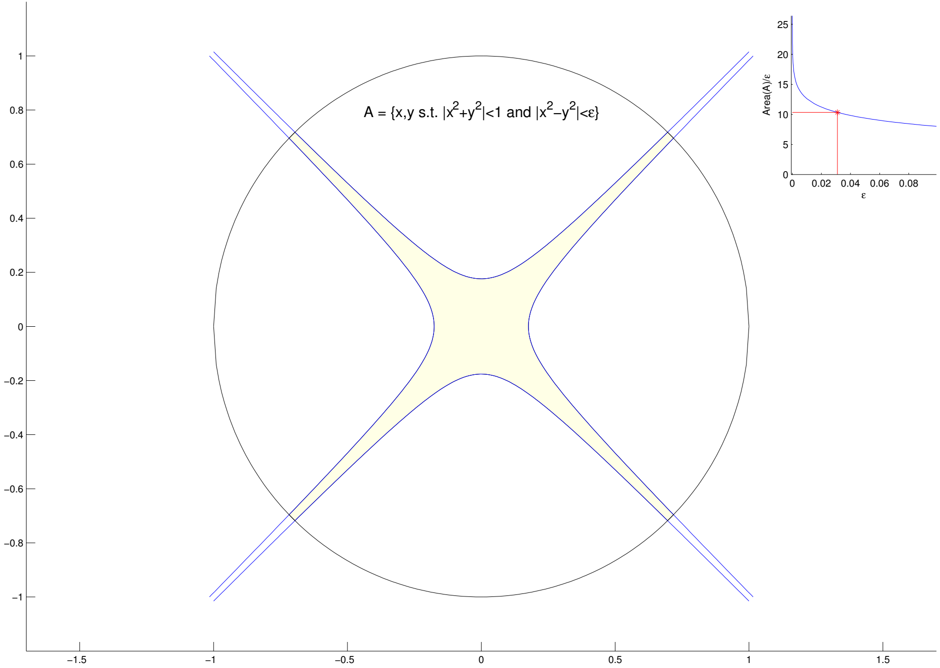
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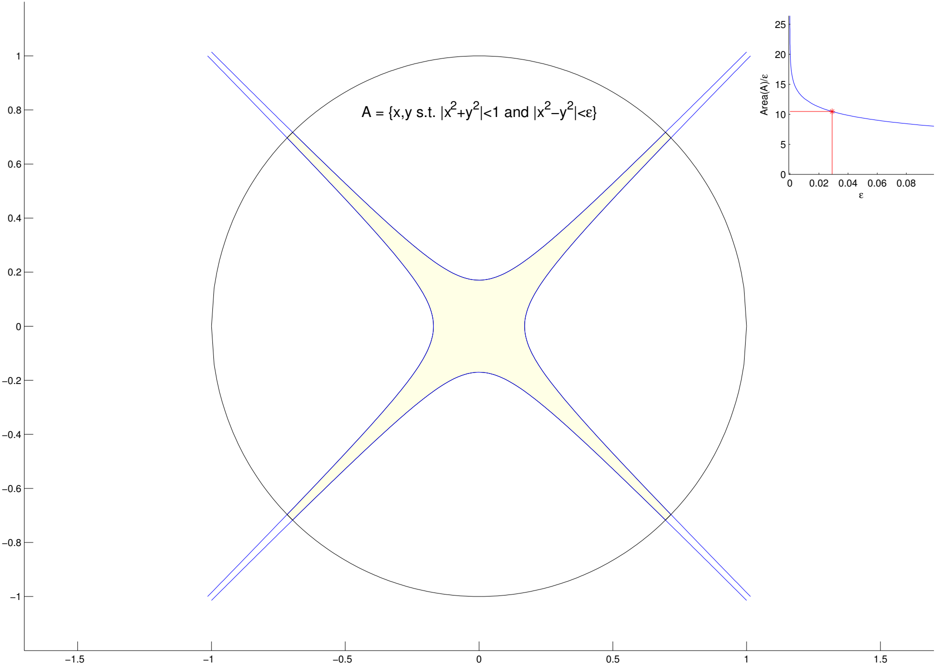
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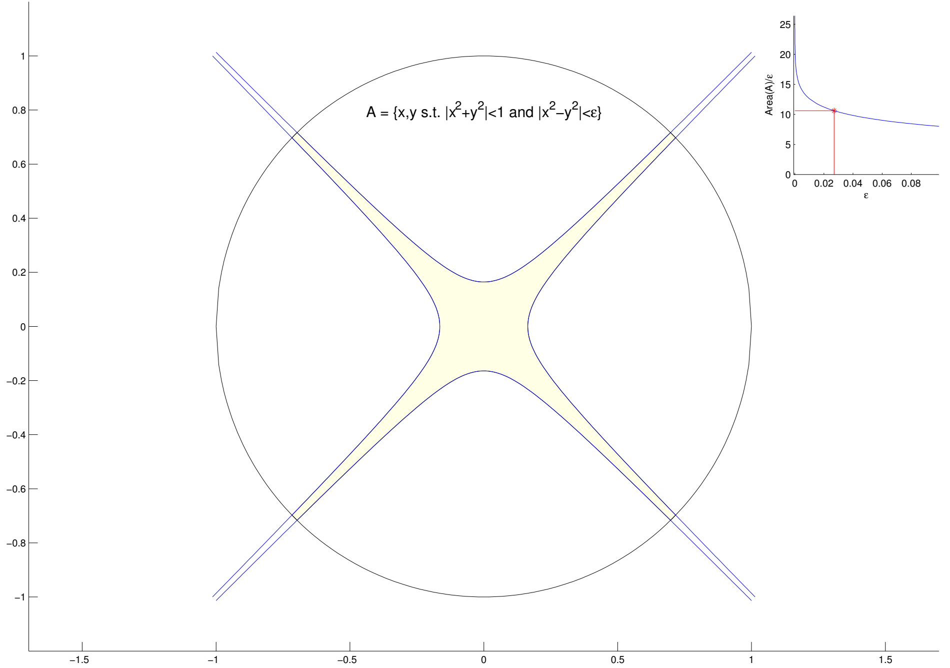
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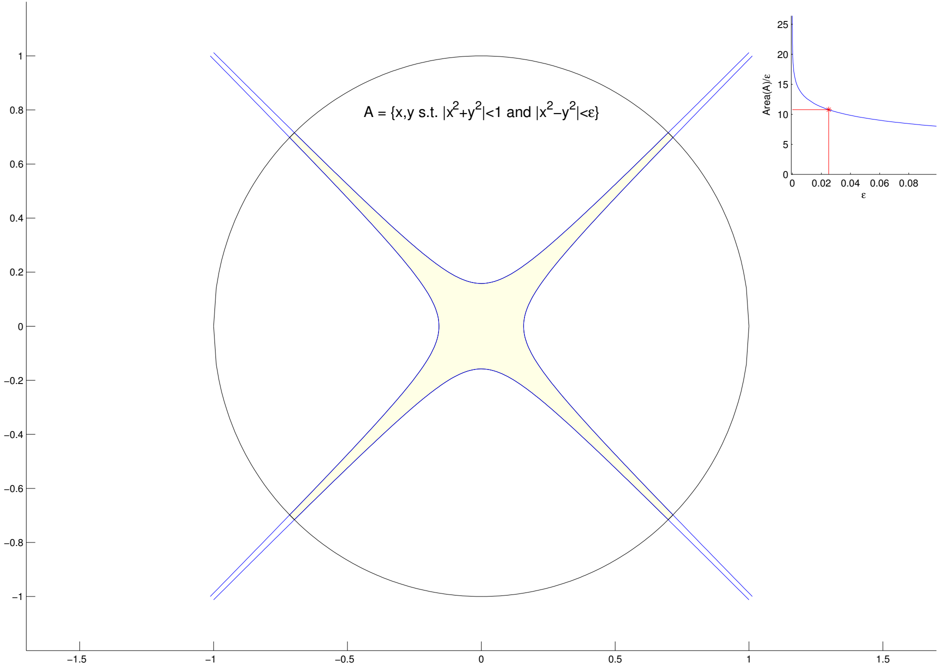
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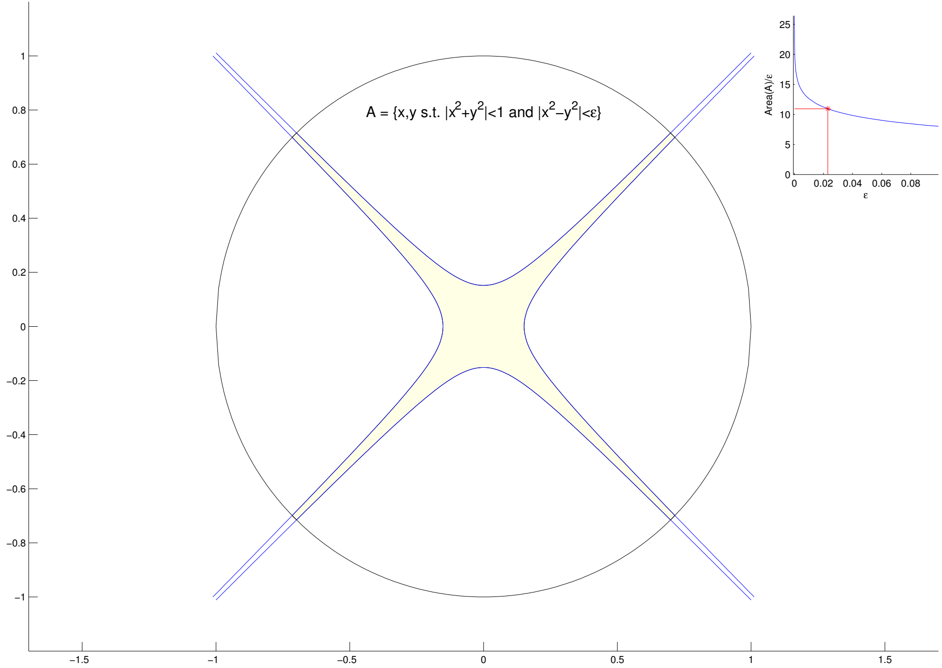
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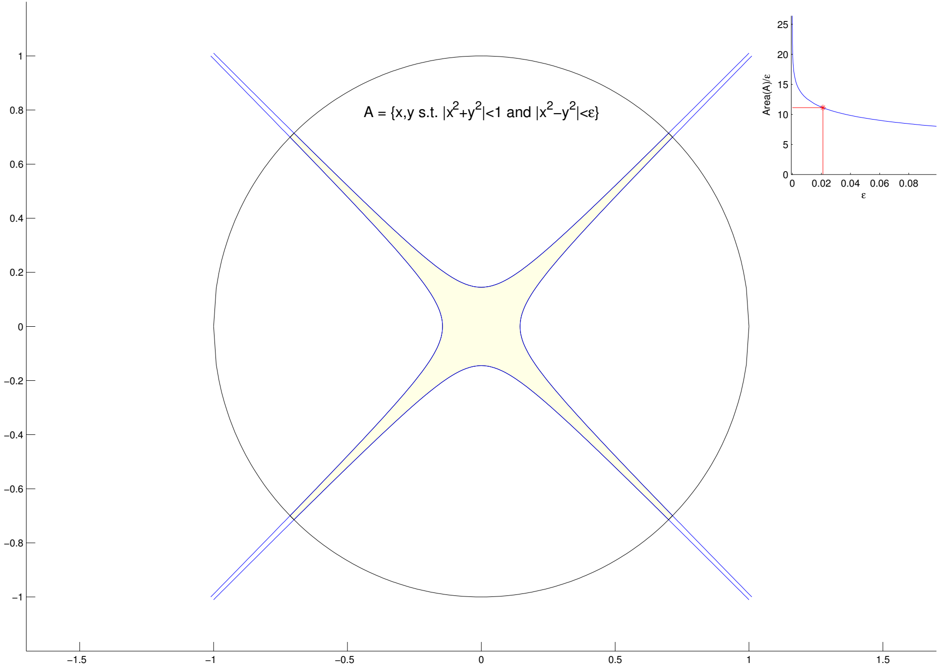
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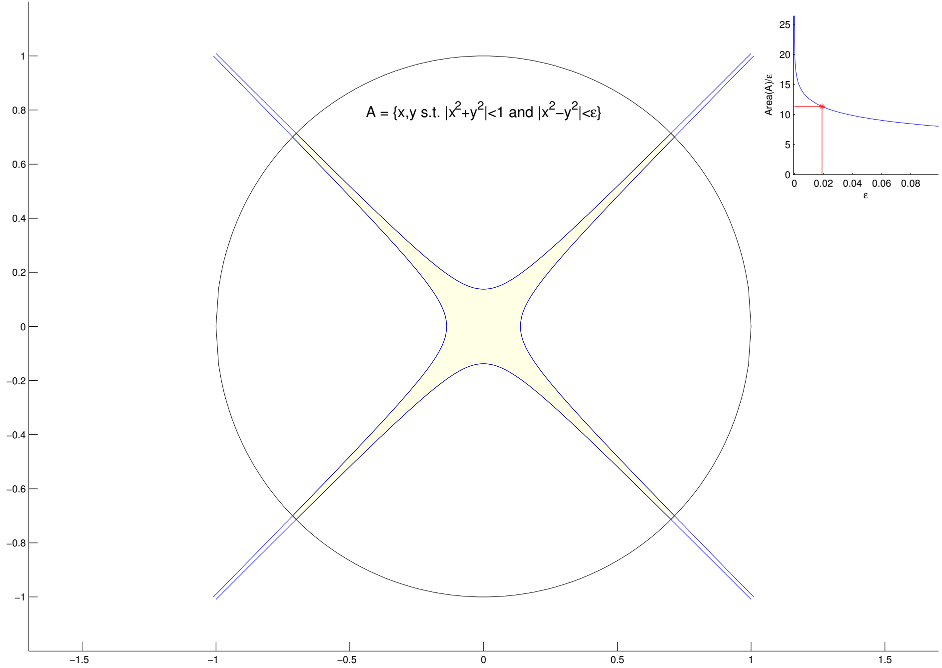
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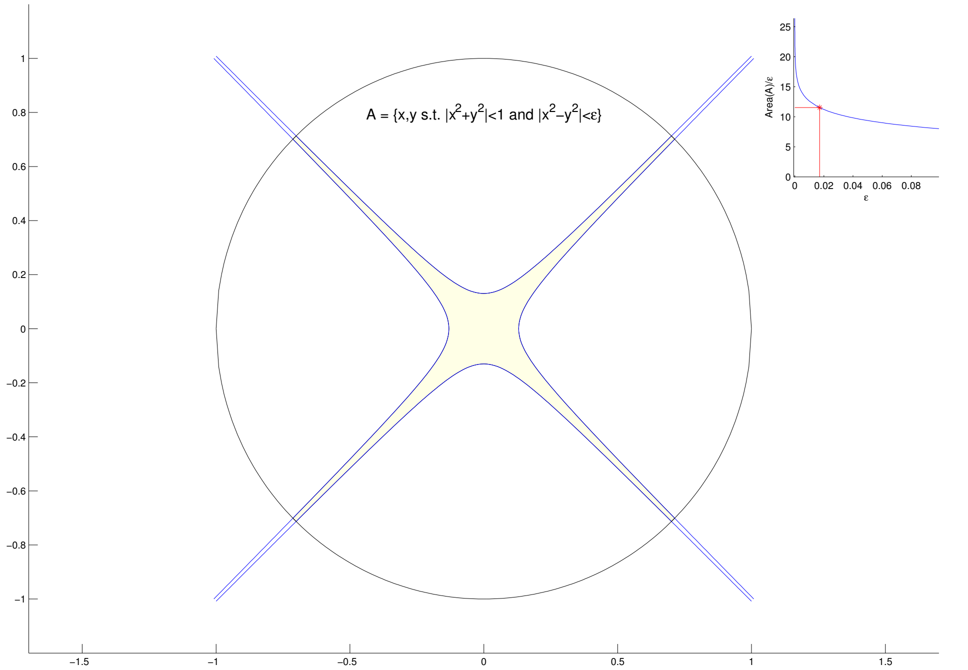
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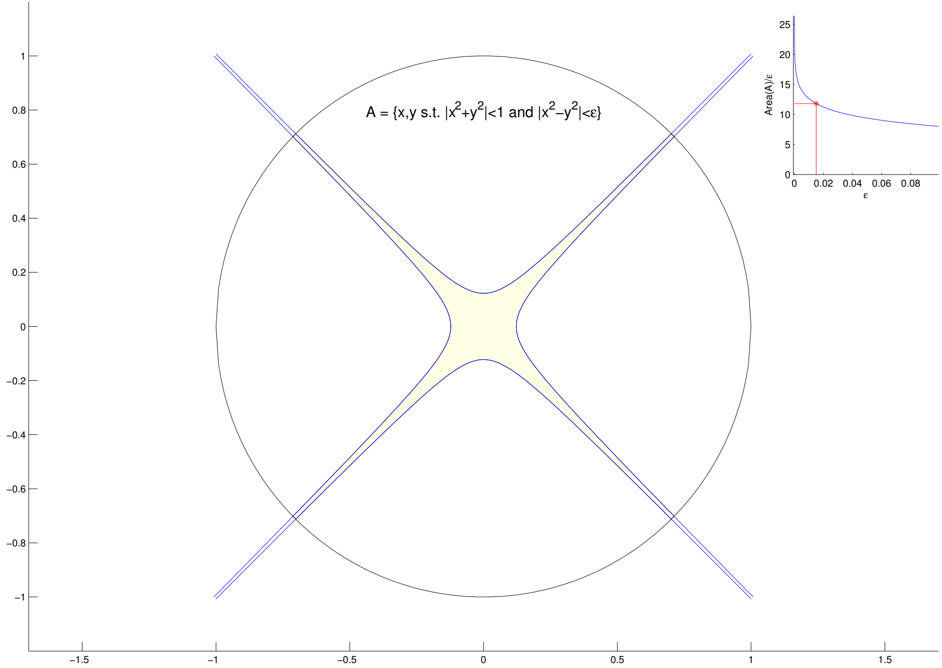
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−1.5
−1
−0.5
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25
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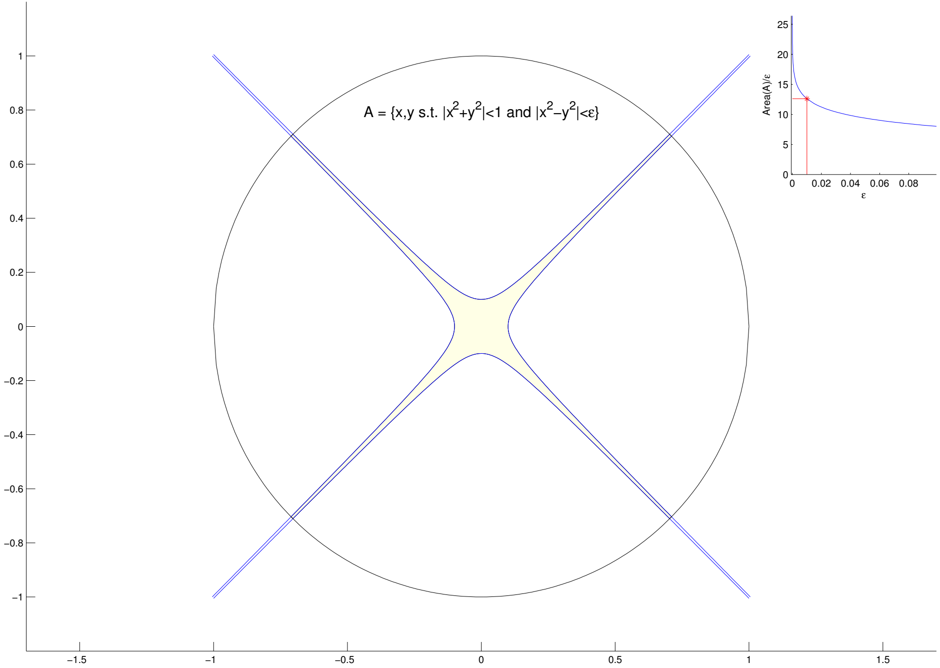

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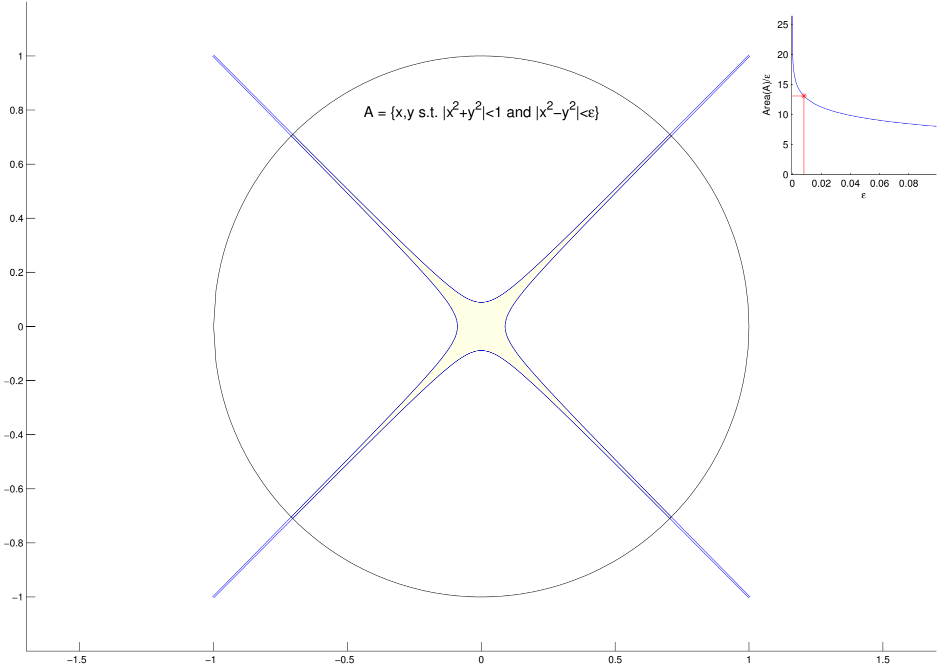
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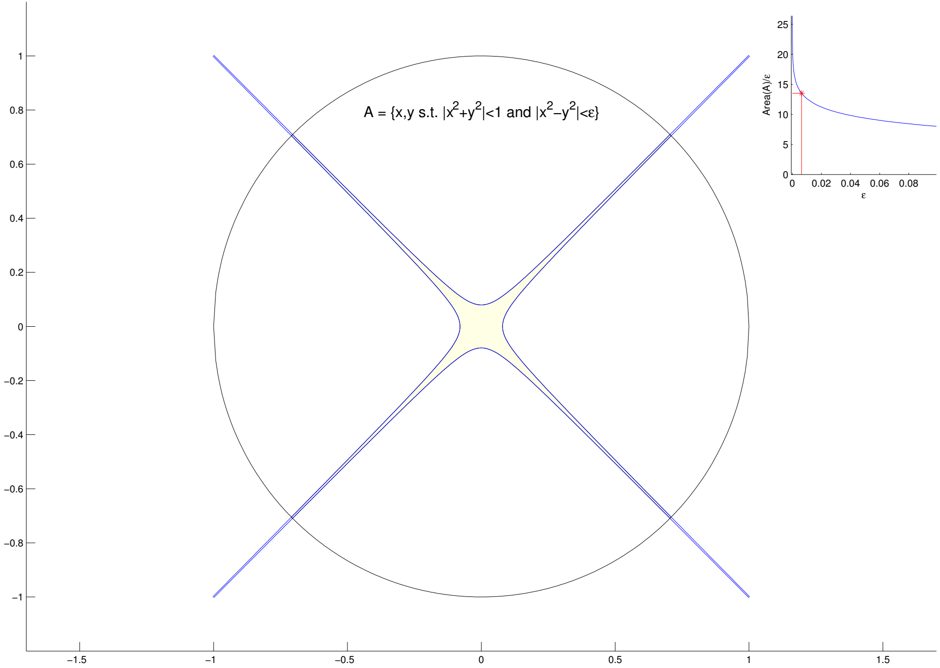

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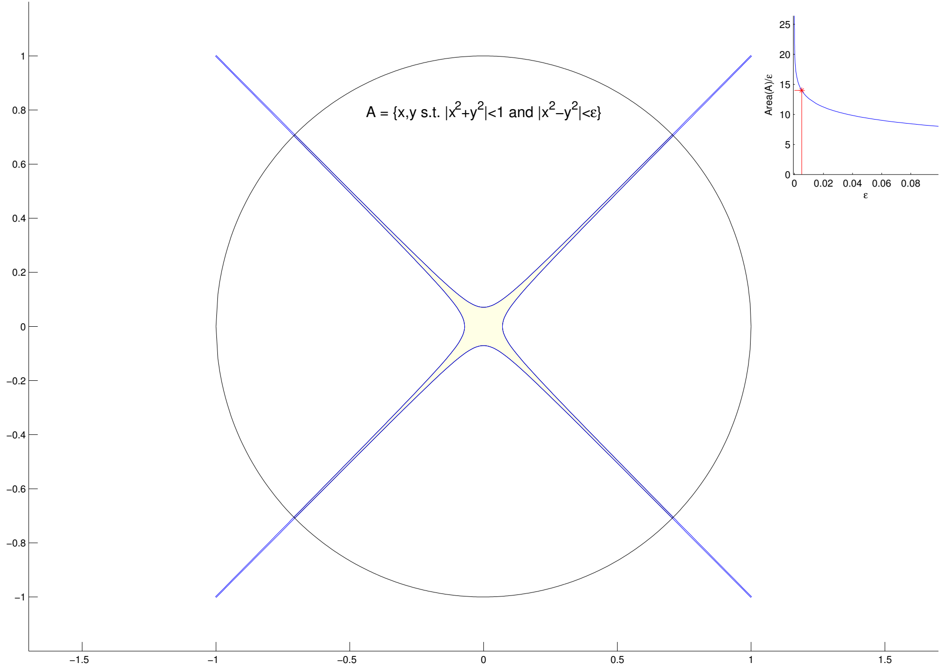
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0.2
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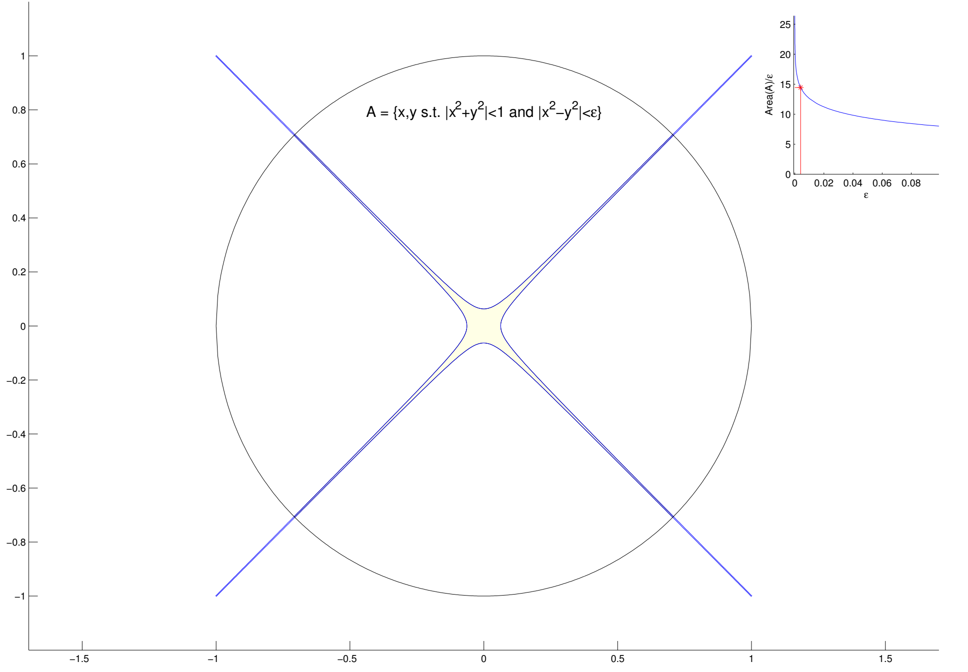
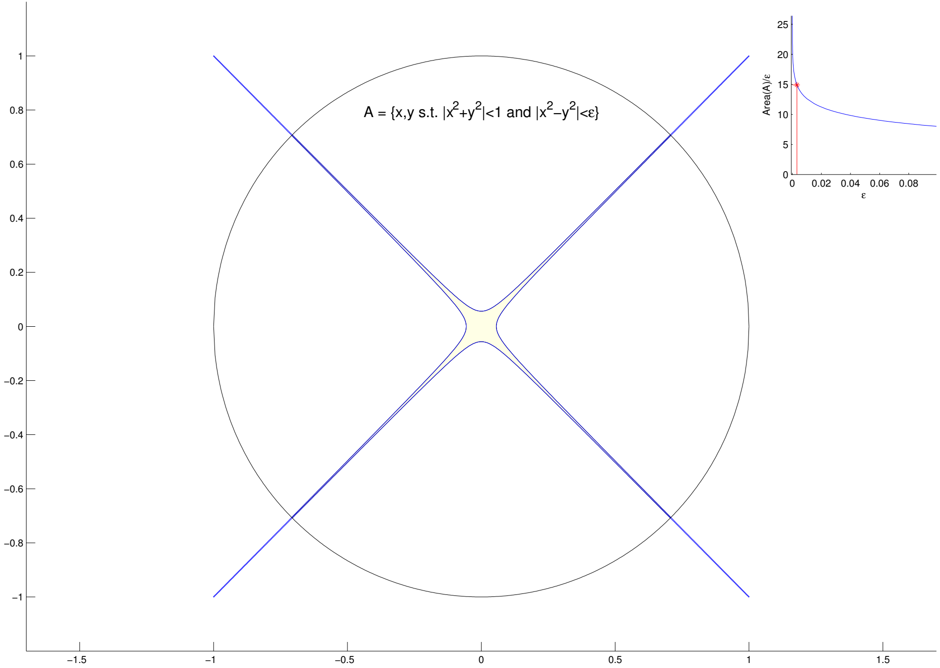
0.8



A = {x,y s.t. |x2+y2|<1 and |x2−y2|<ε}
Area(A)/ε
ε


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
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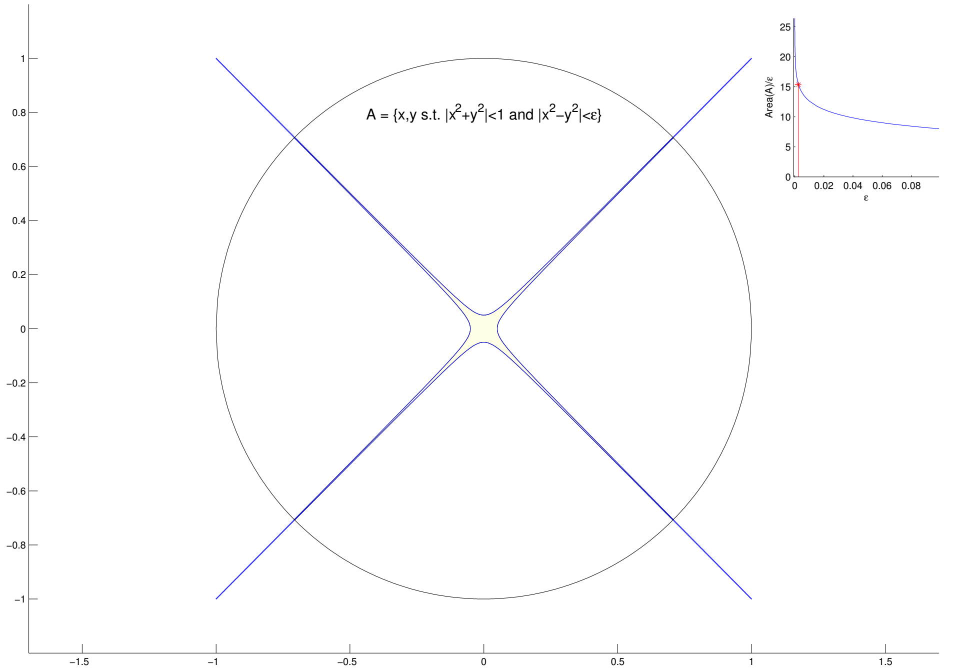
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5


A = {x,y s.t. |x2+y2|<1 and |x2−y2|<ε}
Area(A)/ε
ε
0
0.02
0.04
0.06
0.08
5
10
15
20
25
−1.5
−1
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1.5
0.2
0.4
0.6
0.8
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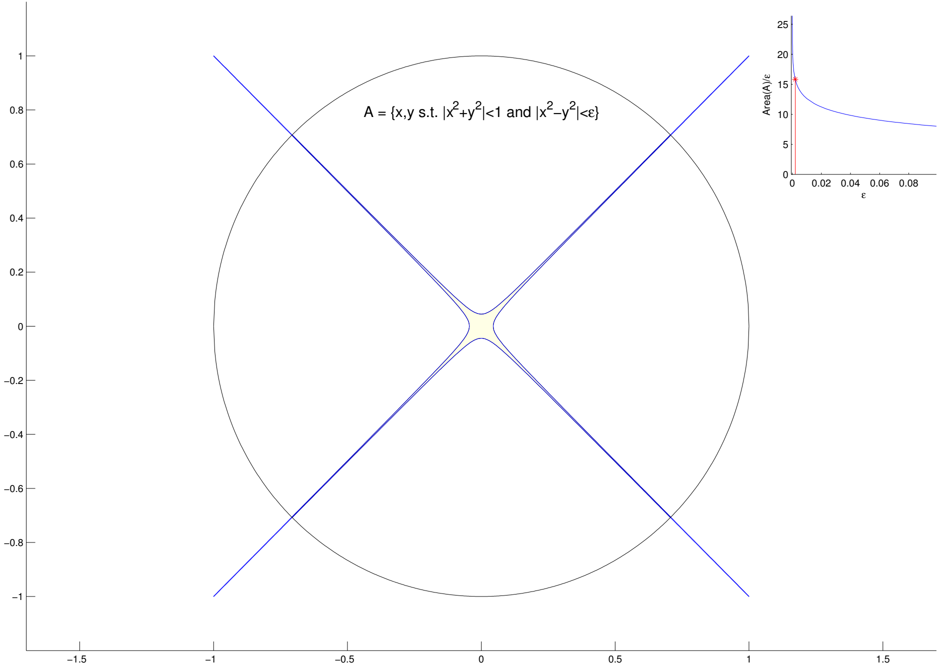
−0.4
−0.6
−0.8


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
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20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
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−1
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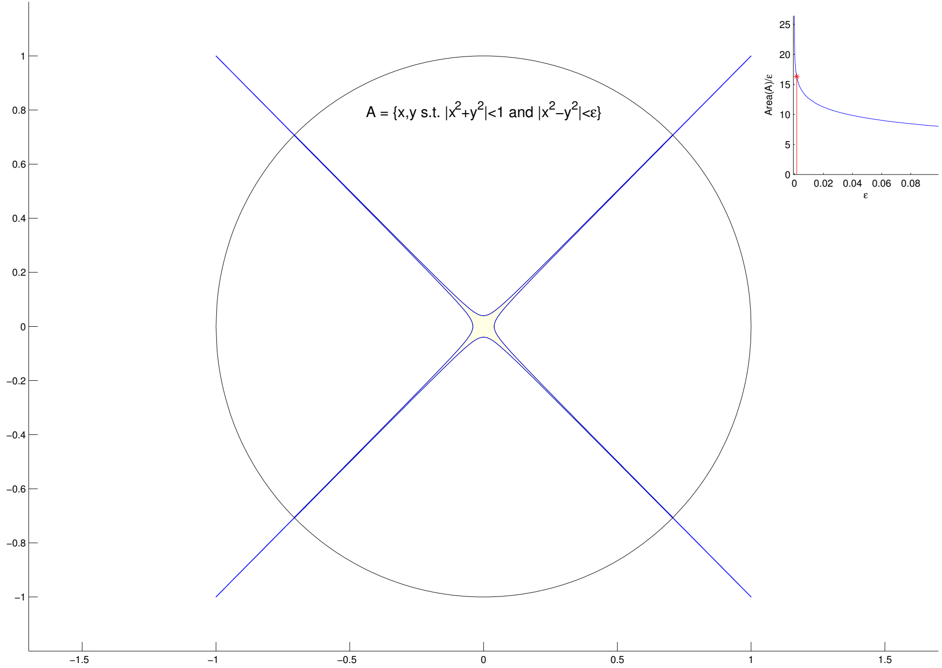
0
0.5
1
1.5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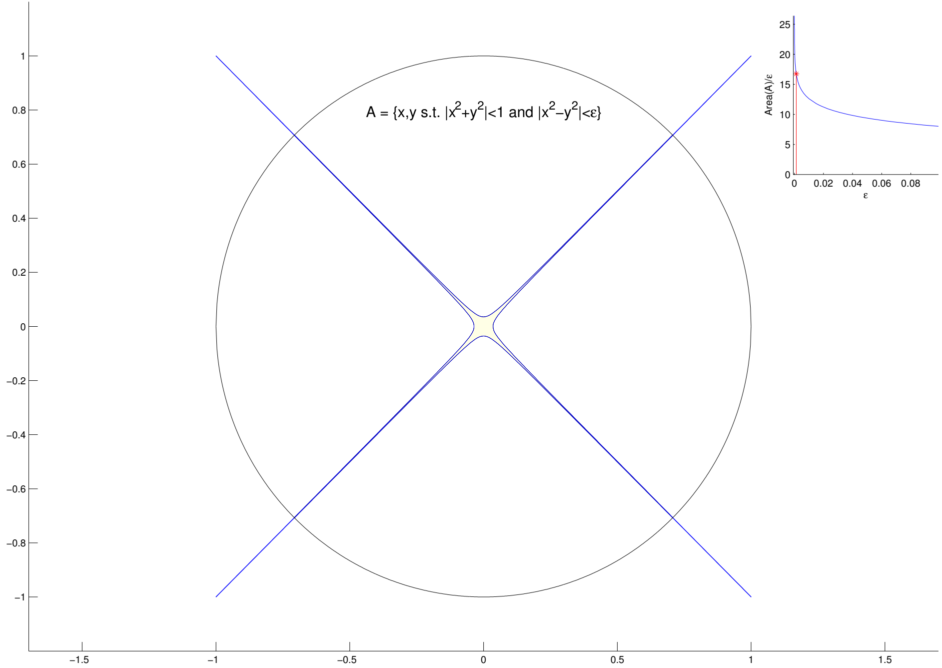
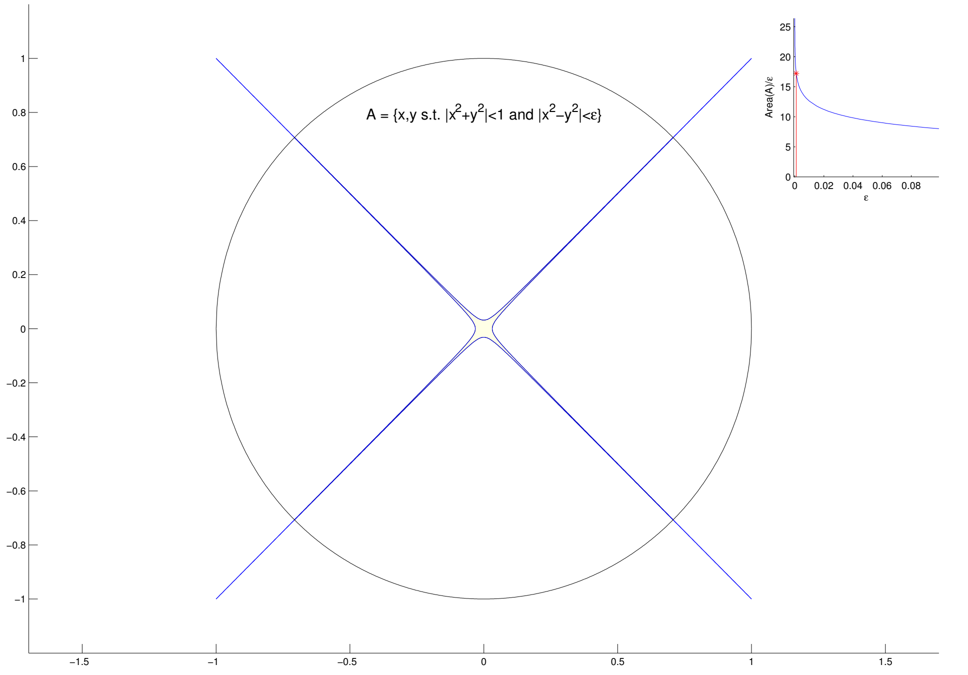
Area(A)/ε
ε


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
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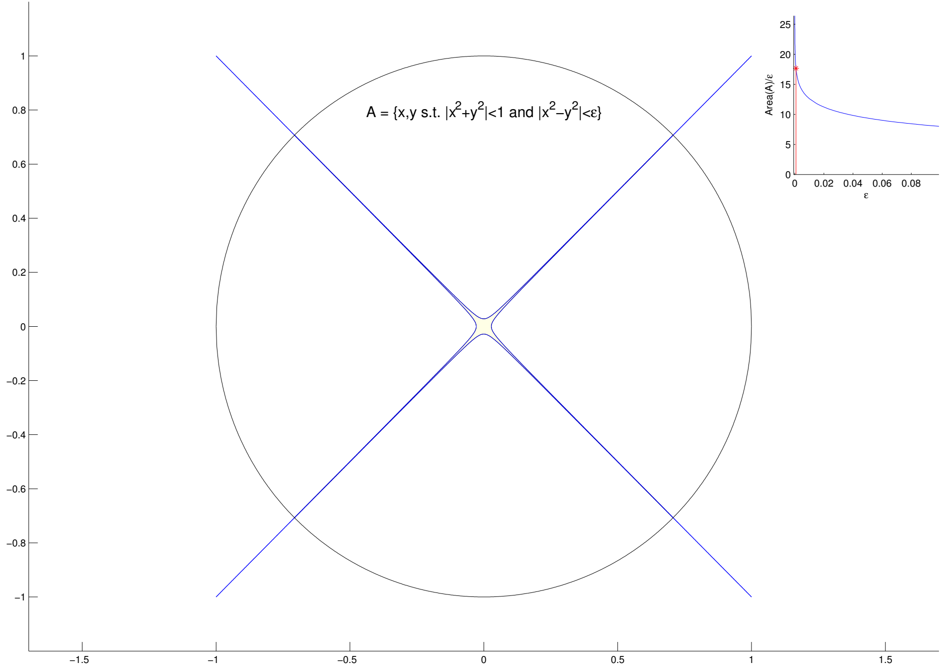
1
1.5
25
20
15
10
5
0
0.02
0.04
0.06
0.08


A = {x,y s.t. |x2+y2|<1 and |x2−y2|<ε}
Area(A)/ε
ε
0
5
10
15
20
25
0.02
0.04
0.06
0.08
1
0.8
0.6
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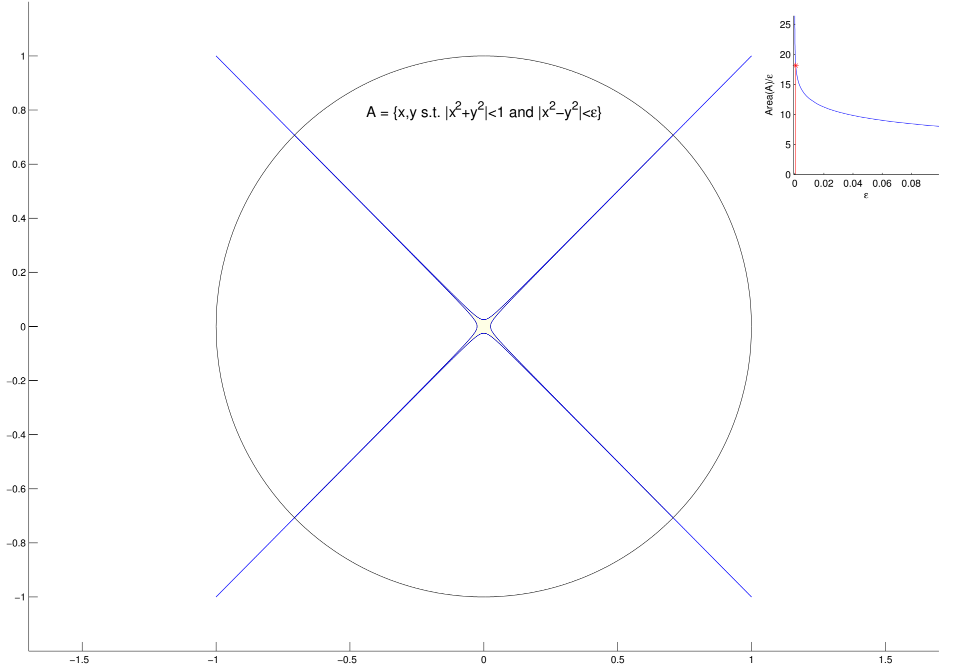
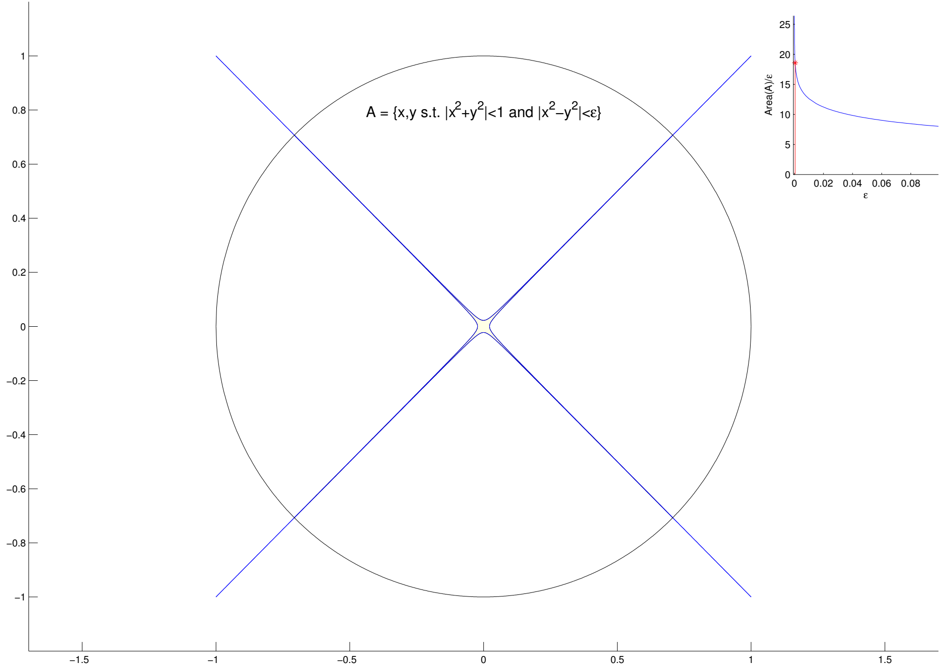
−0.4
−0.6
−0.8
−1
−1.5
−0.5
0.5
1.5



A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0 0.02 0.04 0.06 0.08
0 5 10 15 20 25
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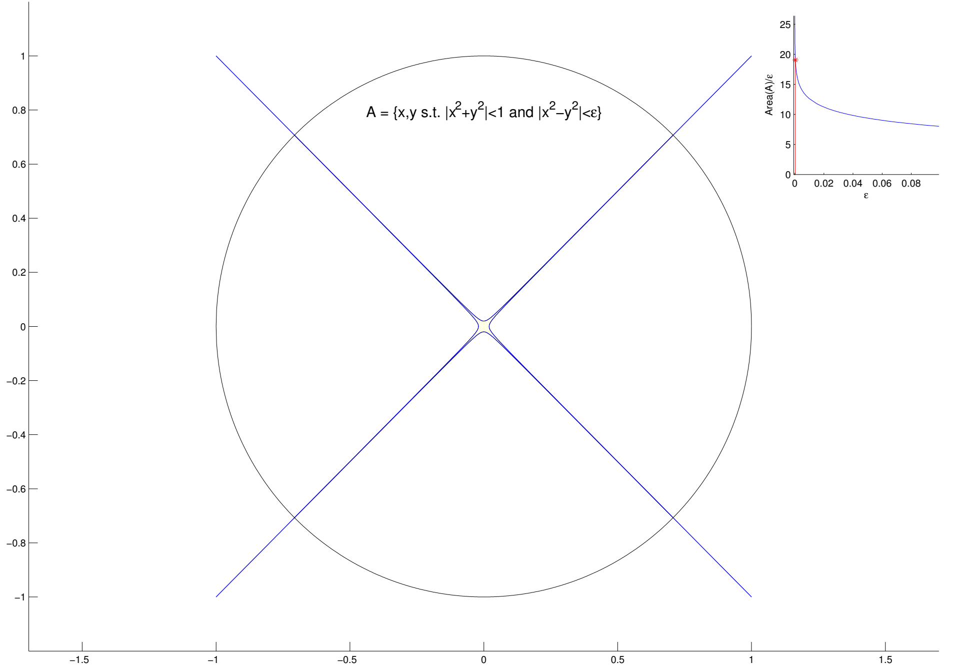
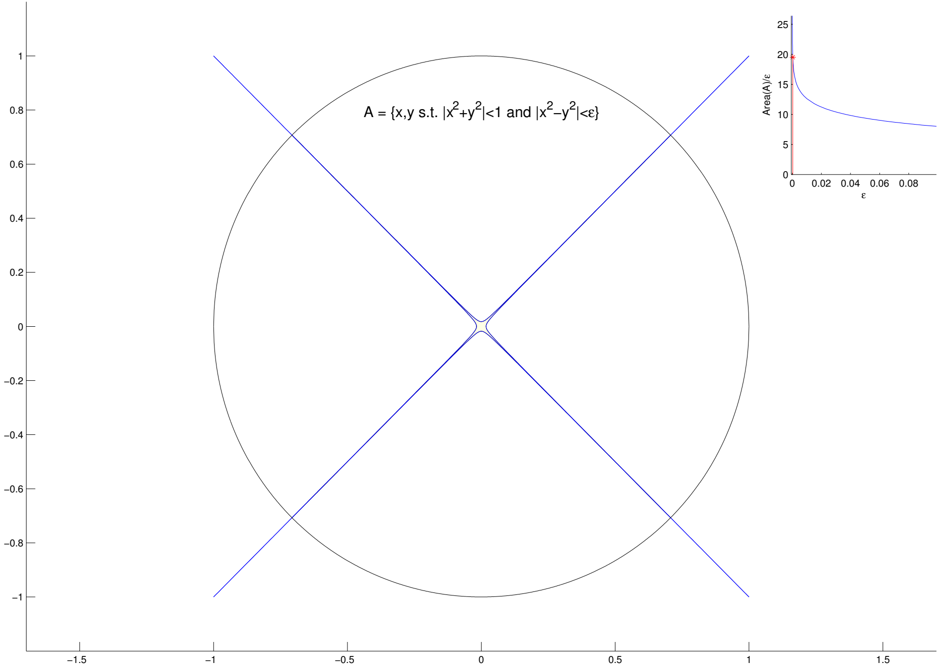
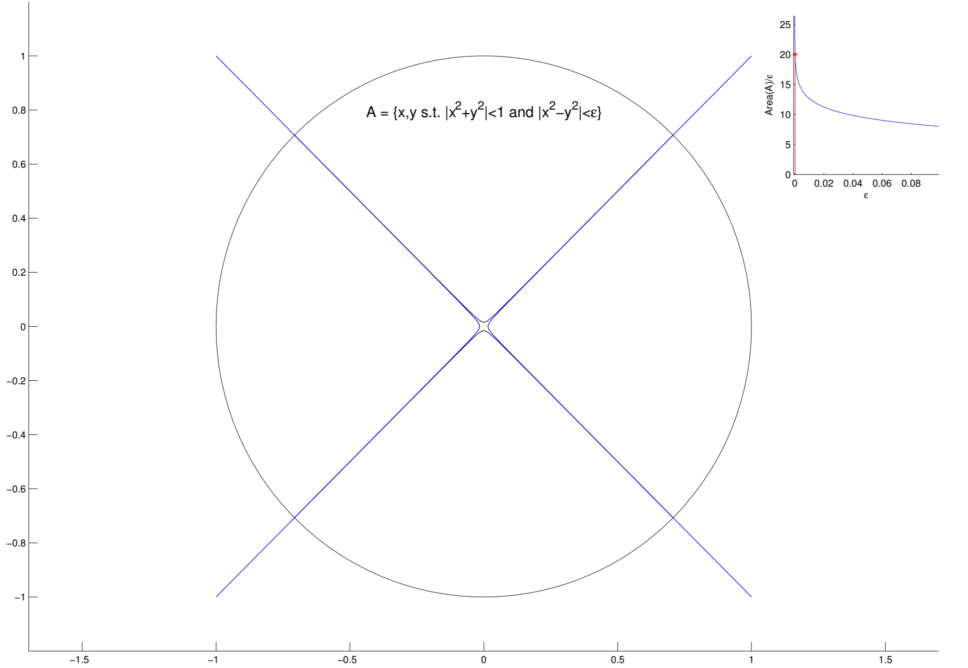
−1 −0.8 −0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8 1

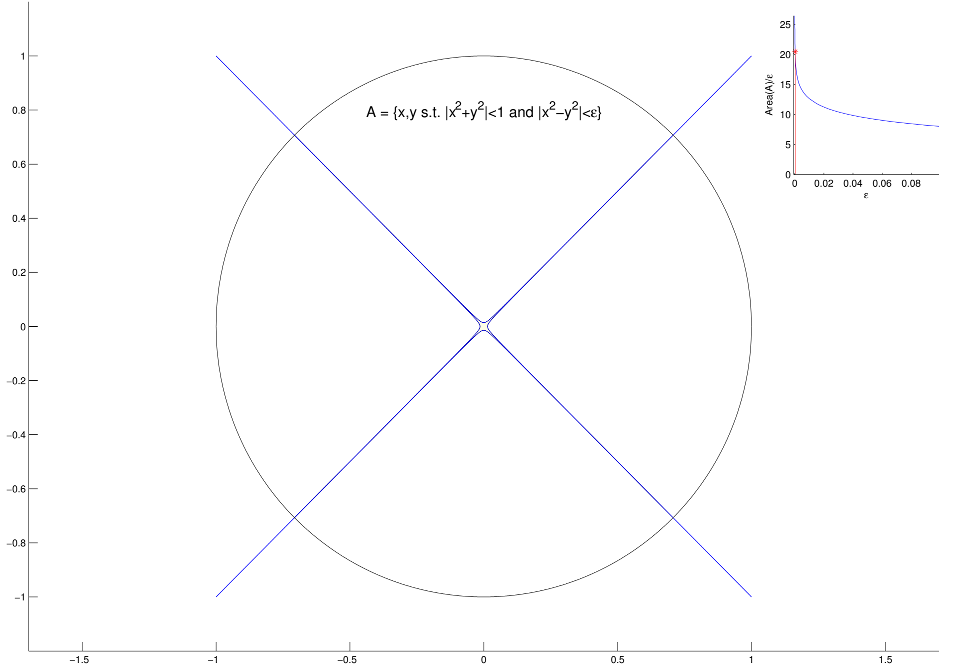


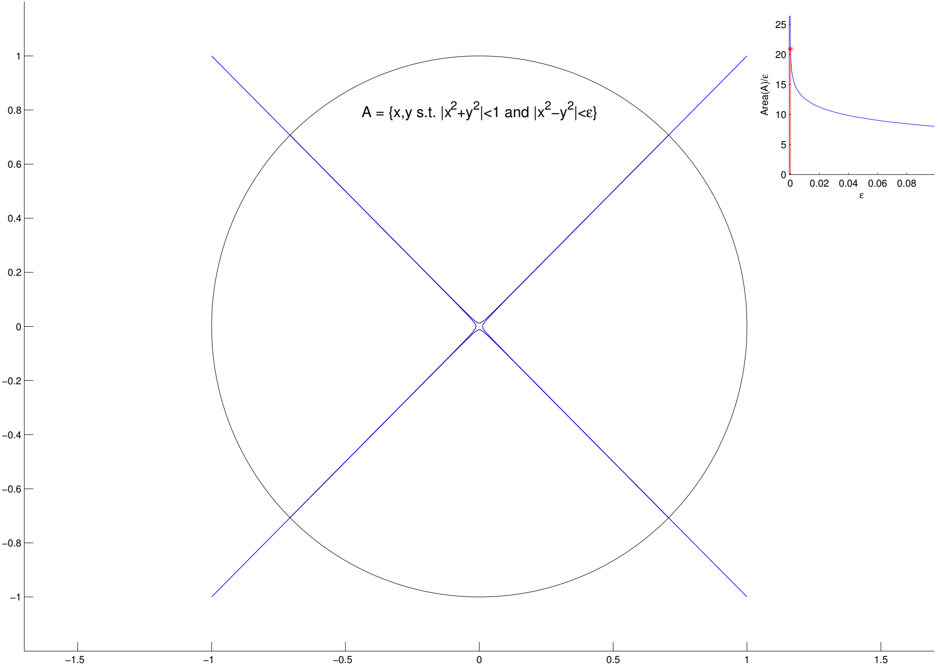
A = {x,y s.t. |x²+y²|<1 and |x²−y²|<ε}
Area(A)/ε
ε
0
5
10
15
20
25
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−0.5
0.5
1.5


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5



A = {x,y s.t. |x²+y²|<1 and |x²−y²|<ε}
Area(A)/ε
ε


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0 0.02 0.04 0.06 0.08
0 5 10 15 20 25
−1.5 −1 −0.5 0 0.5 1 1.5
−1 −0.8 −0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8 1


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
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−0.4
−0.6
−0.8
−1
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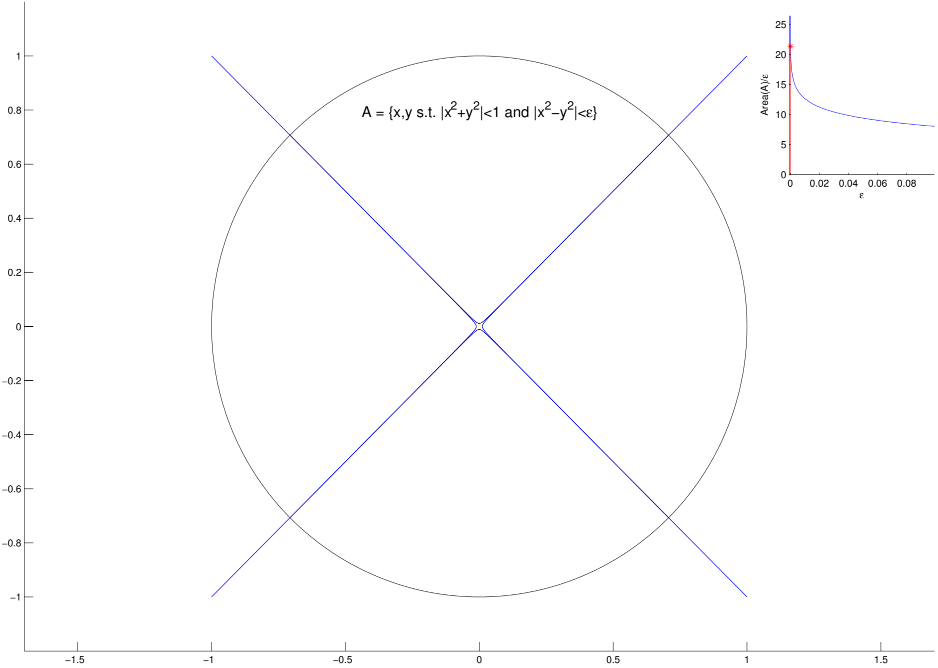
−1
−0.5
0
0.5
1
1.5


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0
0.02
0.04
0.06
0.08
5
10
15
20
25
−1.5
−1
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−0.2
−0.4
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−0.8
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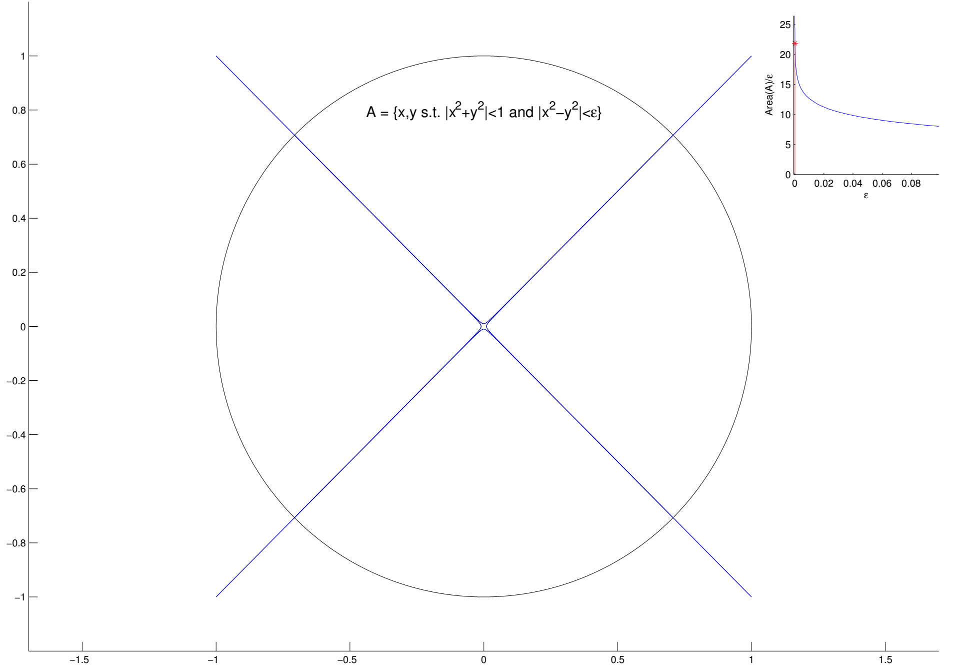
0.4
0.6
0.8


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
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0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
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−0.4
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−0.8
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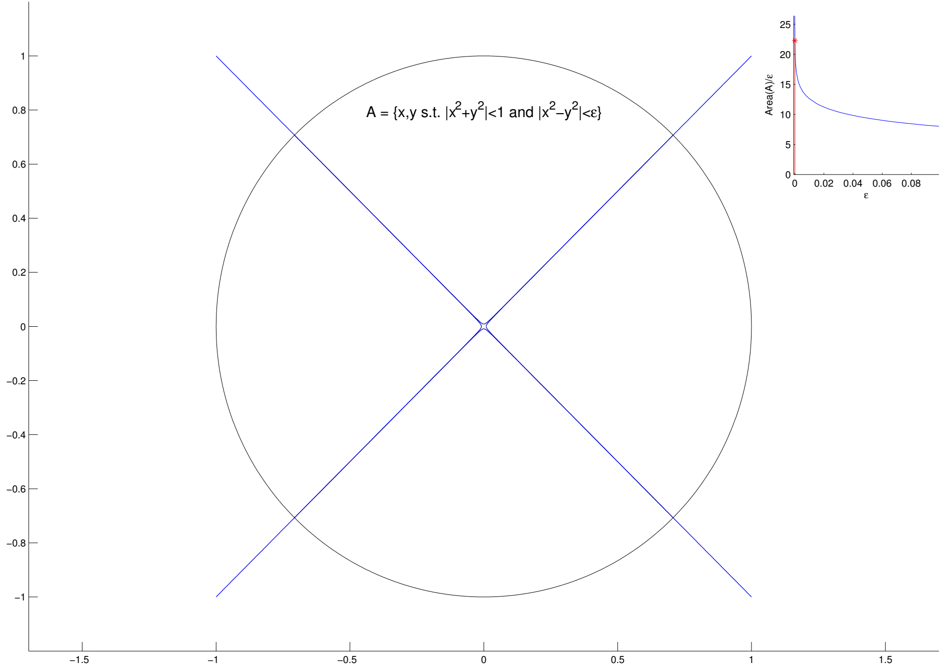
−0.5
0.5
1.5


A = {x,y s.t. |x2+y2|<1 and |x2−y2|<ε}
Area(A)/ε
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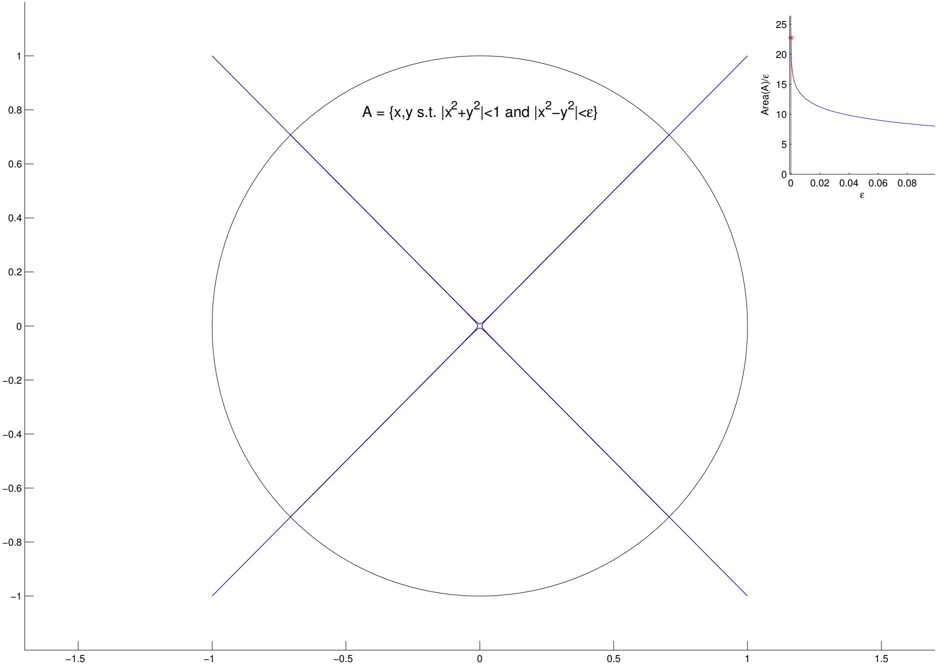
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
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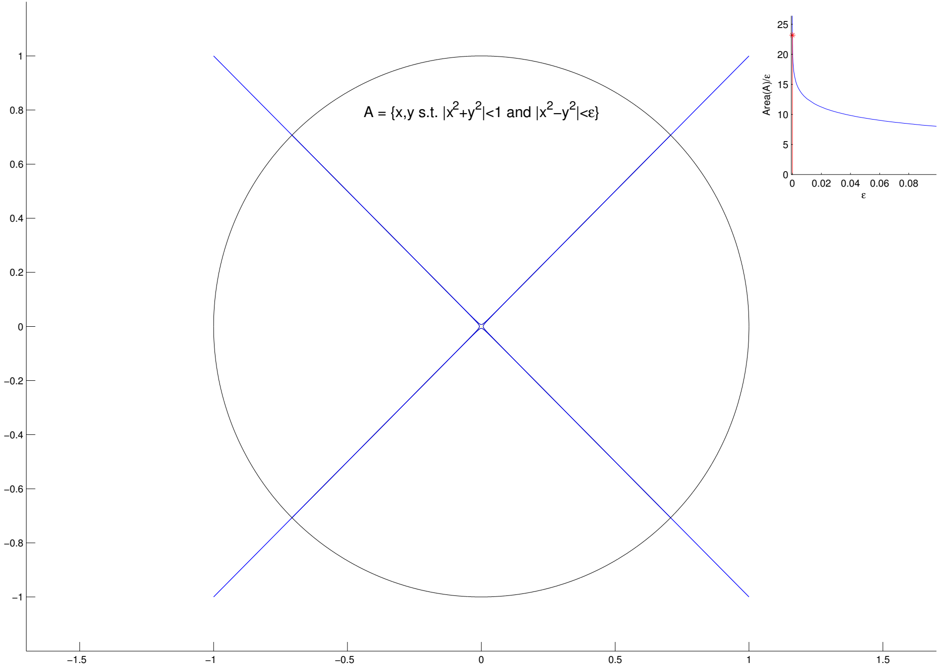
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−0.5
0.5
1.5


A = {x,y s.t. |x²+y²|<1 and |x²−y²|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
−0.2
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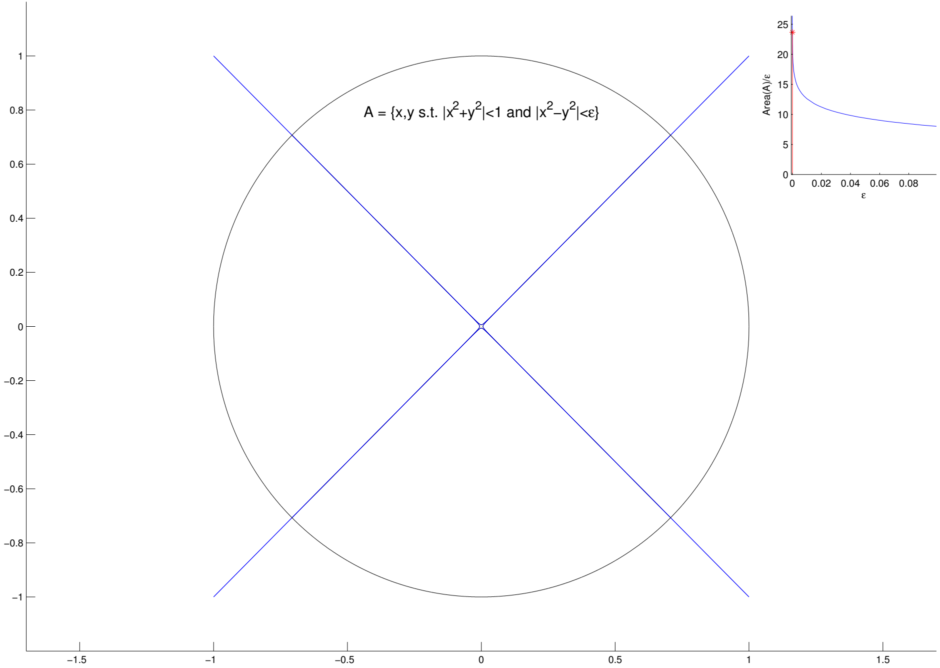
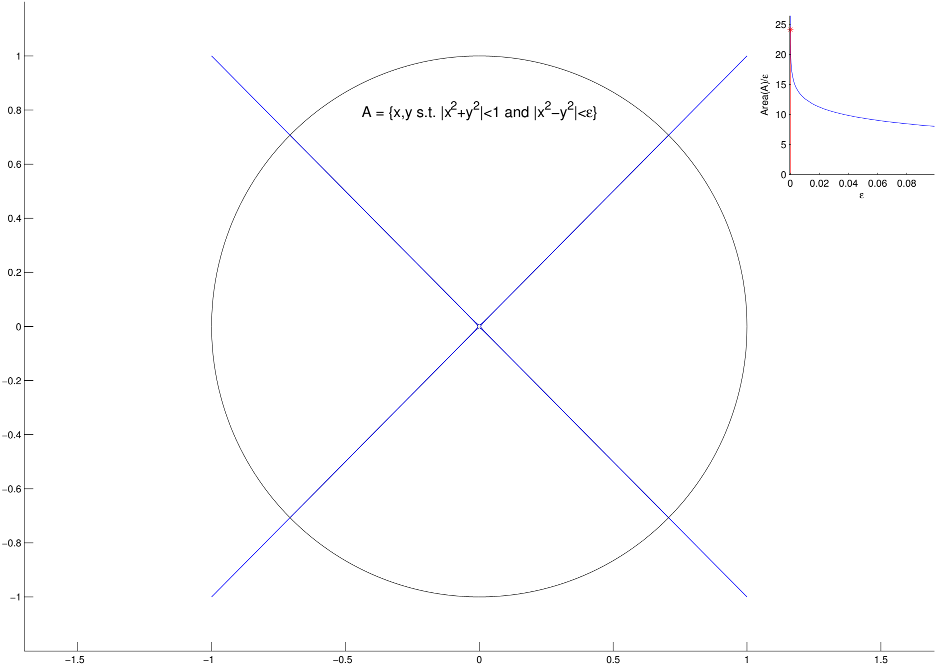
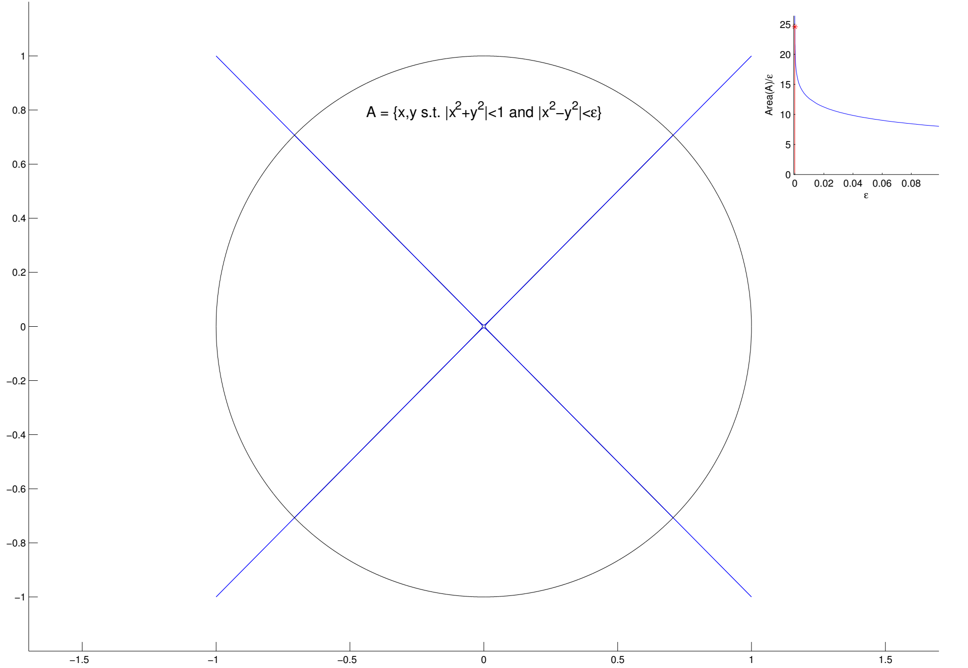
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5

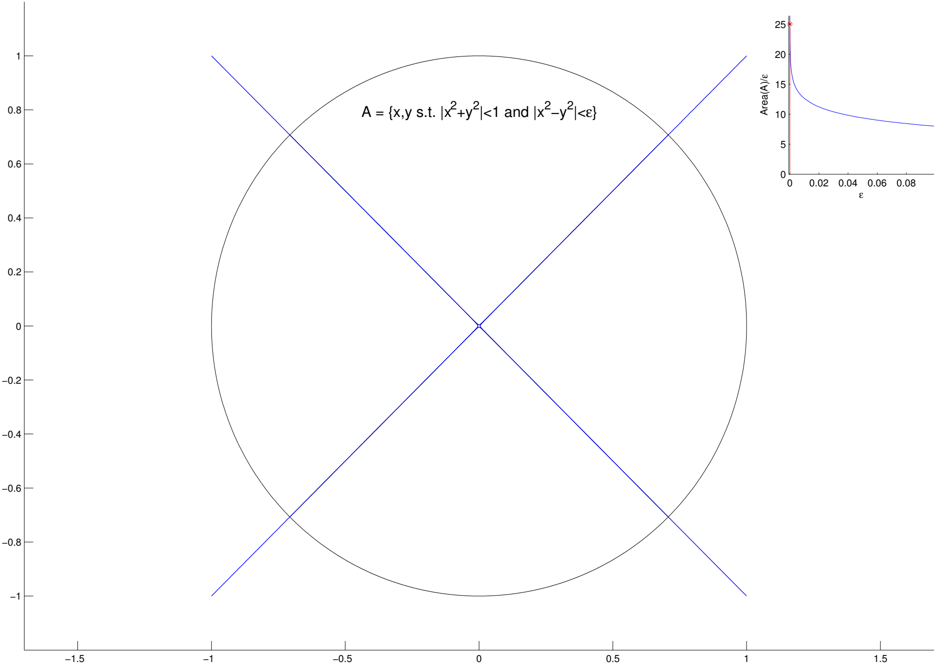


A = {x,y s.t. |x²+y²|<1 and |x²−y²|<ε}
Area(A)/ε
ε


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε



A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε


$A = \{x,y \text{ s.t. } |x^2+y^2|<1 \text{ and } |x^2-y^2|<\varepsilon\}$
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
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−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
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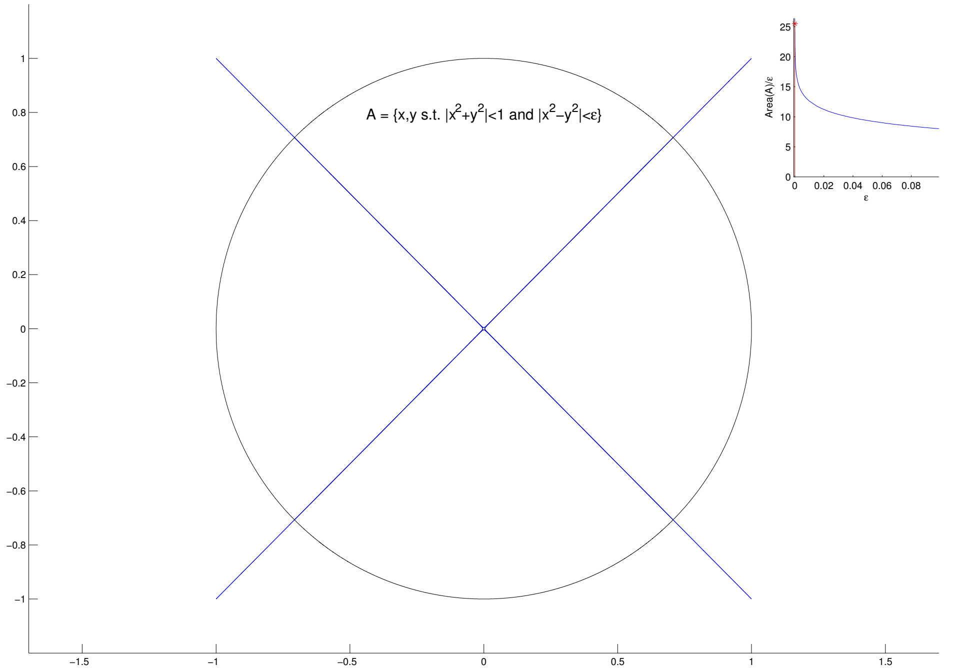
1
1.5


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0
5
10
15
20
25
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
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−0.4
−0.6
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−1.5
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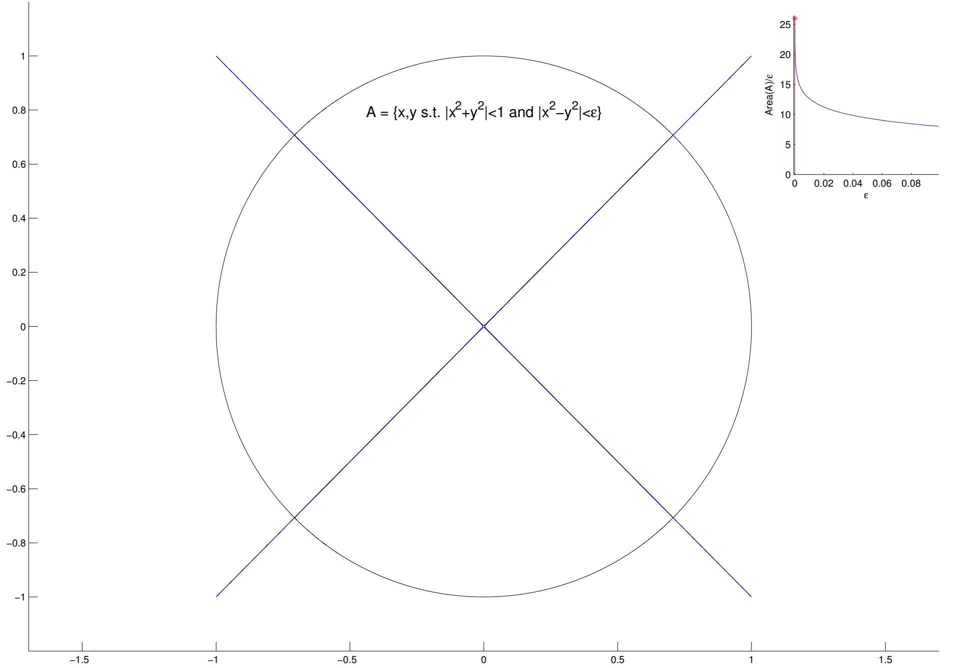
0.5
1.5


A = {x,y s.t. |x²+y²|<1 and |x²−y²|<ε}
Area(A)/ε
ε
0
0.02
0.04
0.06
0.08
5
10
15
20
25
1
0.8
0.6
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−0.4
−0.6
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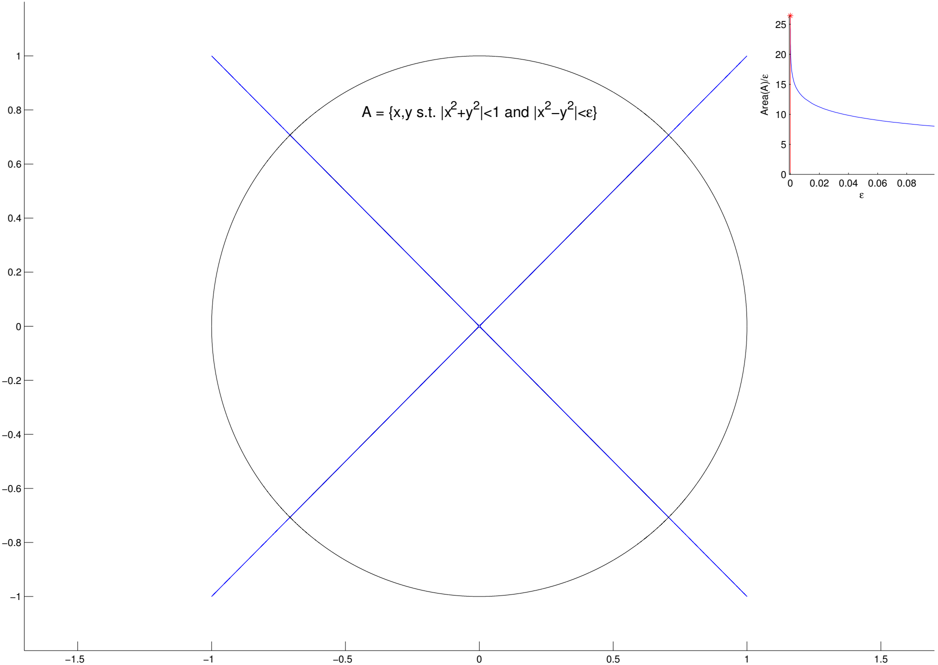
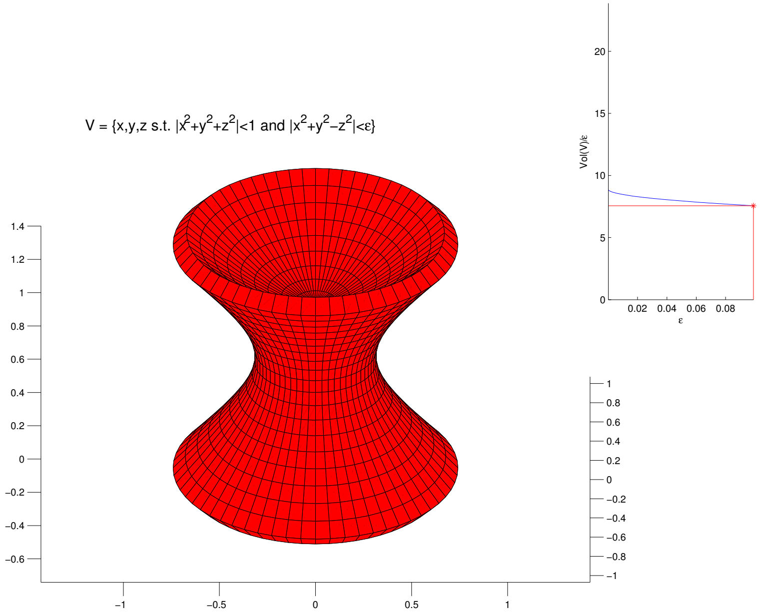
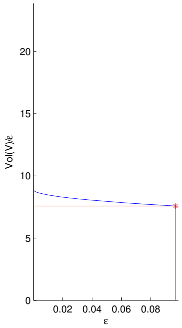
−1.5
−0.5
0.5
1.5

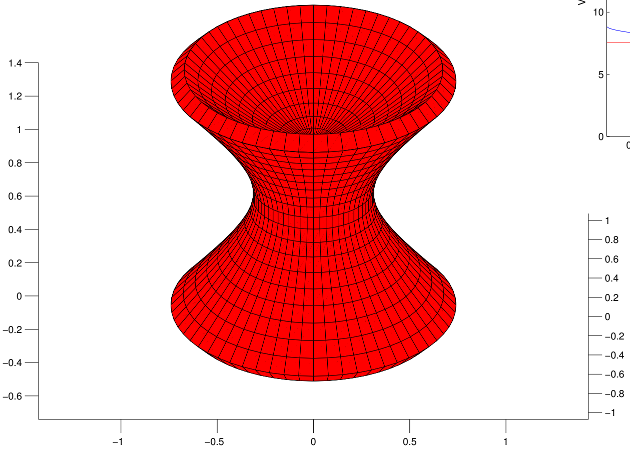

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0
5
10
15
20
25
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−0.5
0.5
1.5



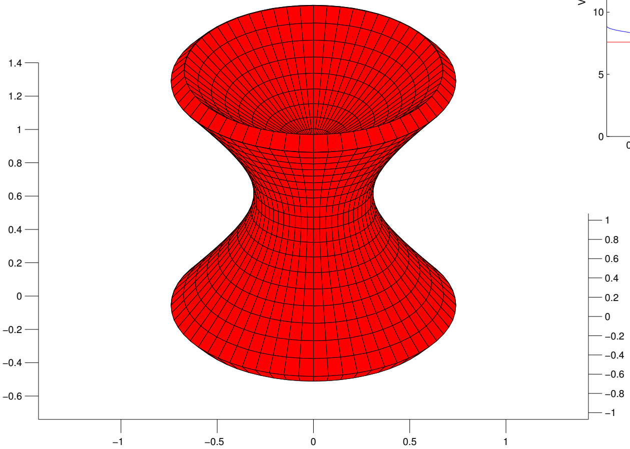
V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε

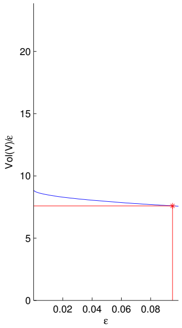
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



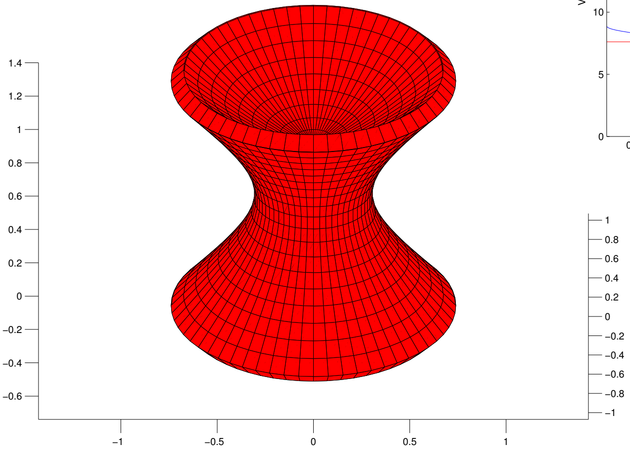


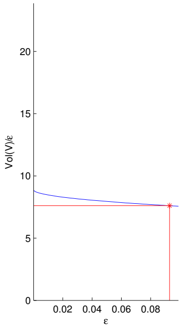
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

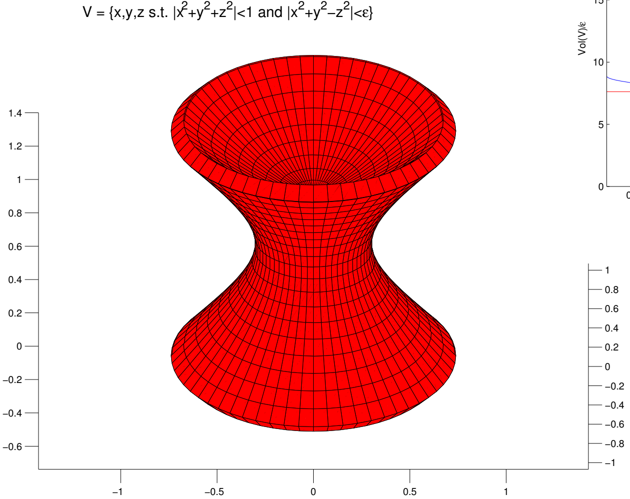


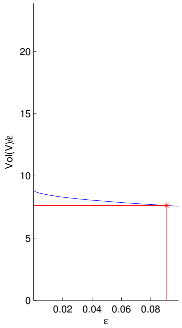


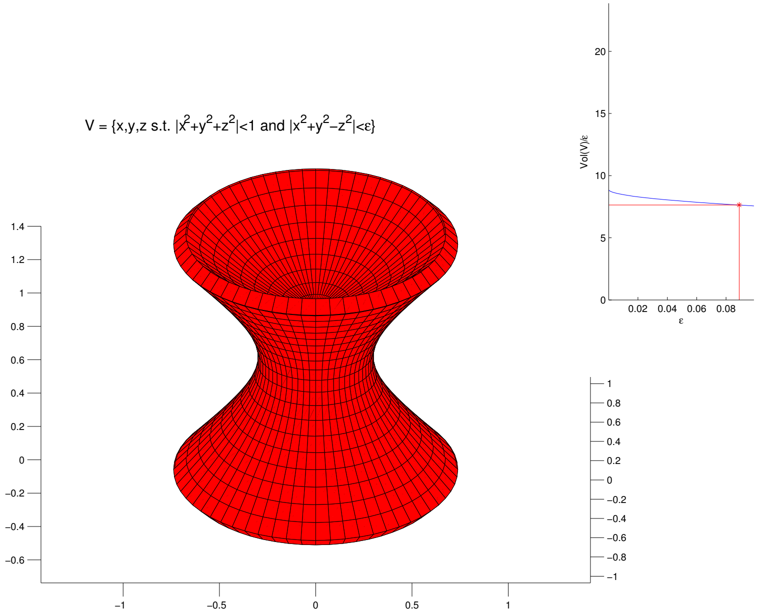
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

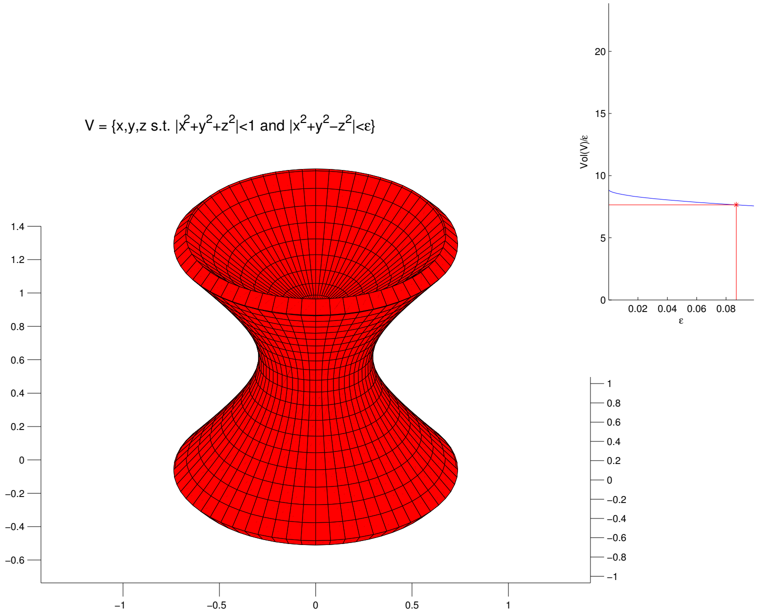


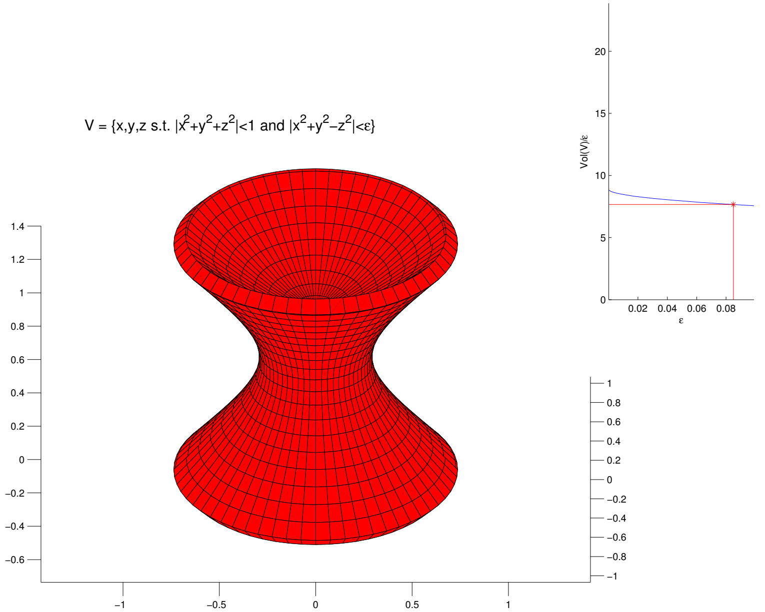



$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

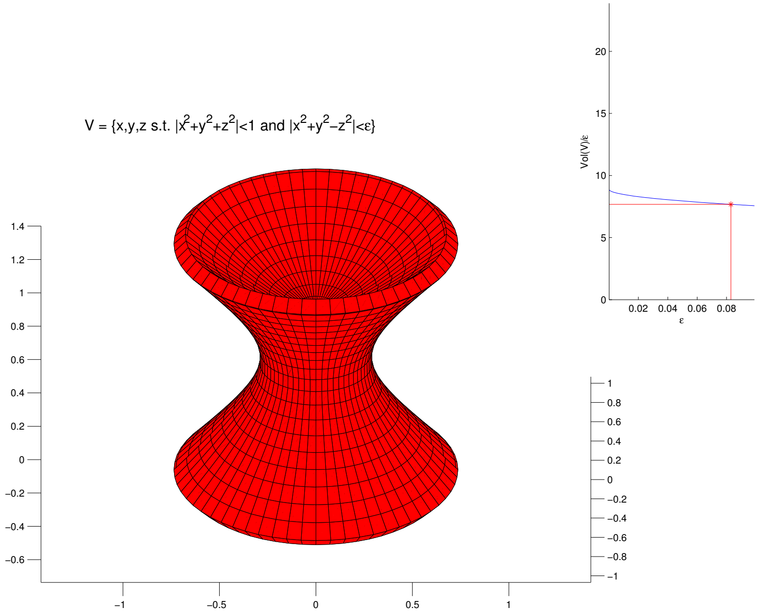

Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0
5
10
15
20
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1


V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0 5 10 15 20
0.02 0.04 0.06 0.08
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1



V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε


V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
20
15
10
5
0
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
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0.5
1
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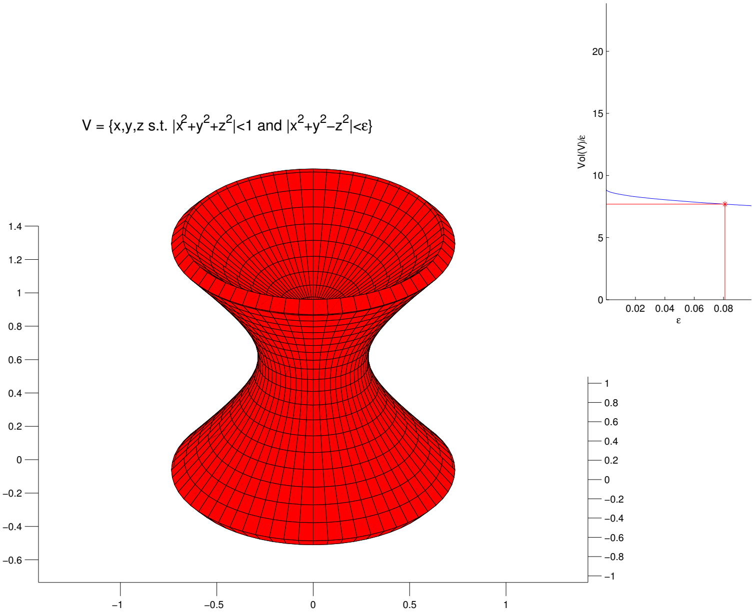
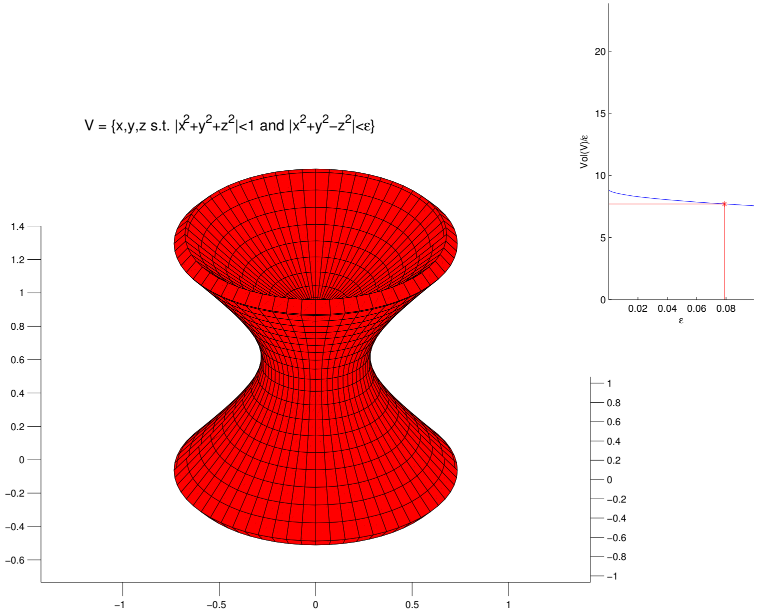
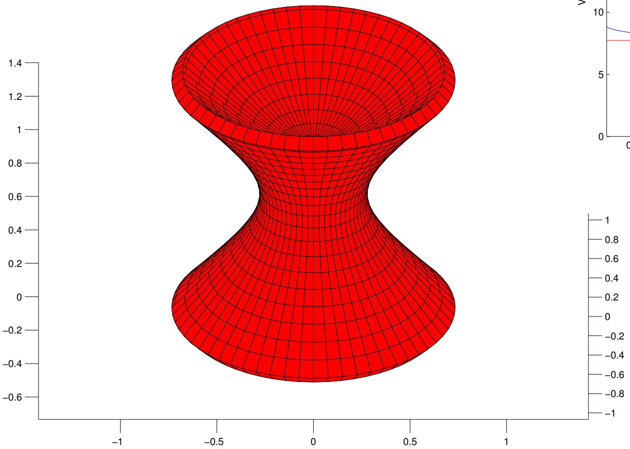
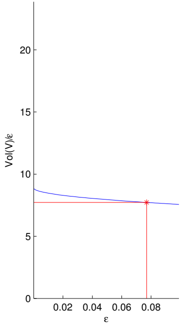
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1


V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

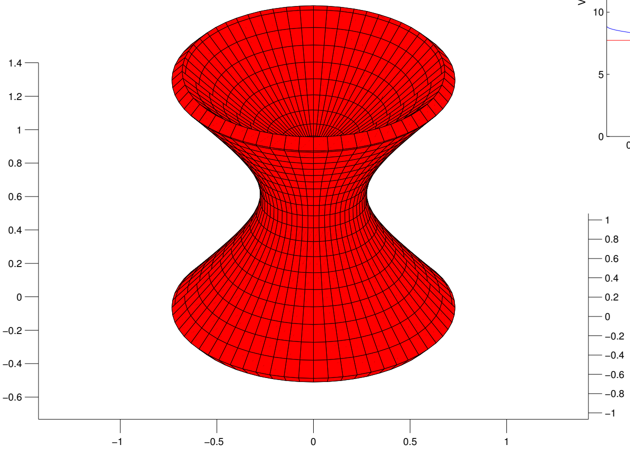

V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0 5 10 15 20
0.02 0.04 0.06 0.08
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

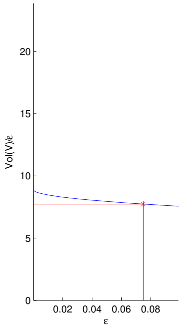
V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}





$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$





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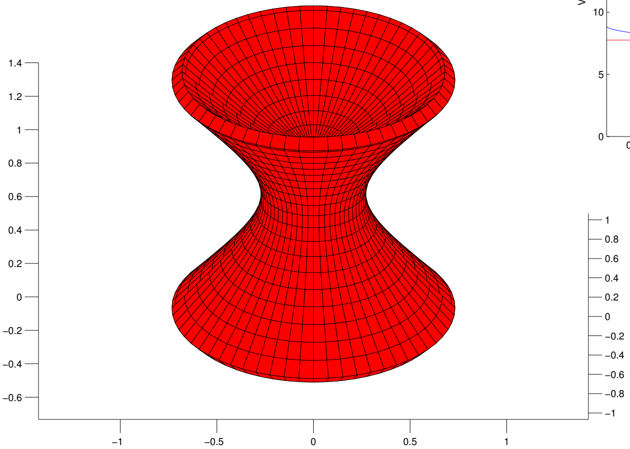
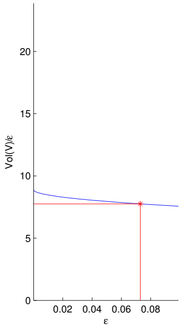
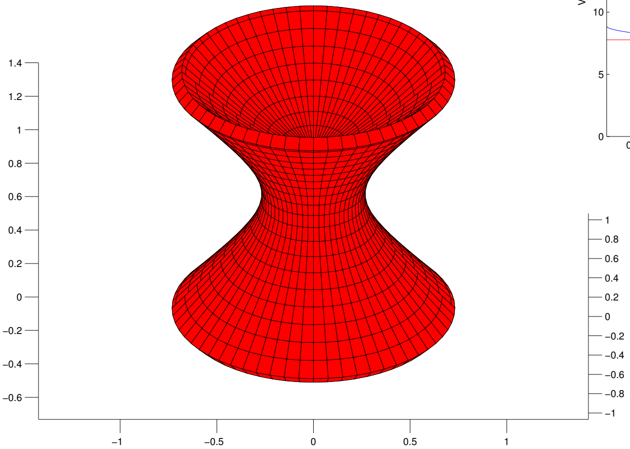
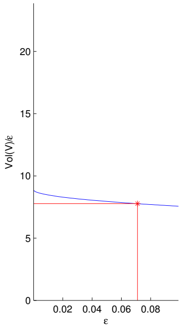




V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}





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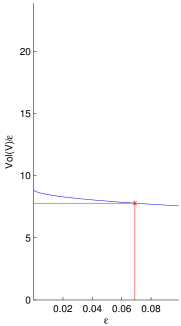
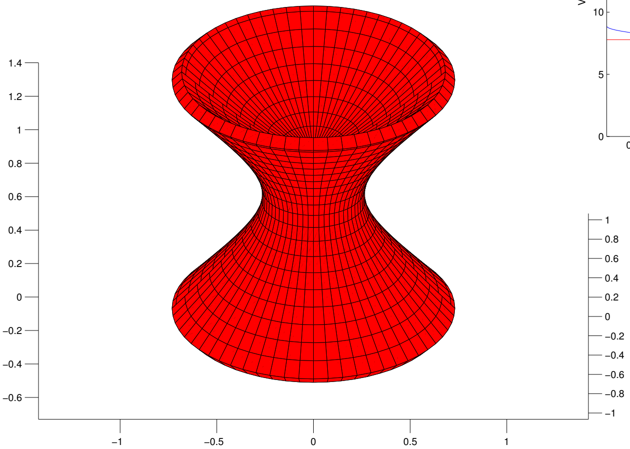
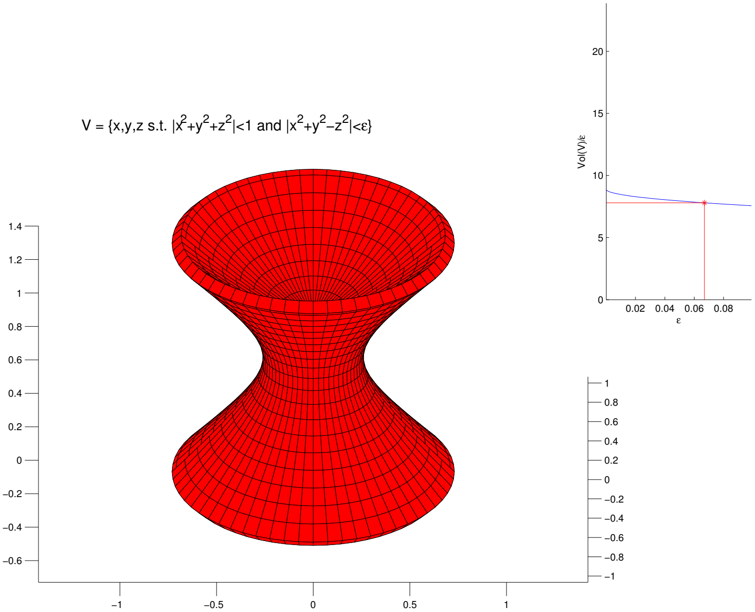
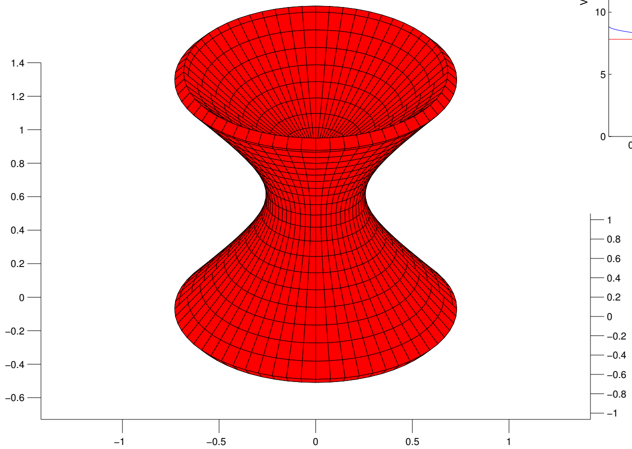
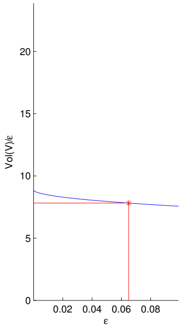


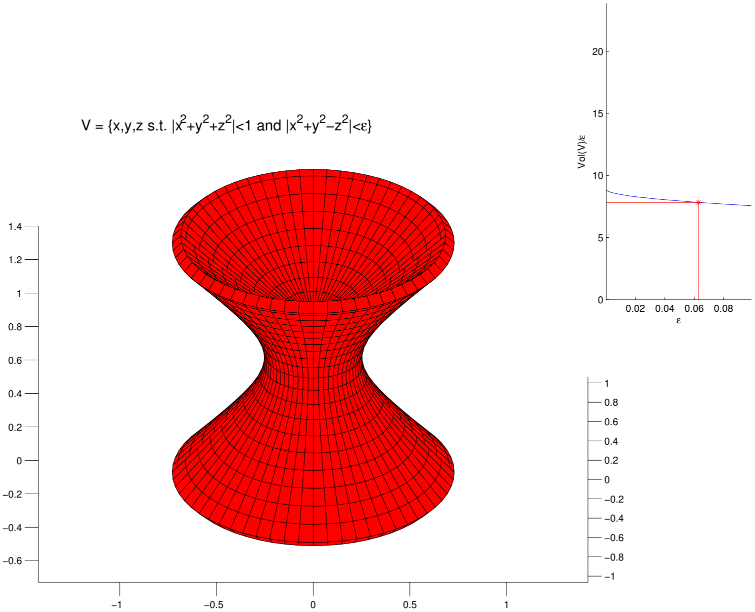



V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε

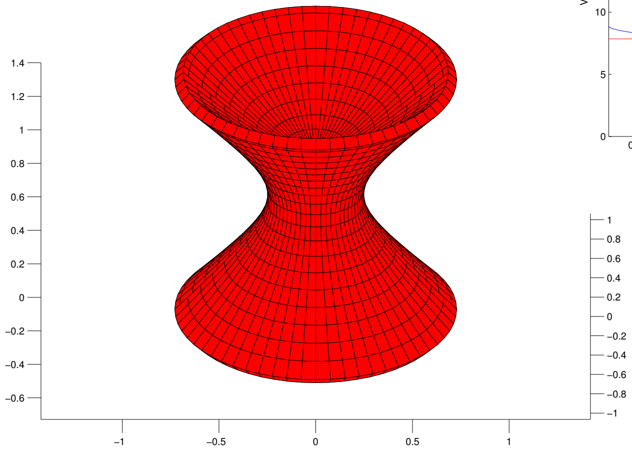
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

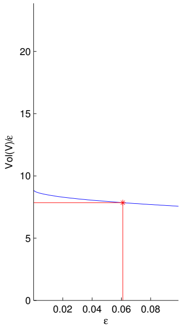




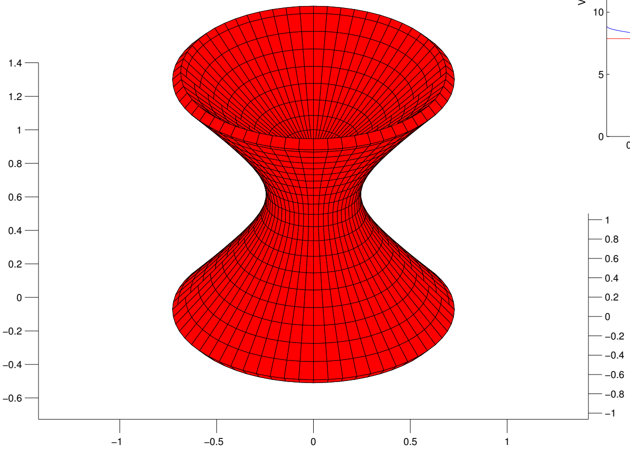

V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

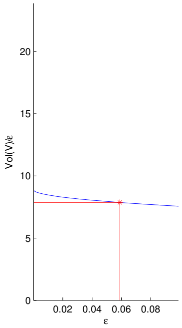
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$





$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$





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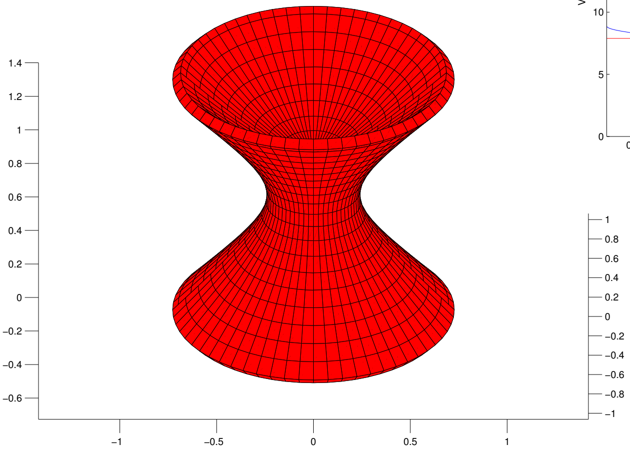

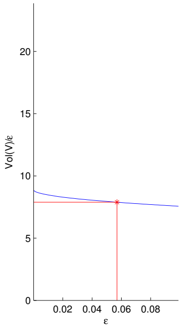

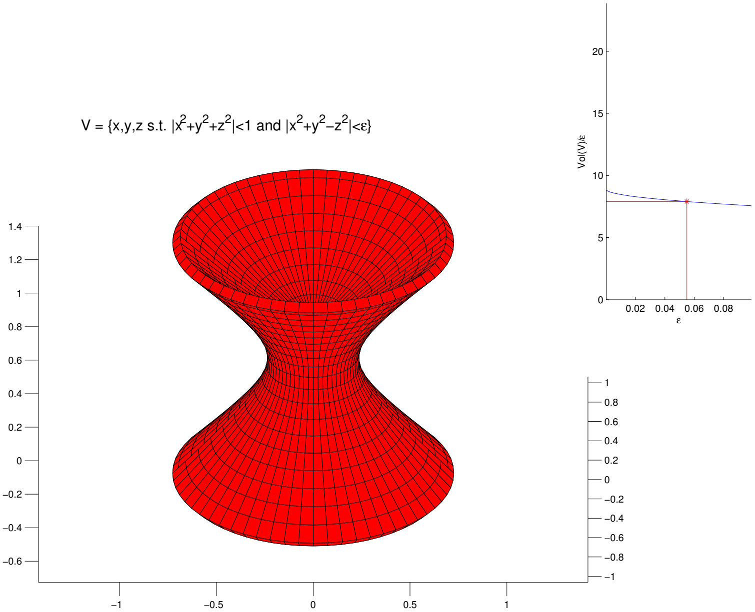
V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
20
15
10
5
0
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
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0.6
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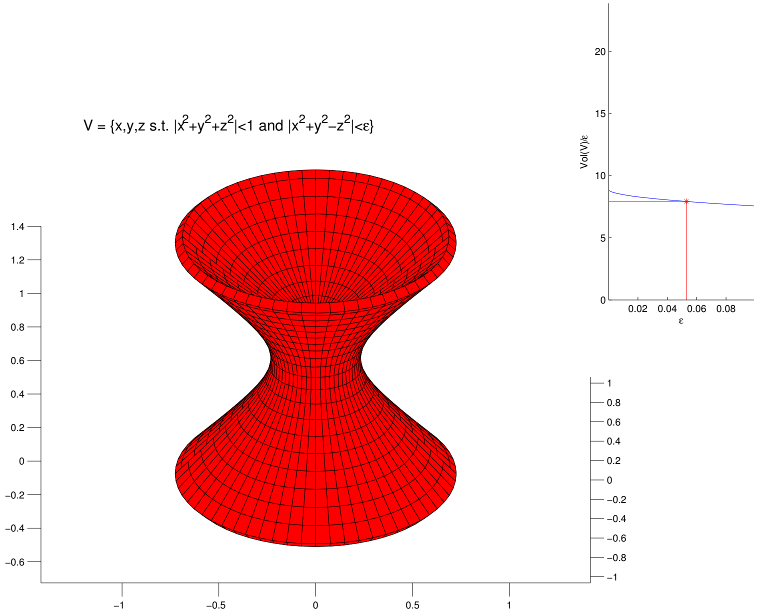
−1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−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
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0
−0.2
−0.4
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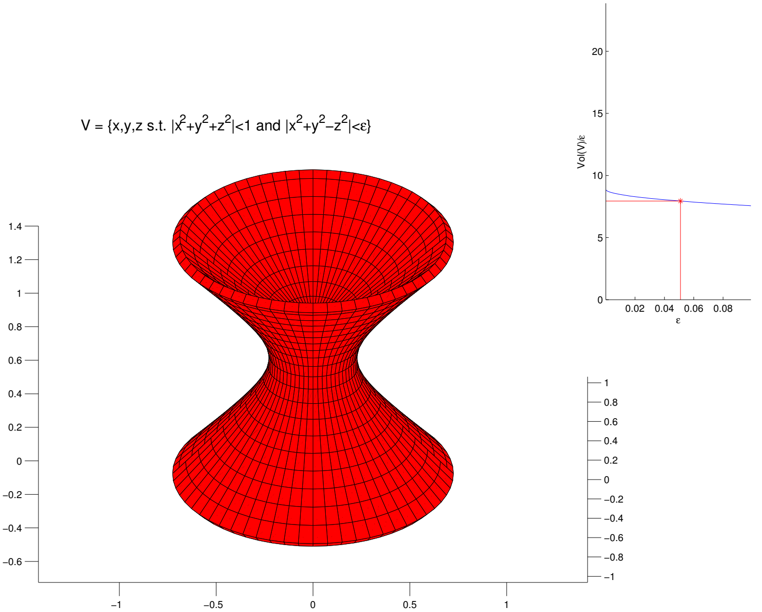
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1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
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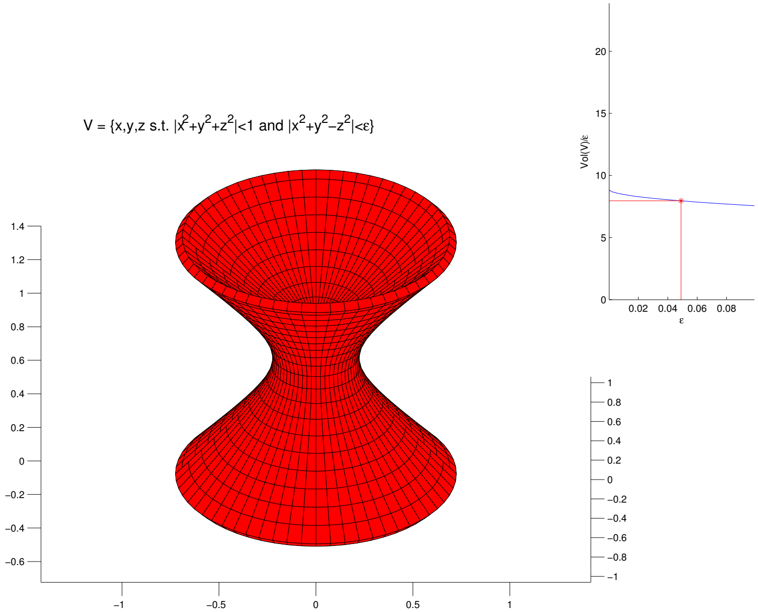
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Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
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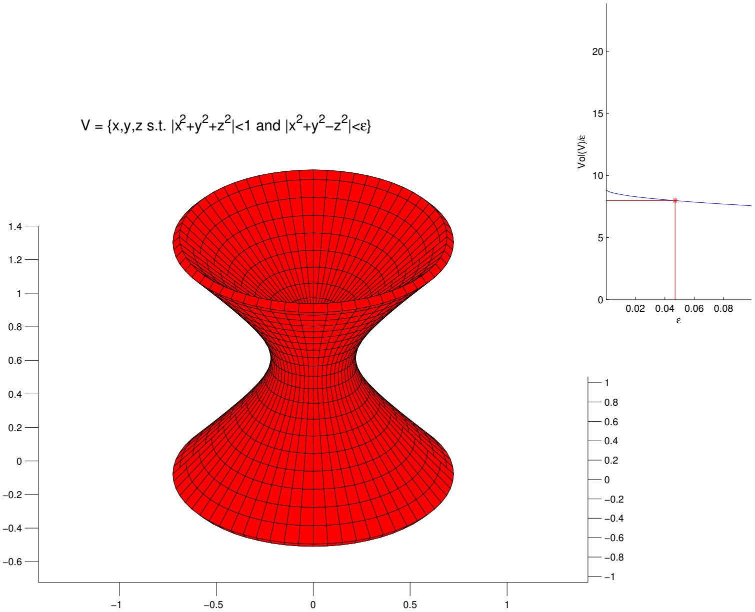
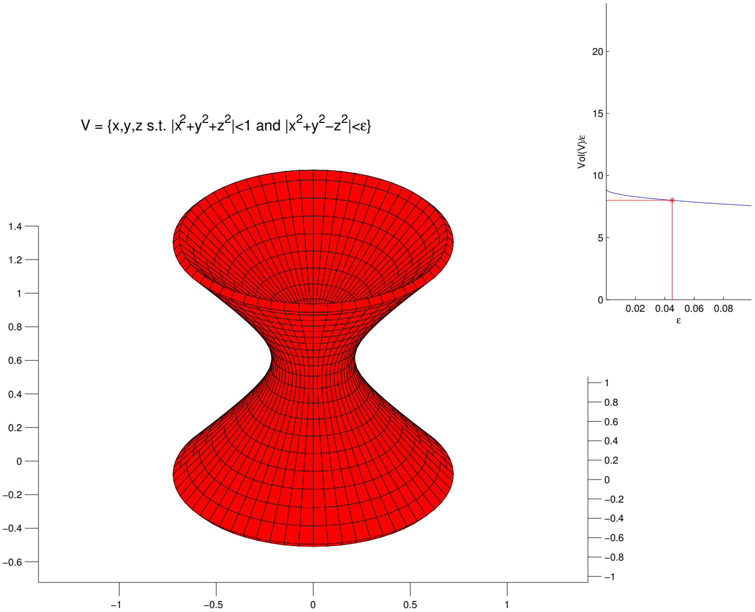
V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}

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ε
20
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0.02
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0.08
1.4
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−0.2
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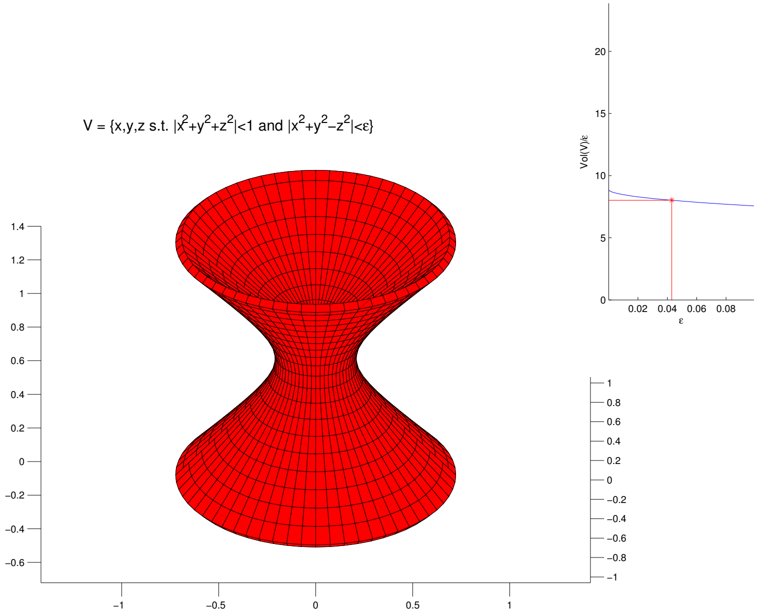
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Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
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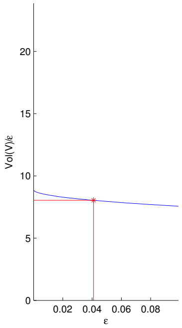

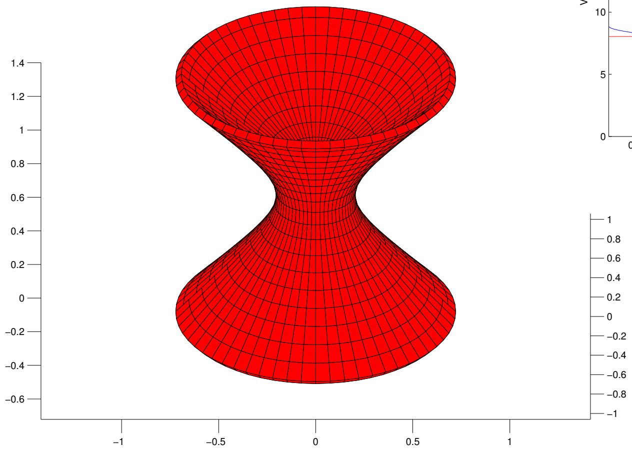

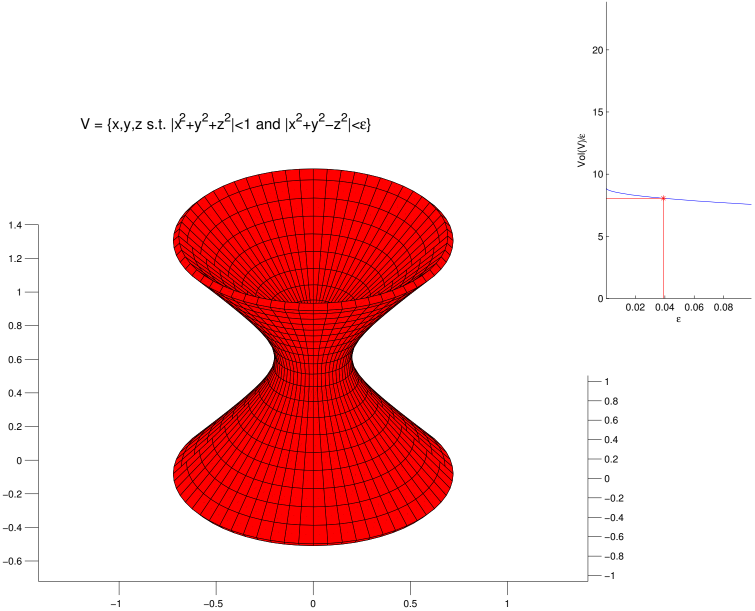
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15
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5
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0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
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0.4
0.2
0
−0.2
−0.4
−0.6
−1
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0
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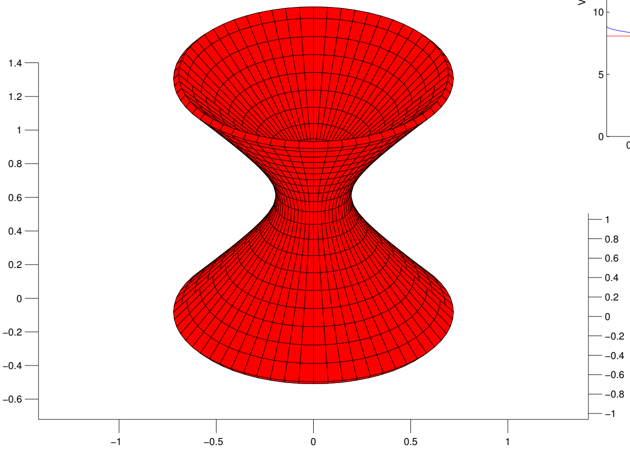
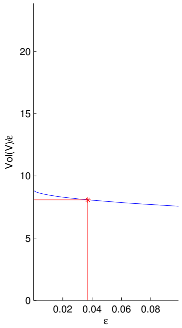
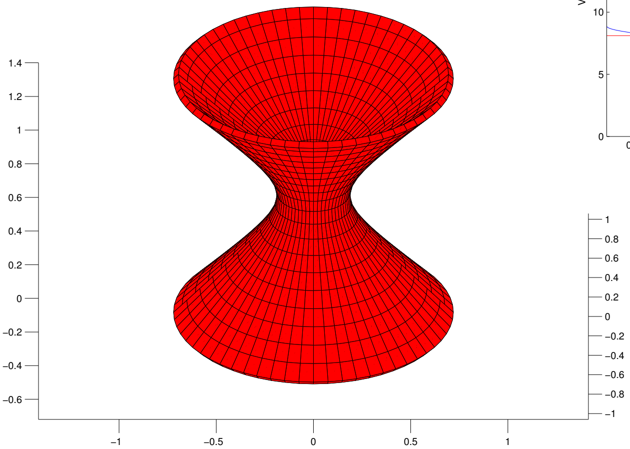
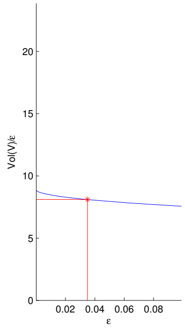
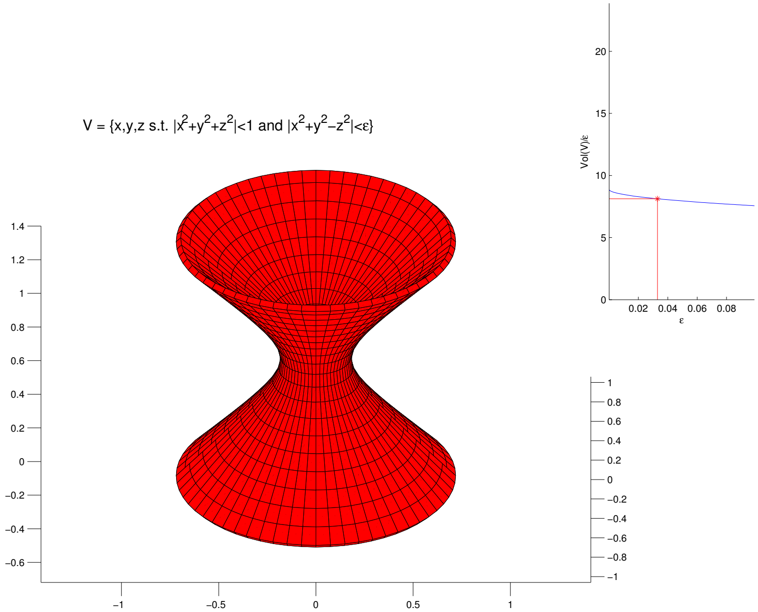




$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$







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1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
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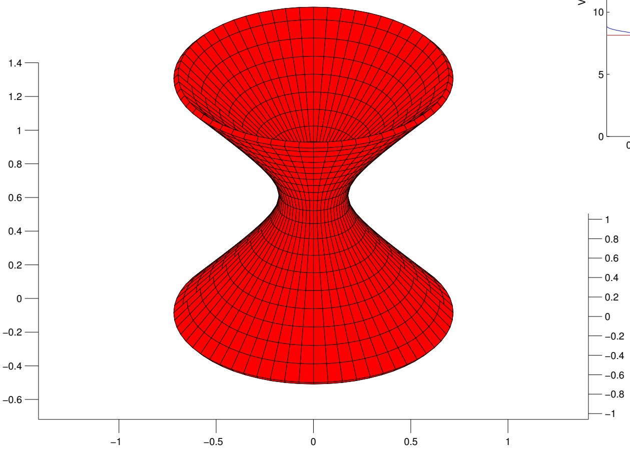

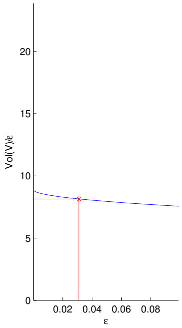

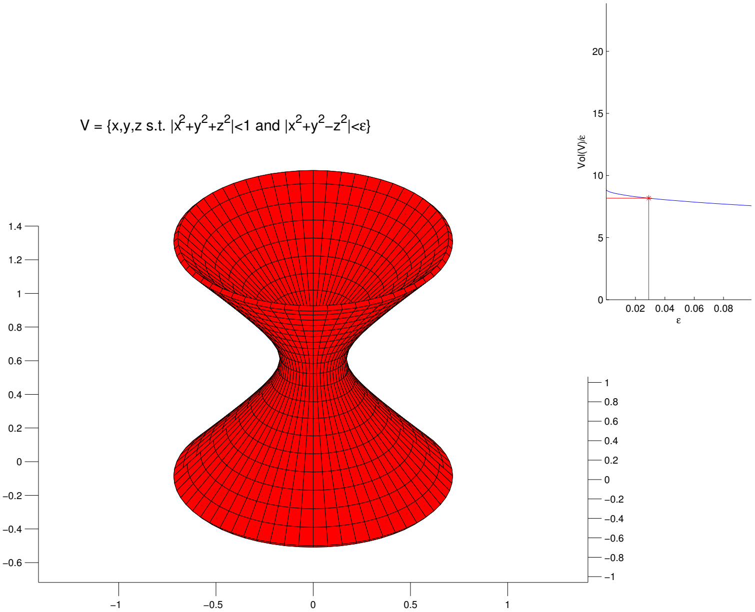
V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
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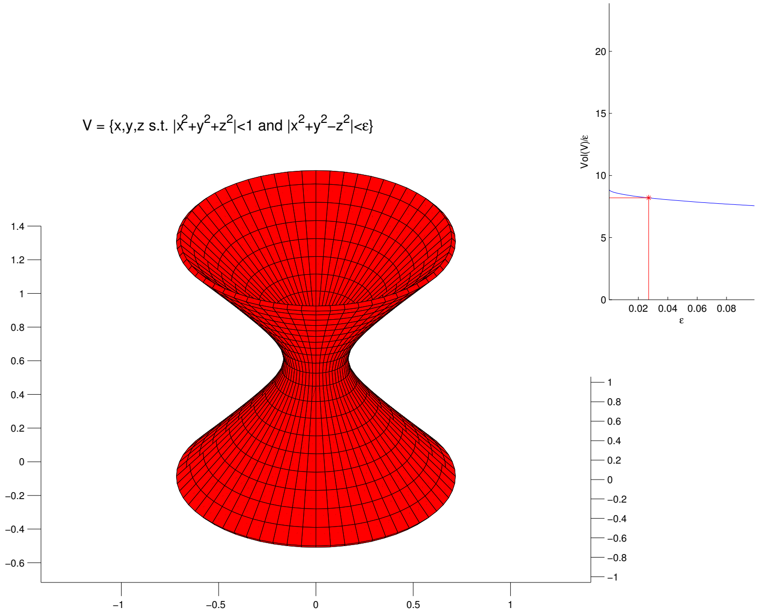
V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
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0.02
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1.4
1.2
1
0.8
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0.5
1
1
0.8
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0
−0.2
−0.4
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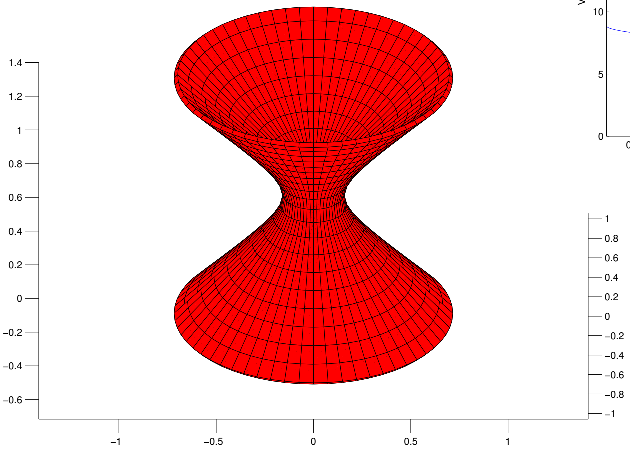
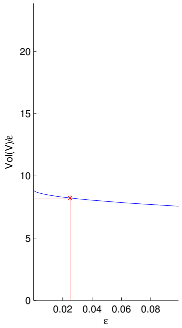
V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}





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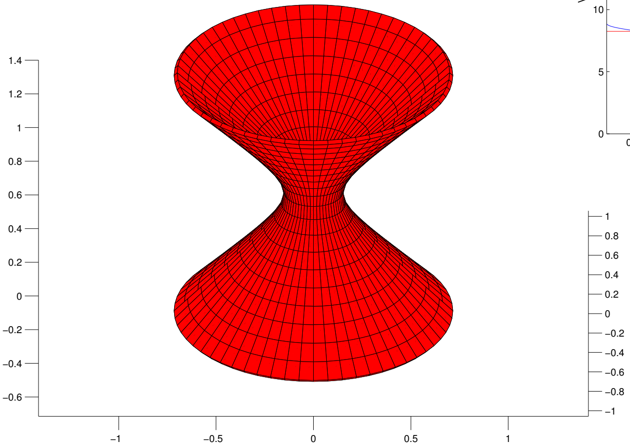
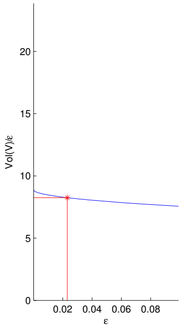
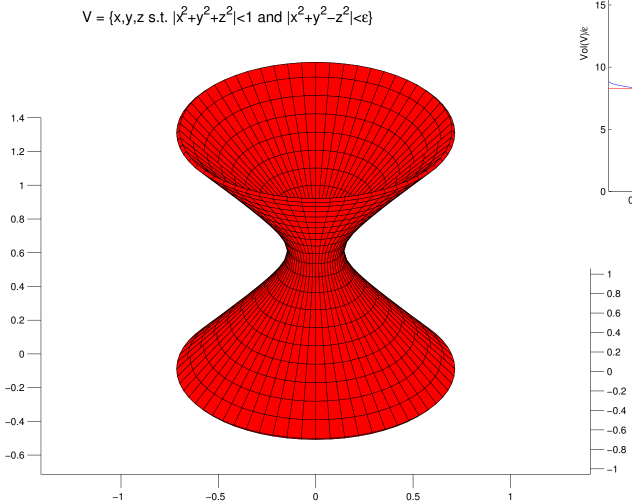
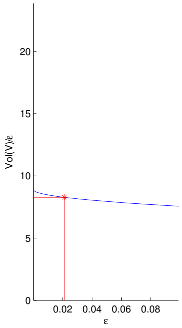
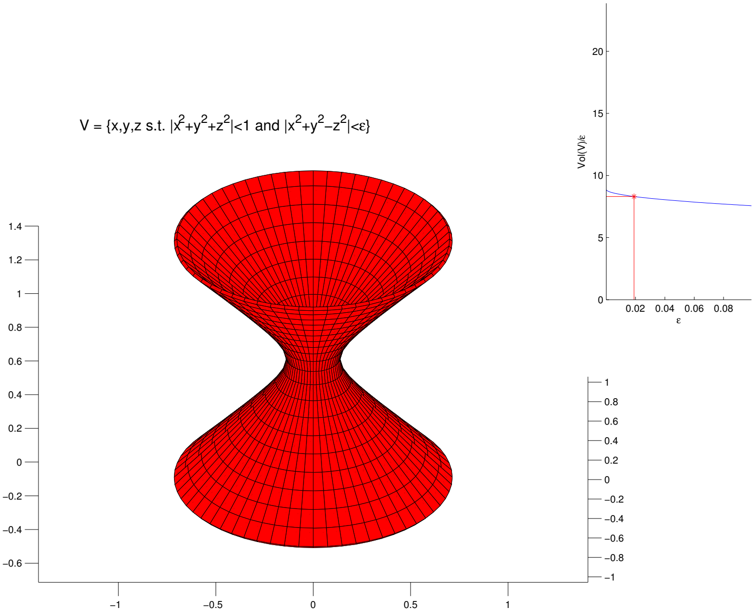





V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1


Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε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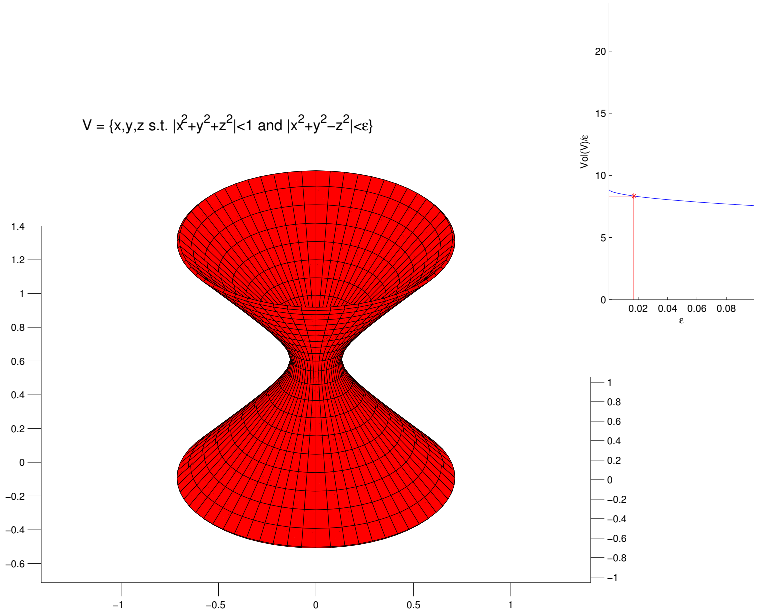
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0 5 10 15 20
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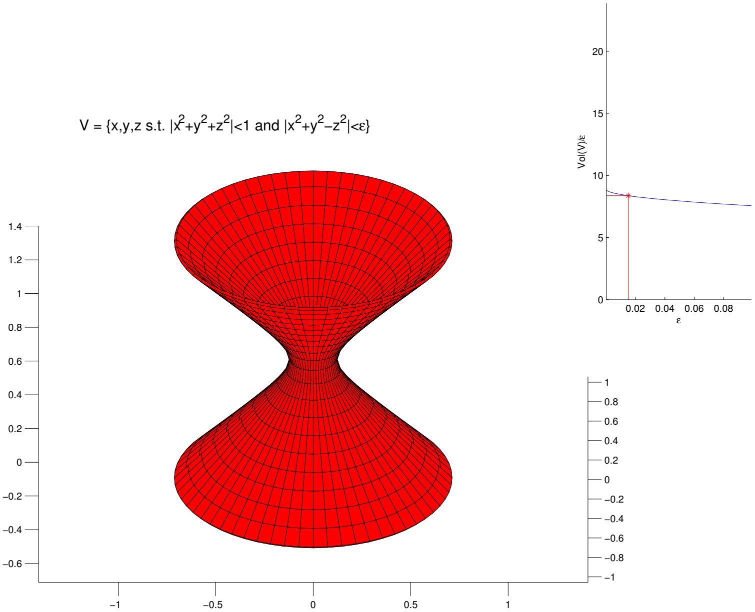

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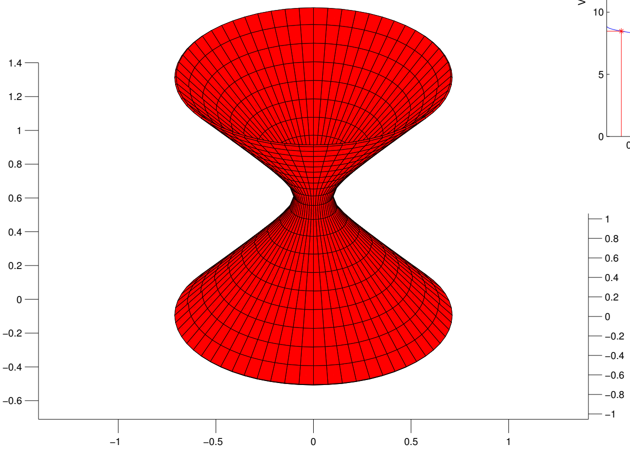

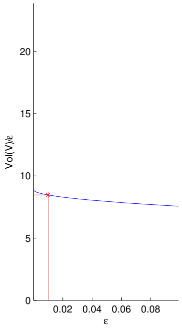

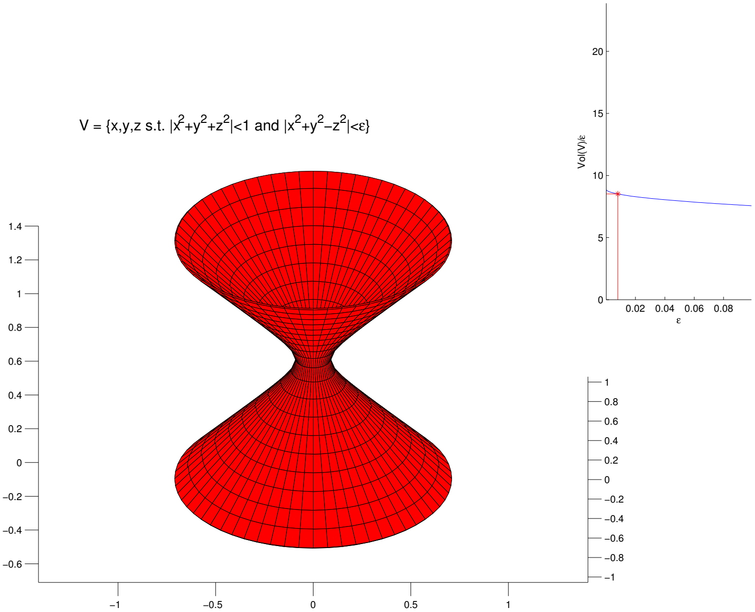

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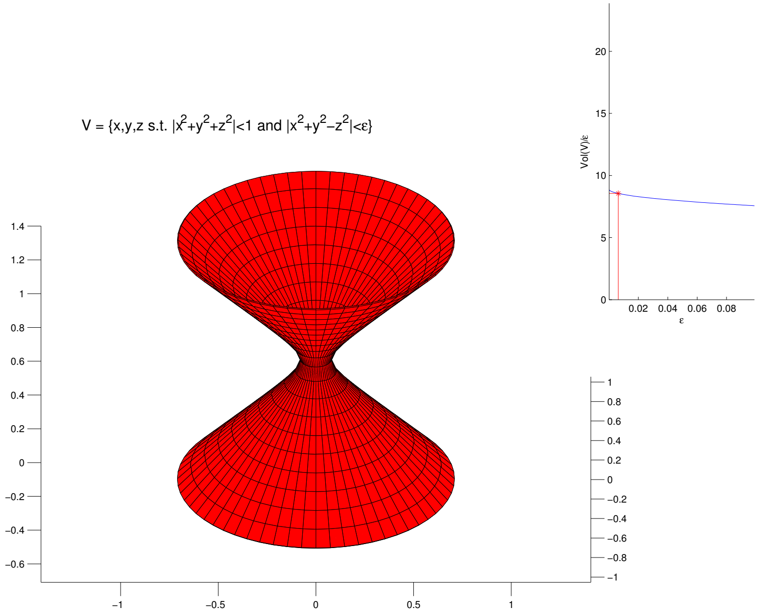

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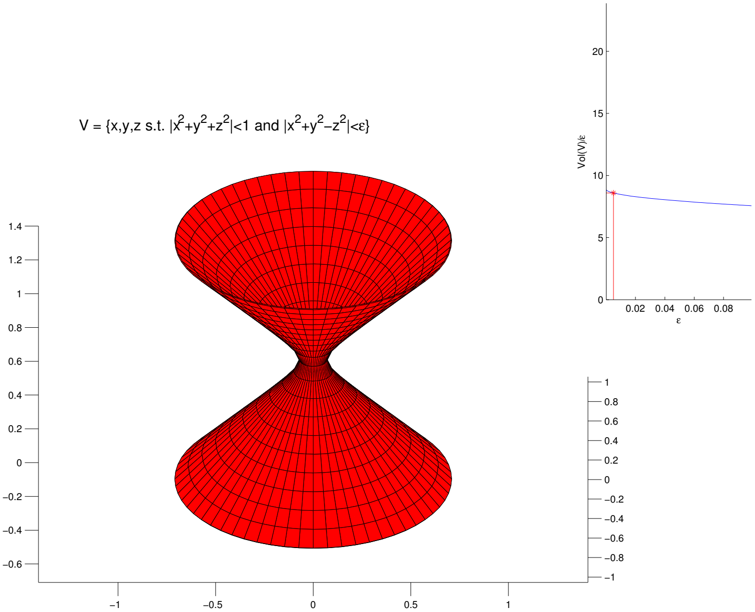
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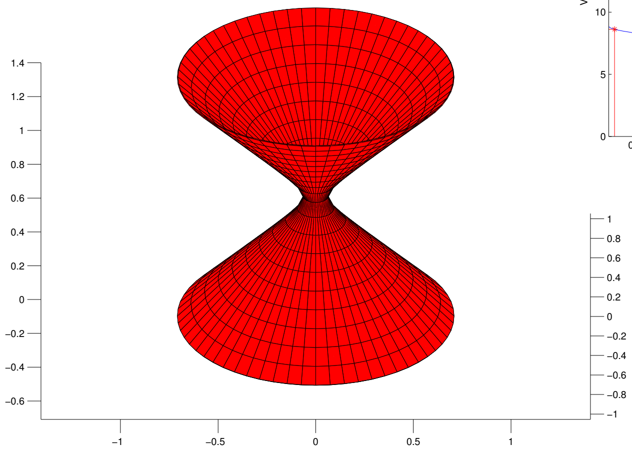

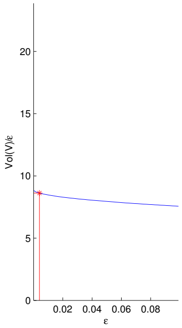

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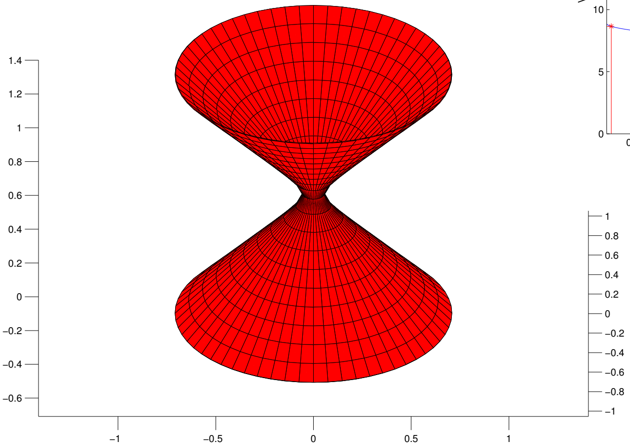

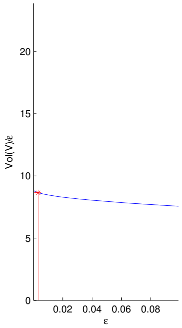

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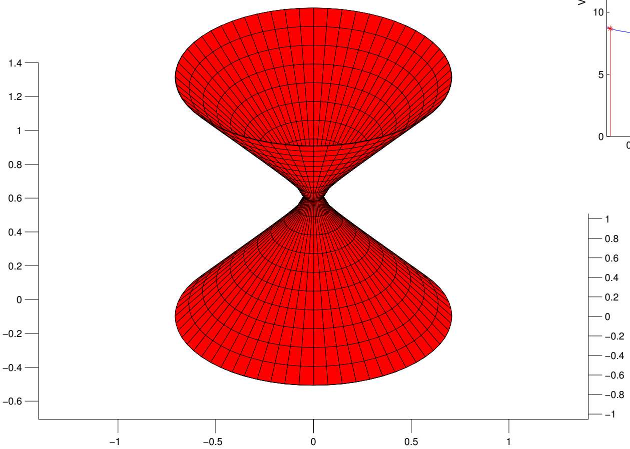

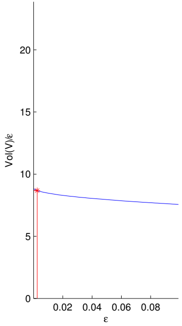

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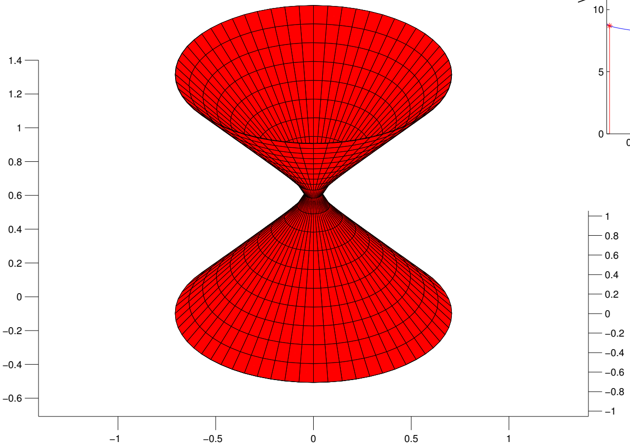

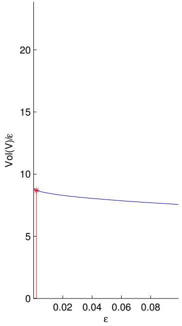

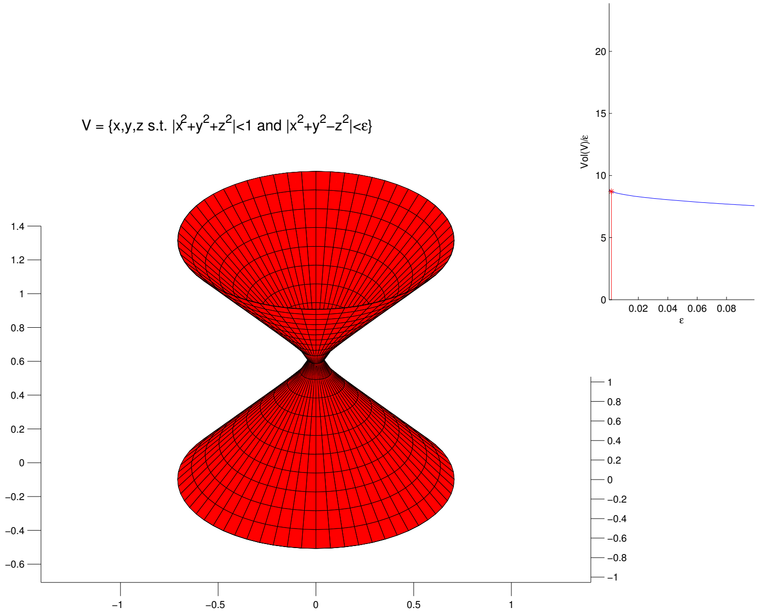

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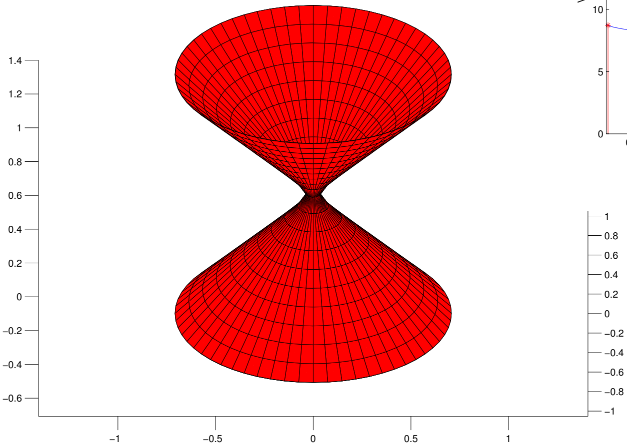

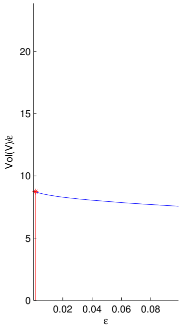

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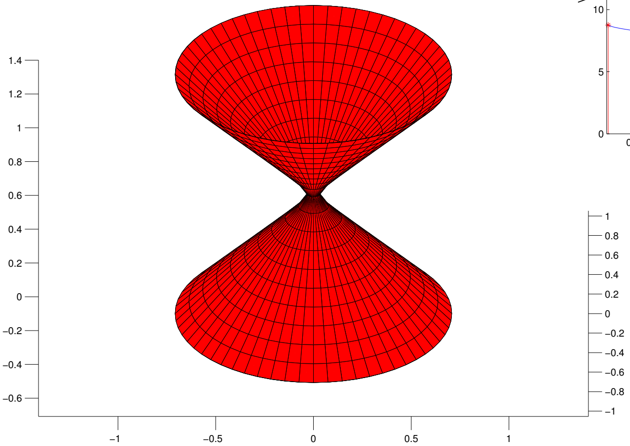

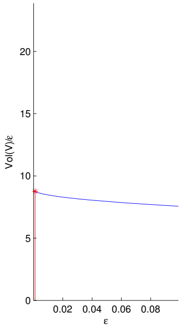

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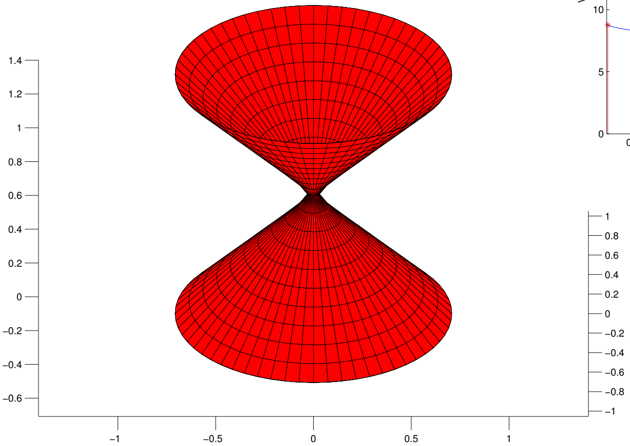

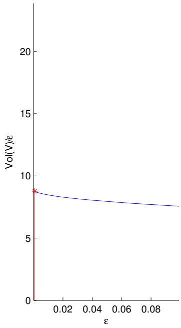

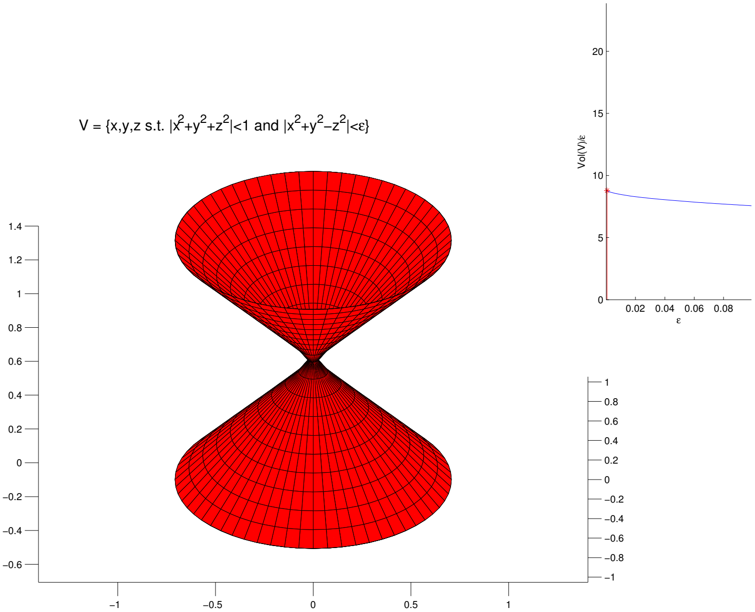
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Vol(V)/ε
ε

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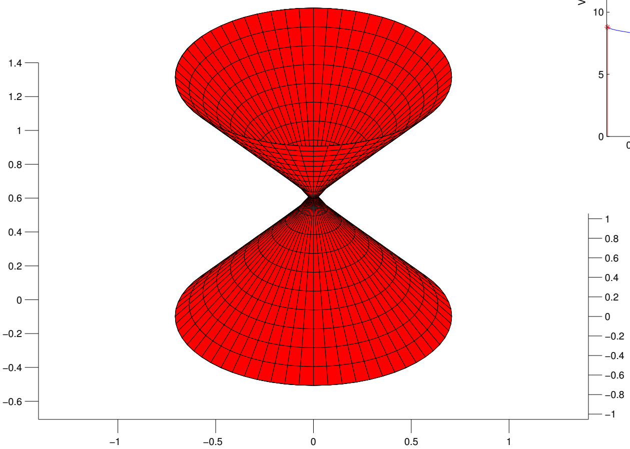

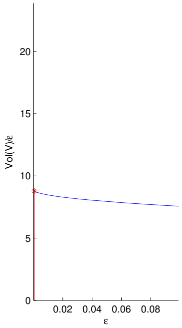

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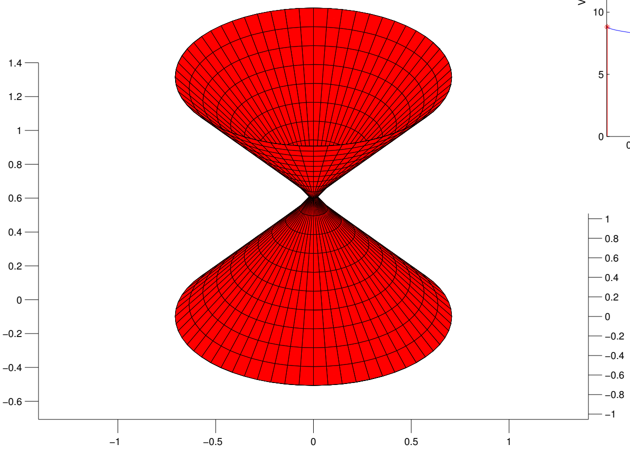

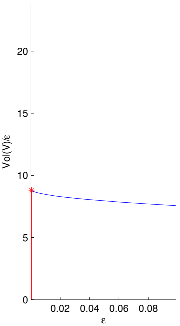

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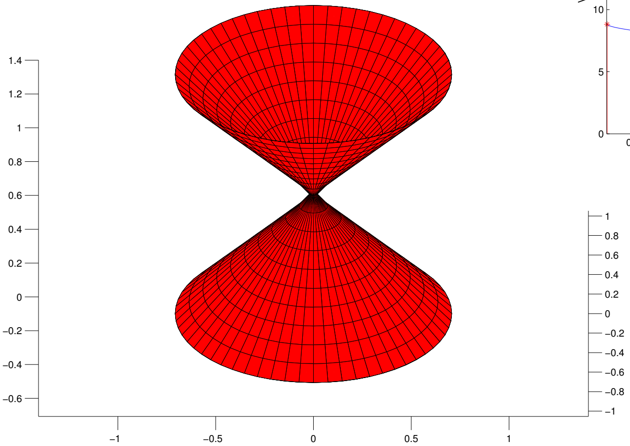

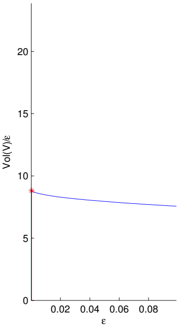

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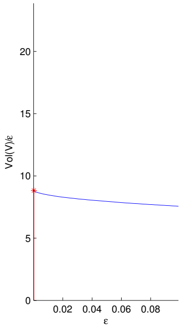

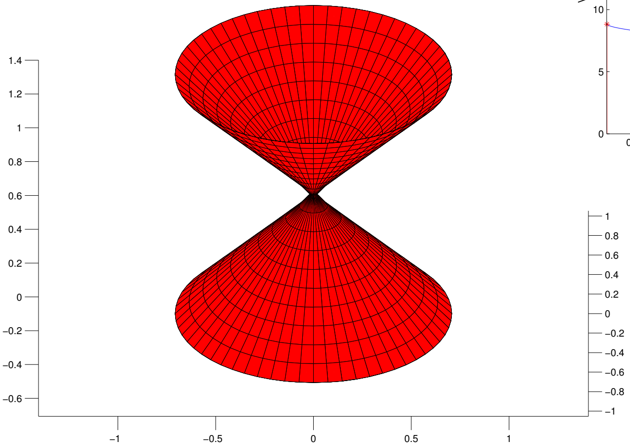

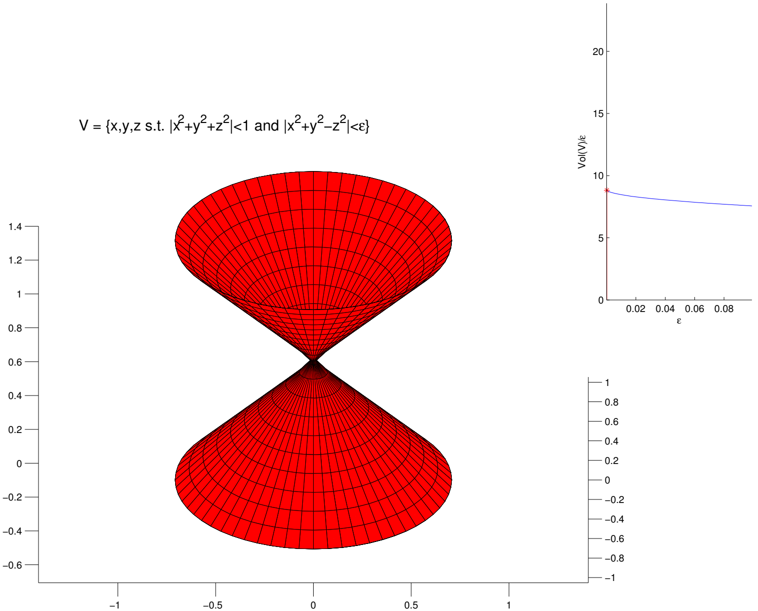

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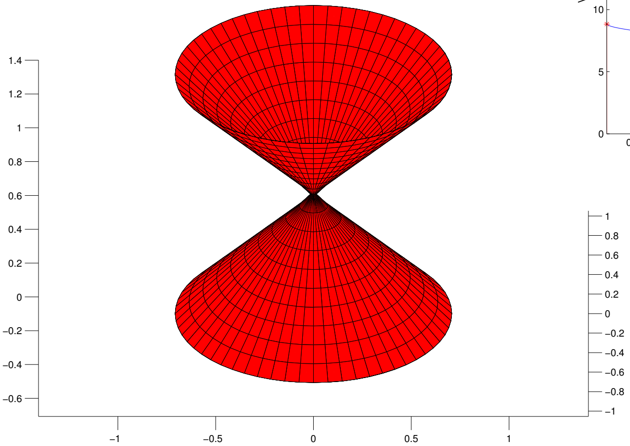

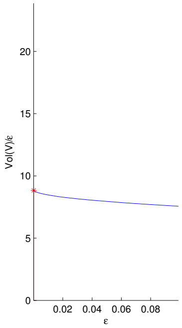

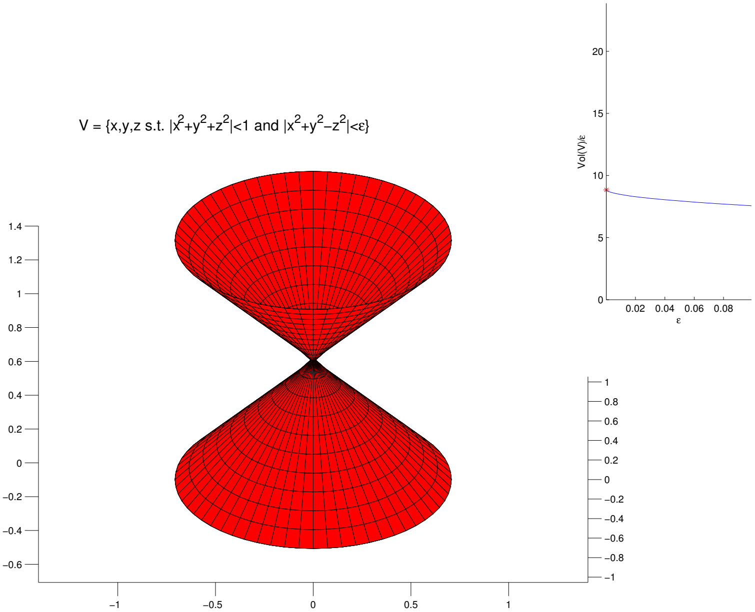
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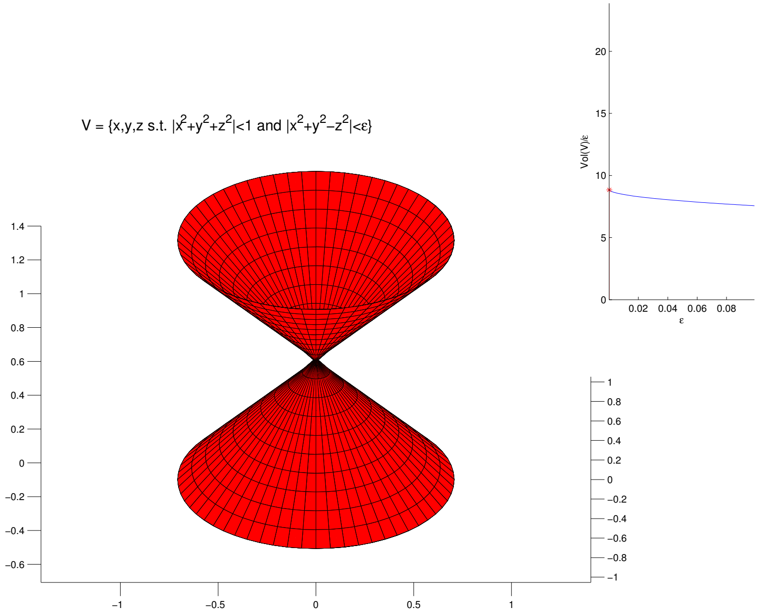

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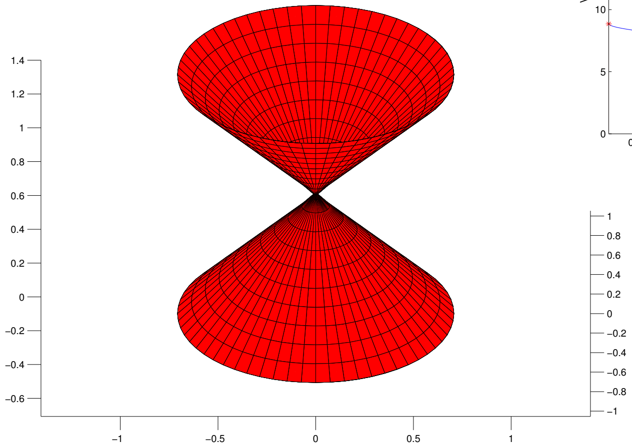

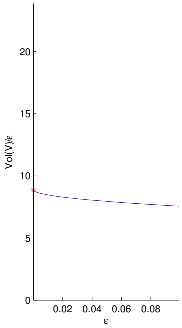

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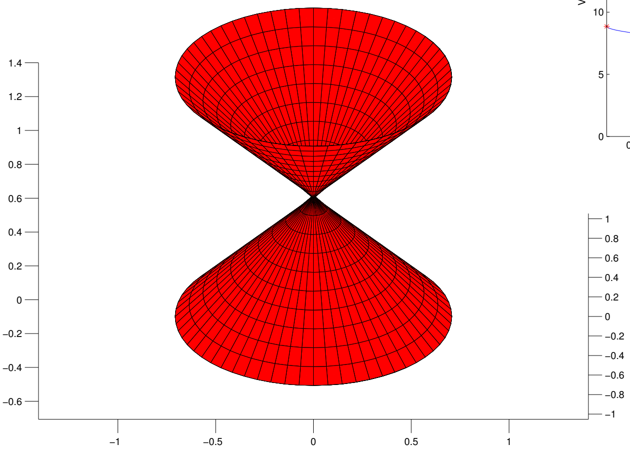

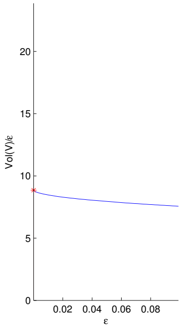

Rationality of the singularities of moduli spaces

Theorem (A.-Avni 2013)

Let $n > 20$. Then $\Phi_{G,n}$ is FRS . In particular, the singularities of the deformation variety $\mathrm{Def}_{G,\Sigma_n} := \Phi_{G,n}^{-1}$ are rational (and complete intersection).

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Corollary (A.-Avni 2013)

The moduli spaces of G local systems on a genus n surface have rational singularities.

Jet schemes and rational singularities

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Definition

For a scheme X defined over k, the jet scheme $jet_n(X)$ is the natural scheme defined over k s.t. $X(k[t]/t^n) \cong jet_n(X)(k)$.

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Theorem (Mustata 2001)

Assume that X is a local complete intersection connected variety. TFAE:

- *X is irreducible and has rational singularities.*
- *The jet schemes of X are irreducible.*

Uniform bounds

Theorem (Cluckers-Loser $\sim$ 2006)

Let X be an irreducible local complete intersection scheme of finite type.

Uniform bounds

Theorem (Cluckers-Loser $\sim$ 2006)

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All of the above happens for $n > 20$.

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$\Downarrow$ CSP, Weyl character formula

$$\zeta_{G(\mathbb{Z})}(2n + \varepsilon) < \infty$$