Counting representations of arithmetic groups and points of schemes.

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Theorem (A.-Avni 2014)

Let G be a semi-simple group defined over $\mathbb{Z}$ whose $\mathbb{Q}$-split rank is > 1. Then $\zeta_{G(\mathbb{Z})}(40)$ converges.

Application of CSP

Theorem (Lubotzky-Larsen 2007)

Let $d > 2$. Any irreducible representation π of $SL_d(\mathbb{Z})$ can be written as

$$\pi = \pi_{fin} \otimes \pi_{alg},$$

where π_{fin} factors through $SL_d(\mathbb{Z}/N\mathbb{Z})$ and π_{alg} extends to an algebraic representation of $SL_d(\mathbb{C})$.

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To show that $\zeta_{G(\mathbb{Z})}(s)$ converges, enough to show that $\zeta_{G(\mathbb{C})}(s)$ converges, and $\zeta_{G(\mathbb{Z}/N\mathbb{Z})}(s)$ is bounded when n varies.

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- $\zeta_H(2n-2) = \frac{\#\{(g_1,h_1,\dots g_n,h_n) \in H^{2n} | [g_1,h_1]\cdots[g_n,h_n]=1\}}{\# H^{2n-1}}$

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$$= Hom(\pi_1(\Sigma_n), G).$$

Then there exists a constant C s.t. for any integer k we have:

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For any $A \subset G(\mathbb{Z}/N\mathbb{Z})$:

$$\mathrm{Prob}([g_1, h_1] \cdots [g_n, h_n] \in A) < C \cdot \mathrm{Prob}(g \in A),$$

for random elements $g, g_1 \ldots g_n \in G(\mathbb{Z}/N\mathbb{Z})$

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- *The only pole of the continued function on the line* $\Re(s) = \dim X_{\mathbb{Q}} + 1$ *is a simple pole at* $\dim X_{\mathbb{Q}} + 1$.

Rationality of the singularities of moduli spaces

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Corollary (A.-Avni 2013)

The moduli spaces of G local systems on a genus n surface have rational singularities.

Sum up

$$\{(x,y,z) | z^2 = x^2 + y^2\} \text{ have rational singularities}$$

$$\Downarrow$$

$$\mathrm{def}_{\mathfrak{g},n} := \{(g_1,h_1,\dots g_n,h_n) \in \mathfrak{g}^{2n} | [g_1,h_1] + \cdots + [g_1,h_1] = 0\}$$

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$$\exists m \text{ s.t.} \#\{(g_1,h_1,\dots g_n,h_n) \in G(\mathbb{Z}/p^k\mathbb{Z})^{2n} | [g_1,h_1]\cdots[g_n,h_n] = 1; g_i = h_i = 1 \mod p^m\} = p^{(2n-1)(k-m)\dim G}(1 + O(p^{-\frac{1}{2}}))$$

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All of the above happens for $n > 20$.

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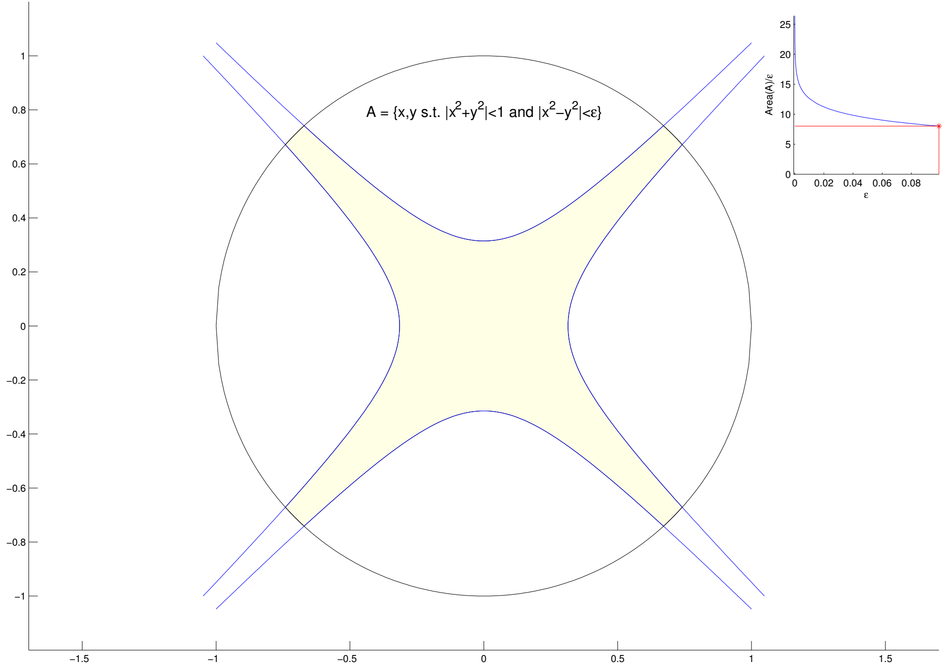
- m *is a Schwartz (i.e. compactly supported locally Haar) measure on* $X(F)$*.*

Then $\phi_*(m)$ *has continuous density.*


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
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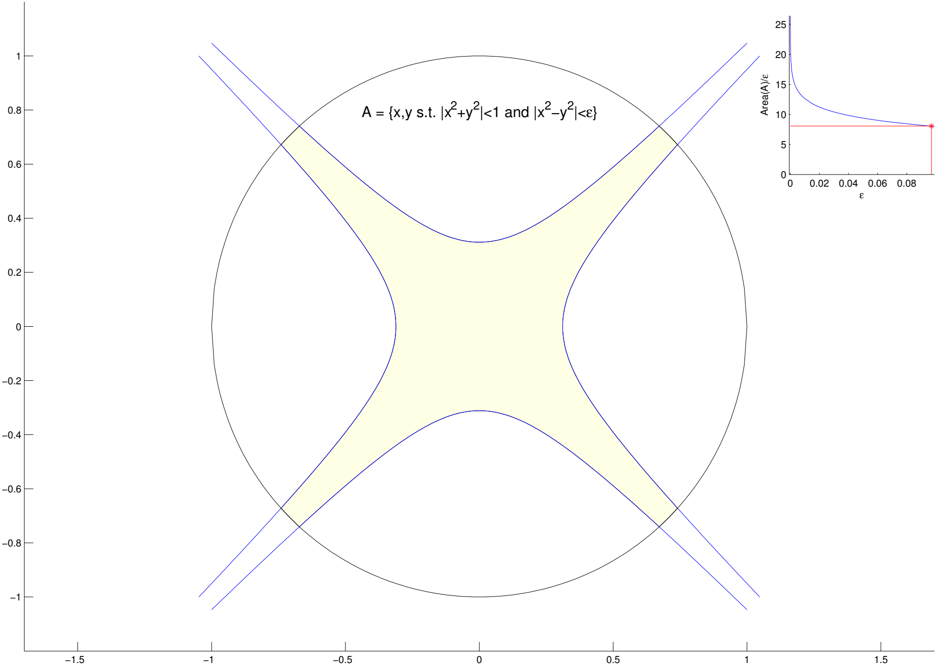
1.5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Area(A)/ε
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0.04
0.06
0.08
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20
25
−1.5
−1
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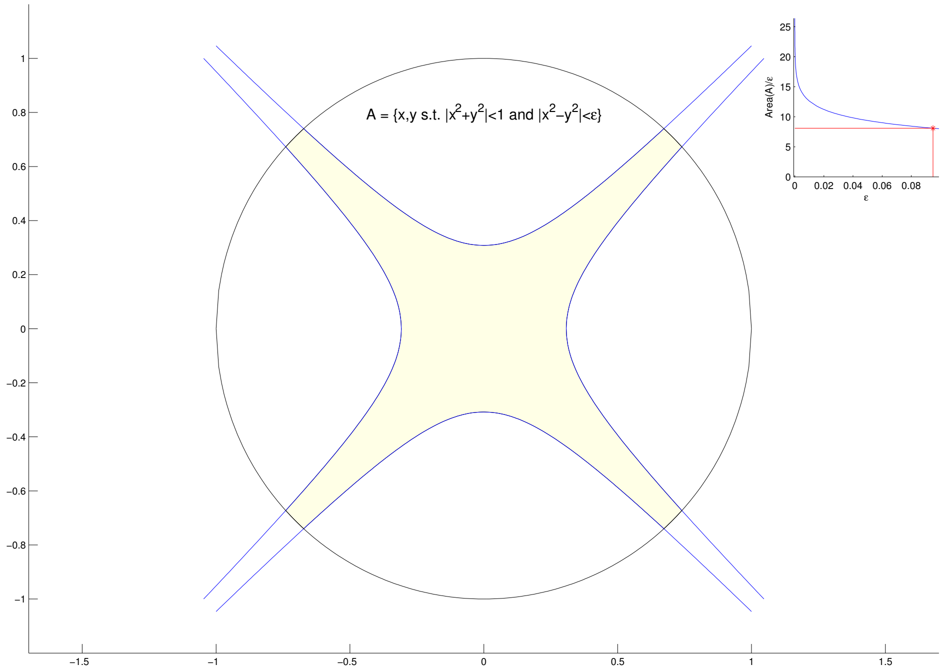

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Area(A)/ε
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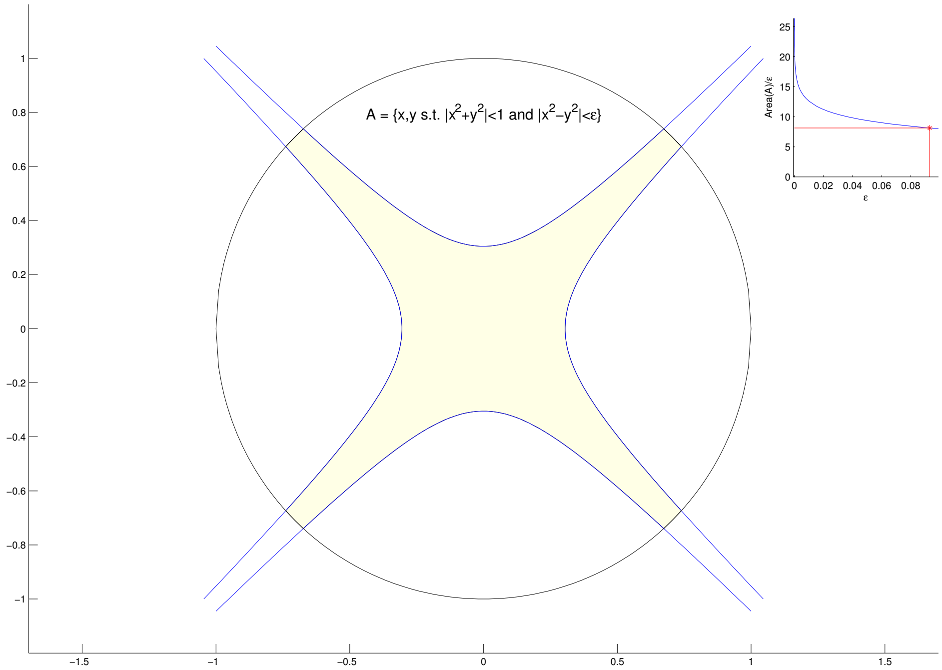
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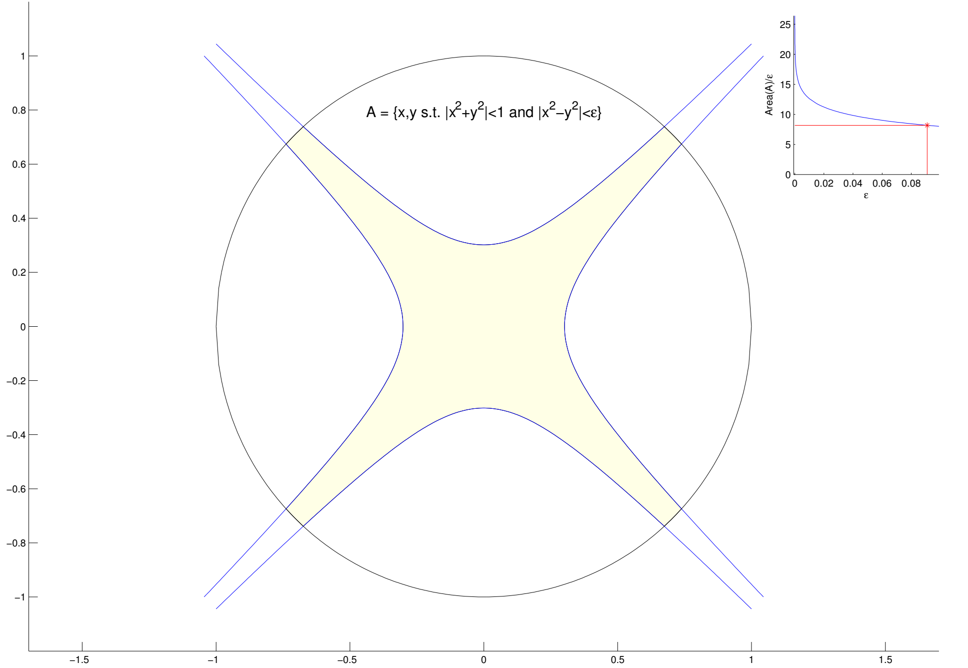
1
1.5



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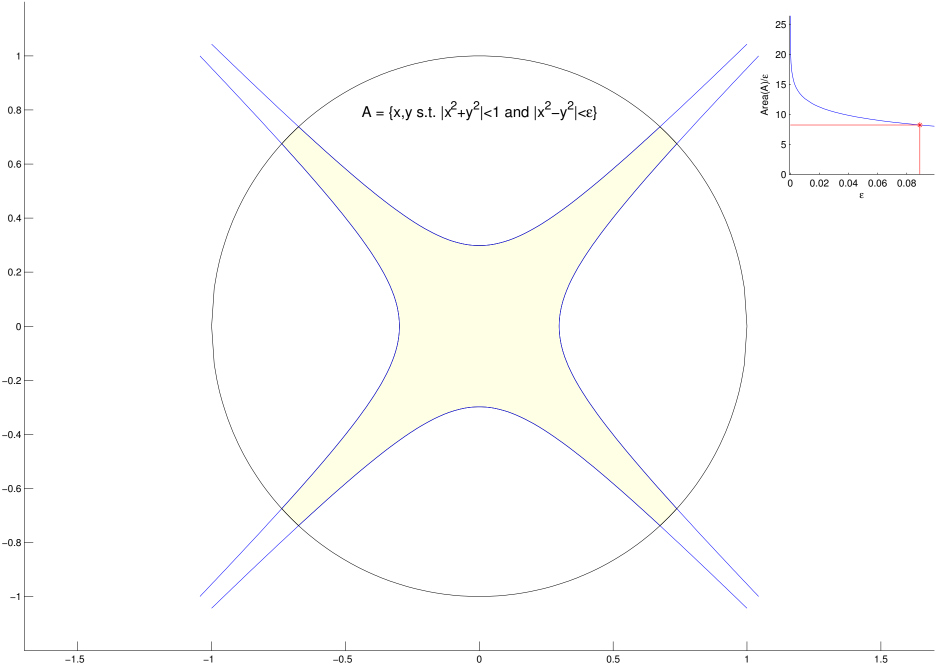

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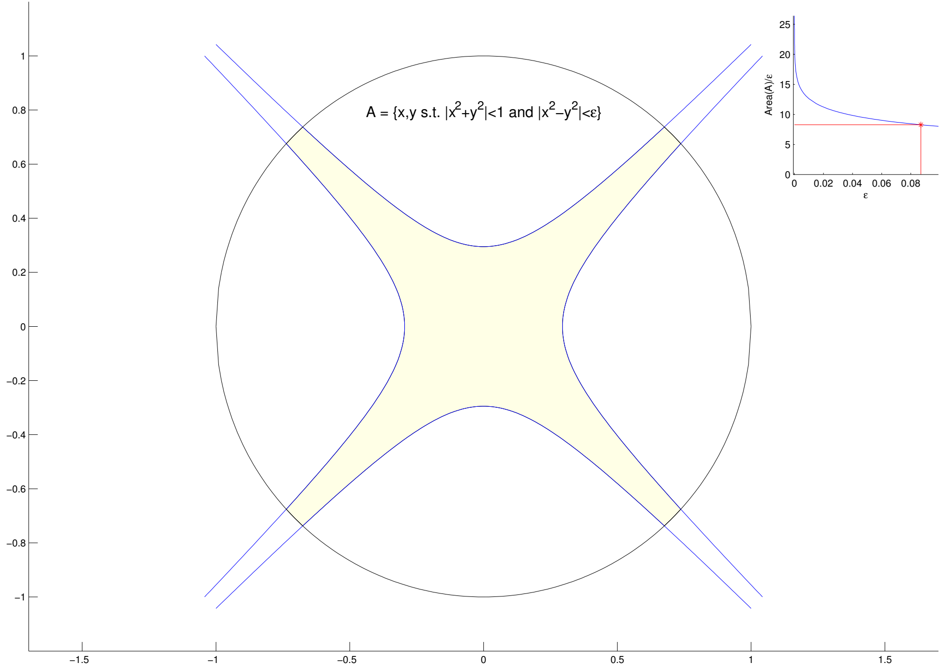
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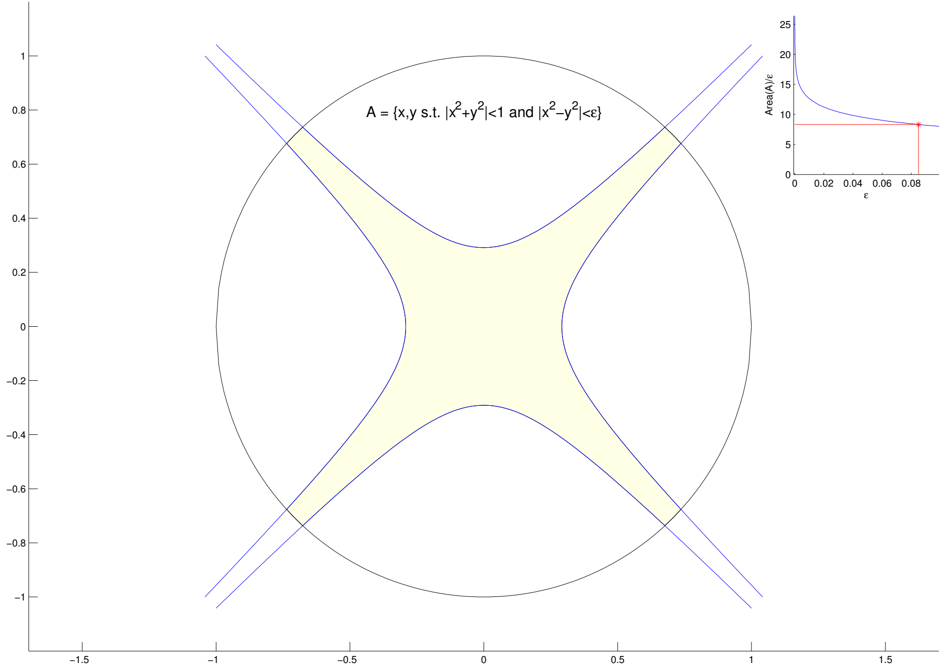
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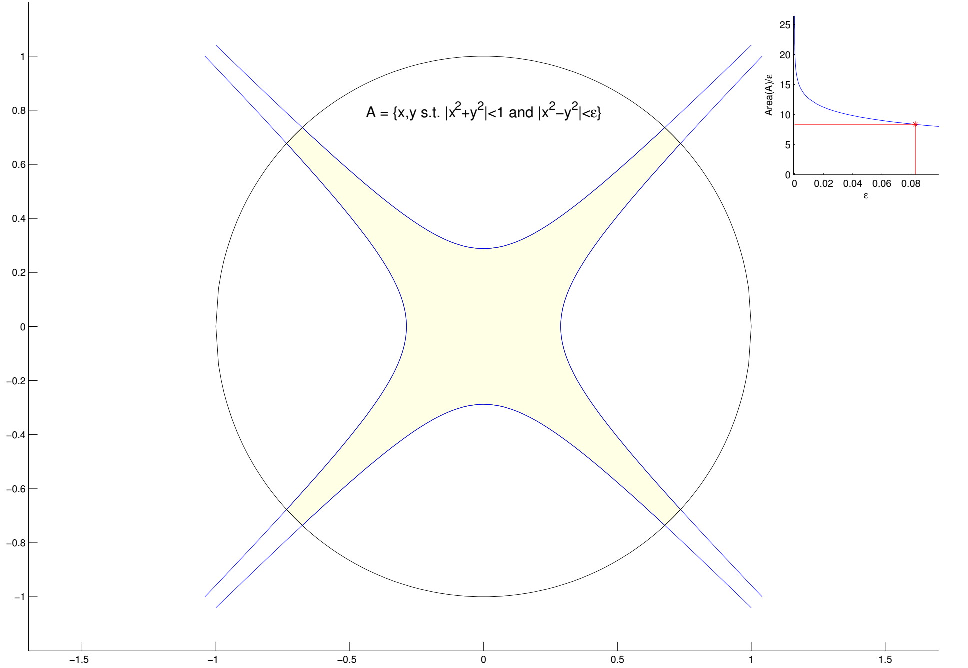
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Area(A)/ε
ε
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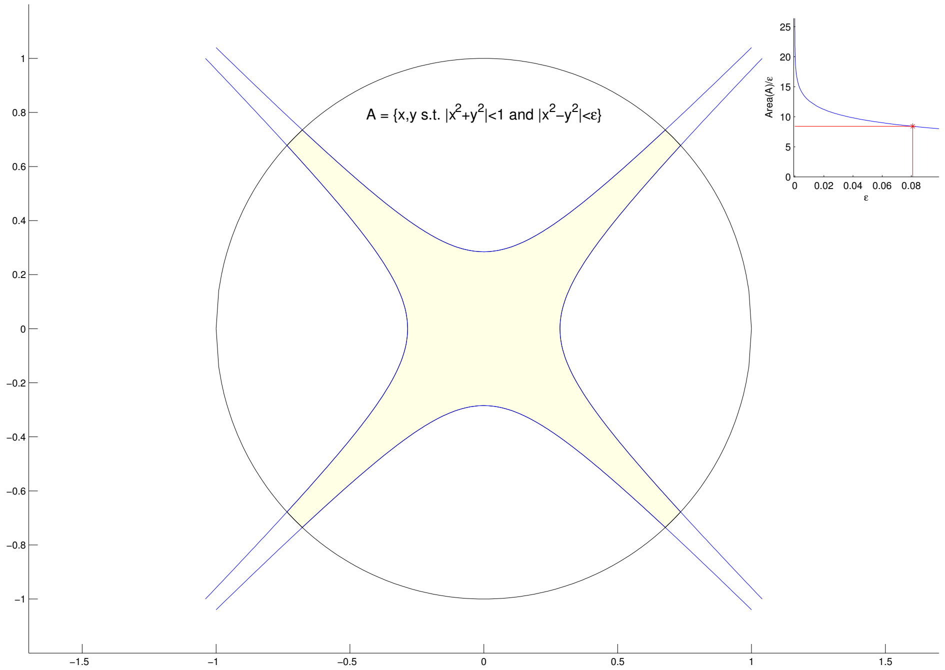
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1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1
0.8
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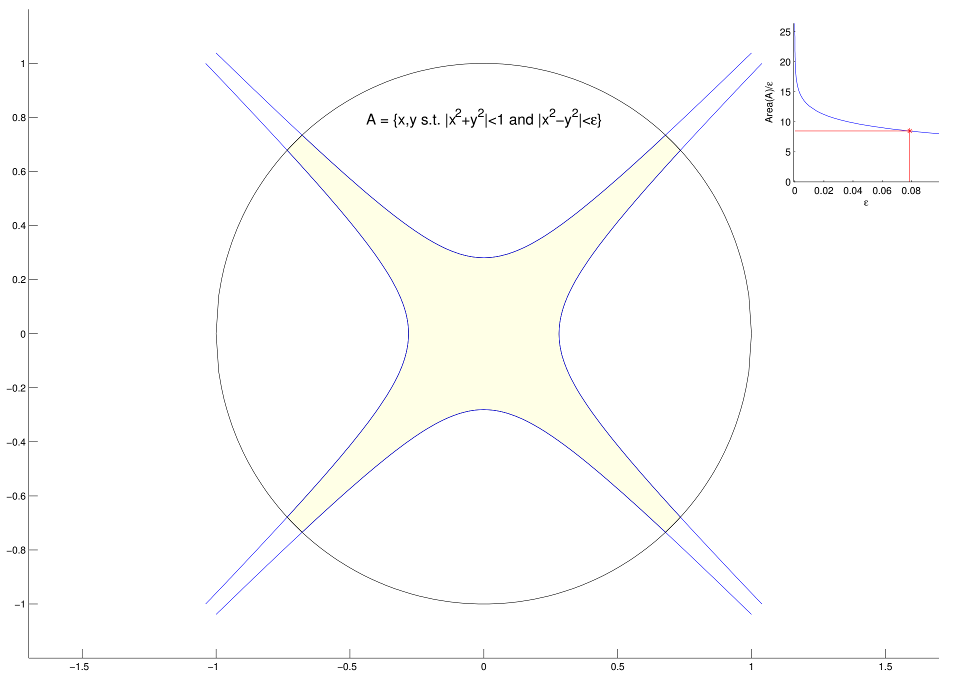
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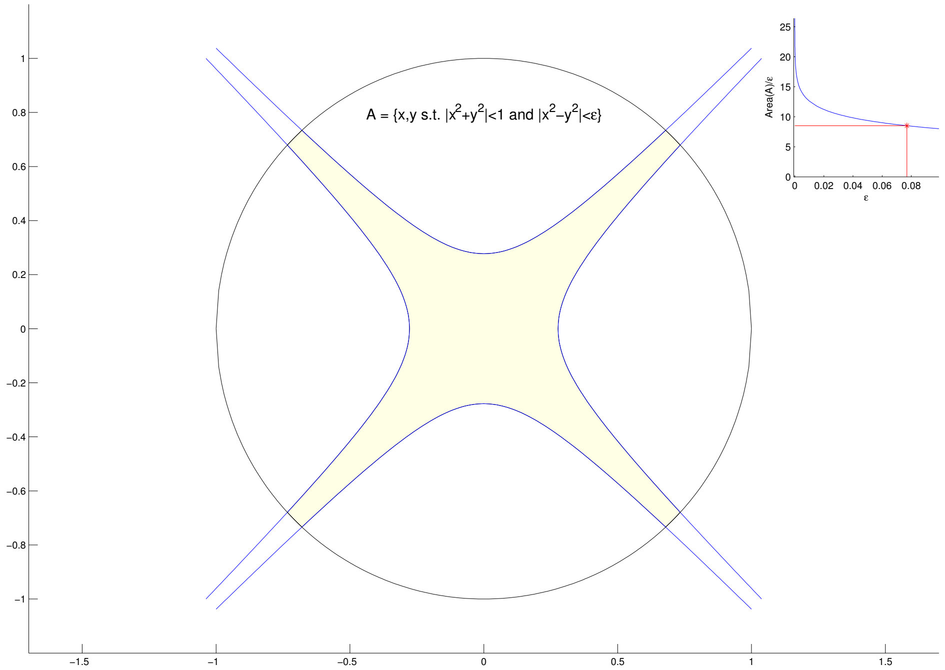

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0.2
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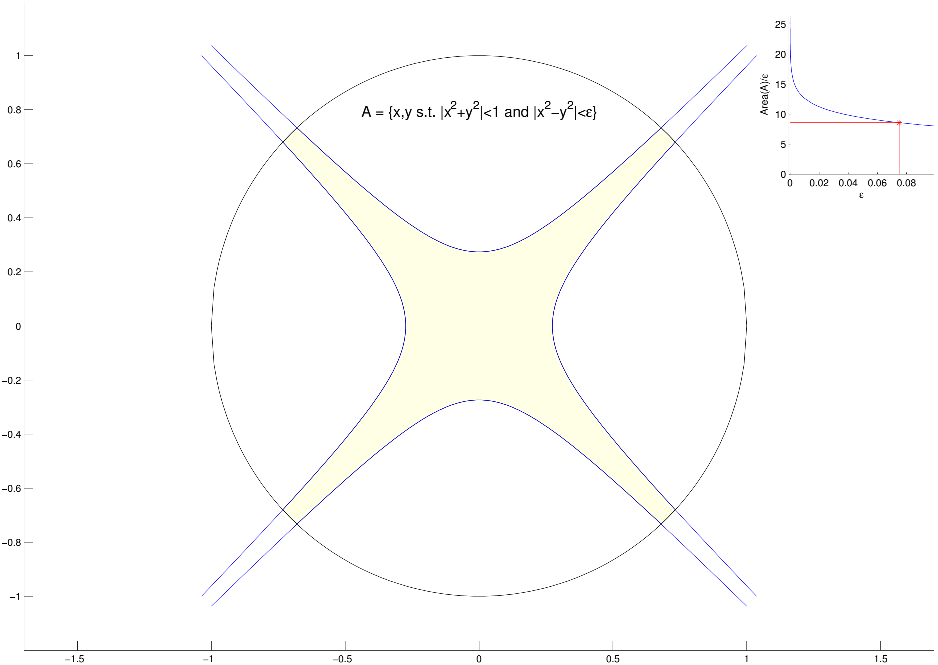
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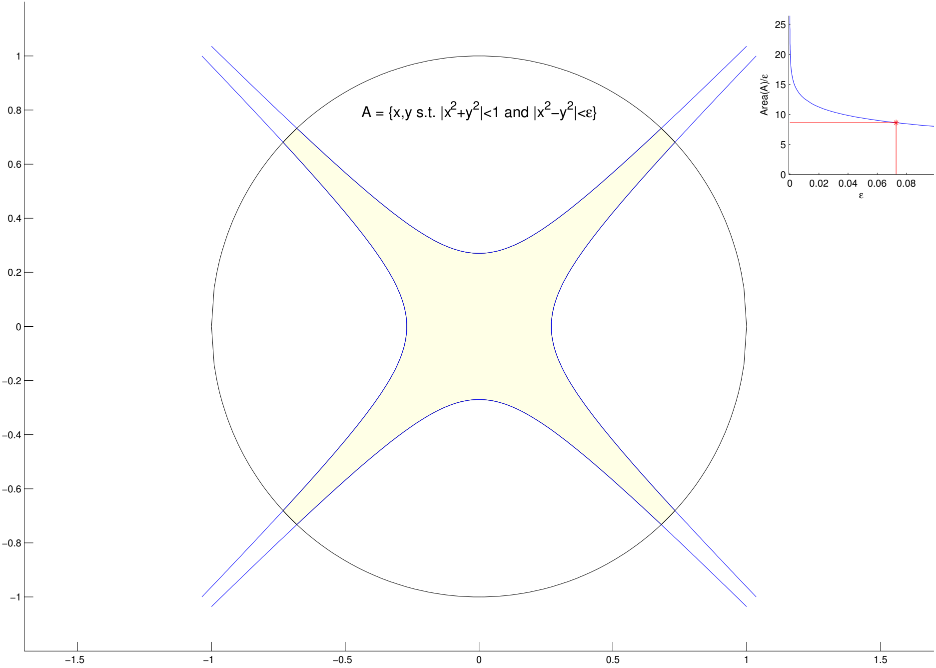

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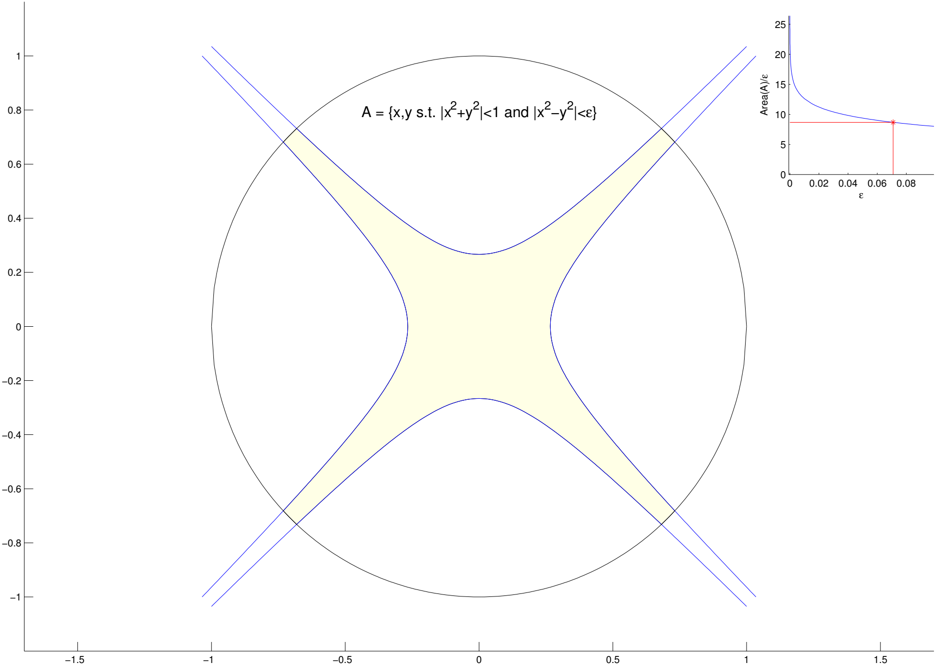

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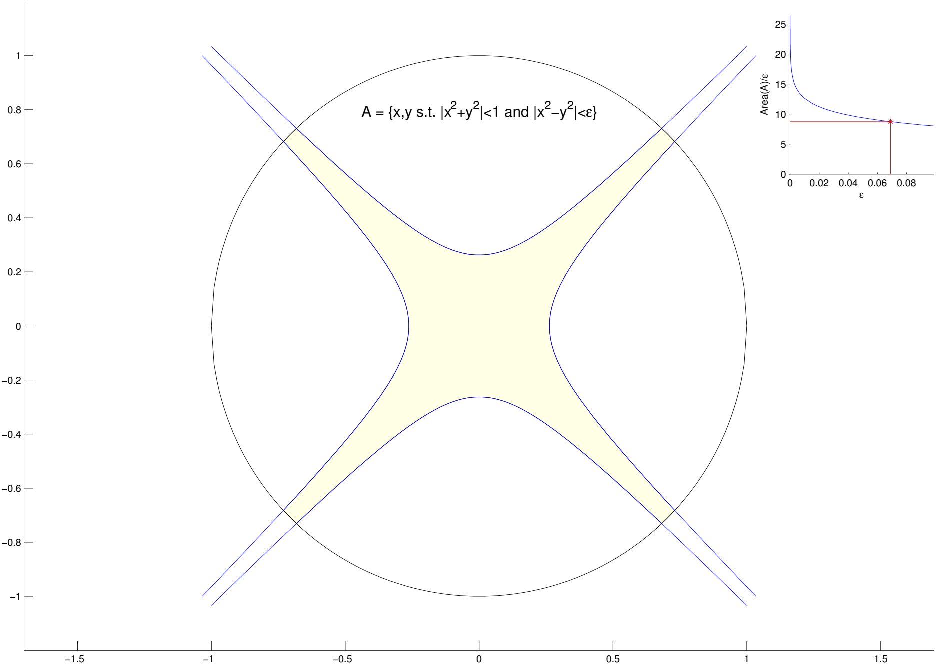
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ε
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0.8
0.6
0.4
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0
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−0.4
−0.6
−0.8
−1
−1.5
−1
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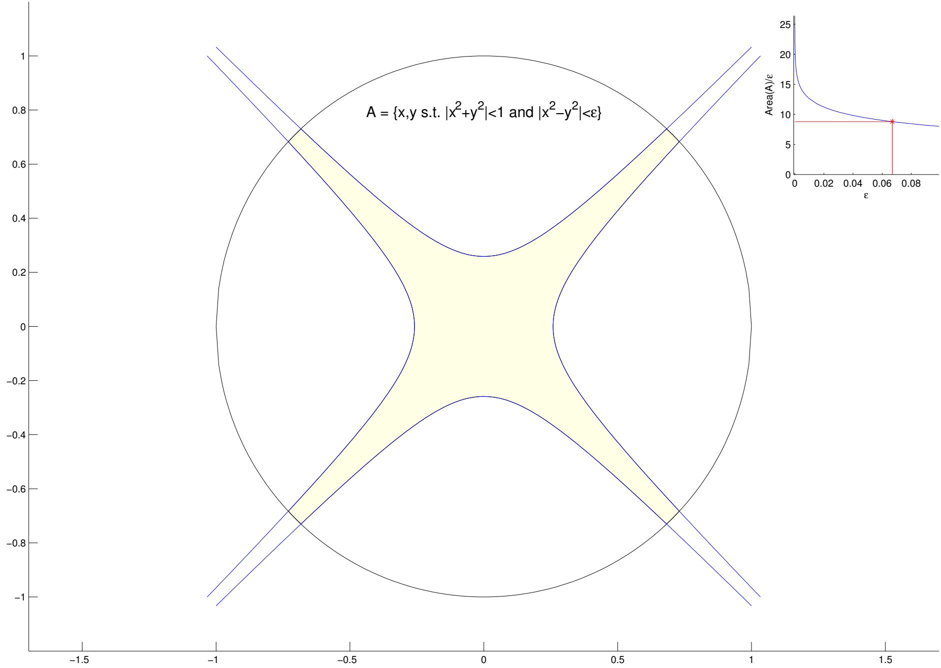
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−1
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0
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Area(A)/ε
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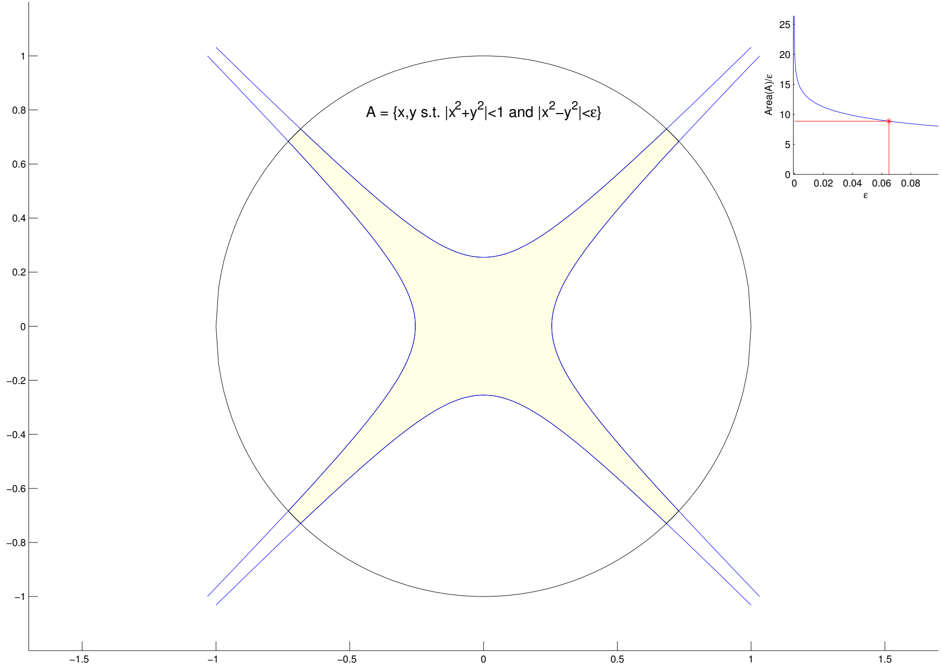
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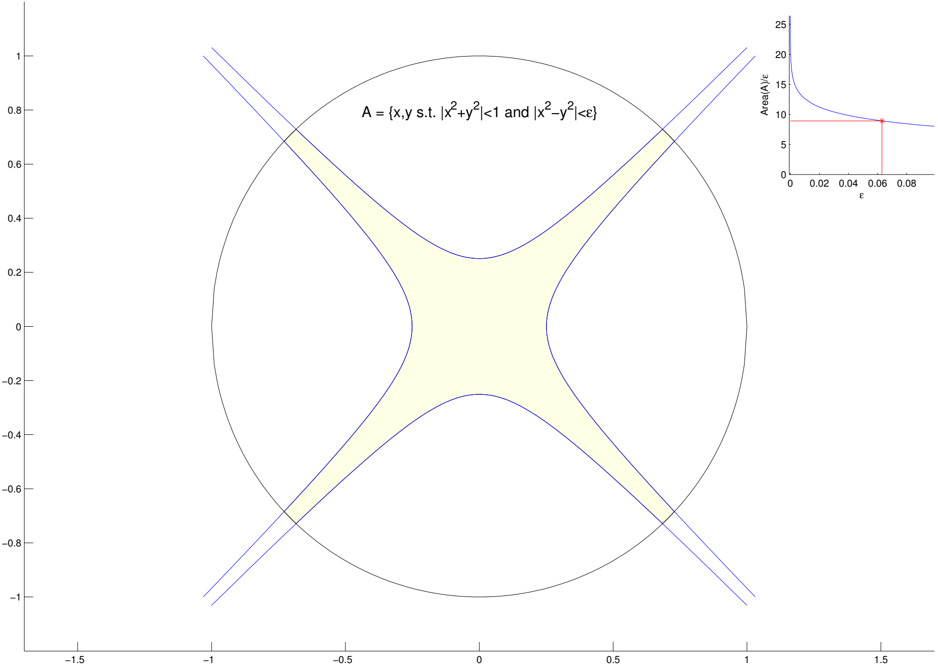

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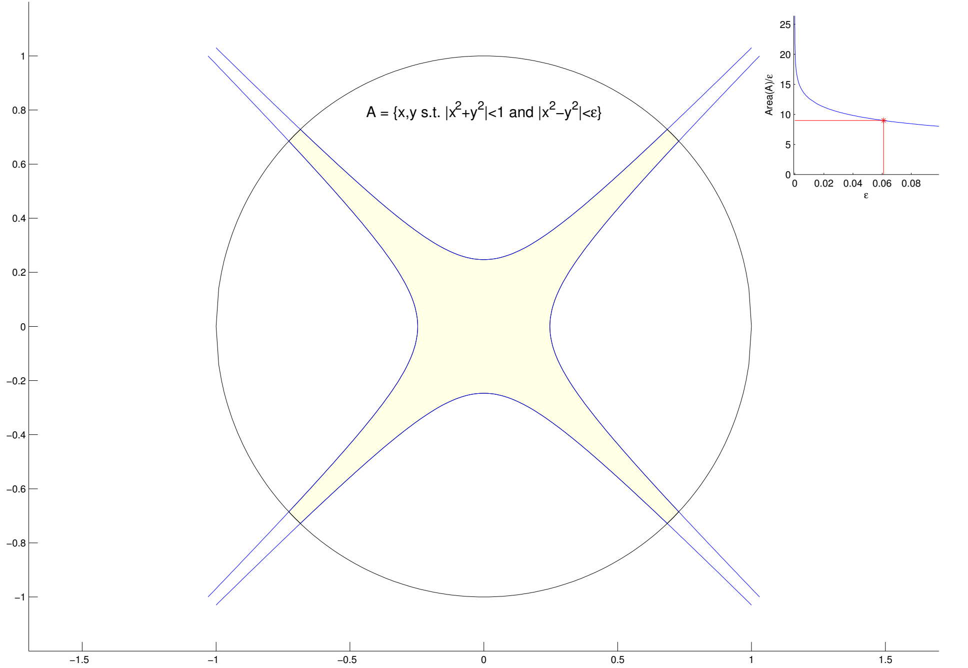
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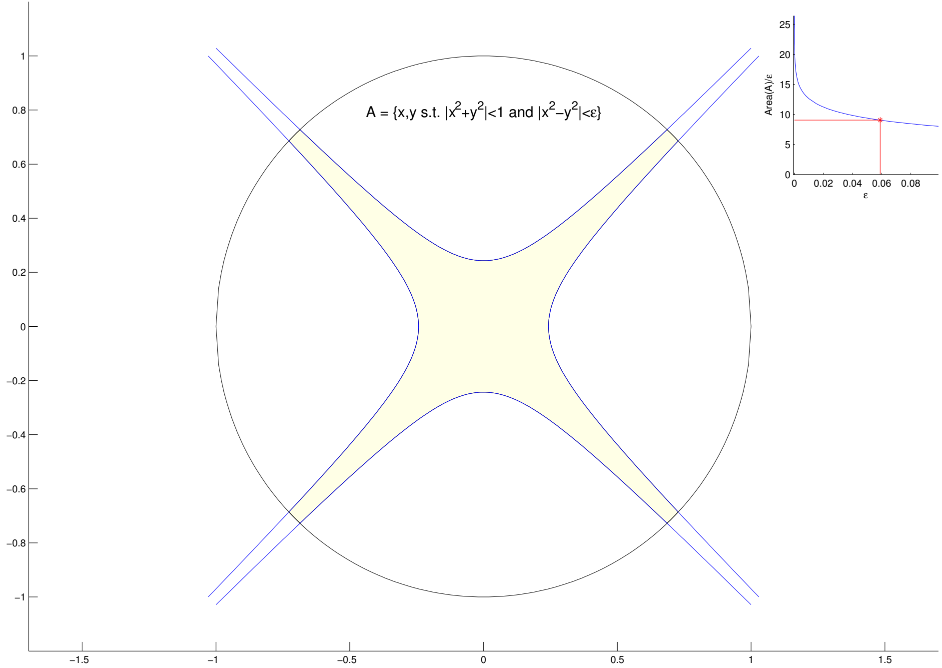
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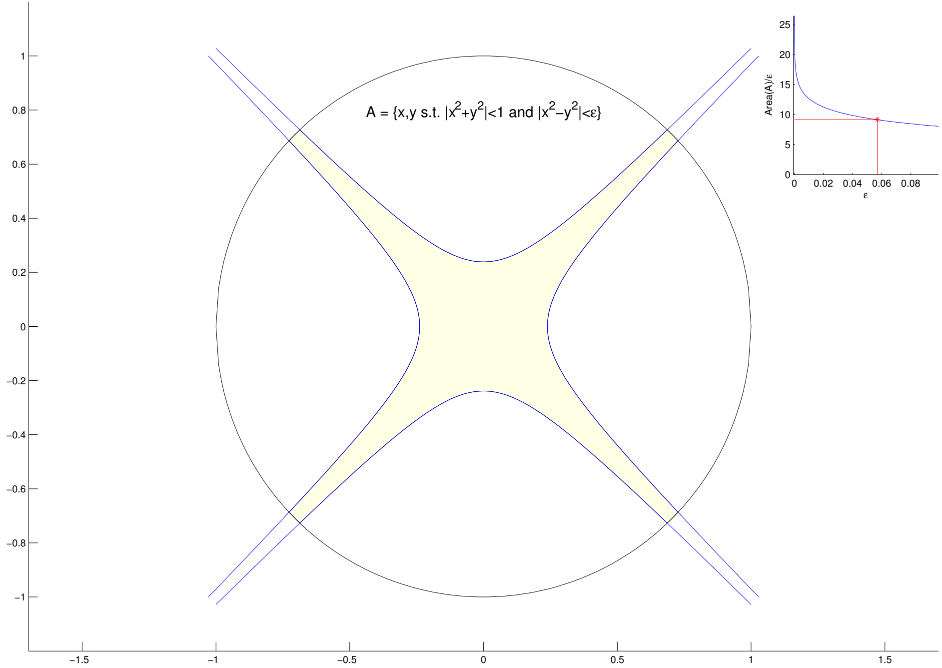

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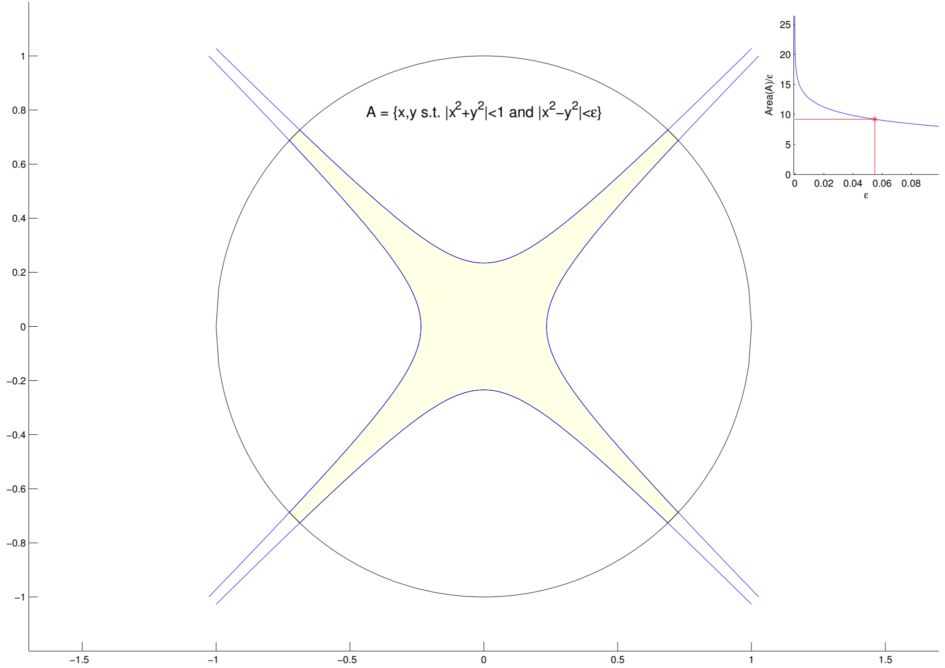
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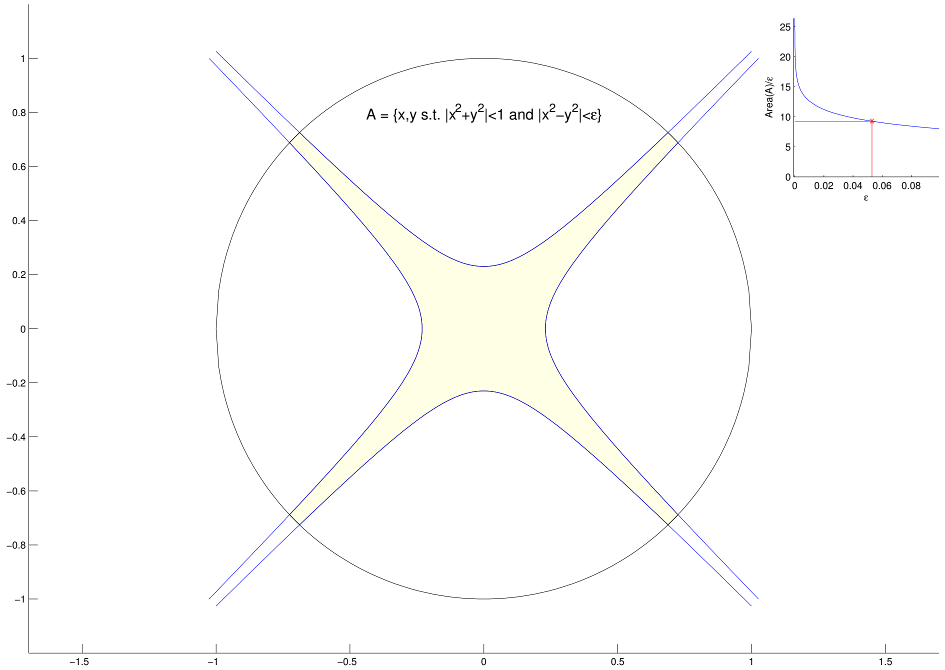
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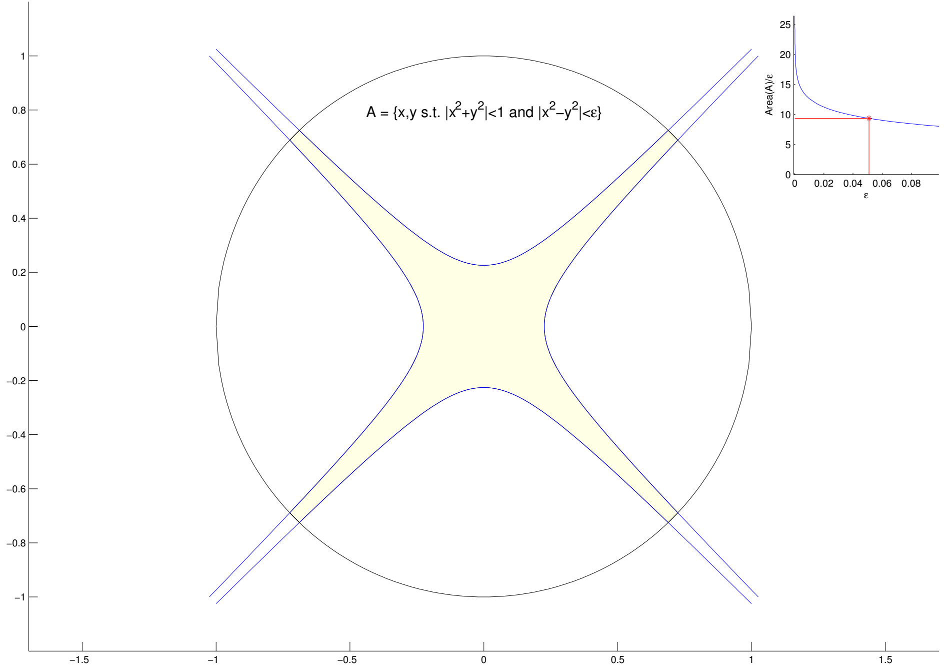

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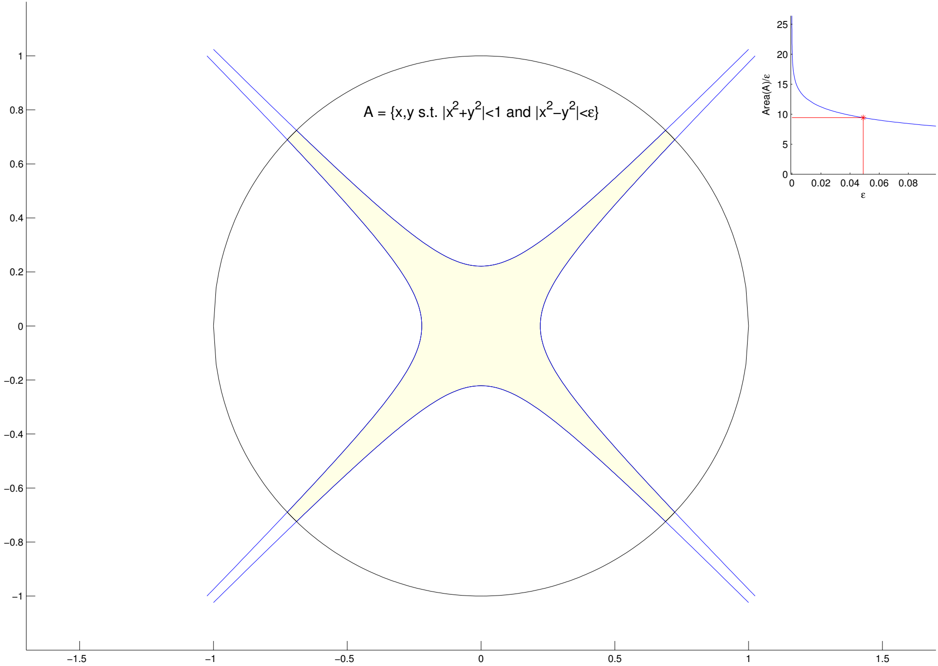
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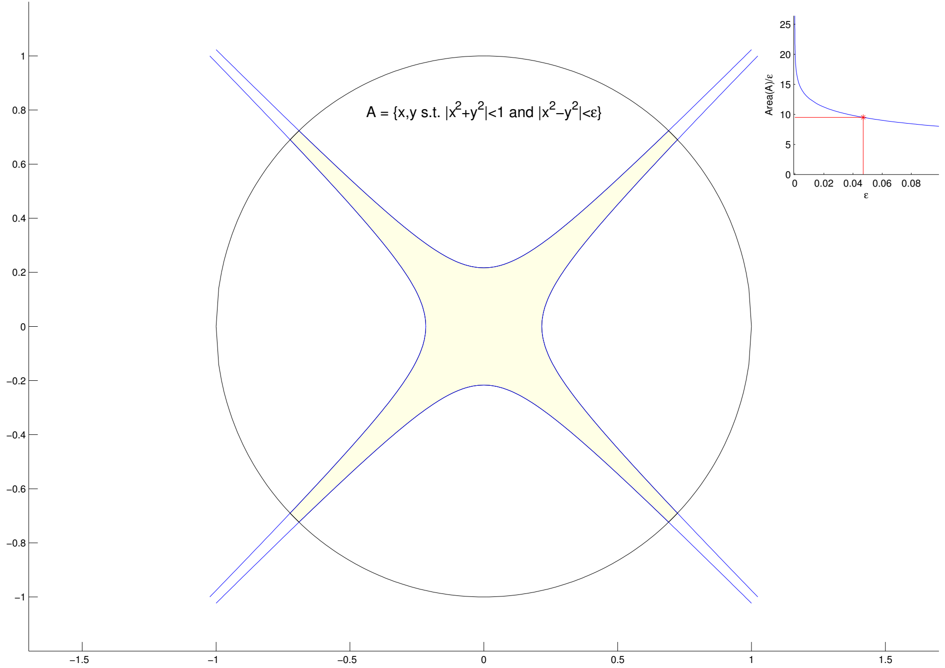
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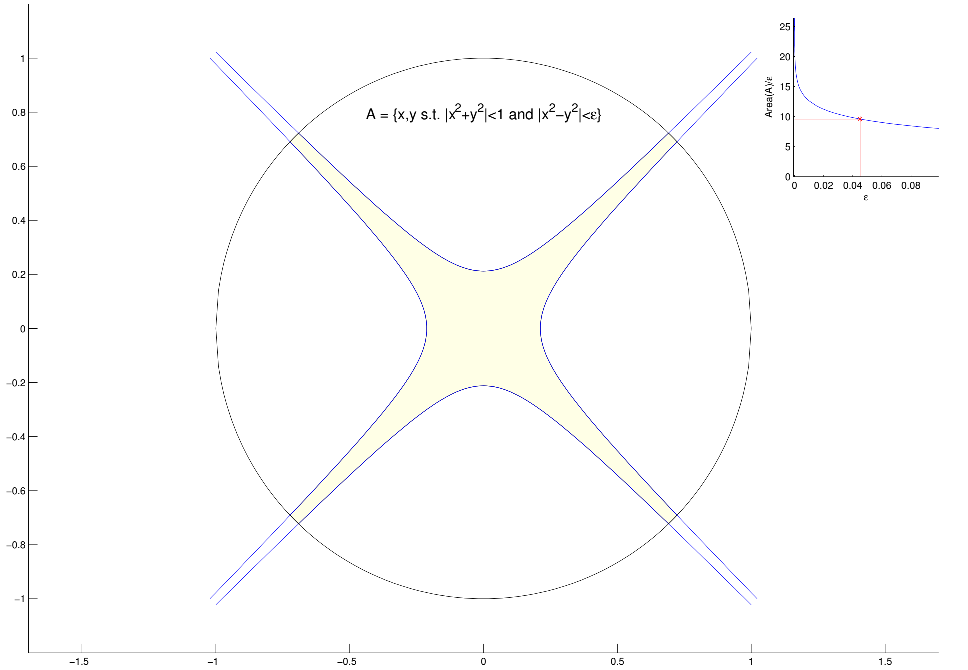
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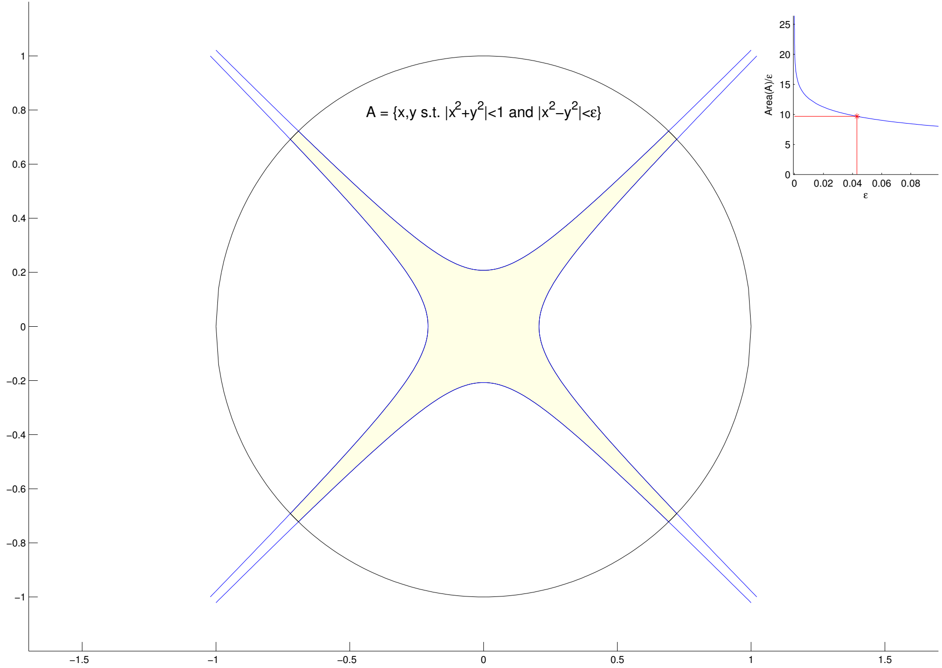

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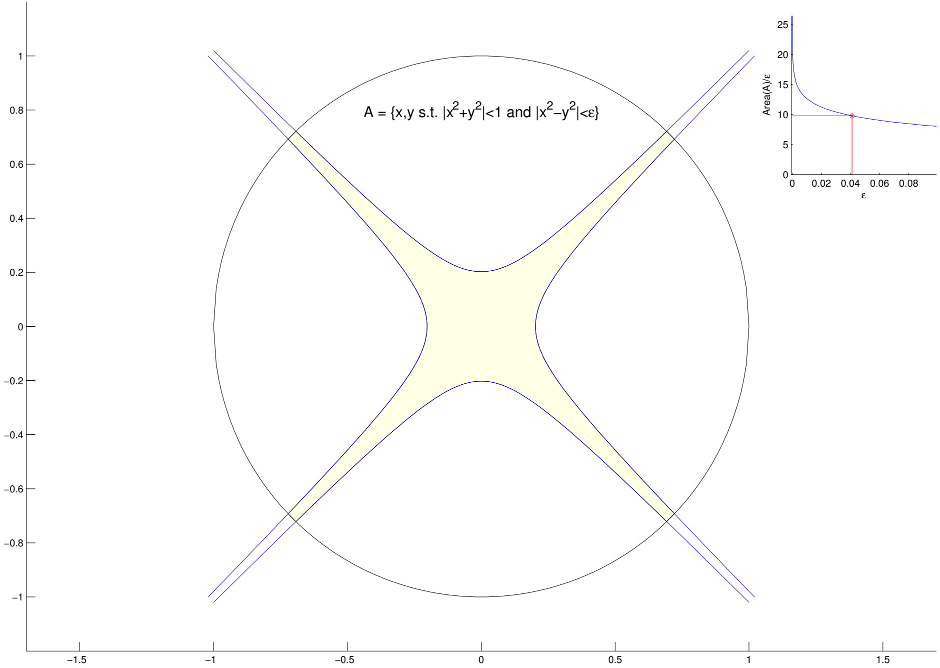

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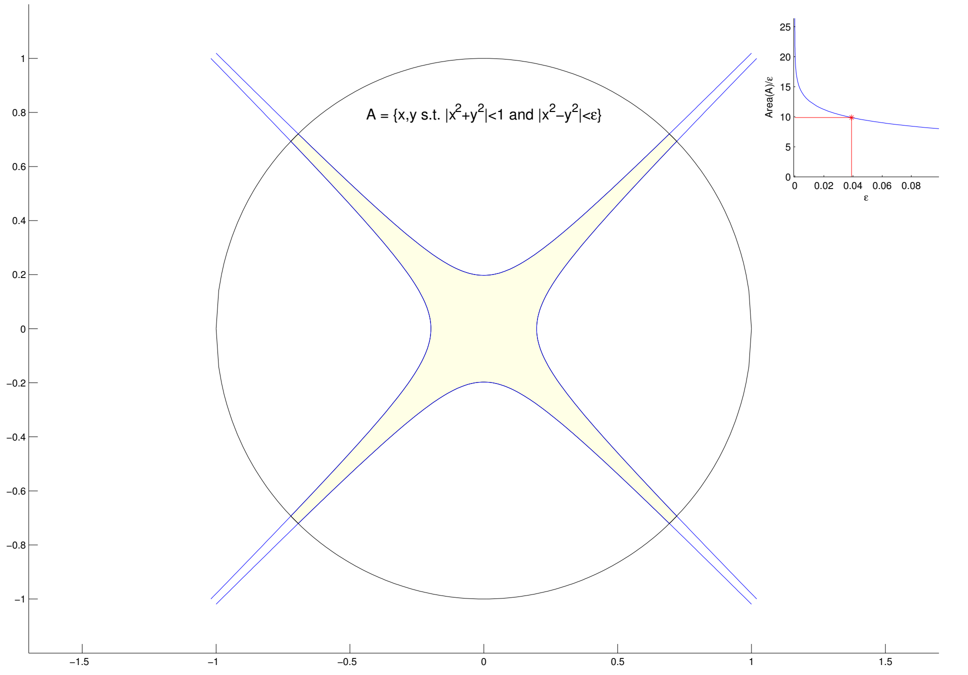
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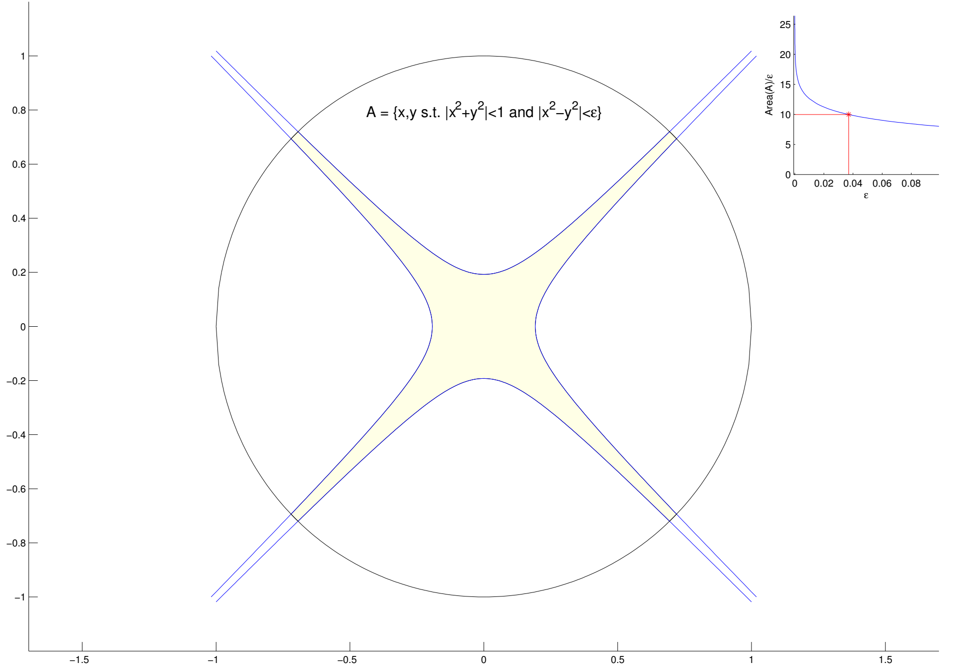
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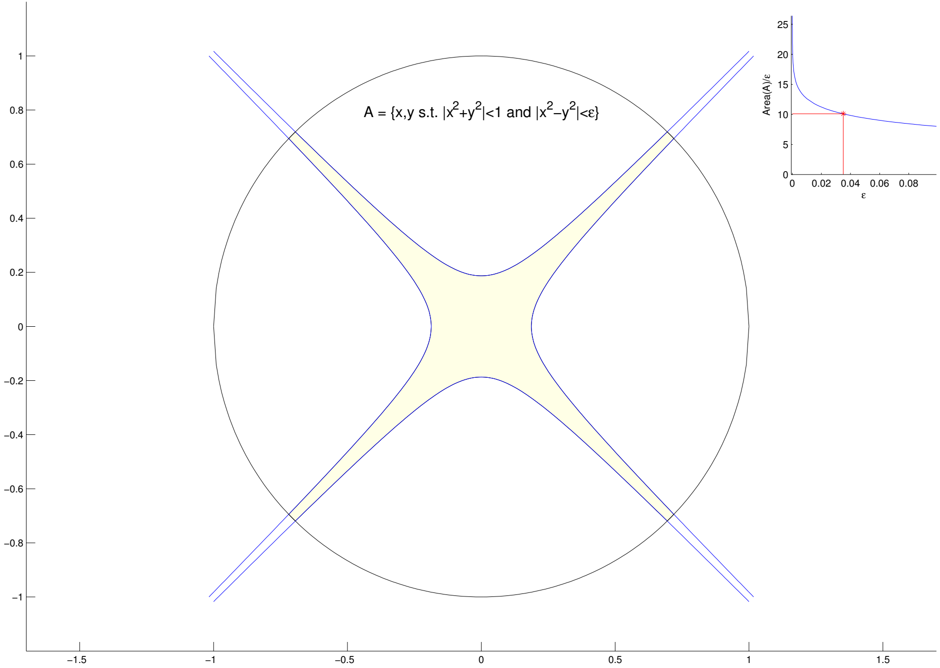
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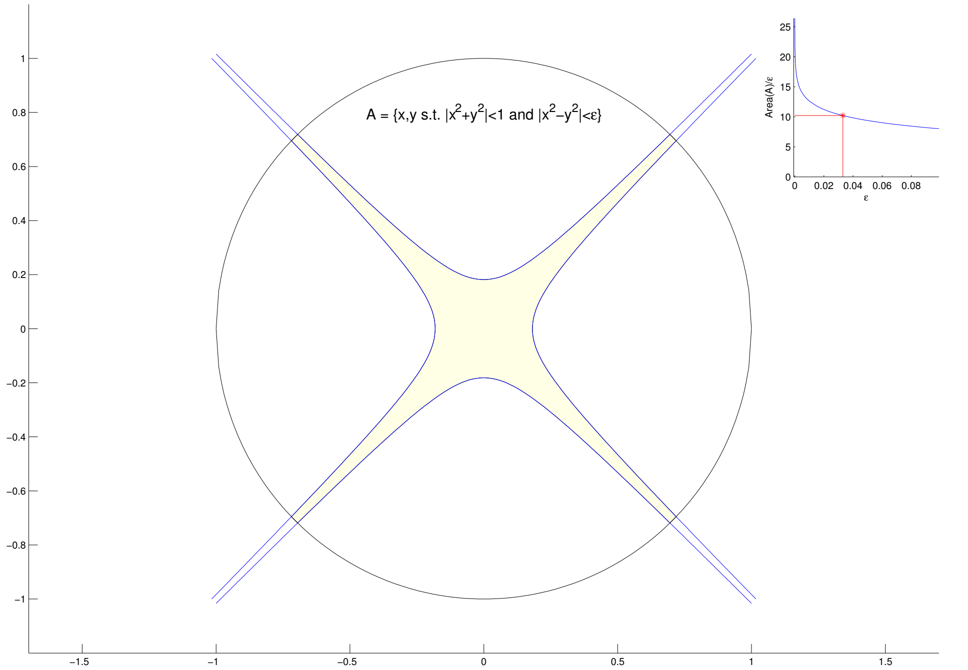
A = {x,y s.t. |x2+y2|<1 and |x2−y2|<ε}
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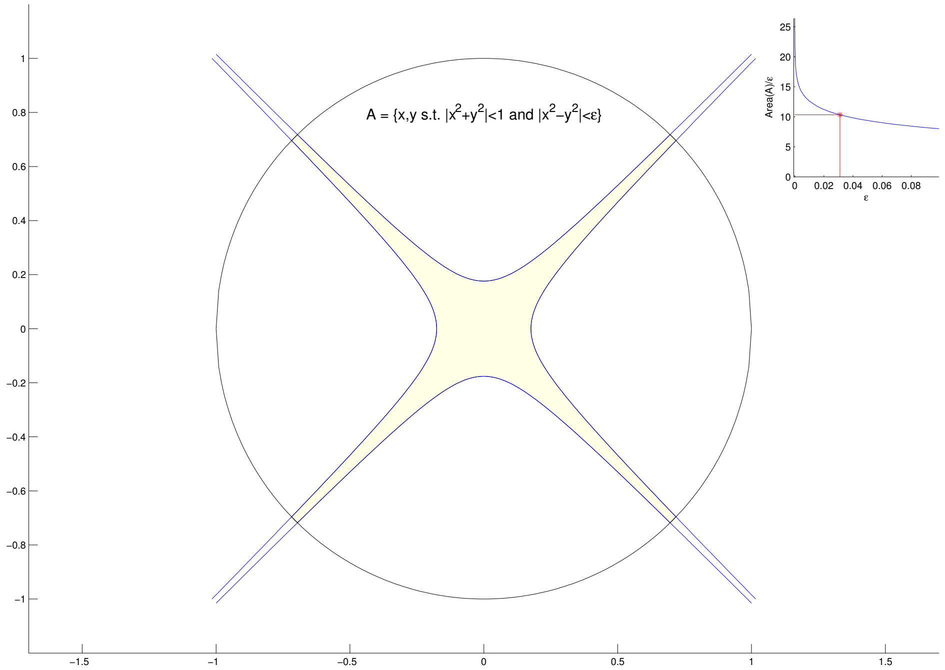

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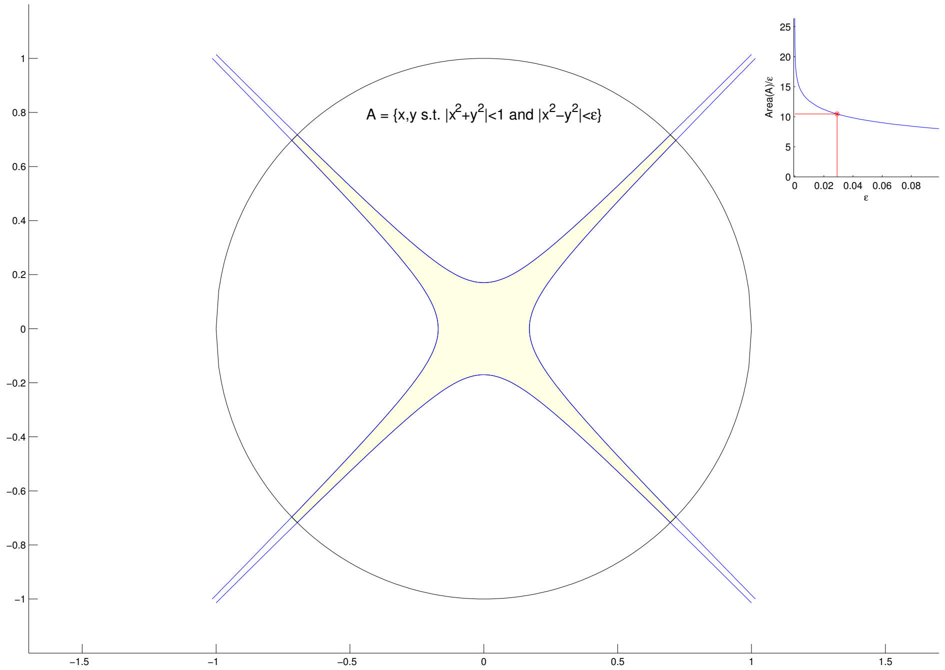

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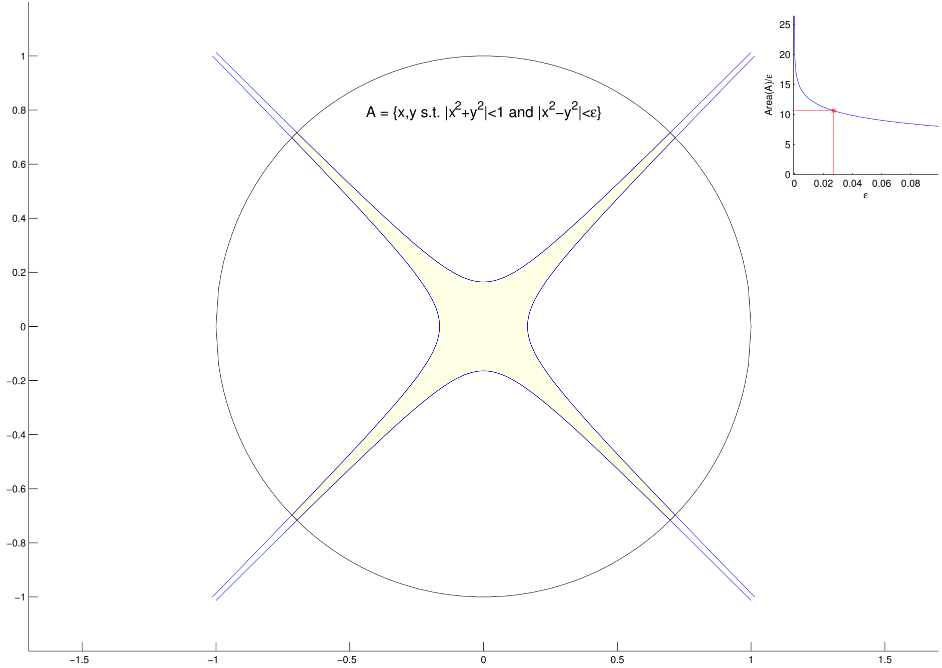
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
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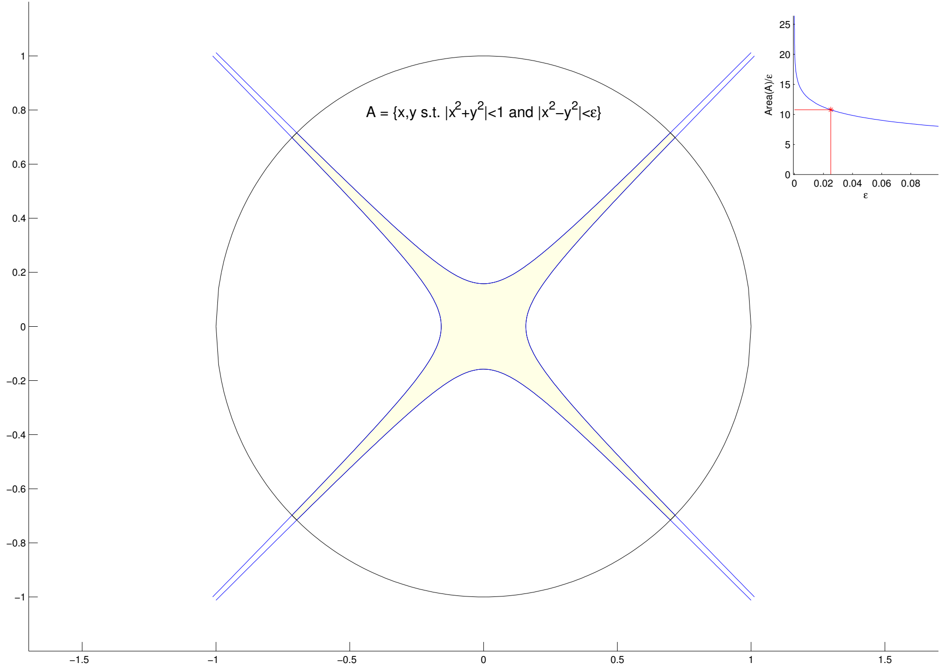
A = {x,y s.t. |x2+y2|<1 and |x2−y2|<ε}
Area(A)/ε
ε
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15
10
5
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0.02
0.04
0.06
0.08
1
0.8
0.6
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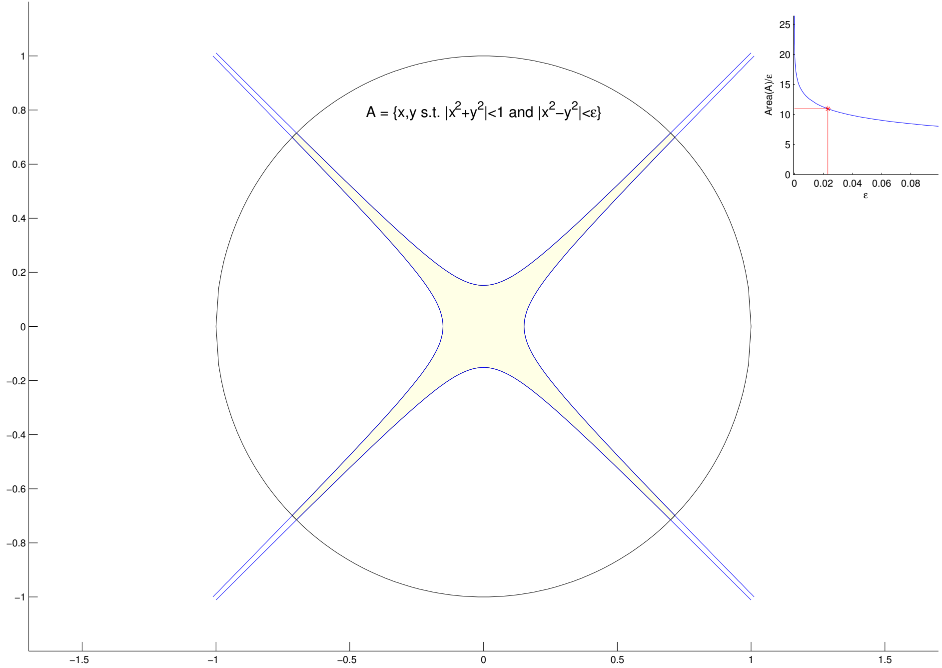

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0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5
Area(A)/ε
ε
25
20
15
10
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0
0
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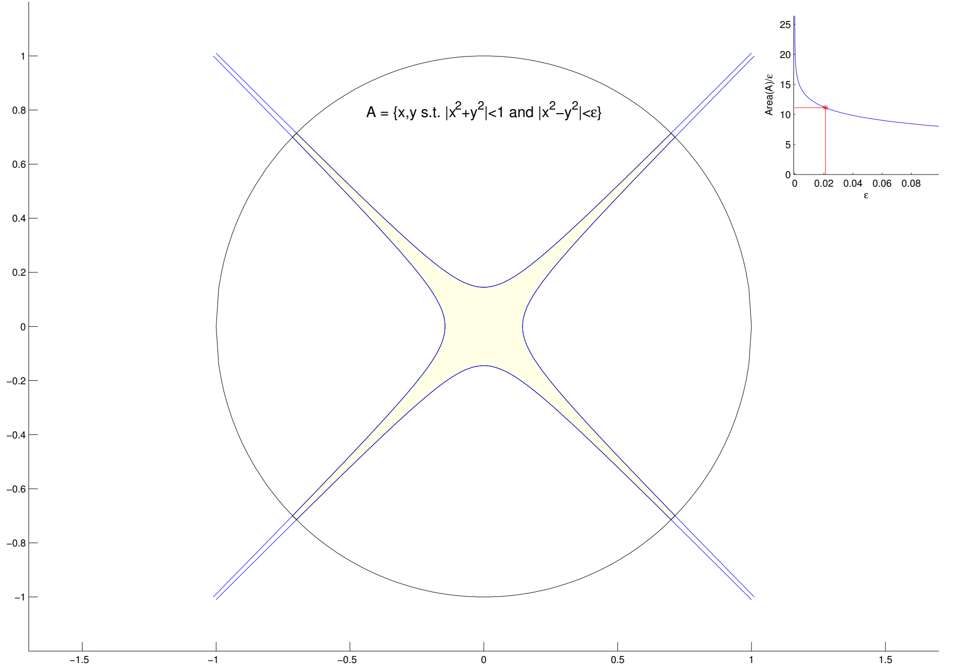
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Area(A)/ε
ε
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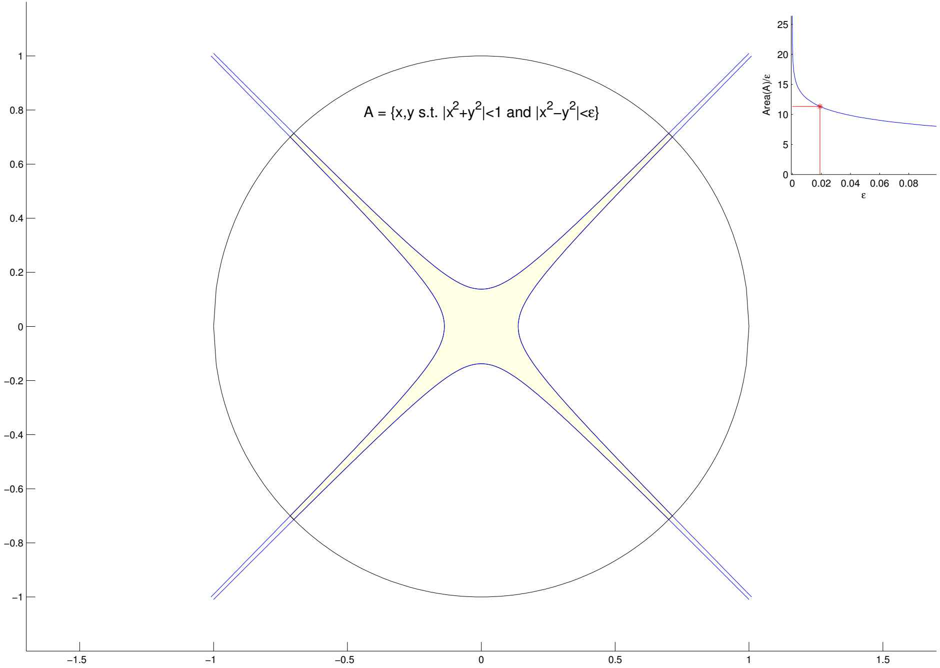
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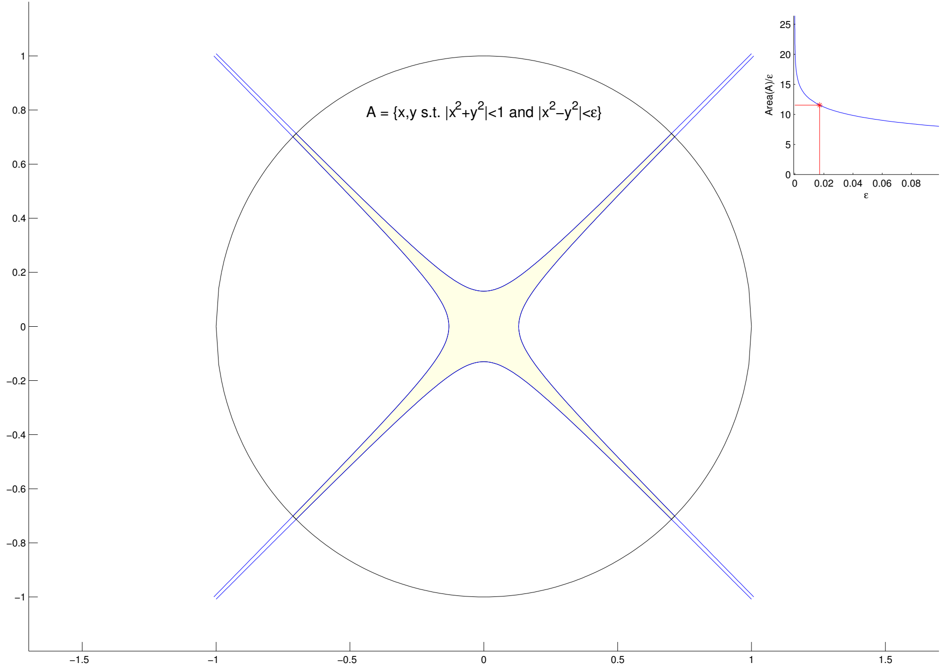
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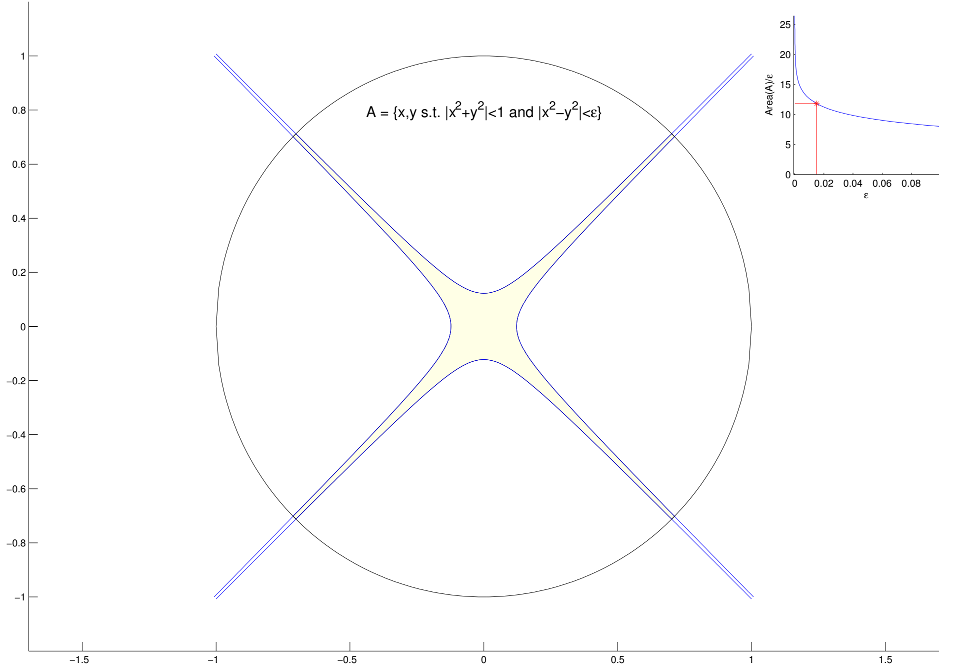
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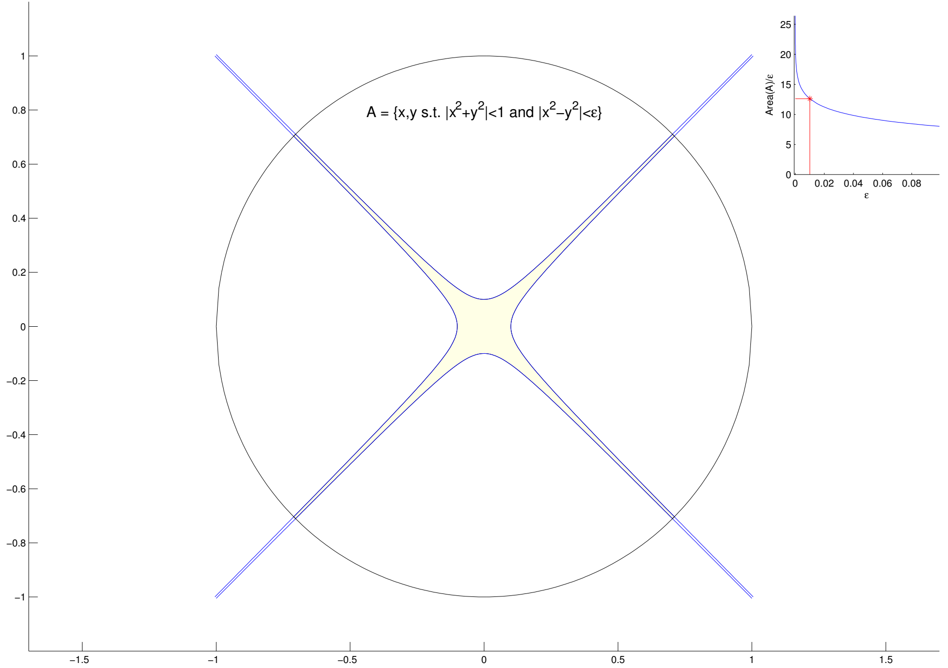
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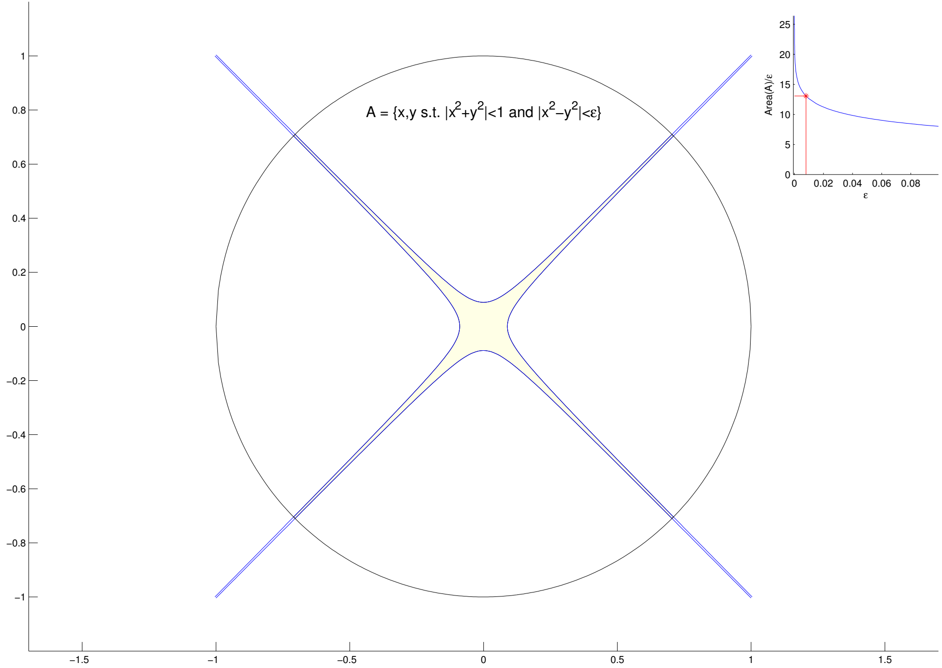

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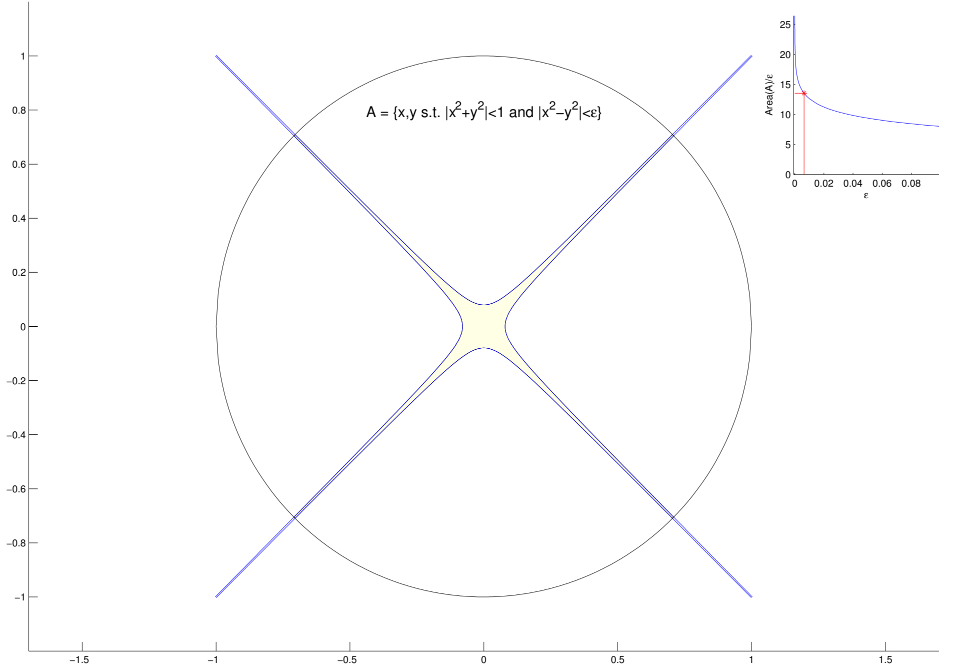
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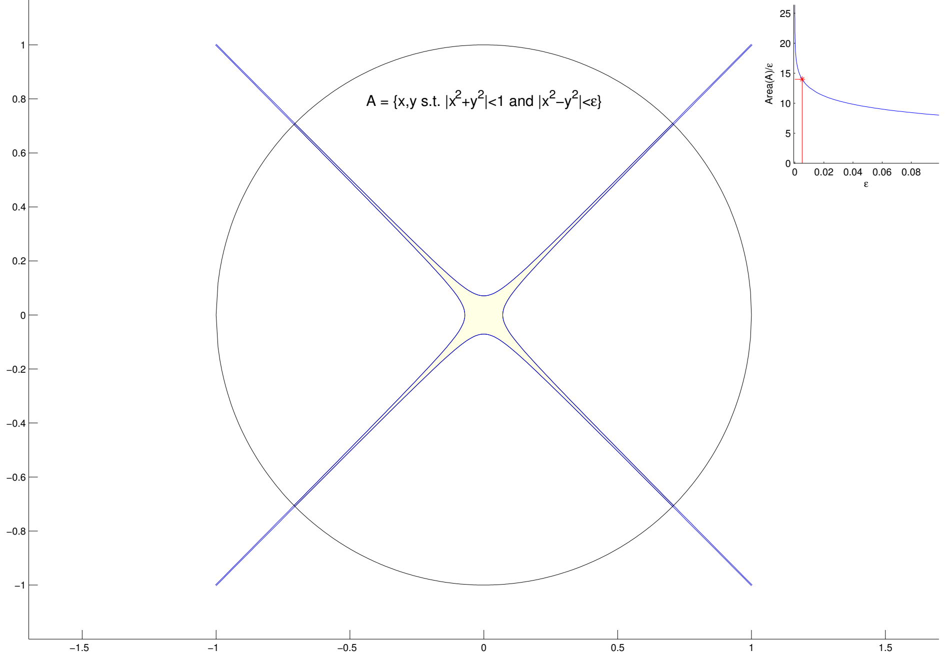
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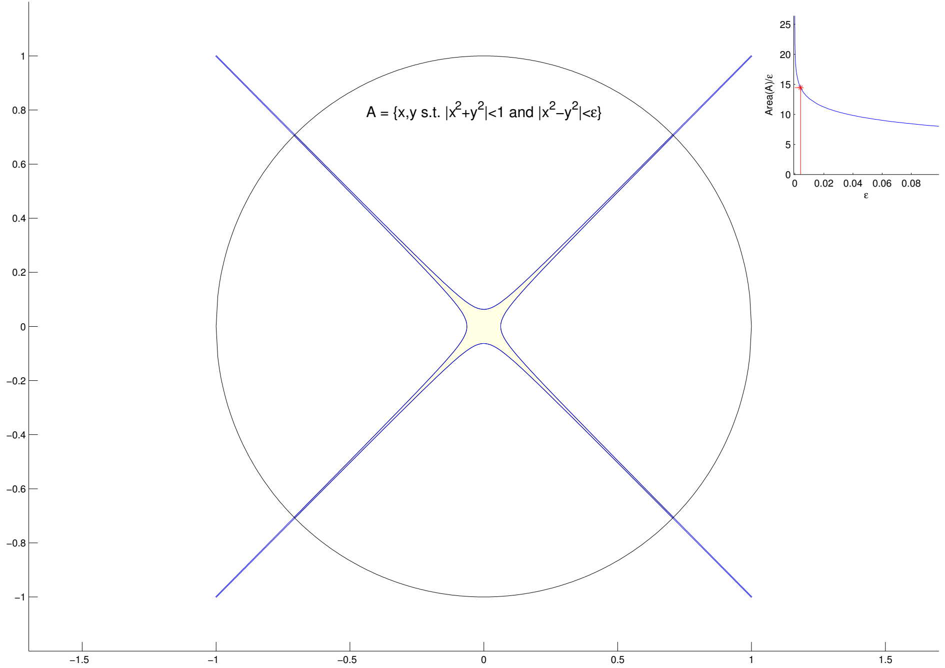
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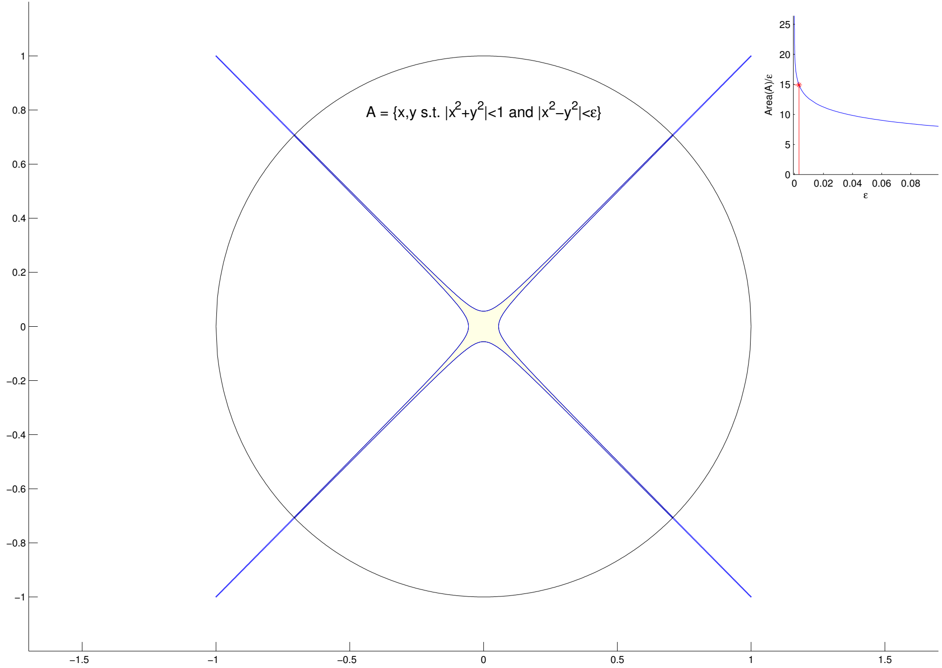
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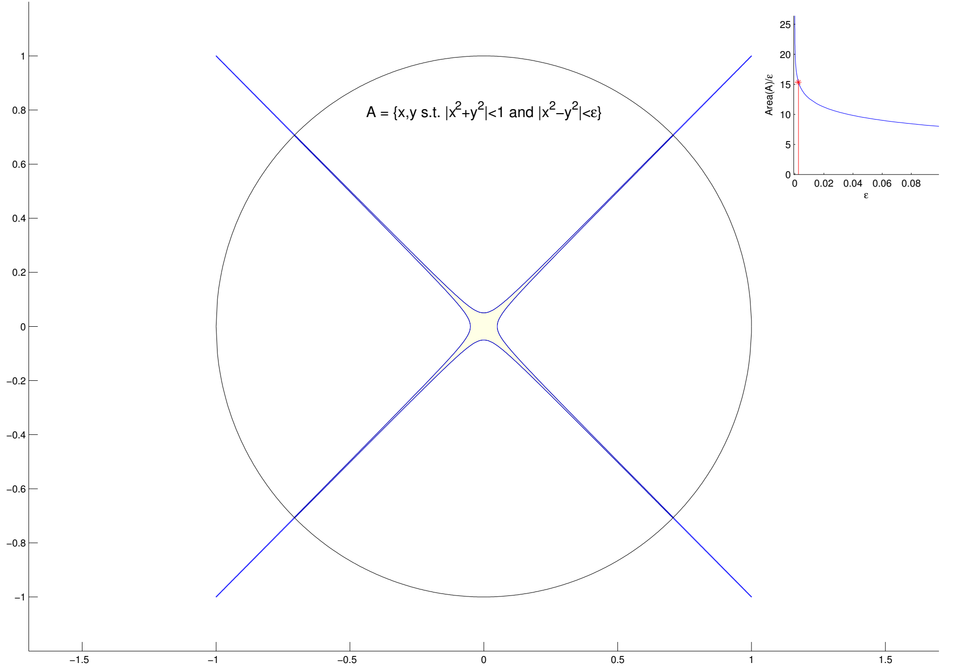

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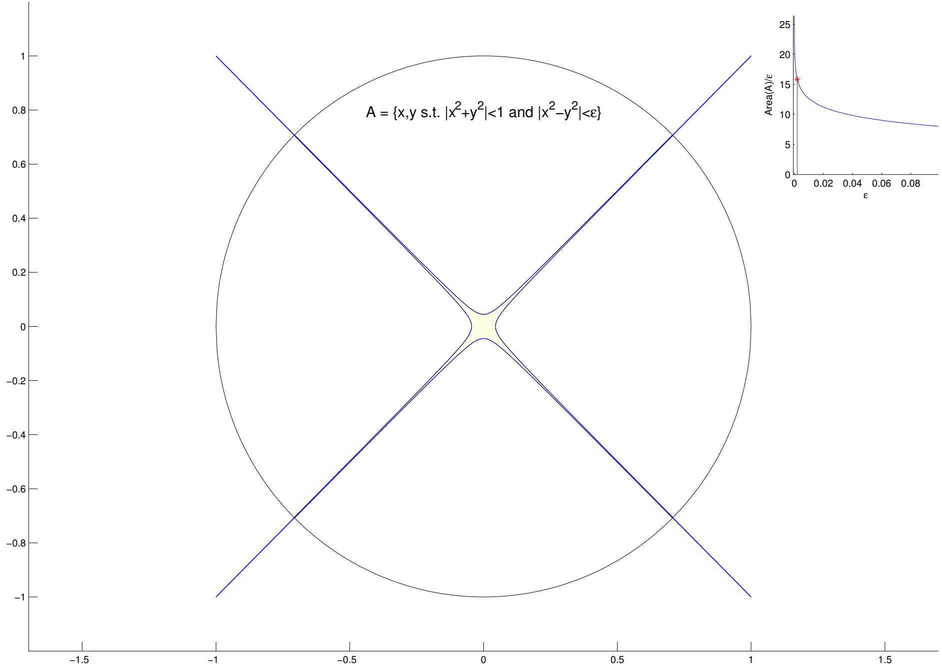
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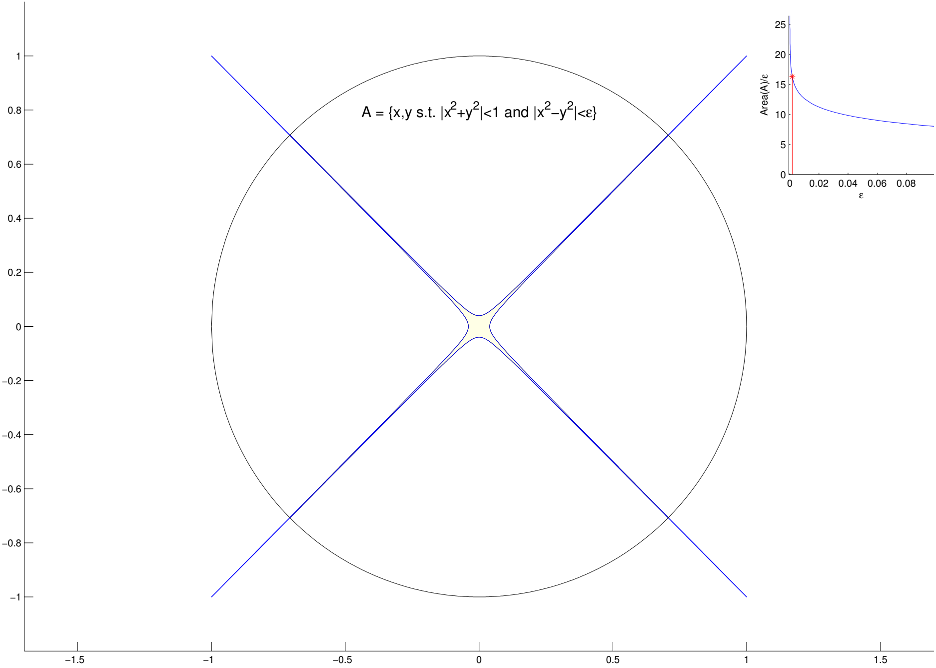
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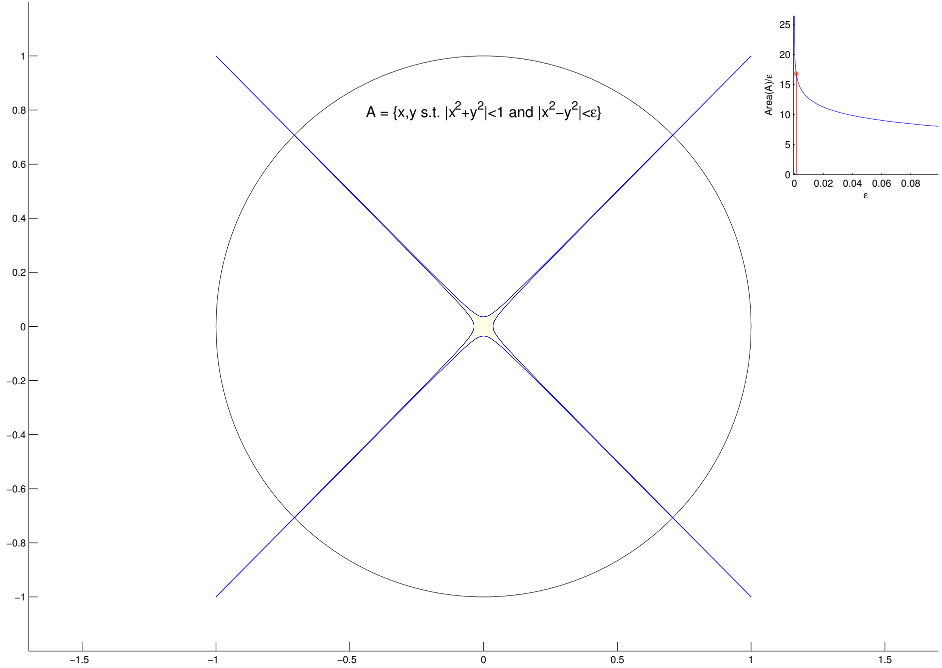
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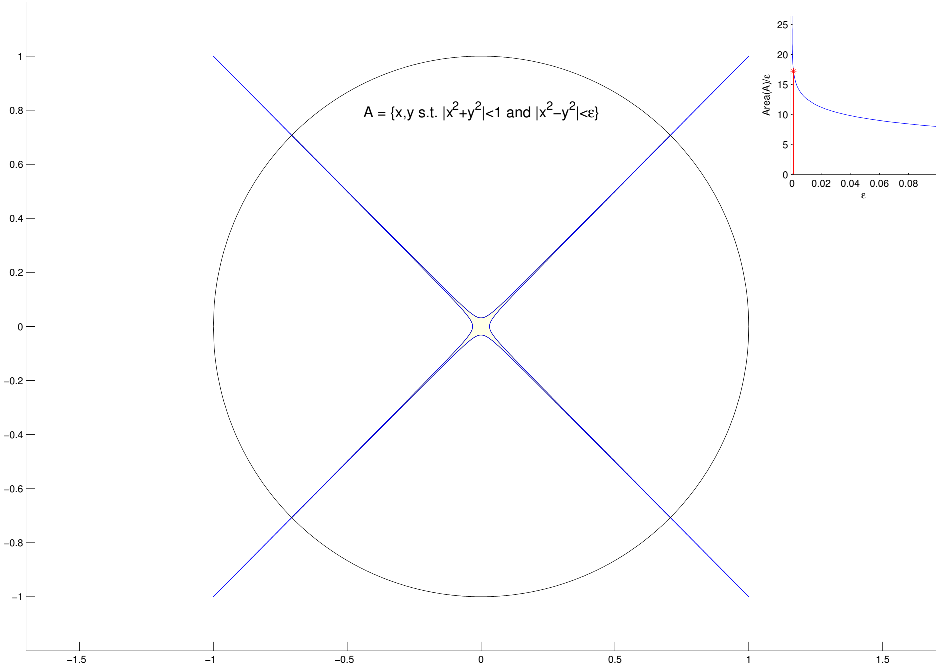

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Area(A)/ε
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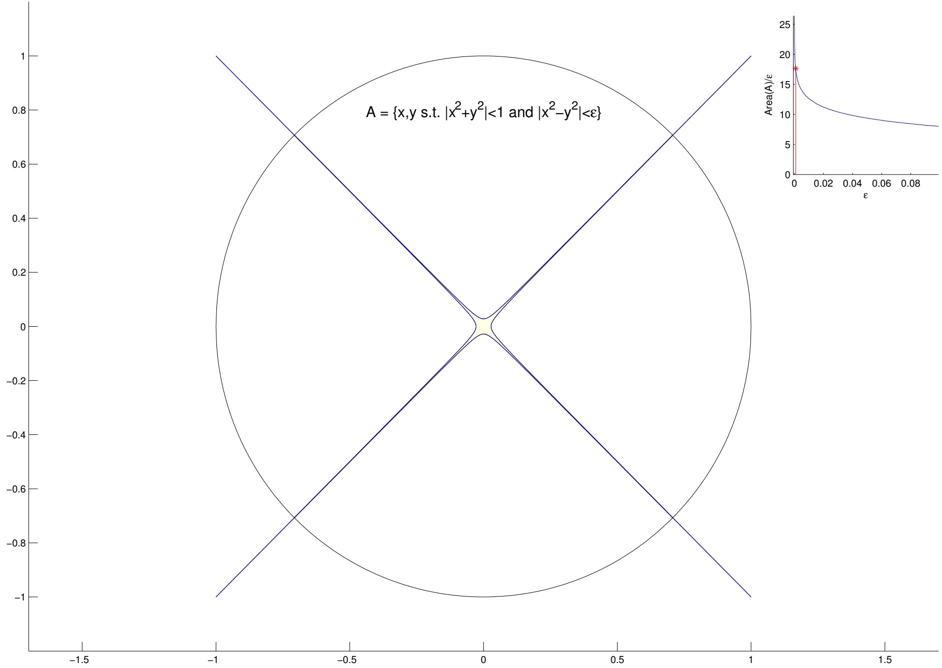

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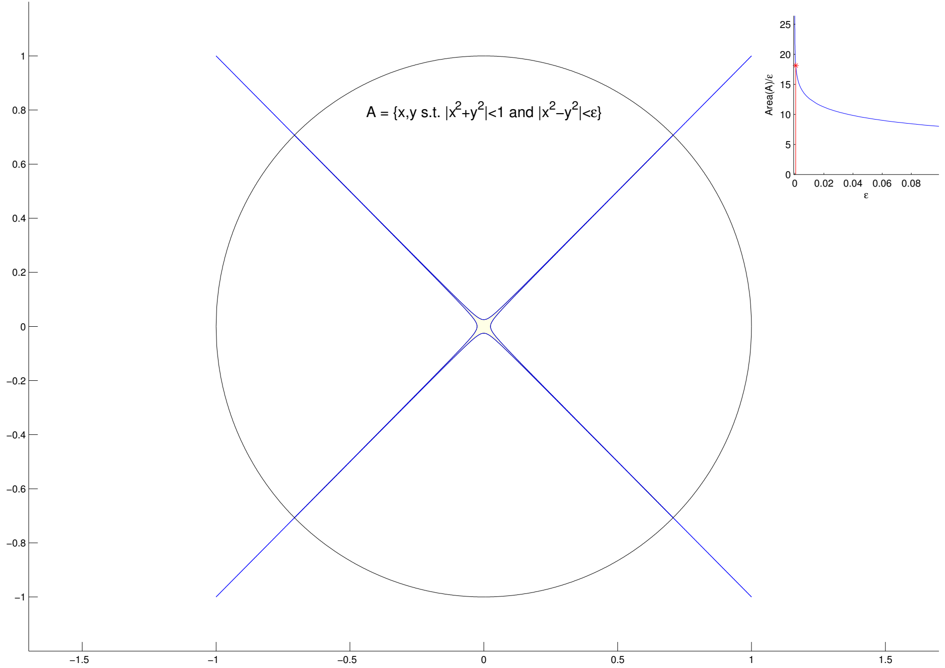

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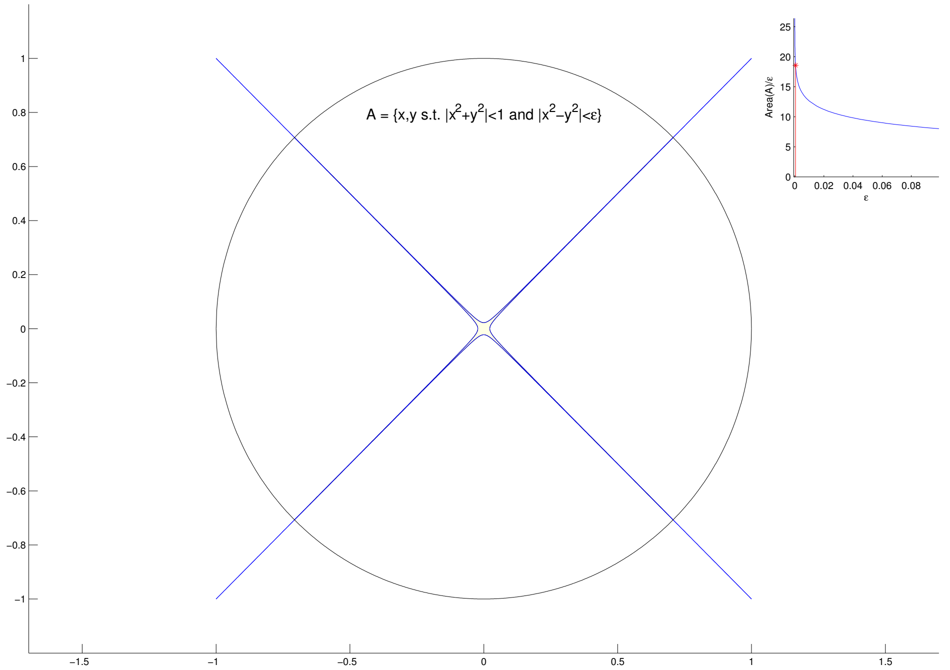
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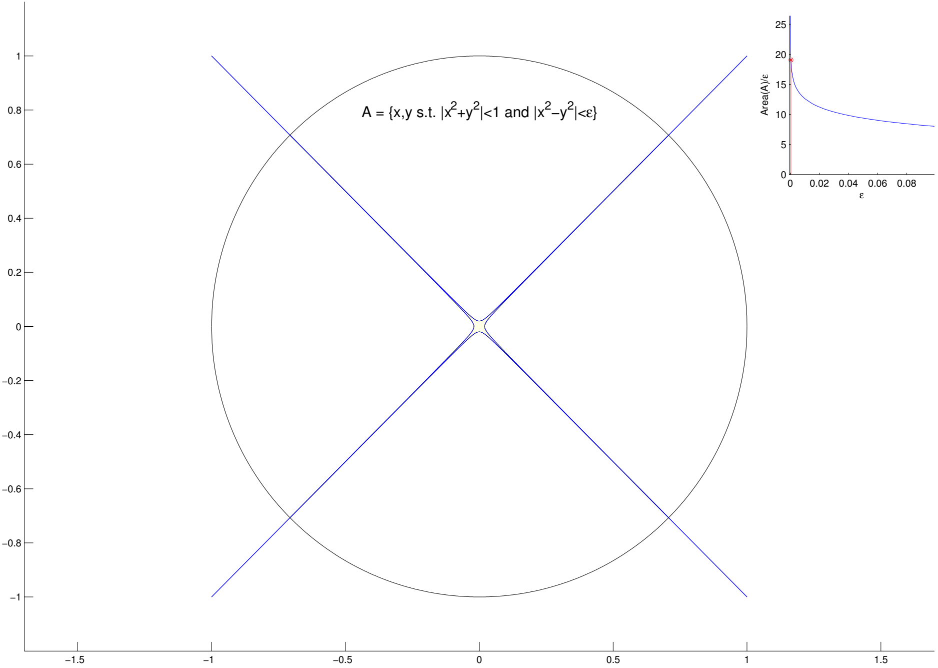
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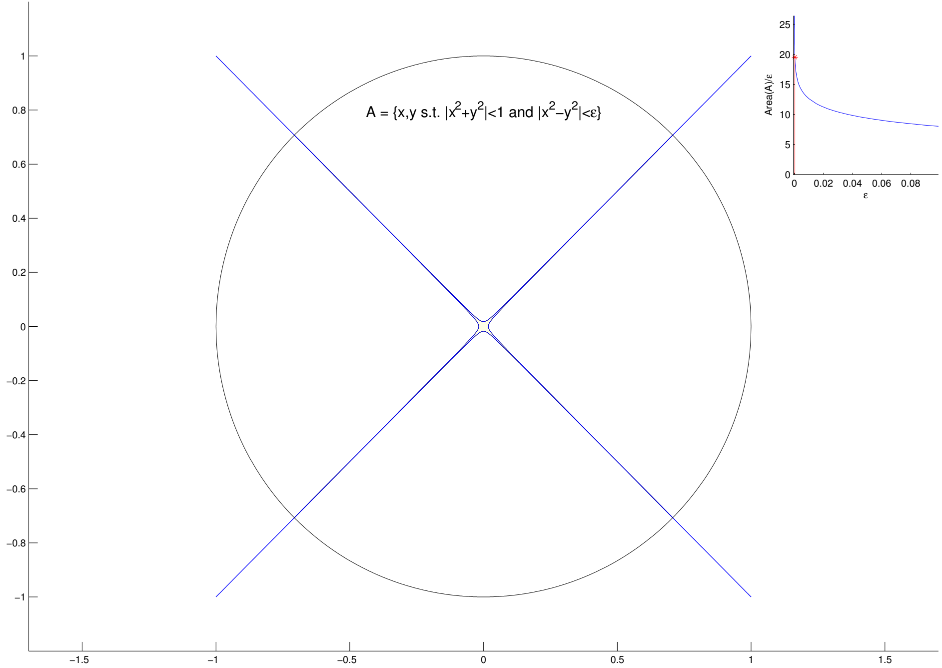

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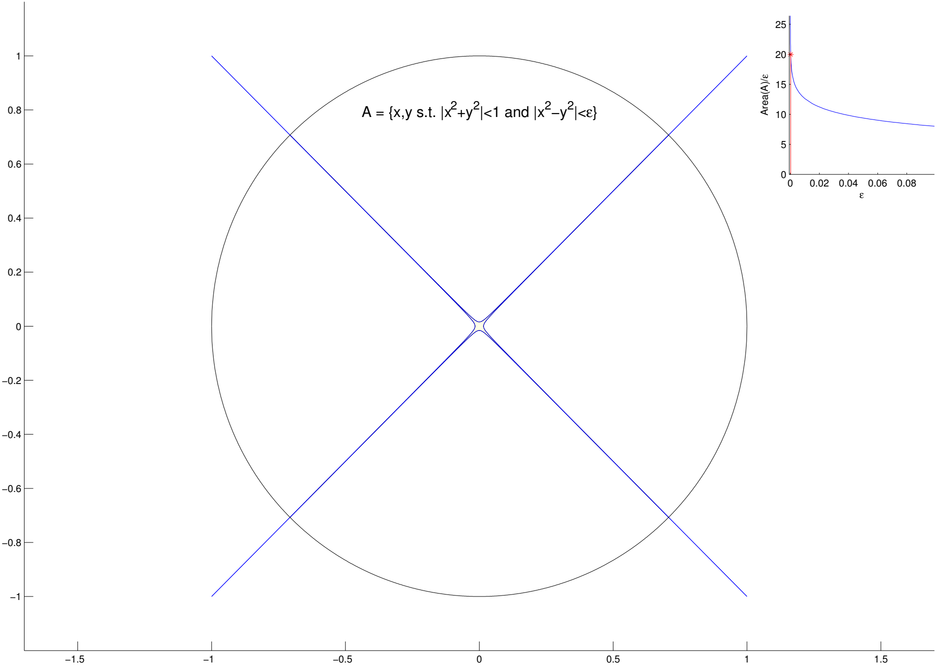

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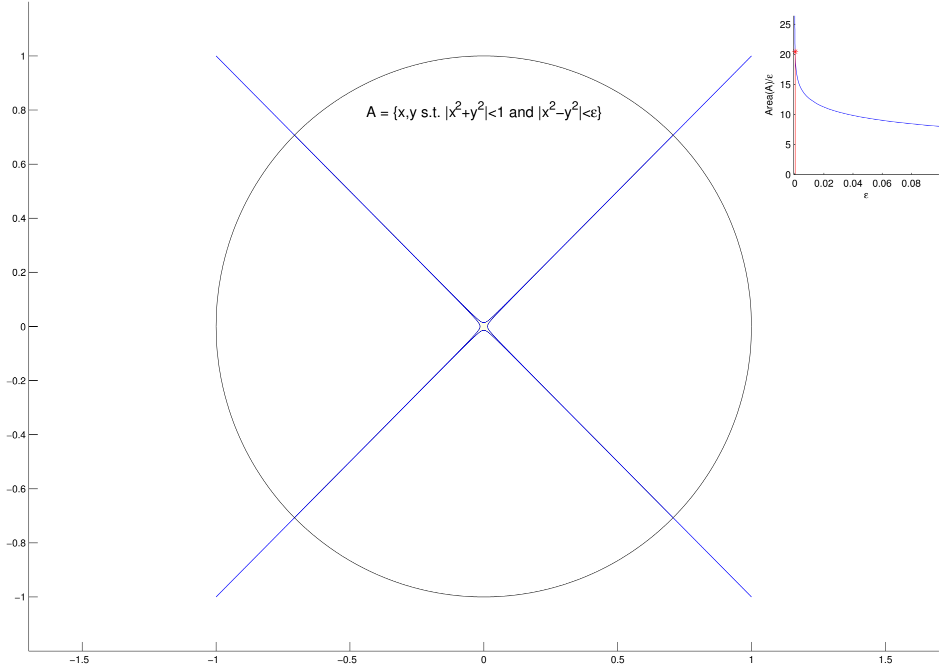
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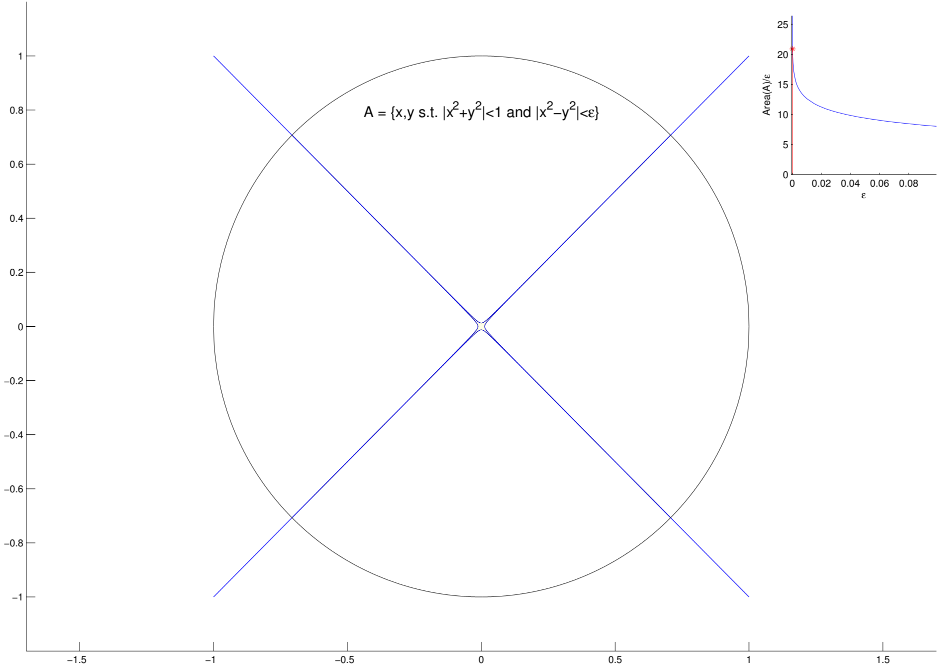
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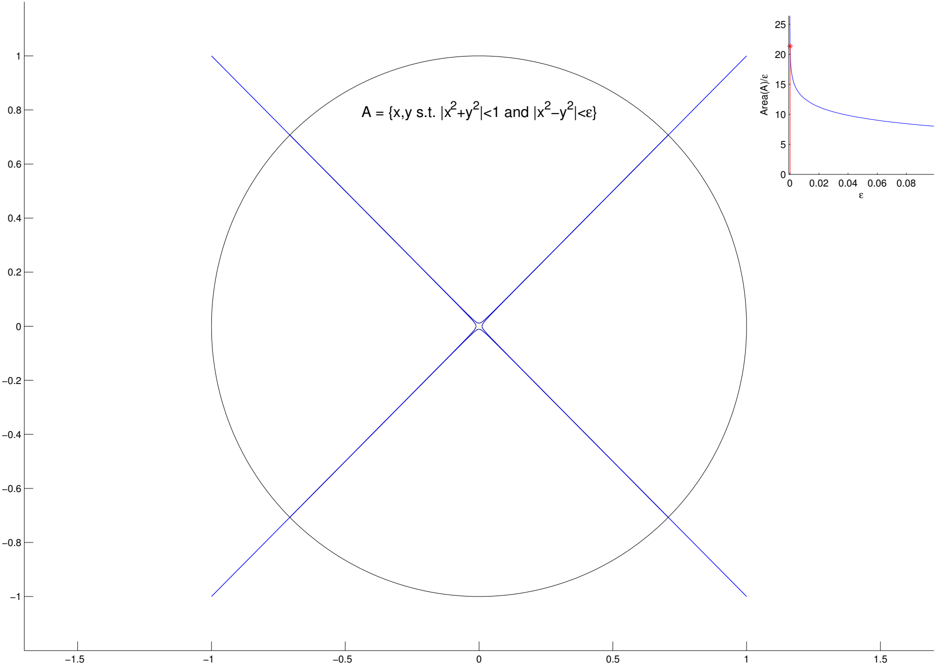

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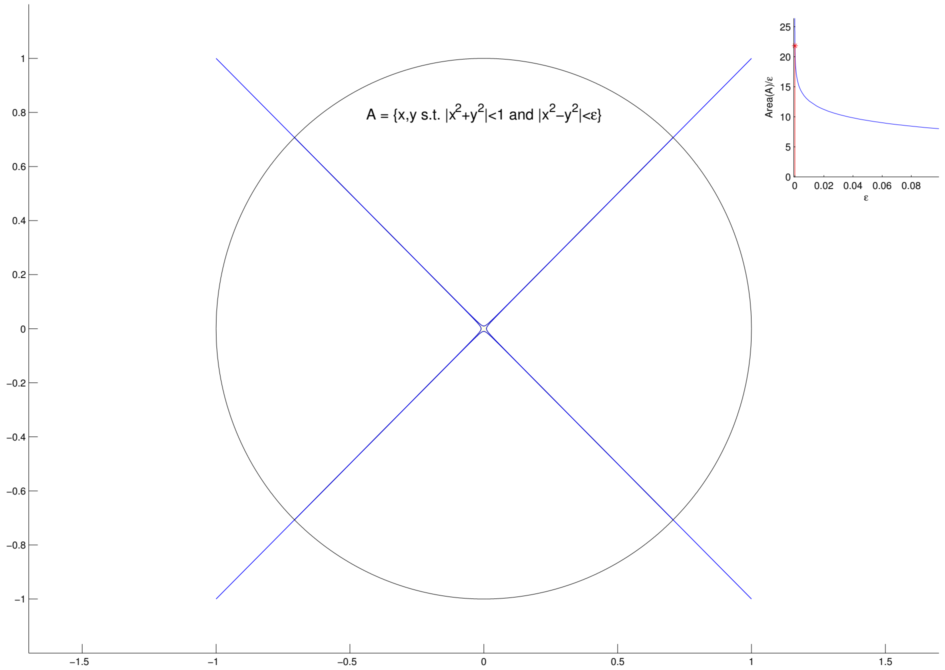
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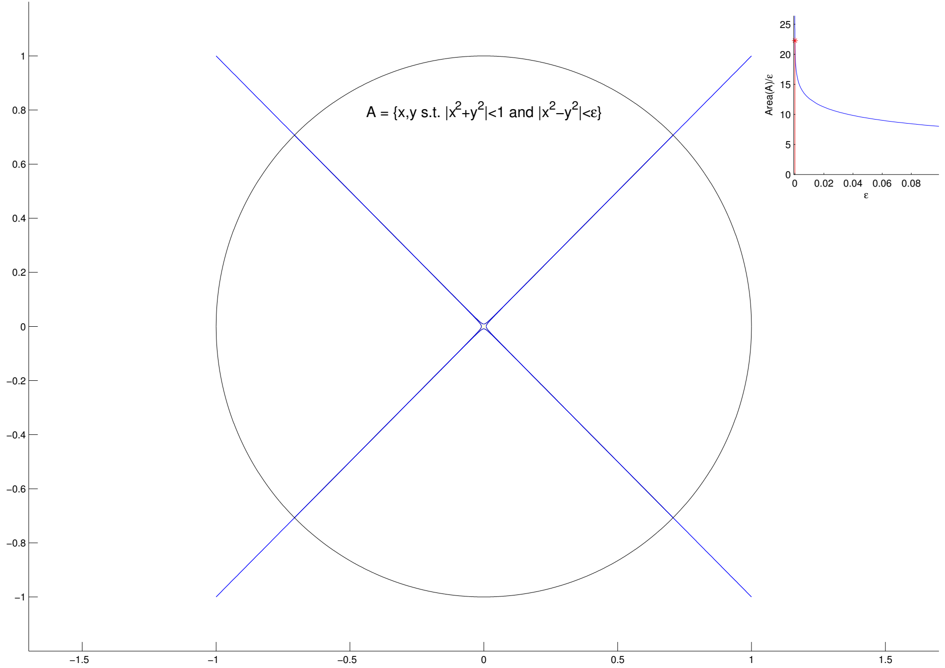
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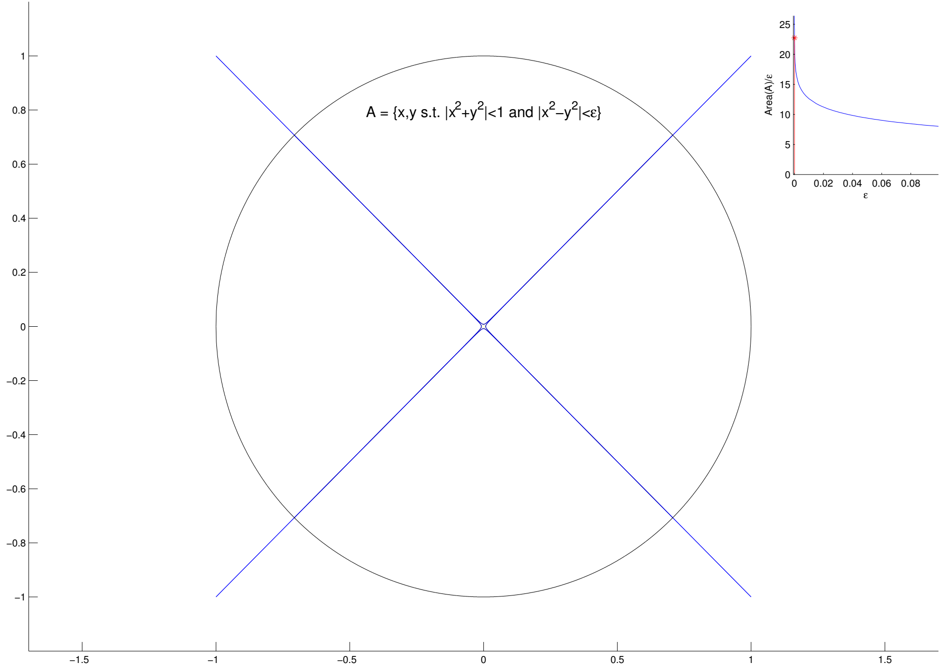
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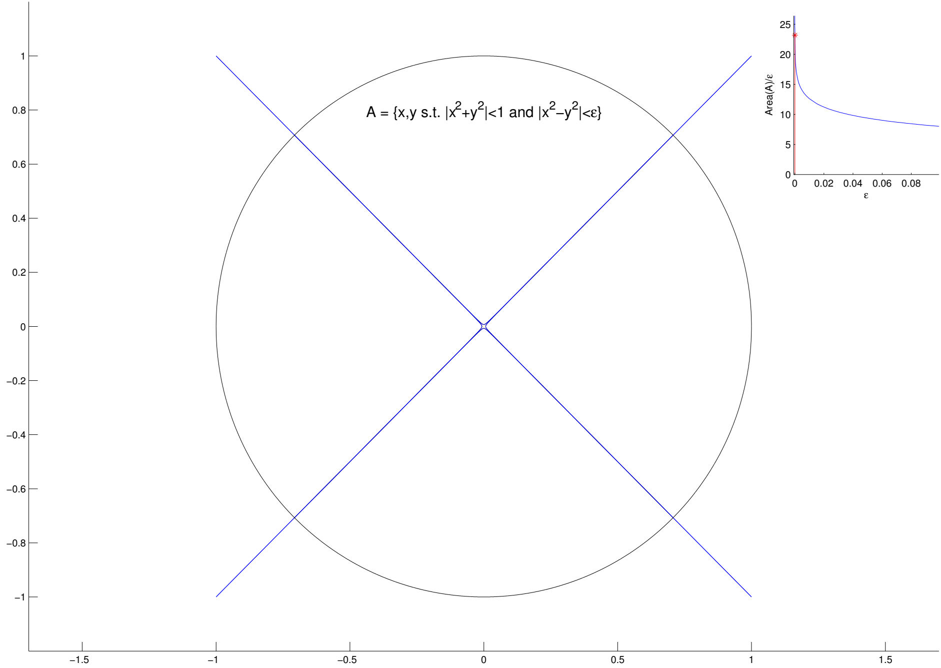

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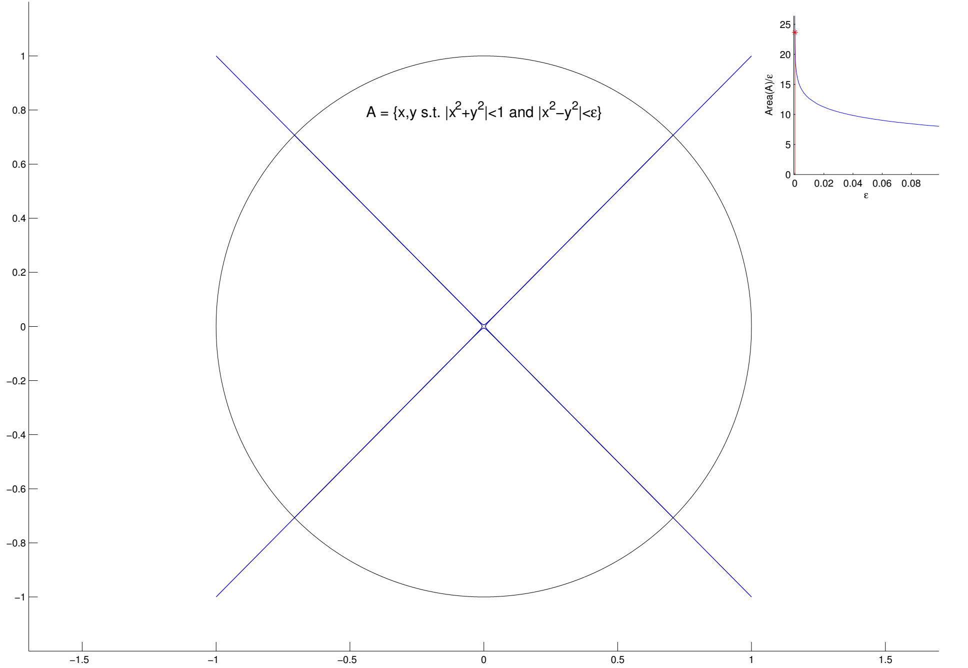

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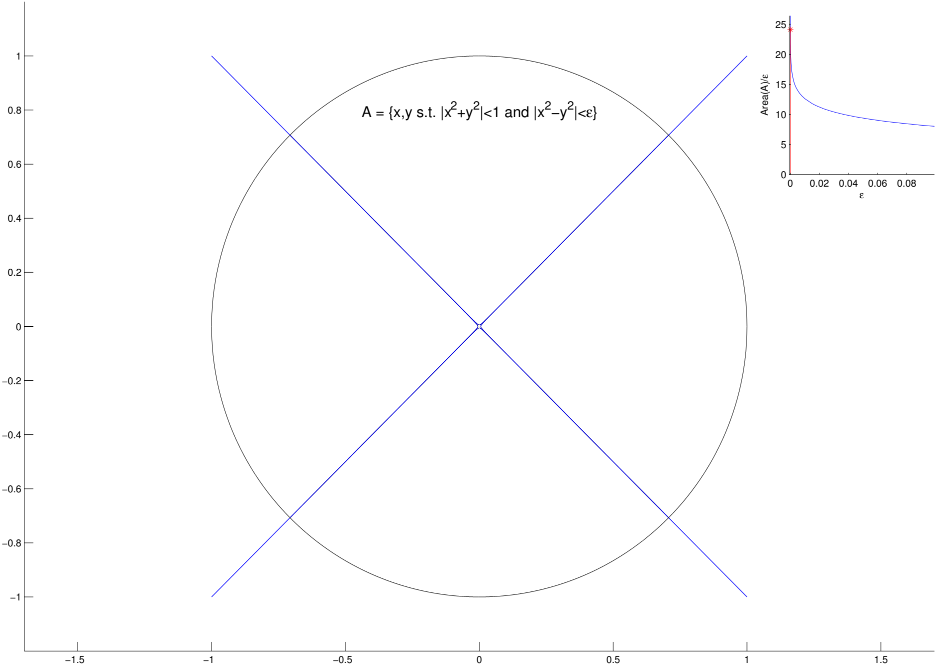

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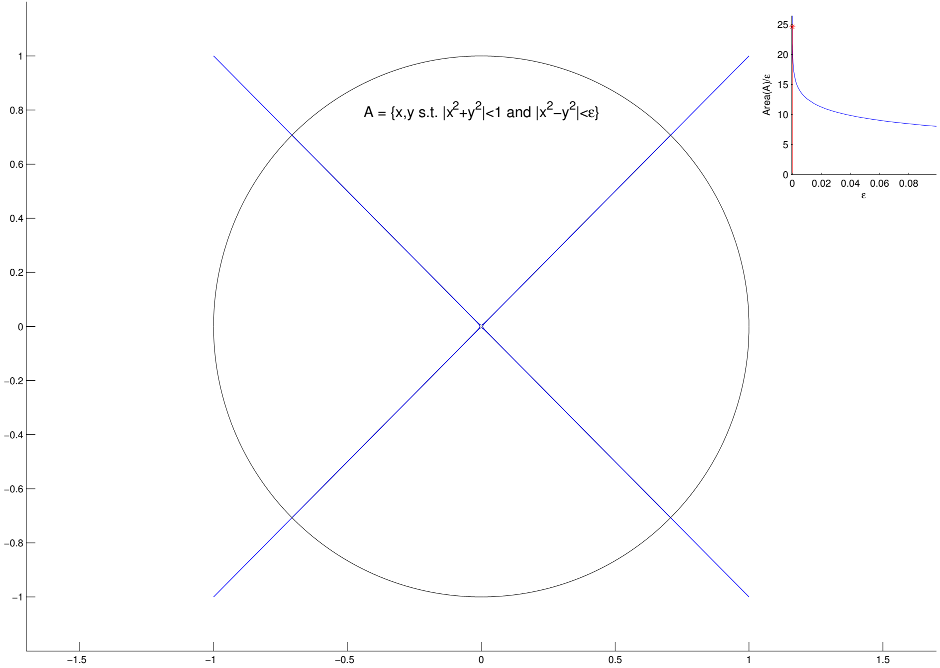

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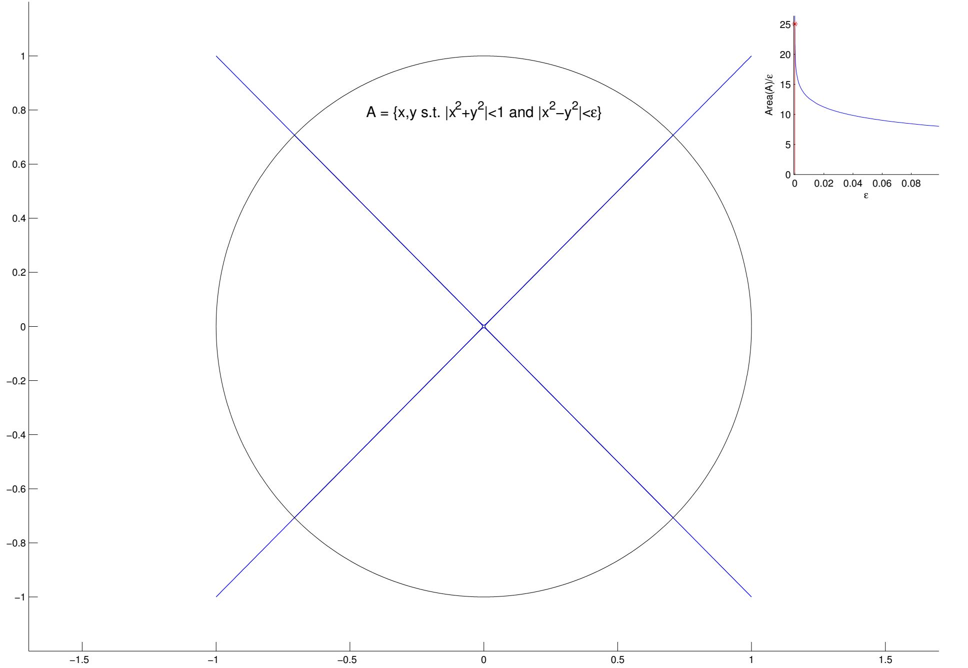
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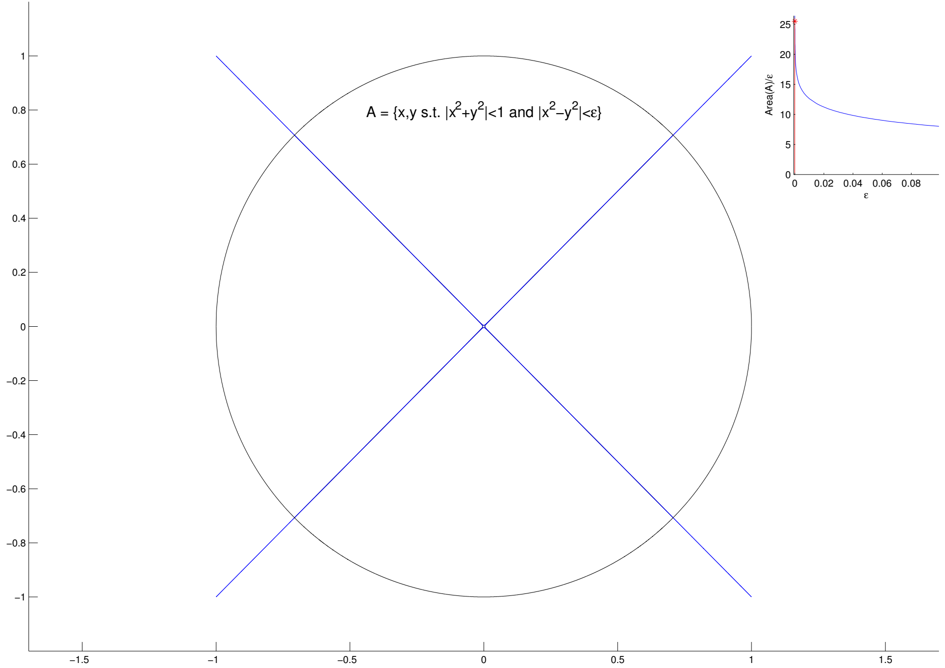
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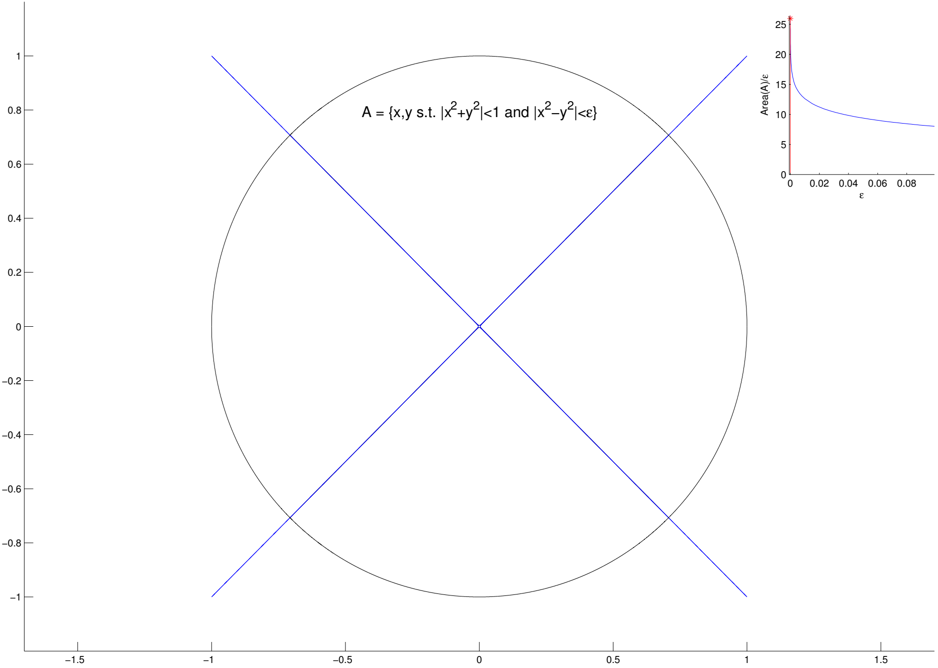
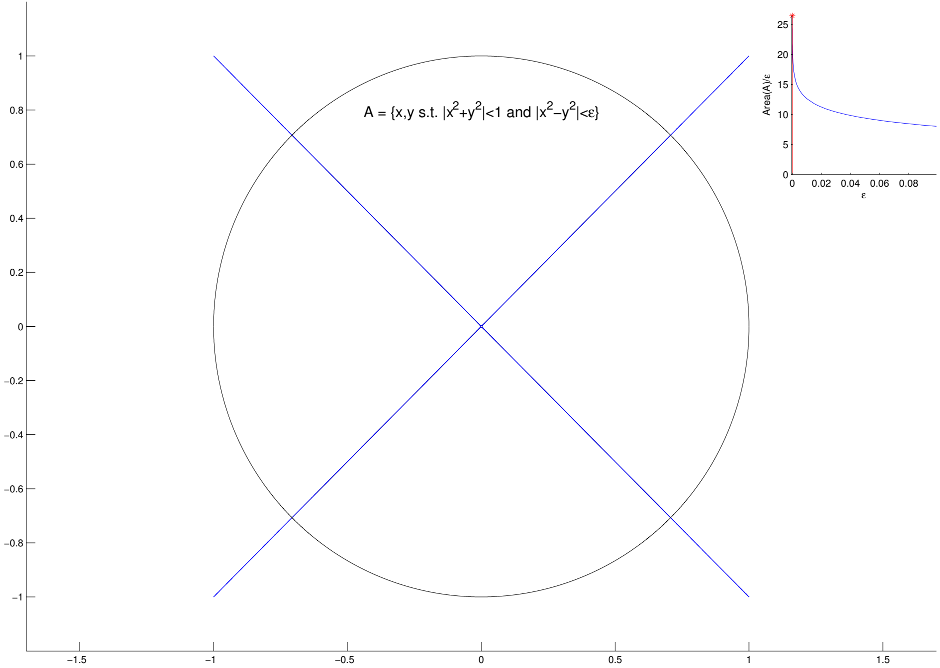
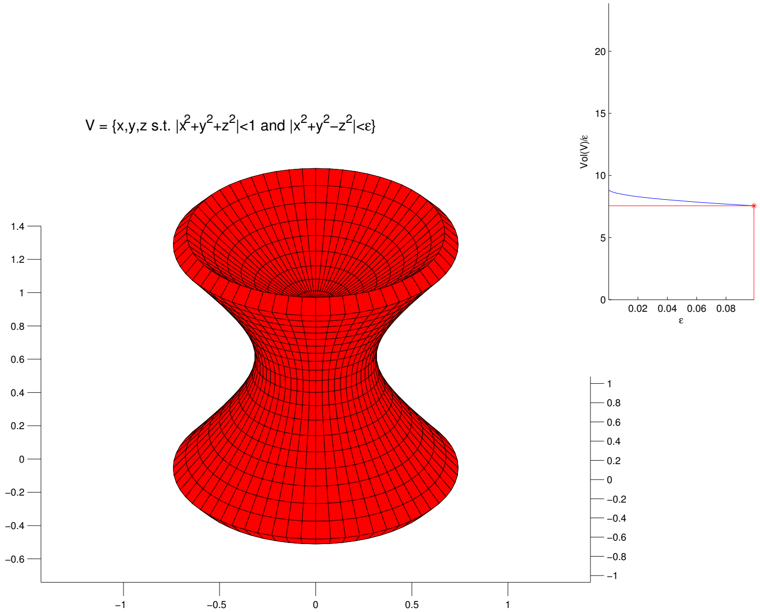
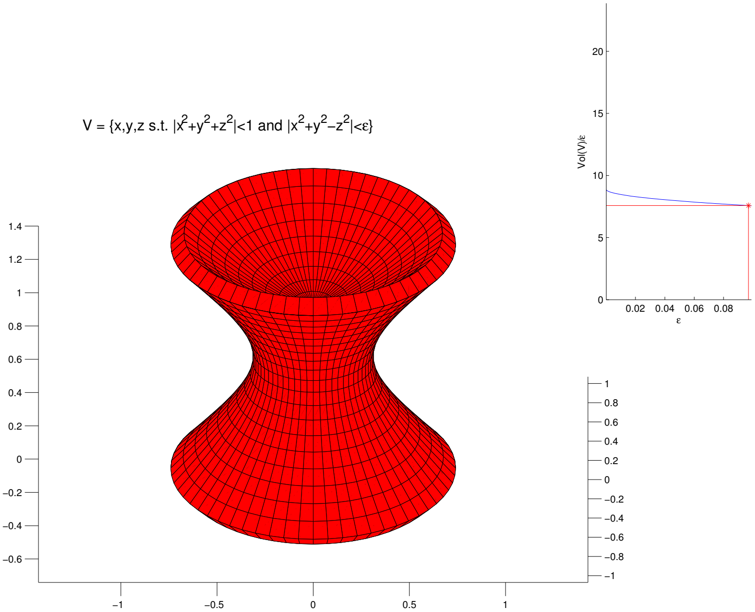


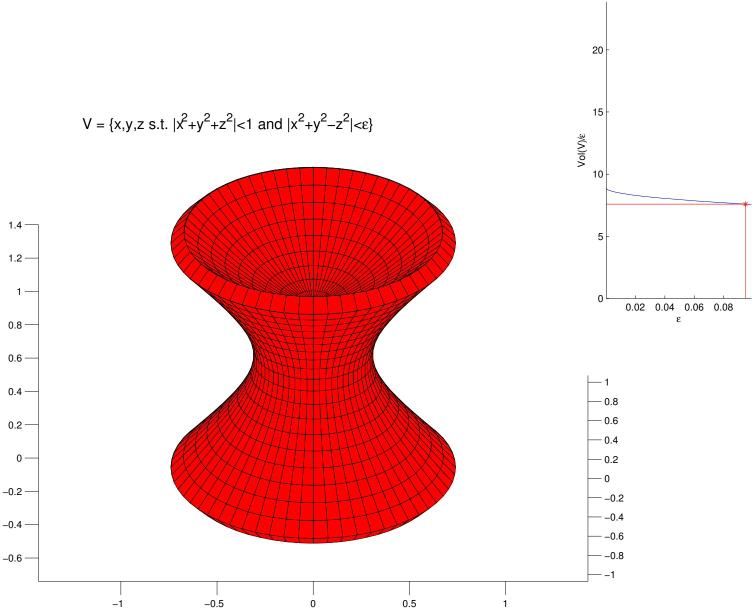
A = {x,y s.t. |x²+y²|<1 and |x²−y²|<ε}
Area(A)/ε
ε


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0
0.02
0.04
0.06
0.08
25
20
15
10
5
−1.5
−1
−0.5
0.5
1
1.5
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8

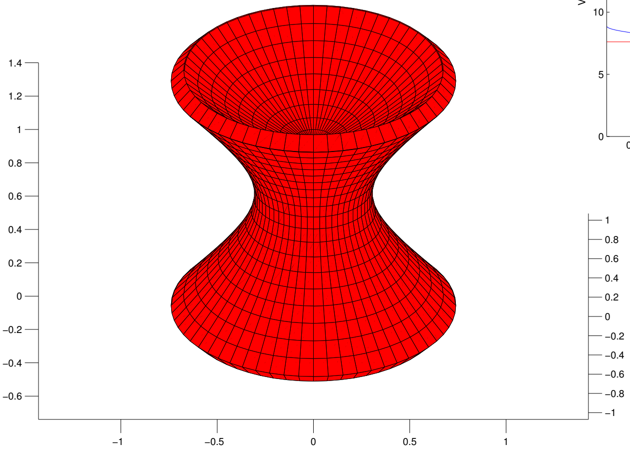

V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

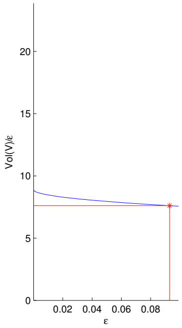

V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0
5
10
15
20
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−0.5
0.5

V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}

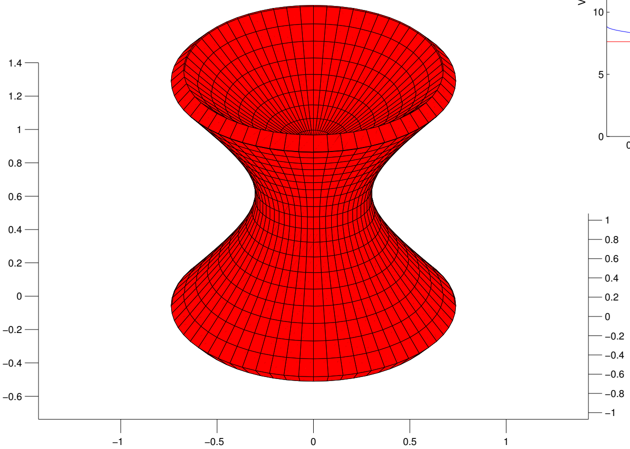


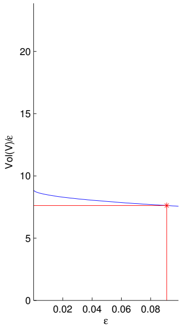
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

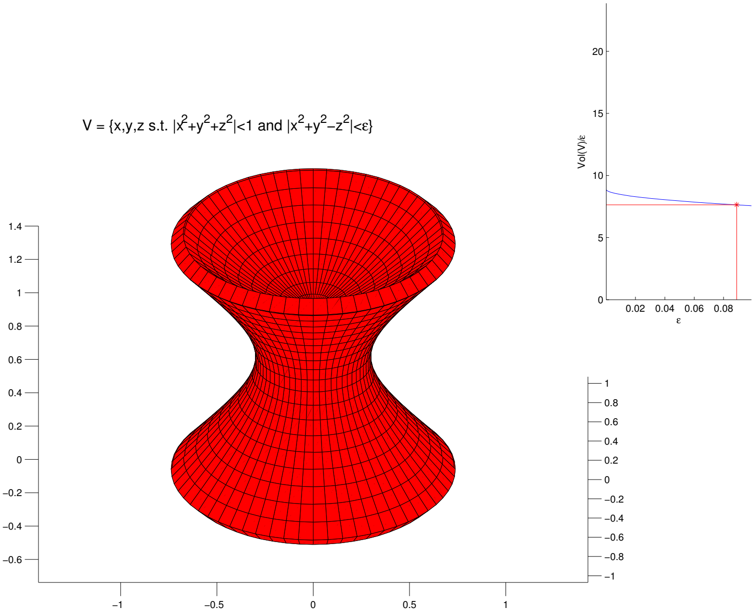




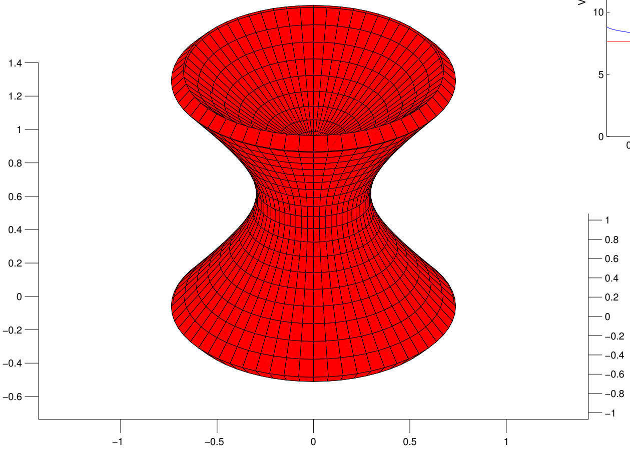
V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}

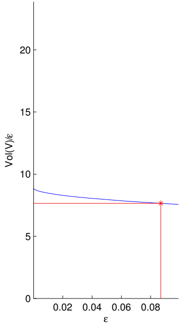




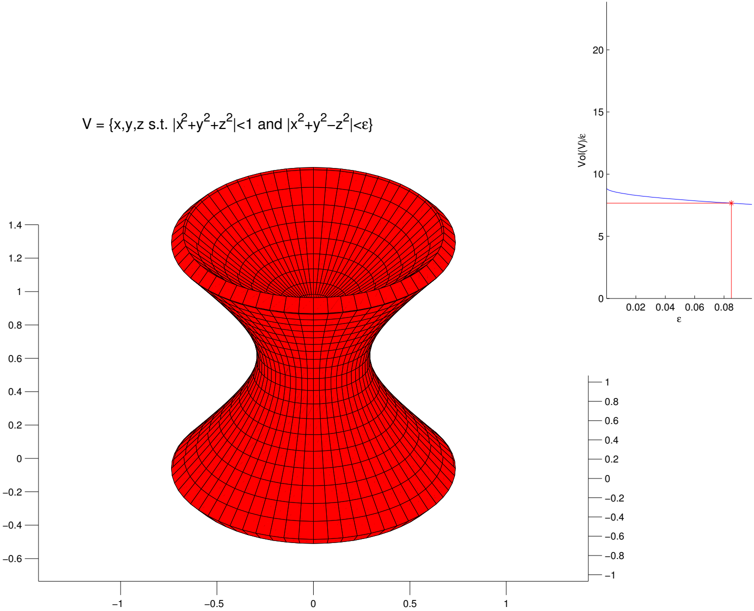

V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε

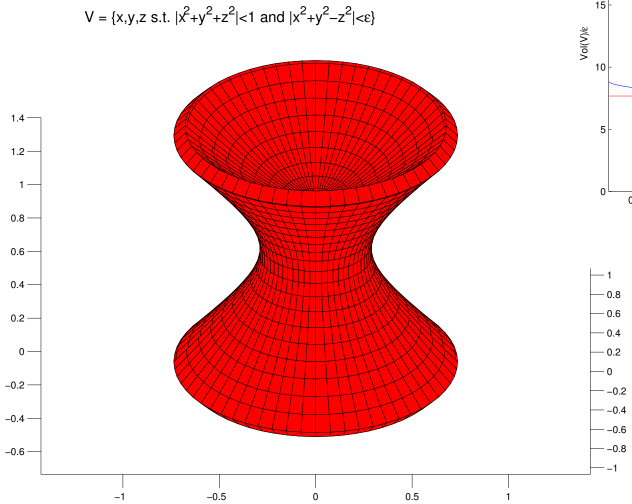
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

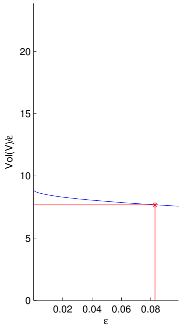


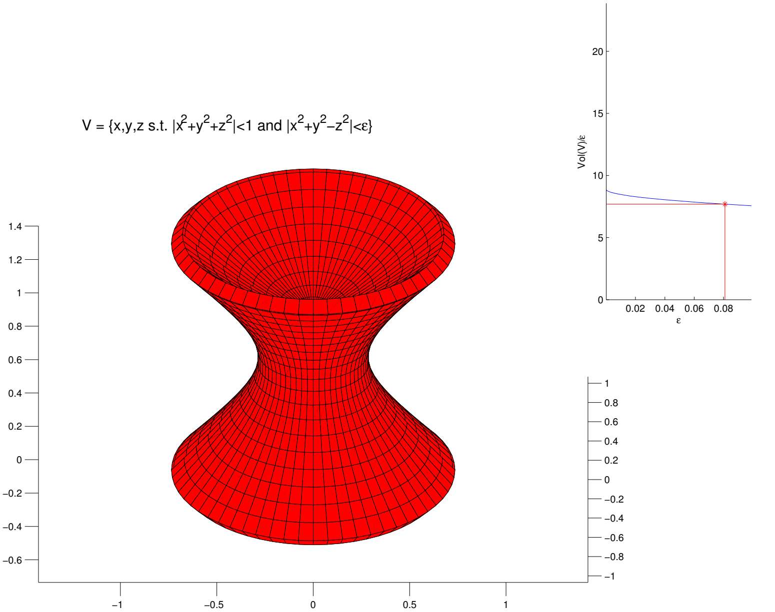


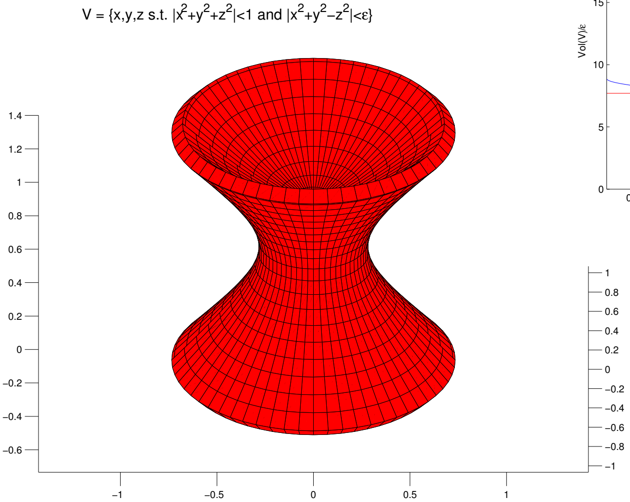
V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}

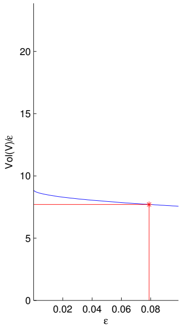



V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

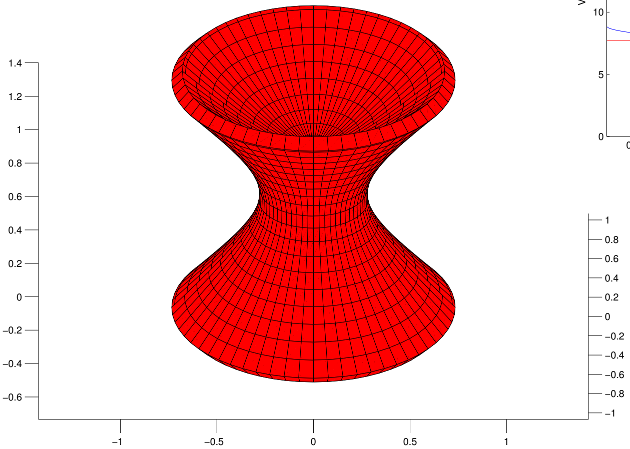

Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε

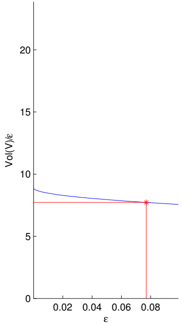

V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1


V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

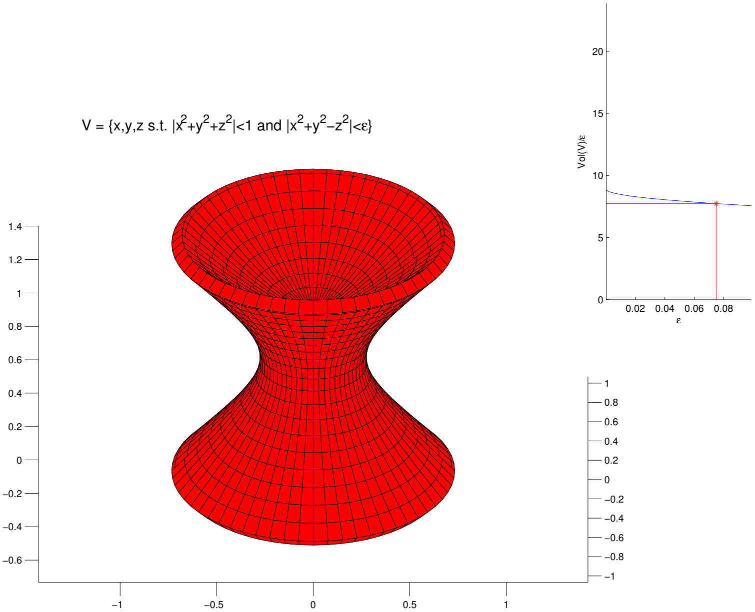

Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε

$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

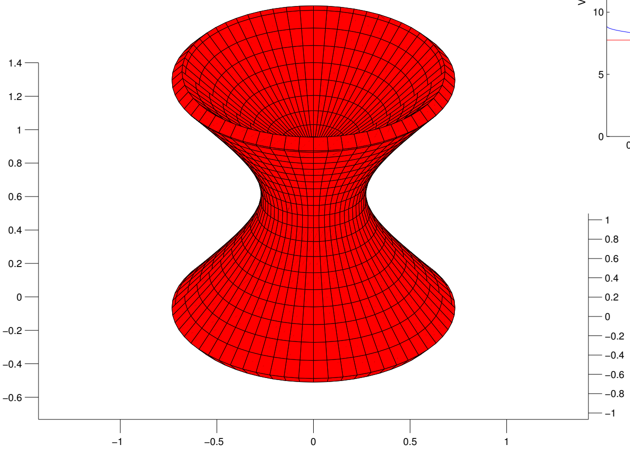


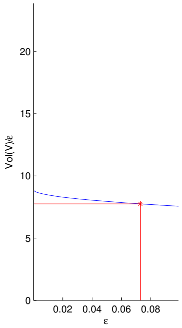


V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}

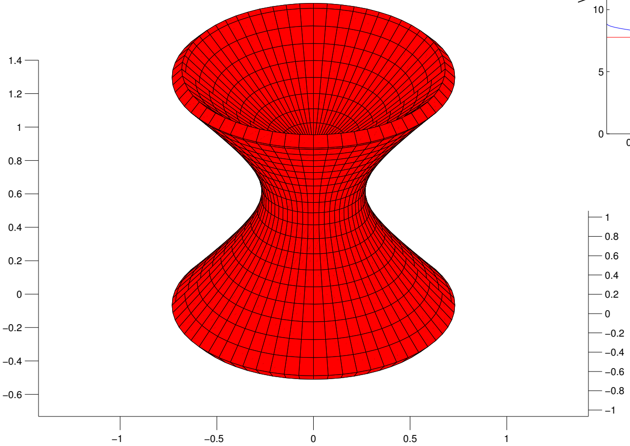


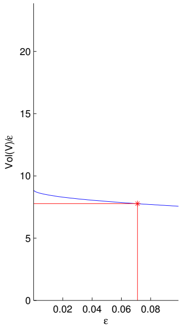
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



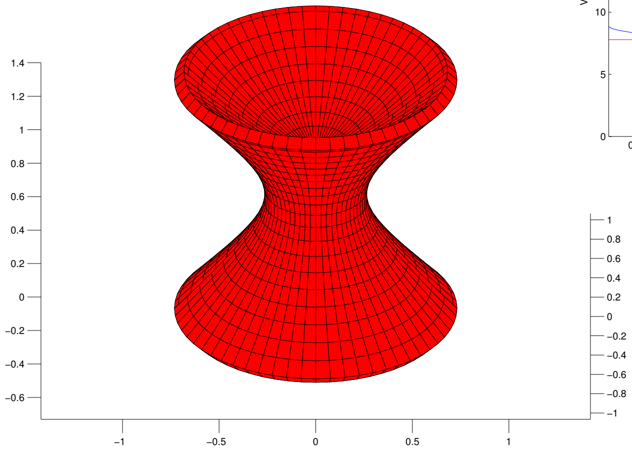


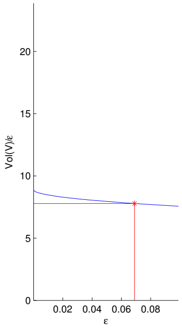
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



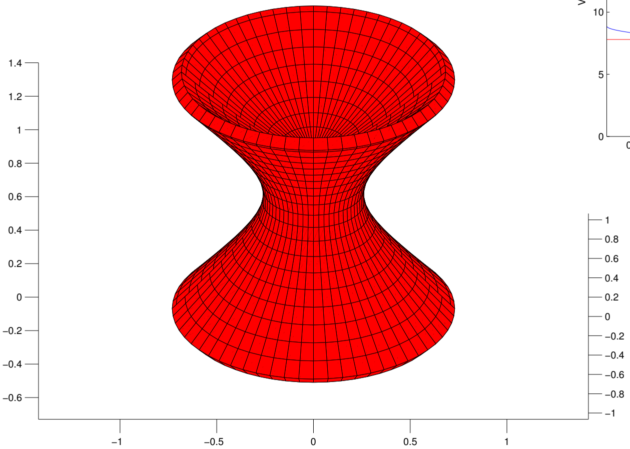


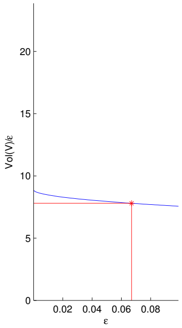
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

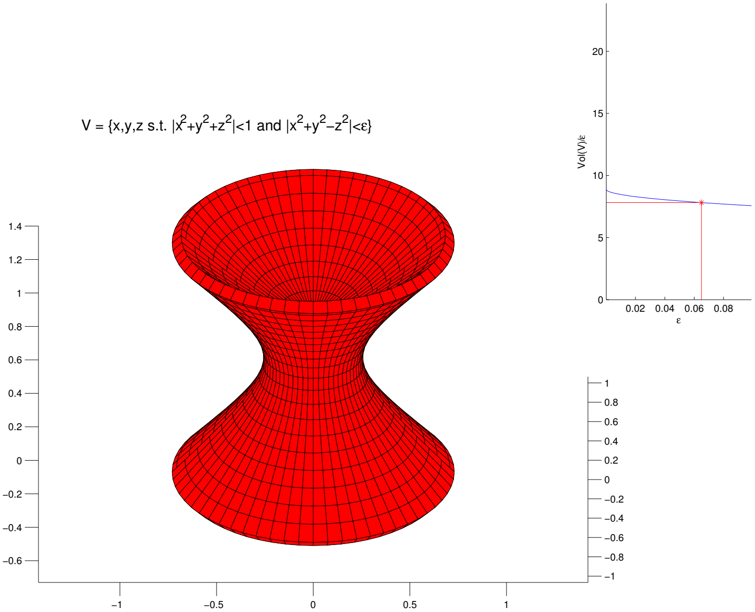




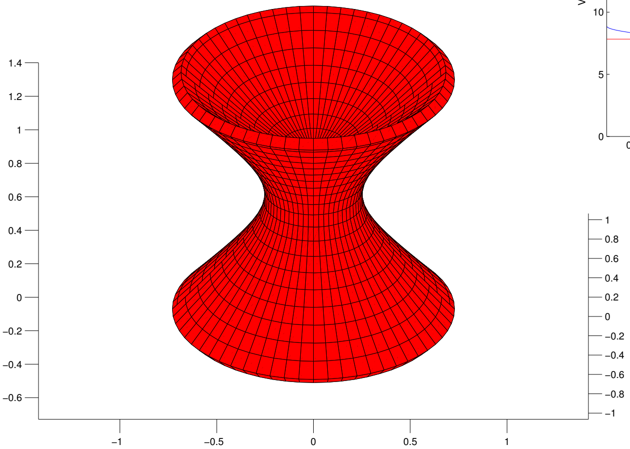
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

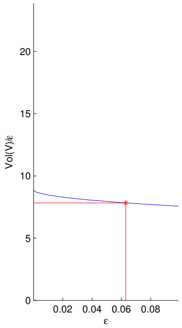


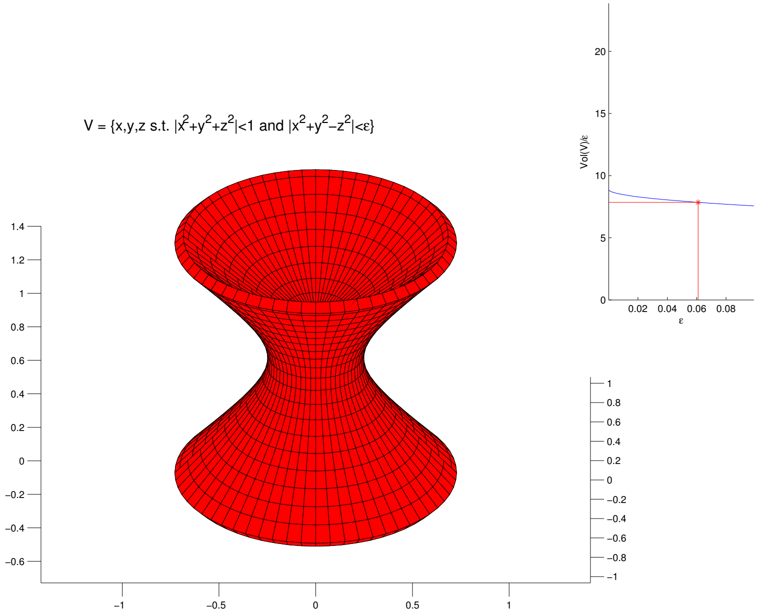



V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε

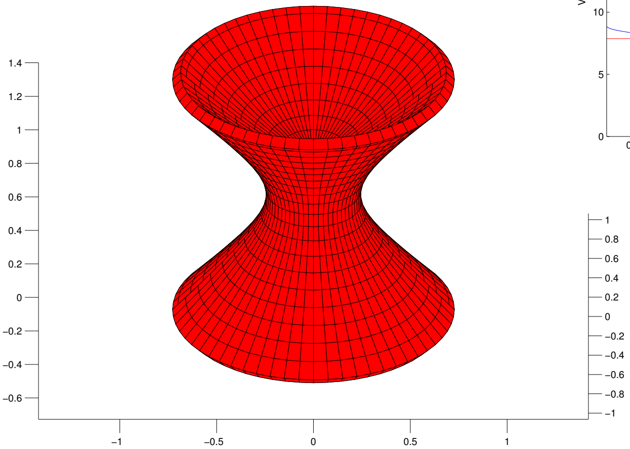
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

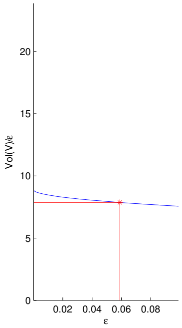


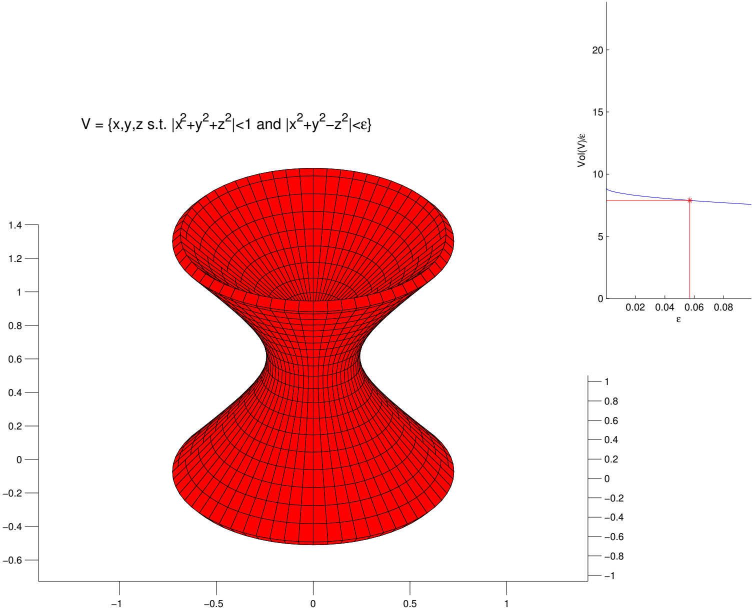


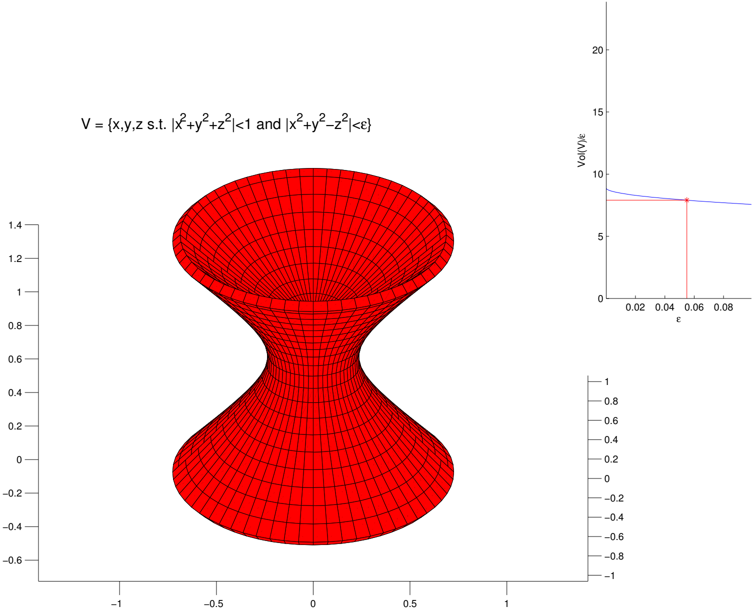

V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0 5 10 15 20
0.02 0.04 0.06 0.08
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

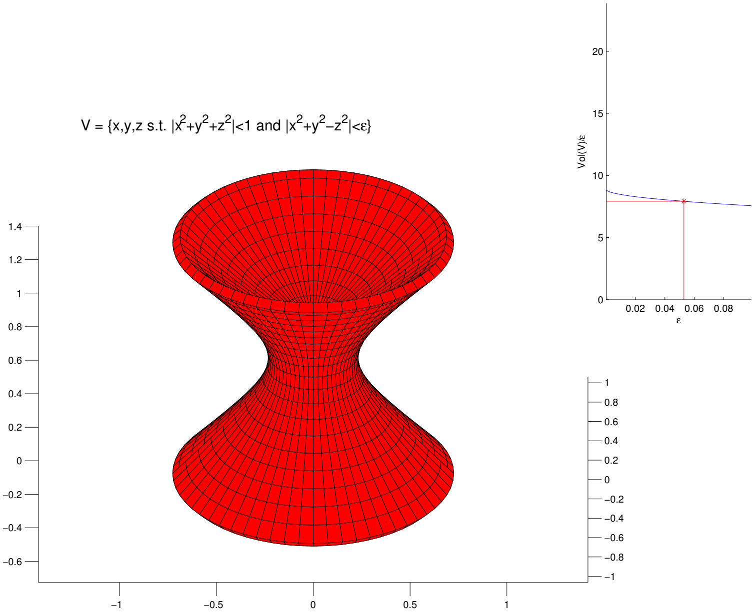
V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}



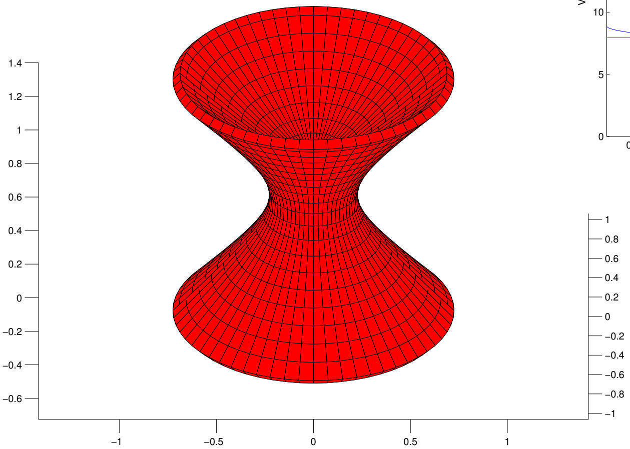


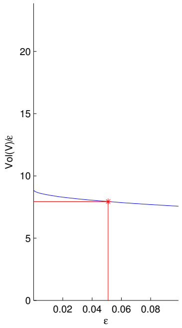

V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}
Vol(V)/ε
ε
20
15
10
5
0
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1


V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0.02
0.04
0.06
0.08
0
5
10
15
20
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
0.8
0.6
0.4
0.2
−0.8

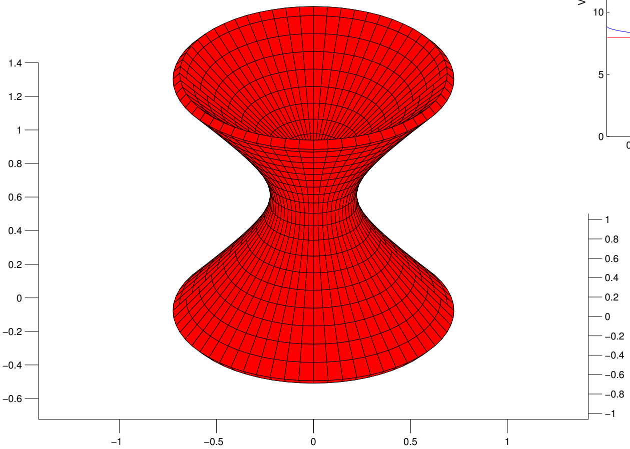

V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0.02
0.04
0.06
0.08
0
5
10
15
20
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0.5
−0.8

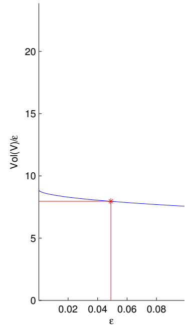
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

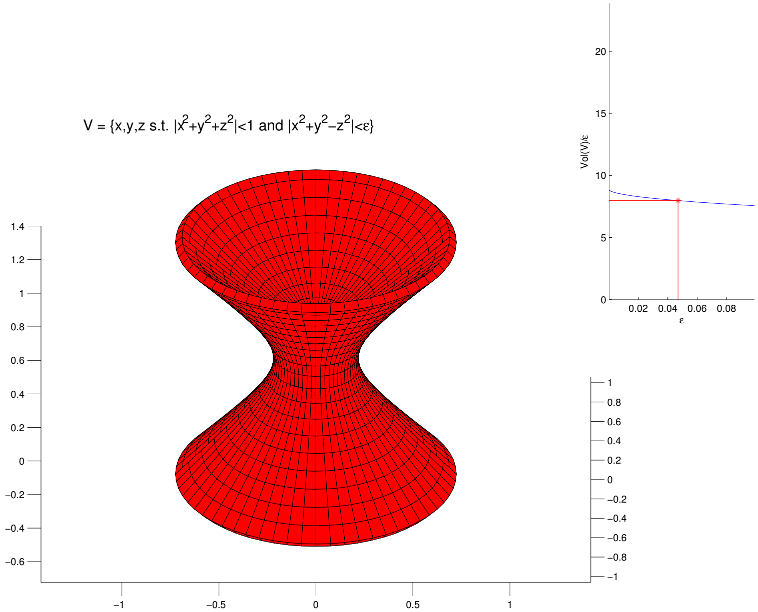




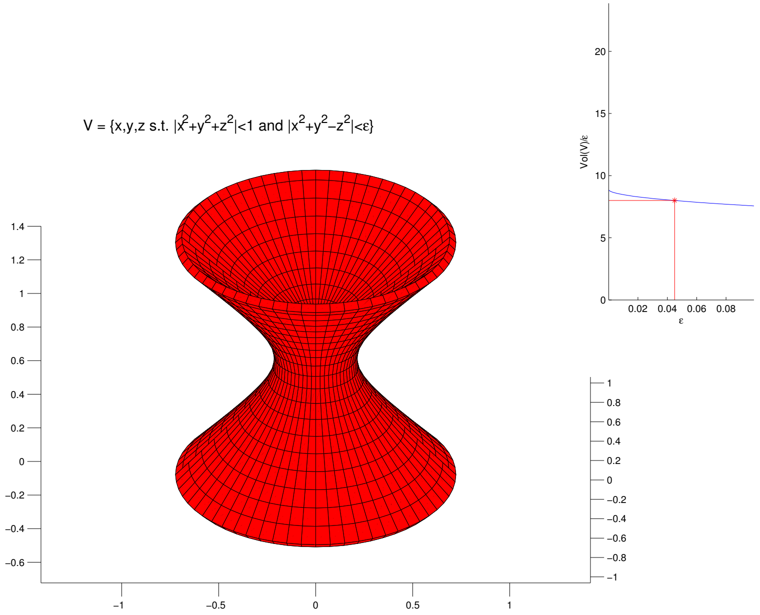
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

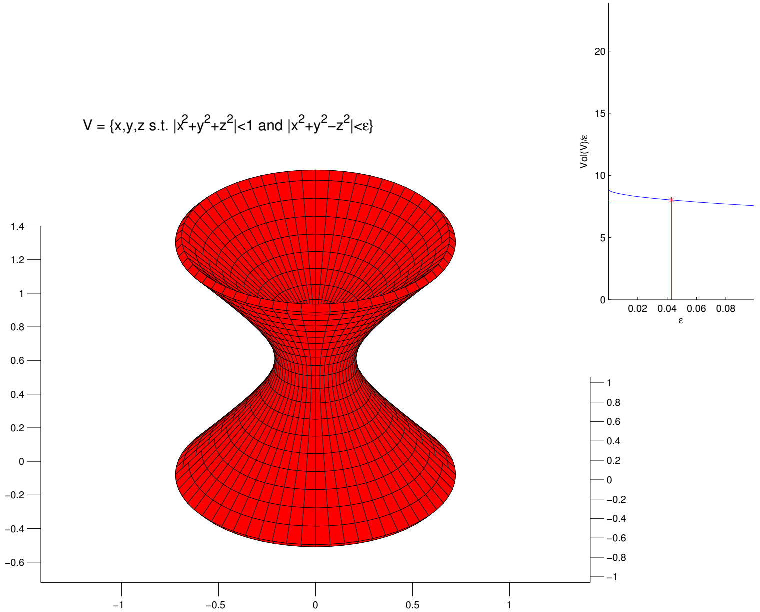


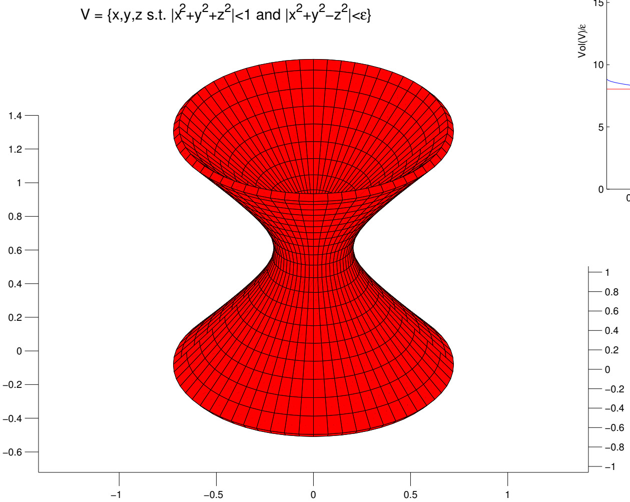


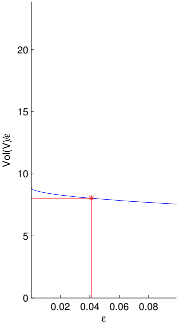

V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
20
15
10
5
0
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
0.8
0.6
0.4
0.2
−0.8

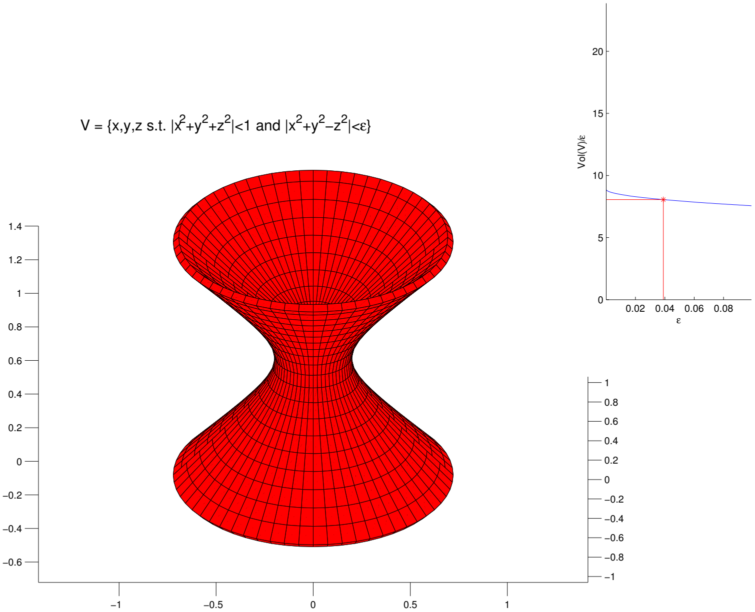
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



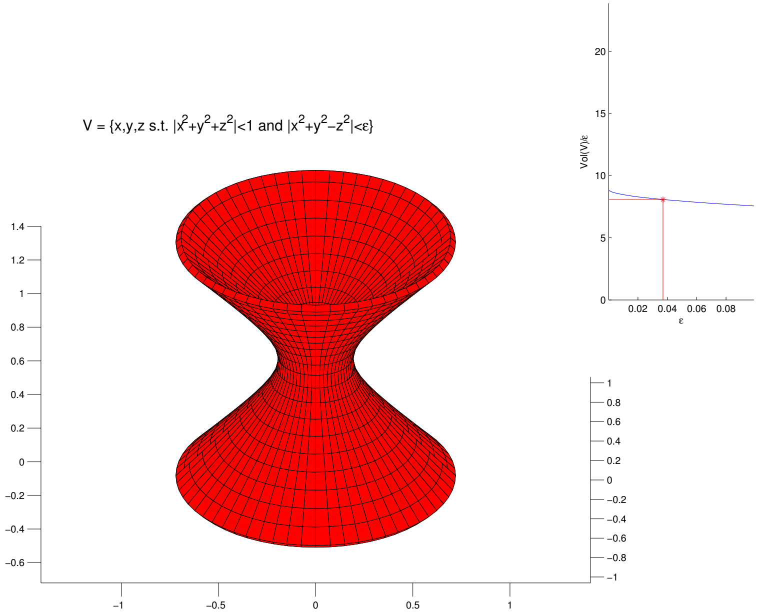

V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}
Vol(V)/ε
ε
20
15
10
5
0
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
0.8
0.6
0.4
0.2
−0.8


V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1


Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε

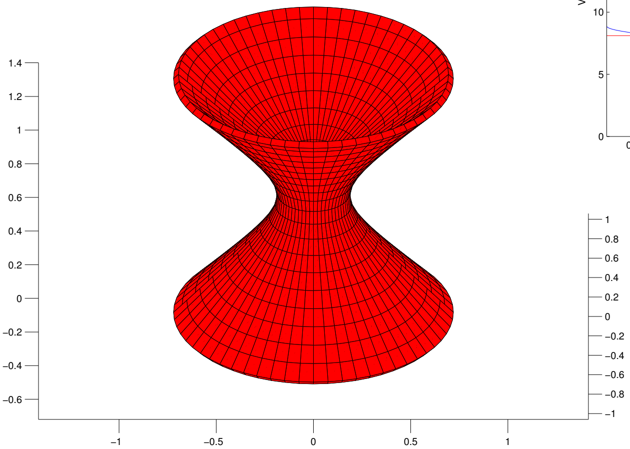


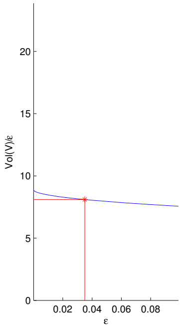
V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε



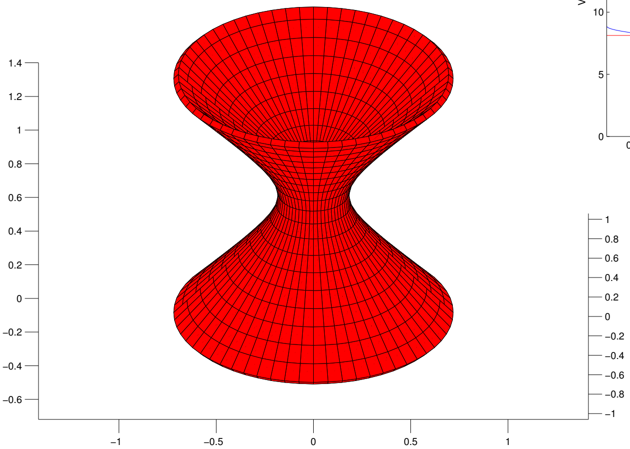
V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε

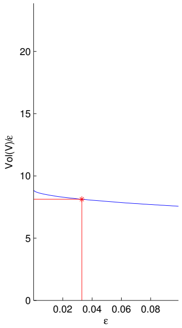
V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}



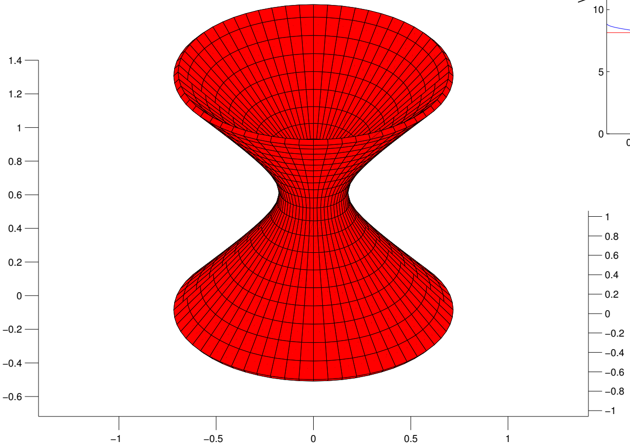


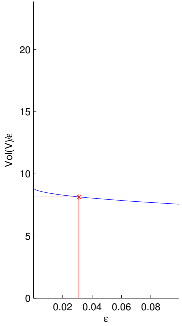
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

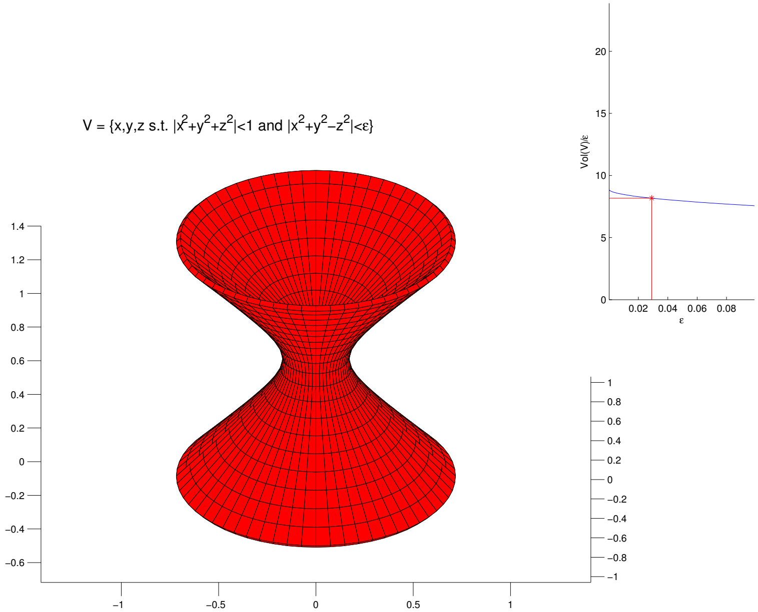


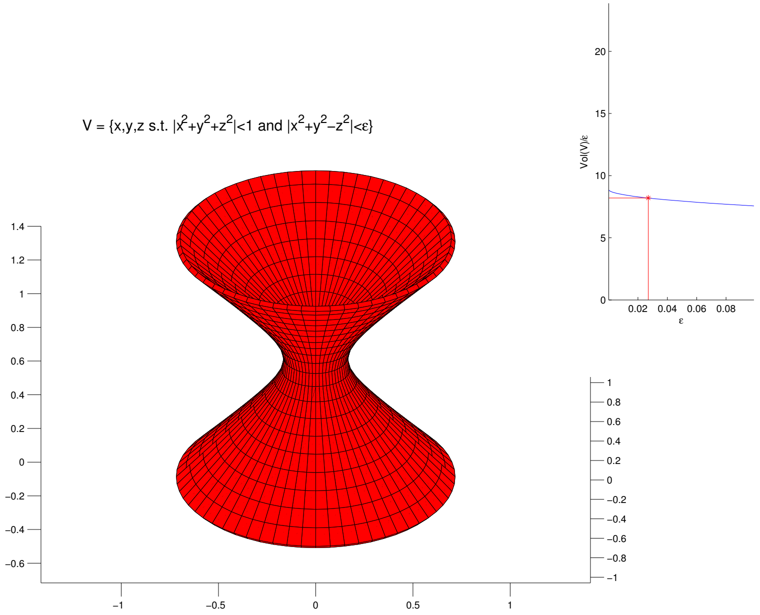


$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$






V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε


V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
20
15
10
5
0
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
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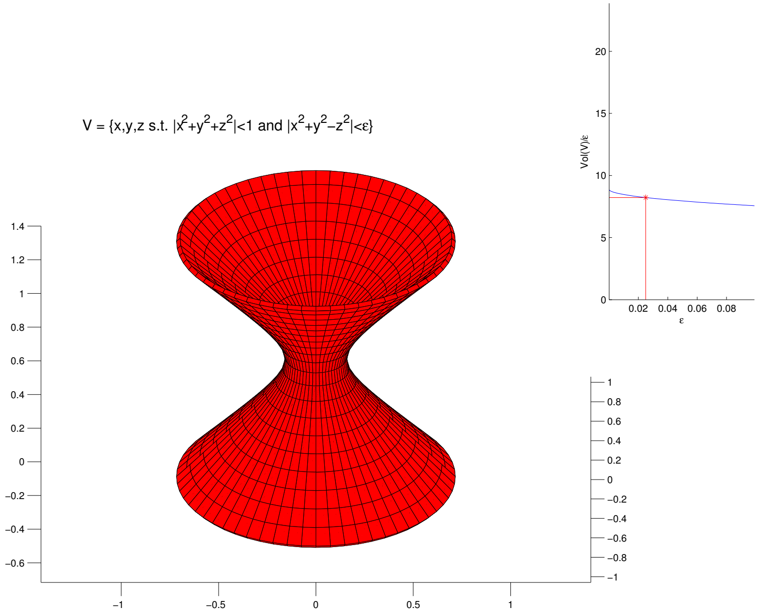
−0.4
−0.6
−0.8
−1


V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0
5
10
15
20
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
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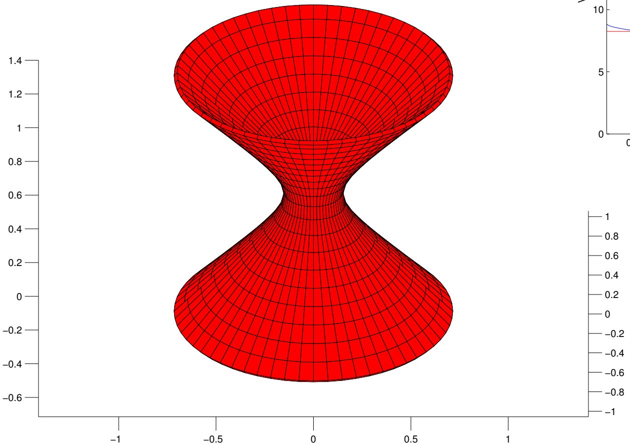
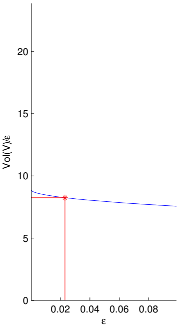
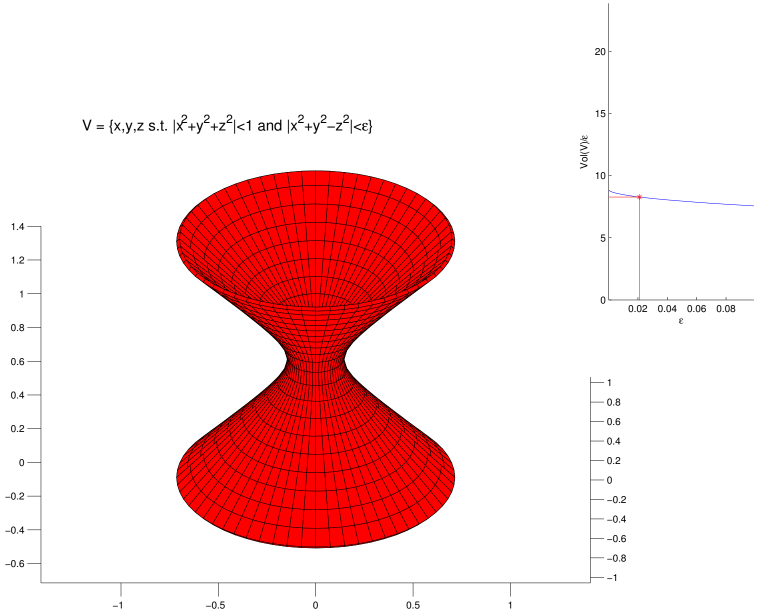
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

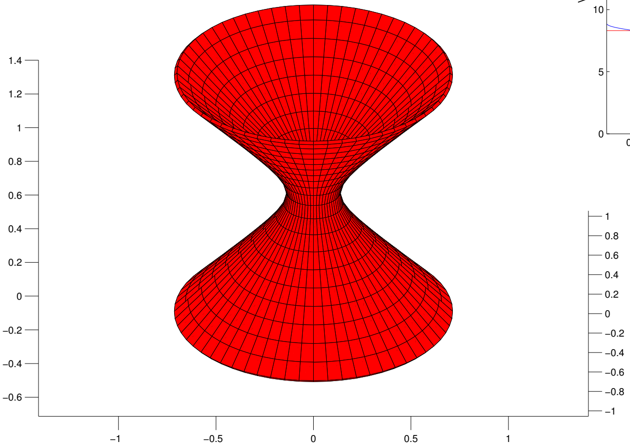


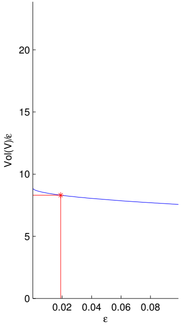


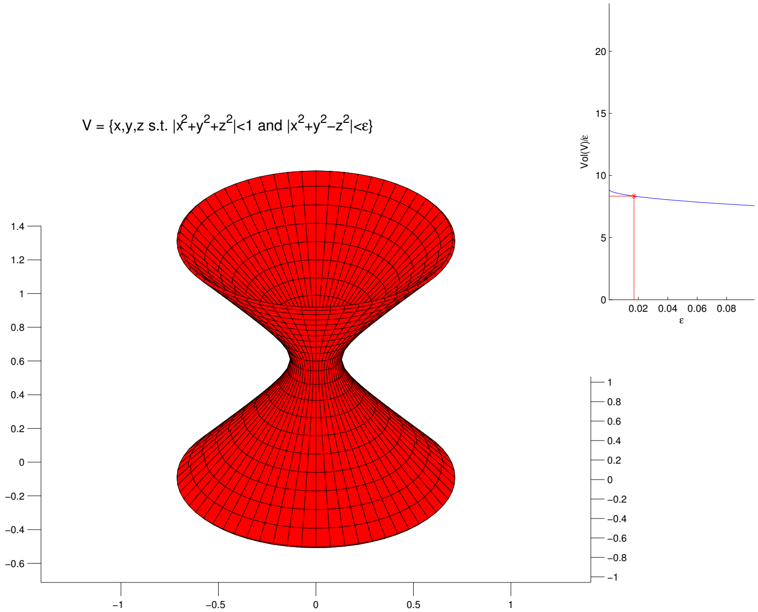


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε

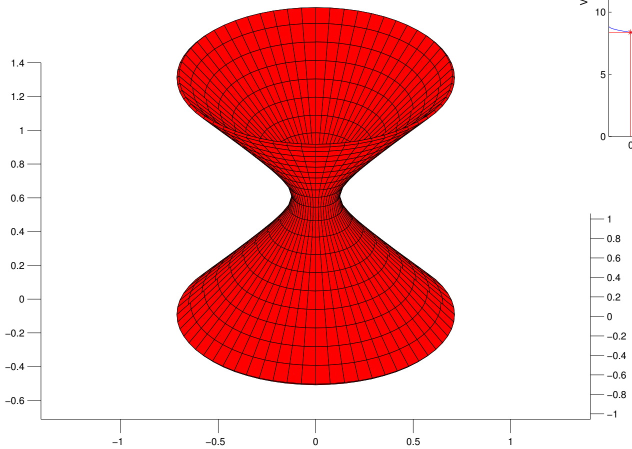
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

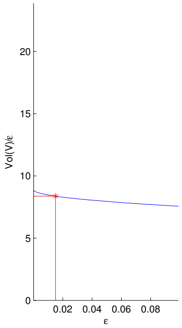


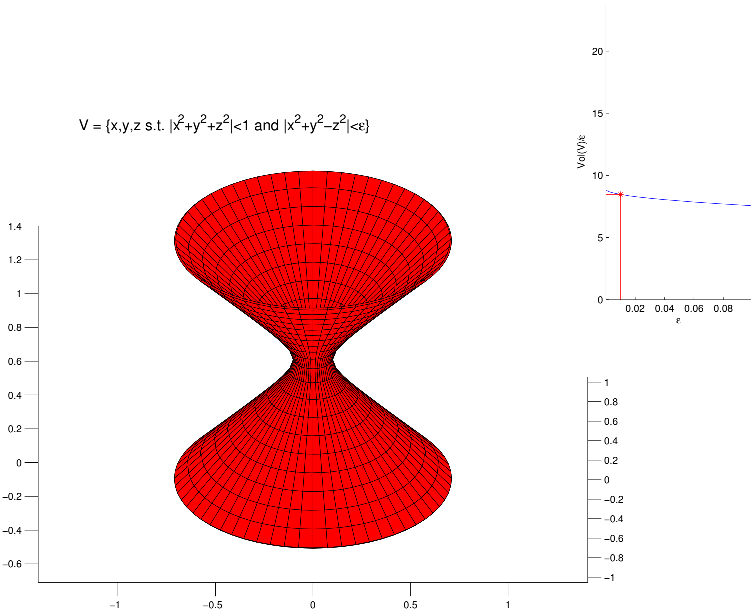



V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε

$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$







V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0.02
0.04
0.06
0.08
0
5
10
15
20
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
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1
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0.2
0
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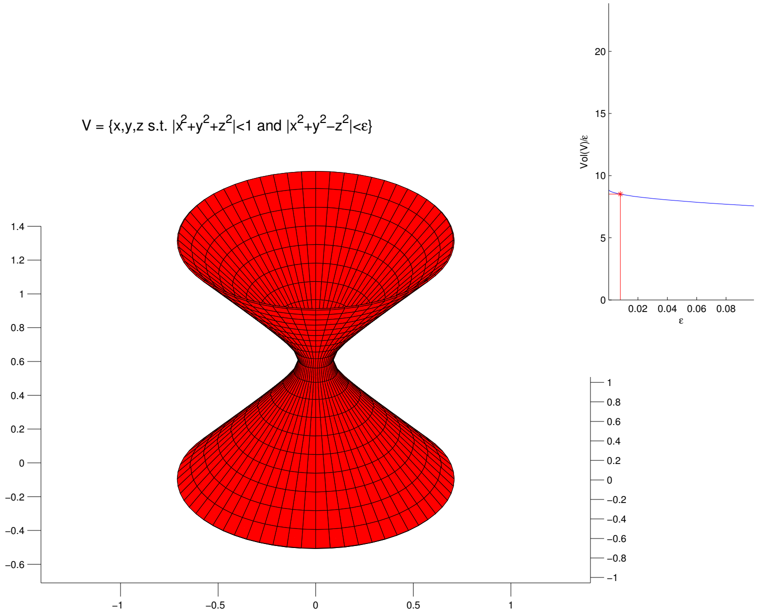
−0.4
−0.6
−0.8
−1


V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
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0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
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0.5
1
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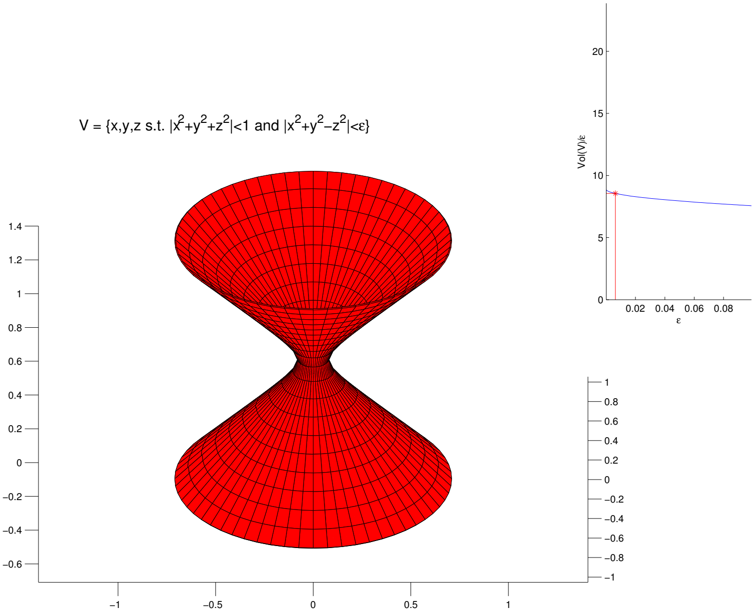
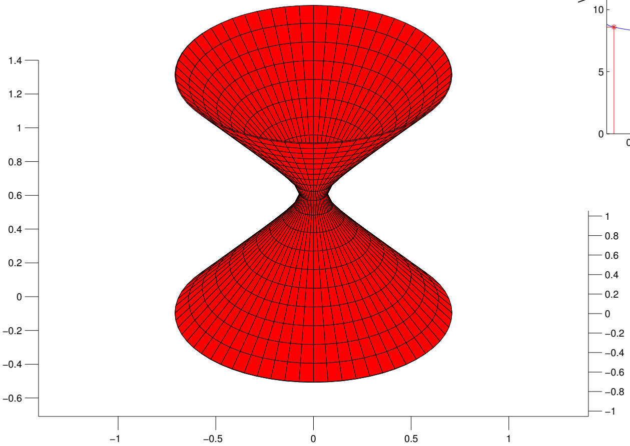
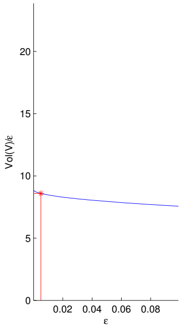
0.6
0.4
0.2
−0.8

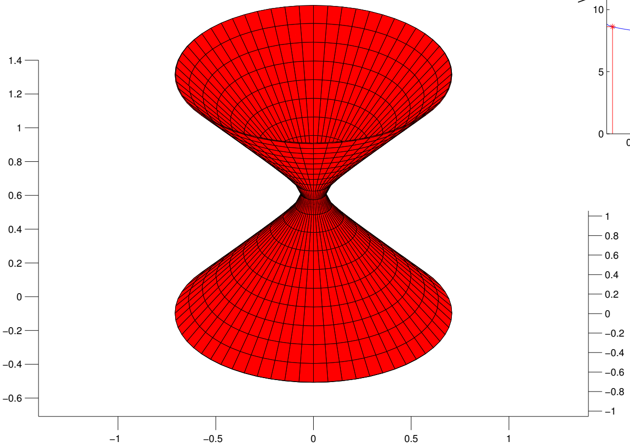

V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
0
5
10
15
20
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0.5
−0.8

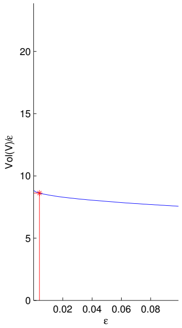
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



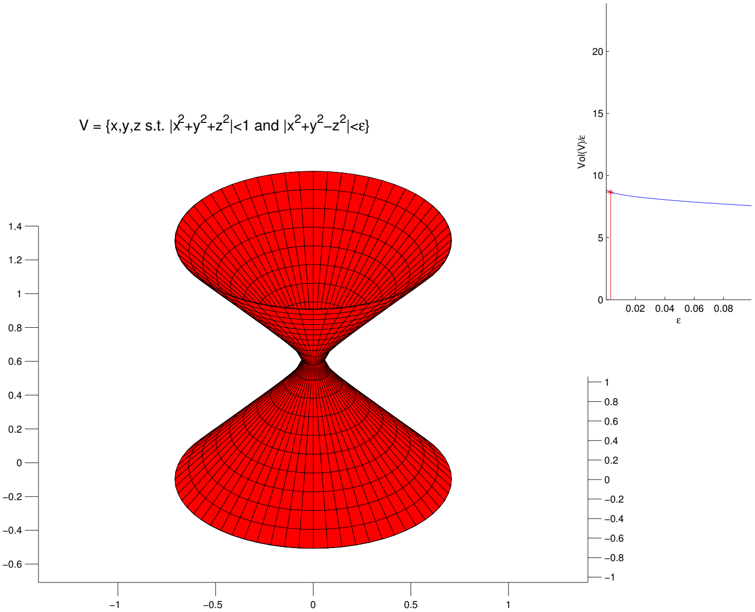


$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

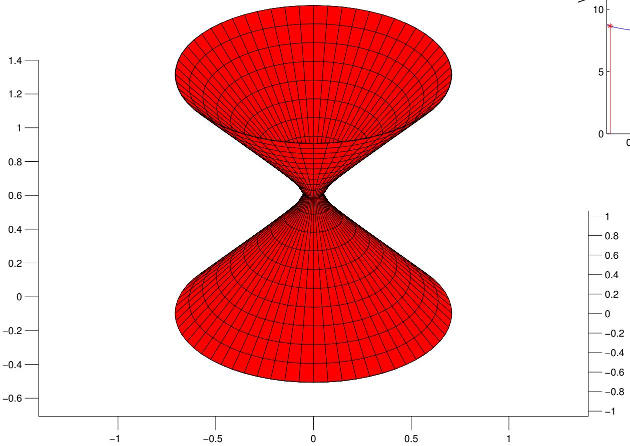


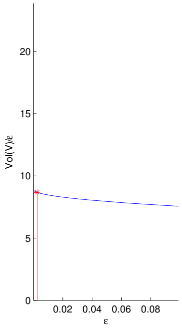


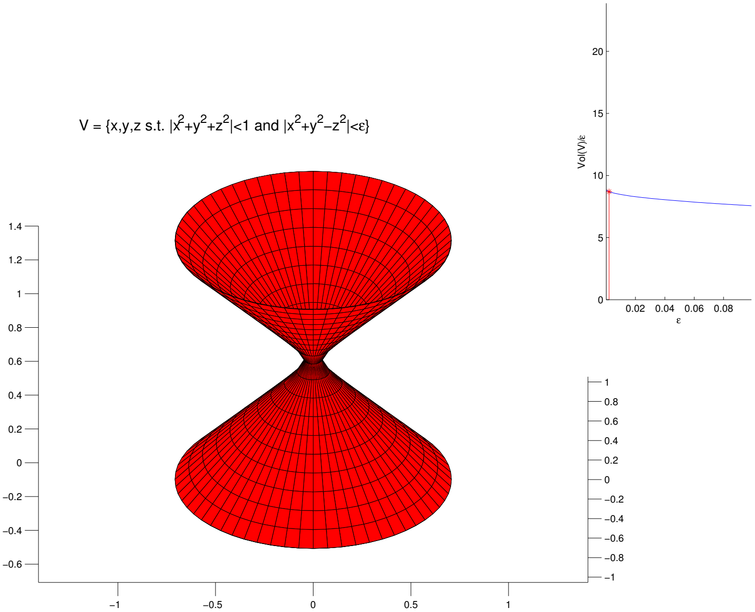
V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}



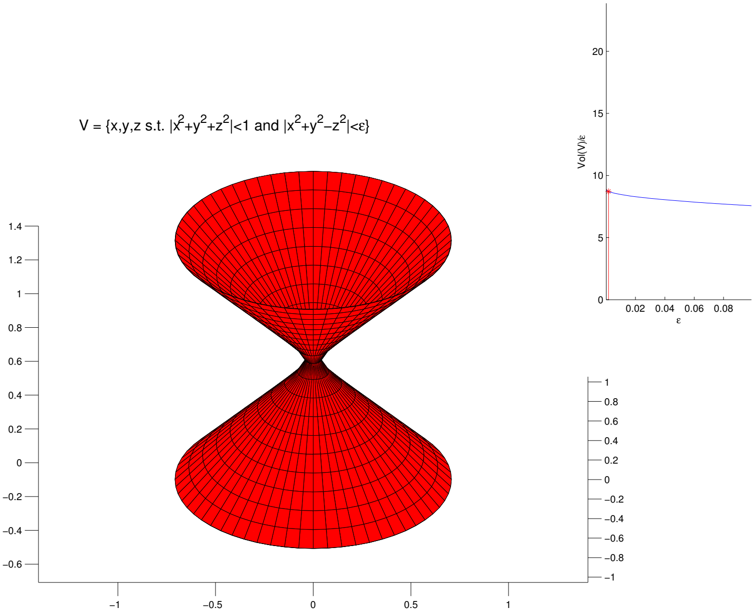
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

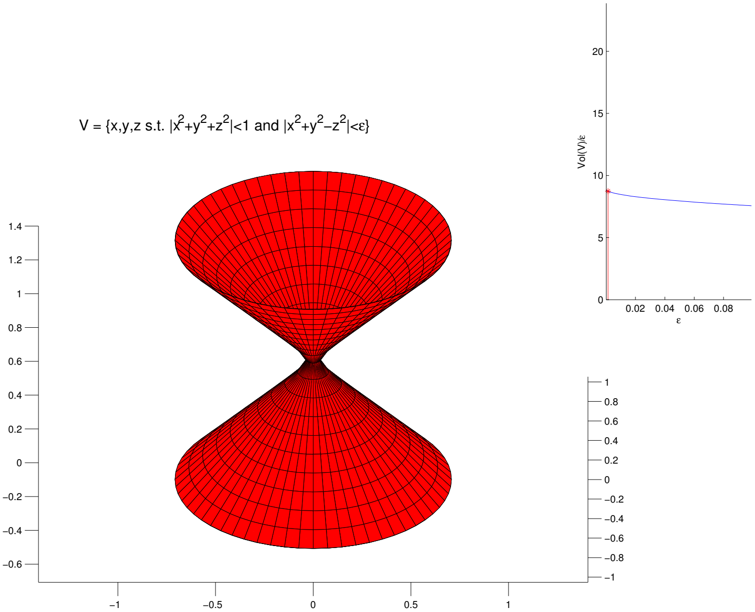





V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
20
15
10
5
0
0.02 0.04 0.06 0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1 −0.5 0 0.5 1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

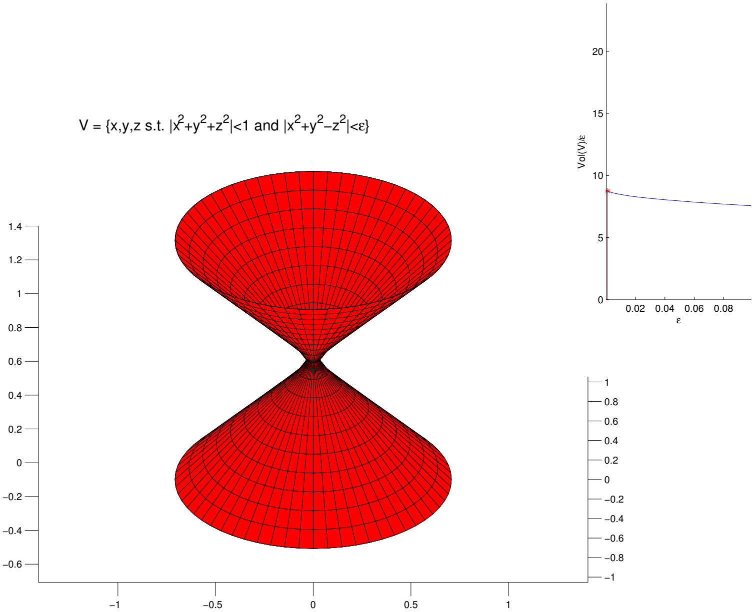
V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}

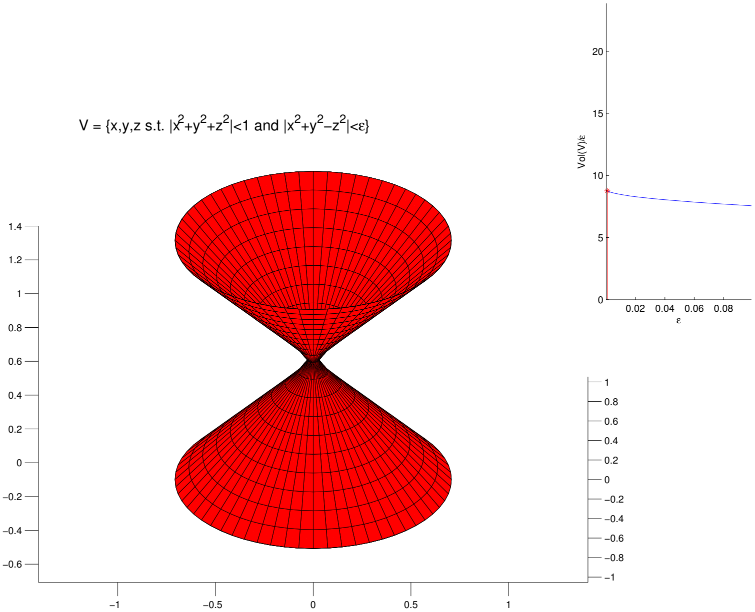


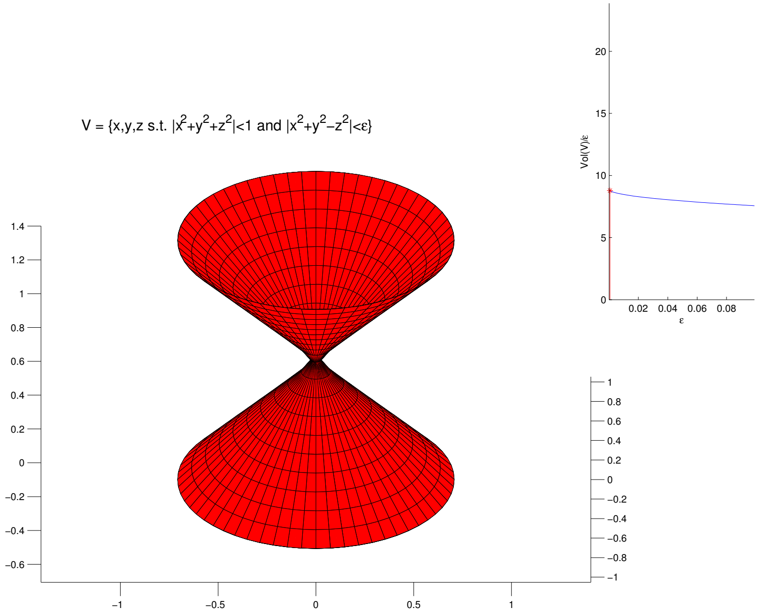


V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε

V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}




V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε


V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0.02
0.04
0.06
0.08
0
5
10
15
20
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
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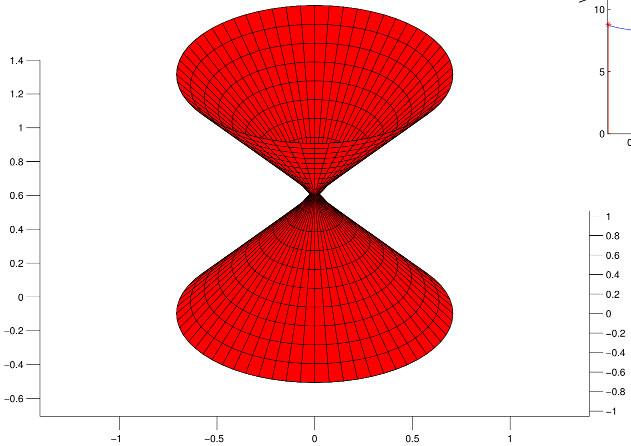
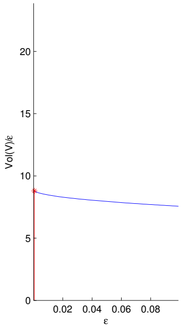
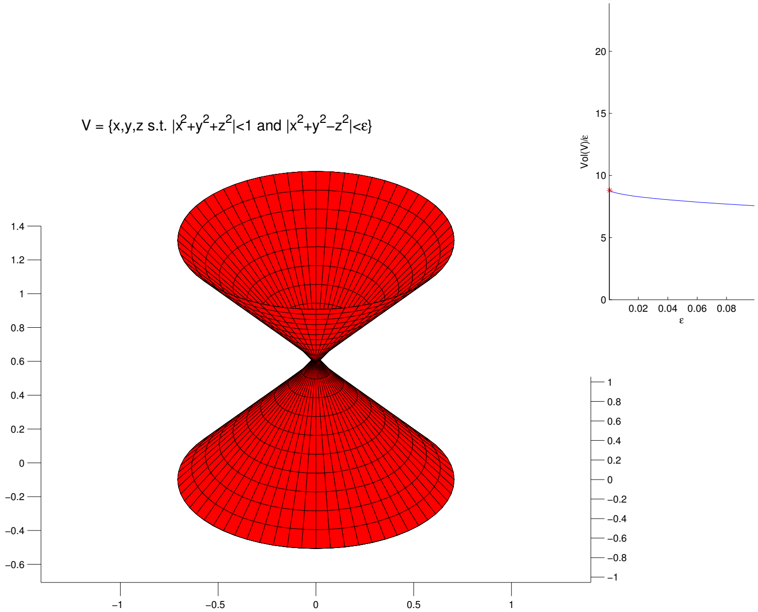
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$






V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
20
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0.02
0.04
0.06
0.08
1.4
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1
0.8
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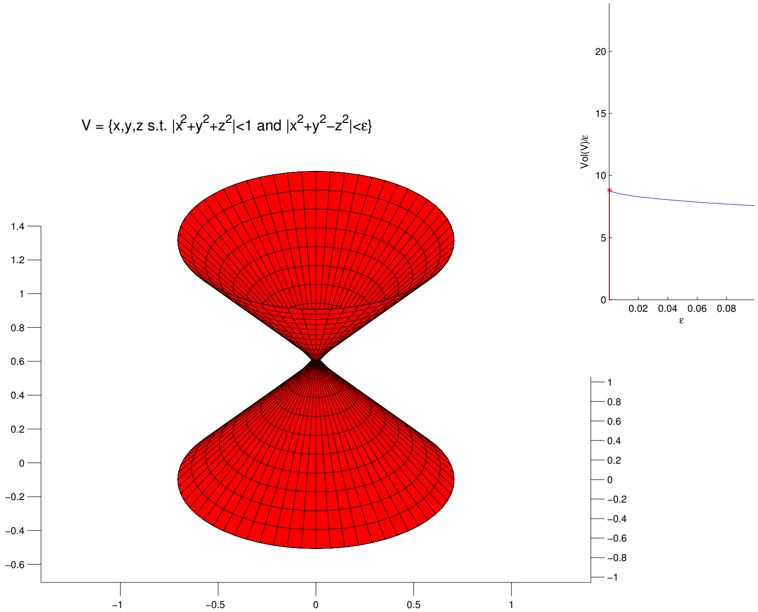

V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
20
15
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0.04
0.06
0.08
1.4
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1
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−0.5
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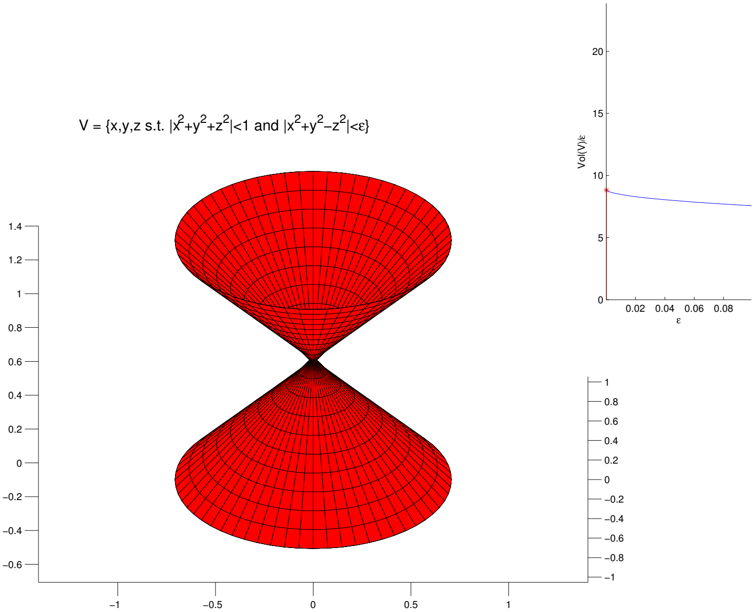
V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
0
5
10
15
20
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
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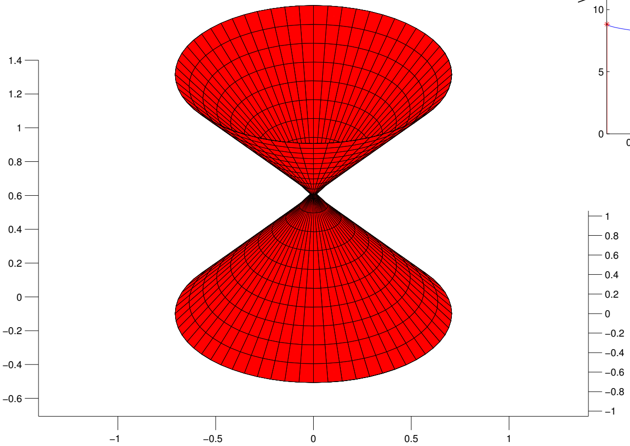
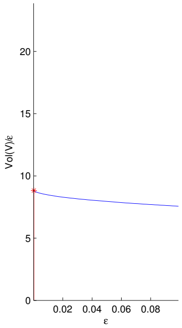
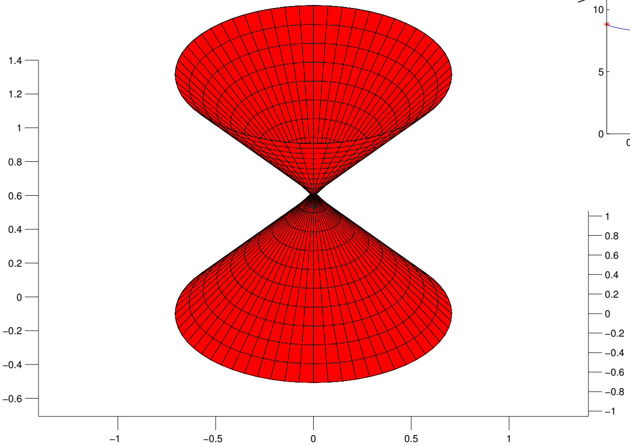
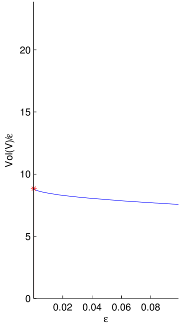
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

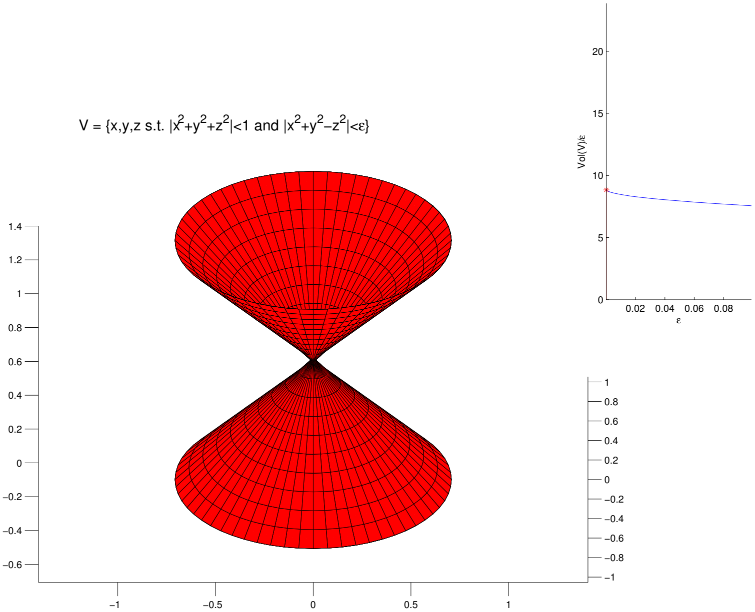




V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}






V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
20
15
10
5
0
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
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−1
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0
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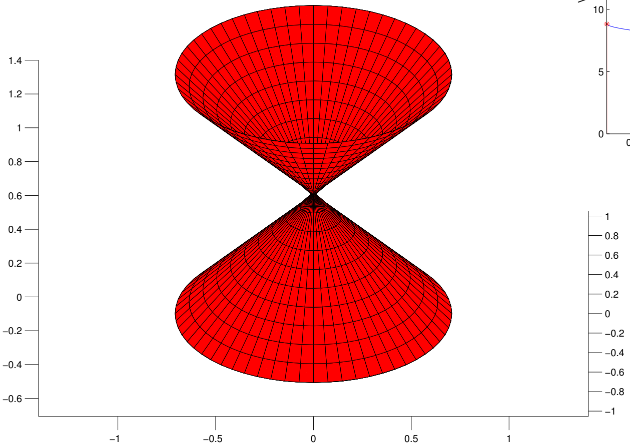

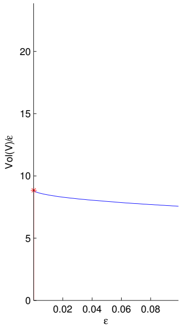

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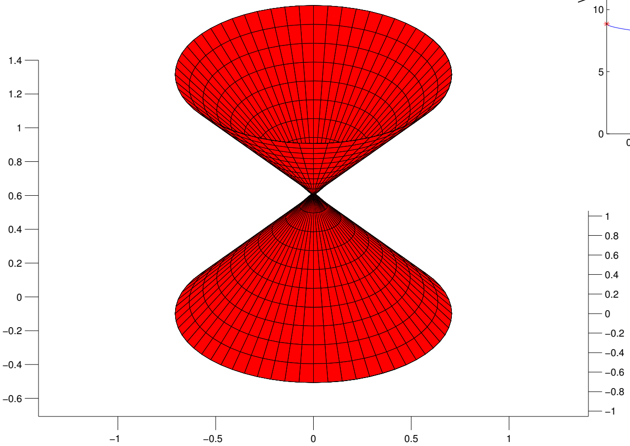

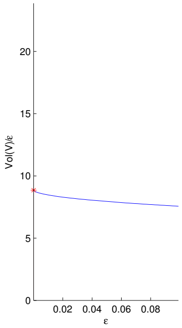

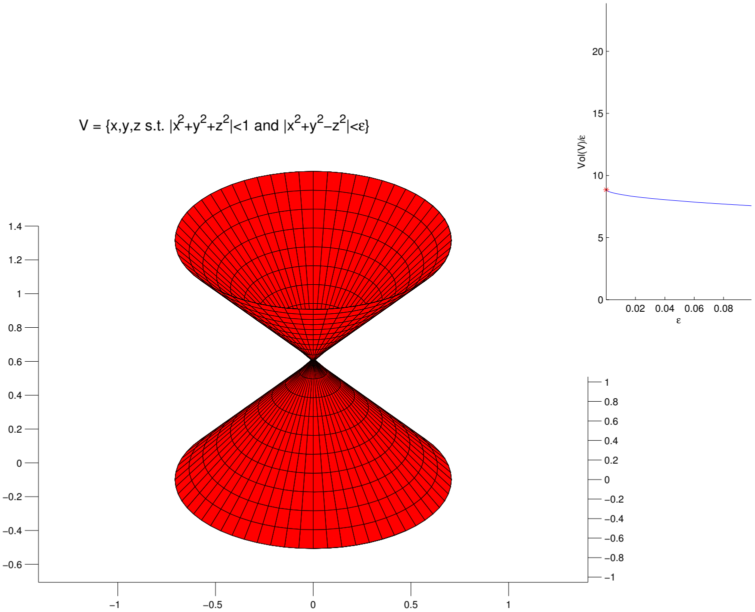
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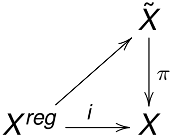
Rational singularities and conversion of integrals



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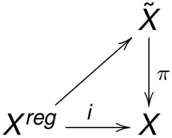

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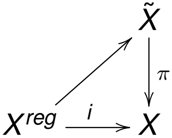

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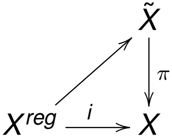

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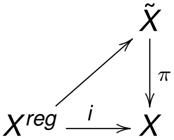

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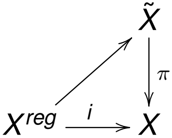

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Application: Hinich Theorem $\Longrightarrow$ Deligne Ranga-Rao Theorem

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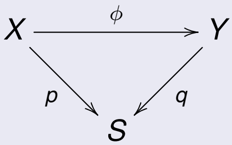

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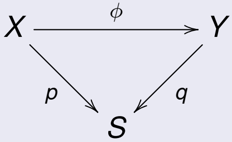

with p, q – flat and X, Y, S – smooth. Then

$$\{x \in X | \phi_{p(x)} : X_{p(x)} \to Y_{p(x)} \text{ is FRS at } x\} \text{ is open.}$$

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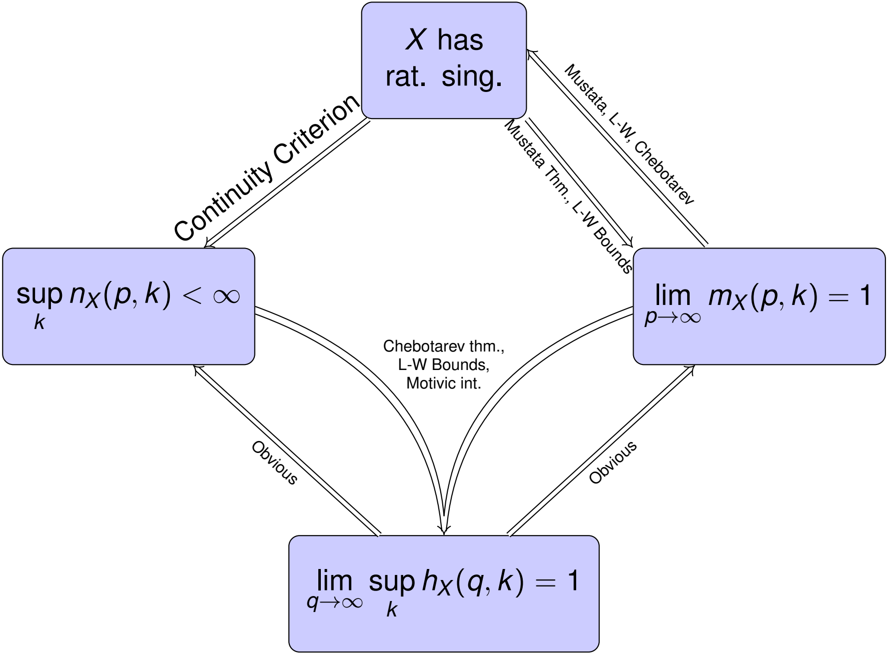
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Mustata, L-W, Chebotarev
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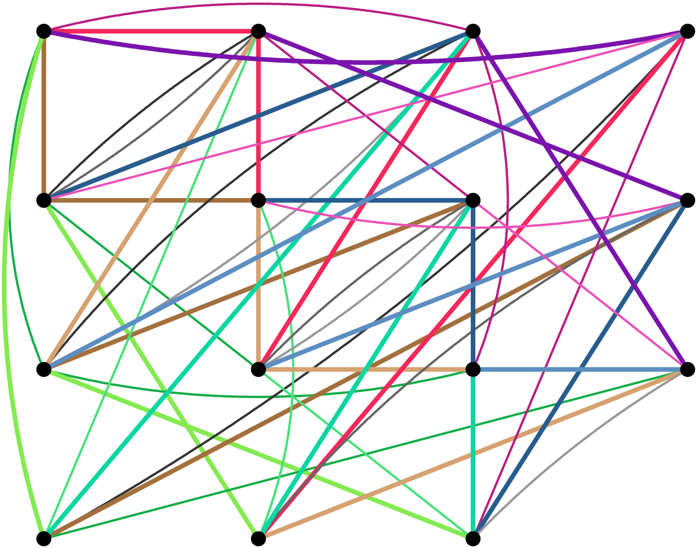

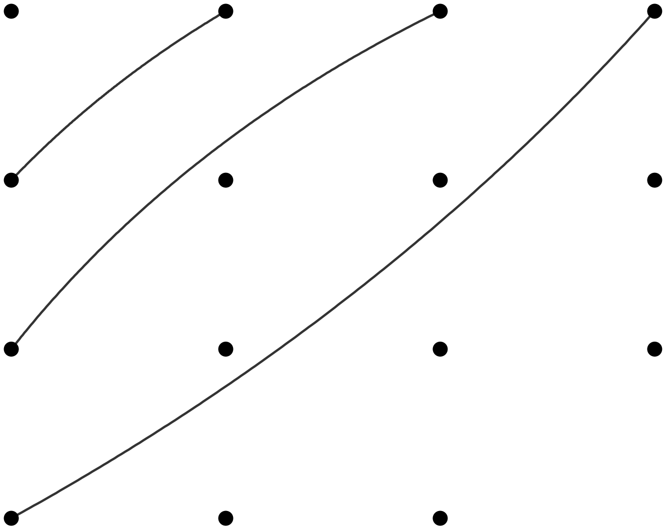

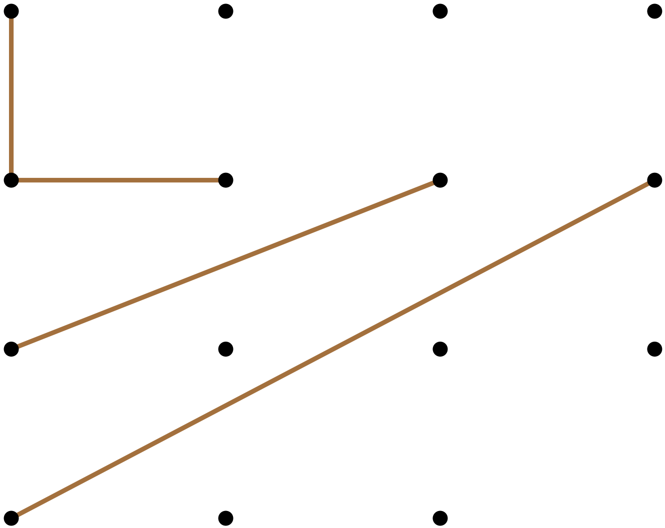

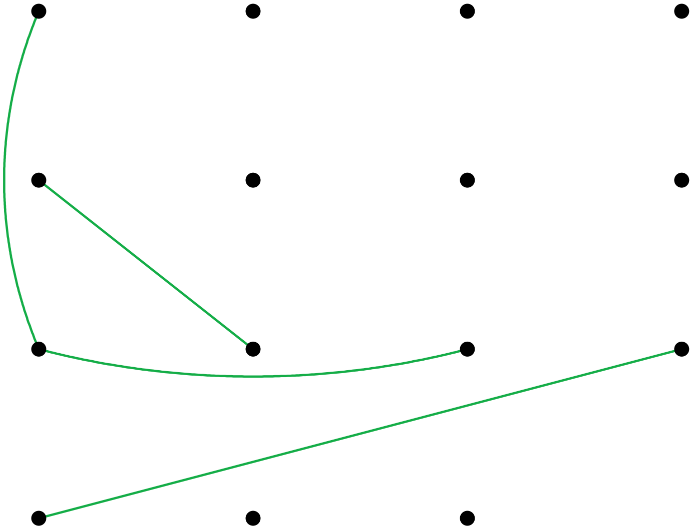

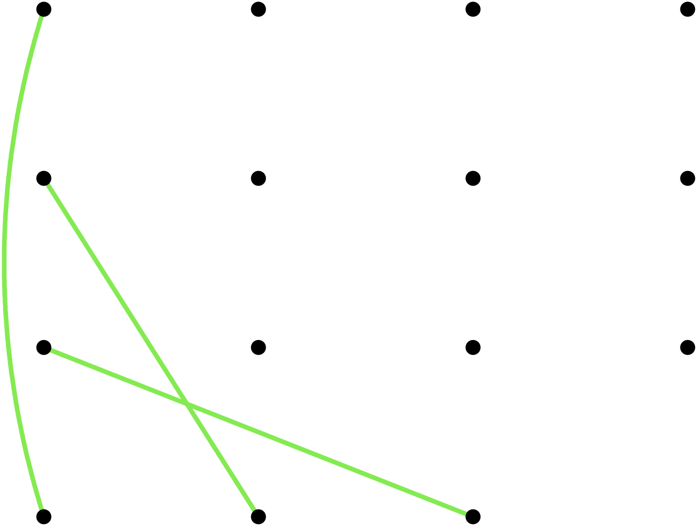

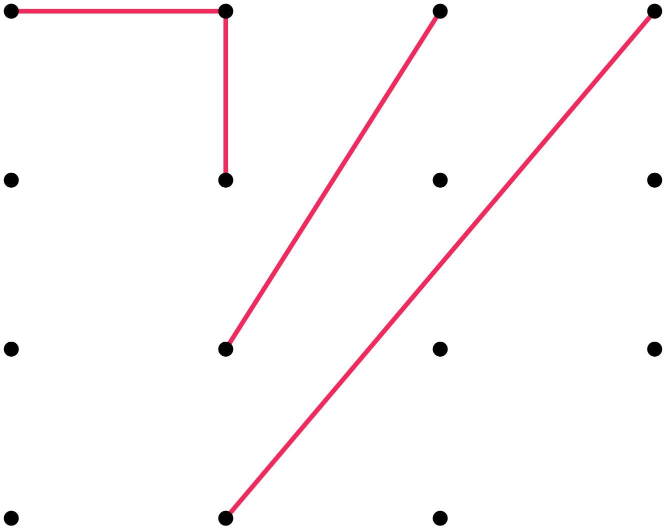

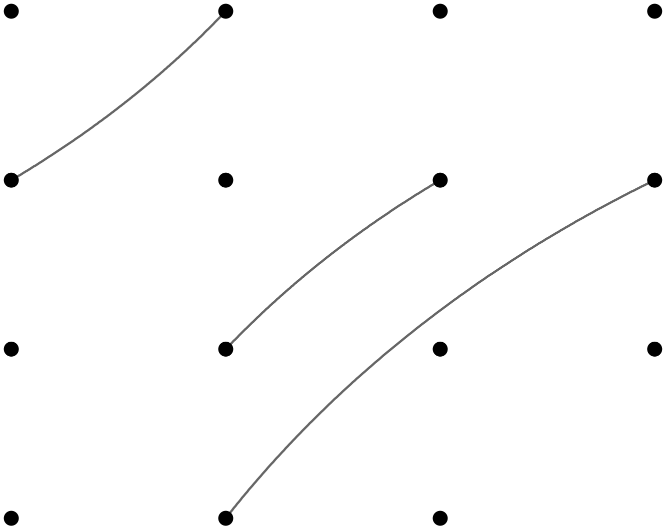

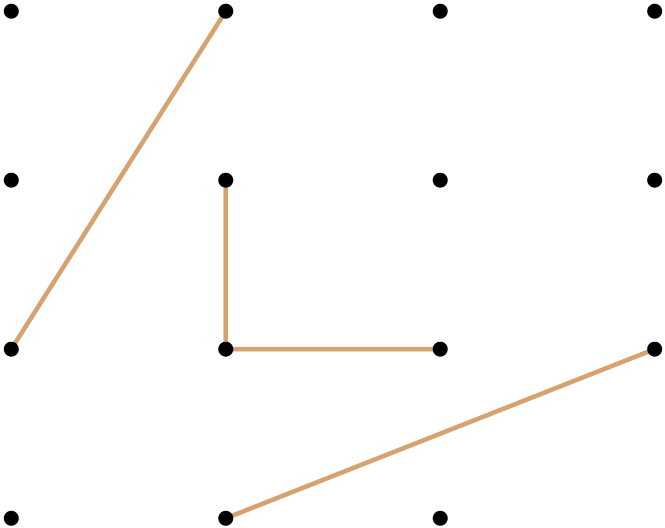

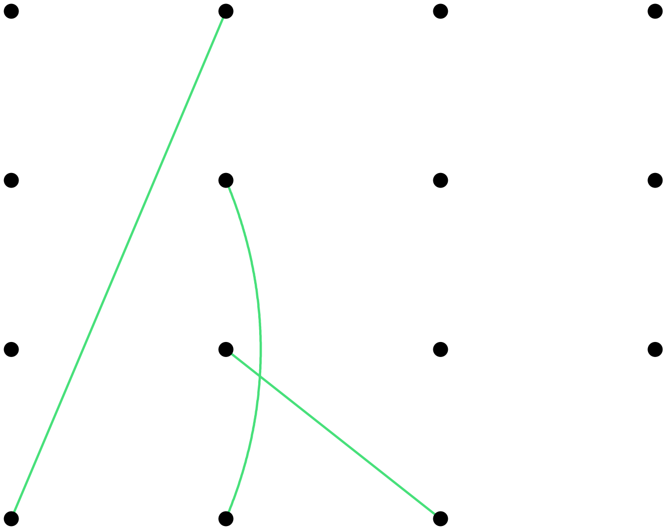

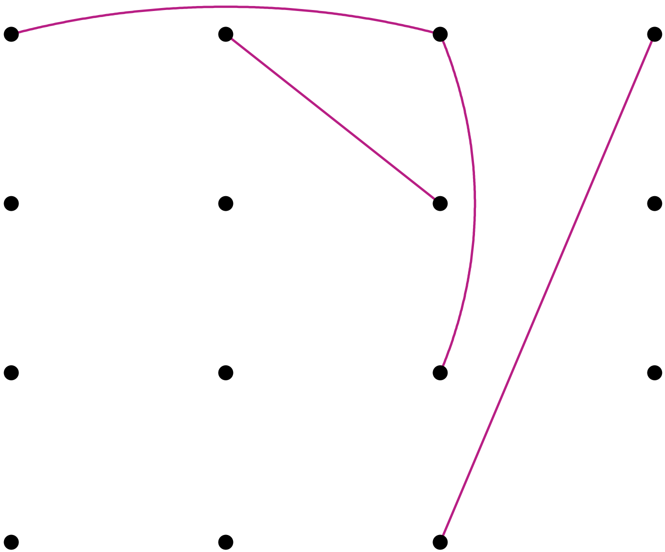

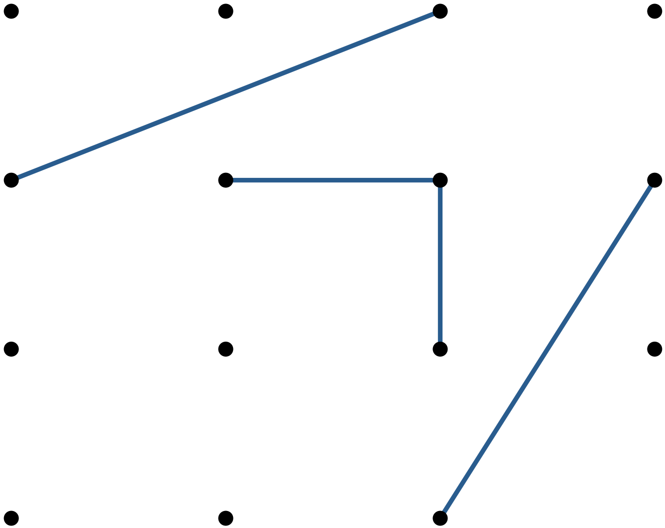

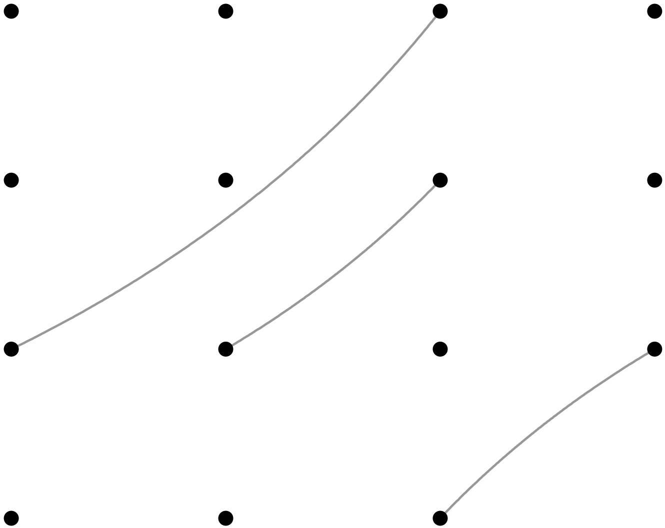

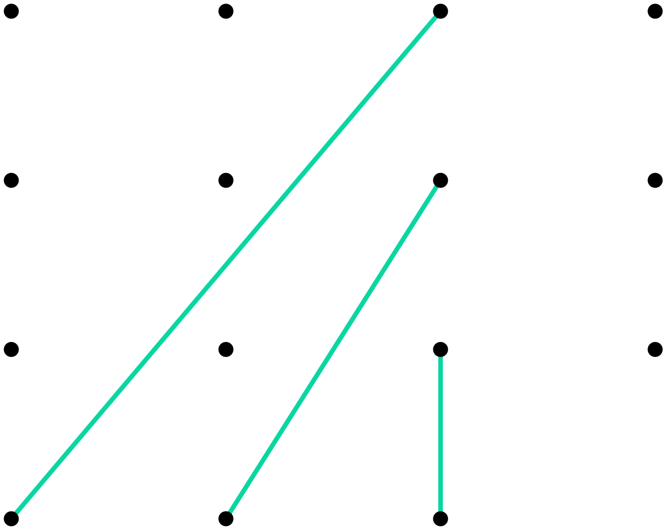

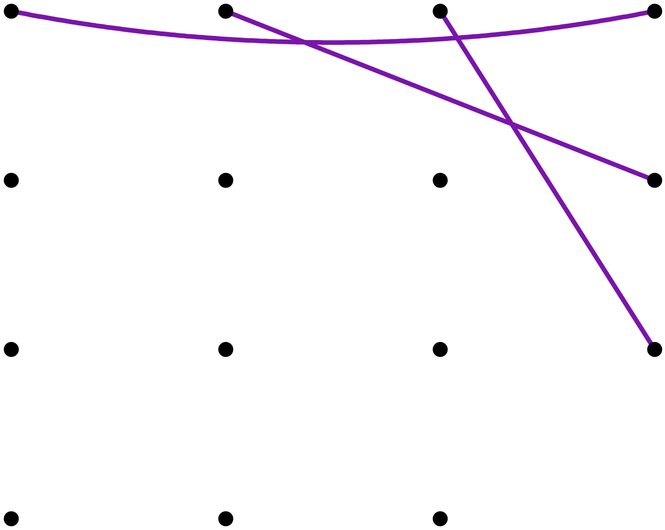

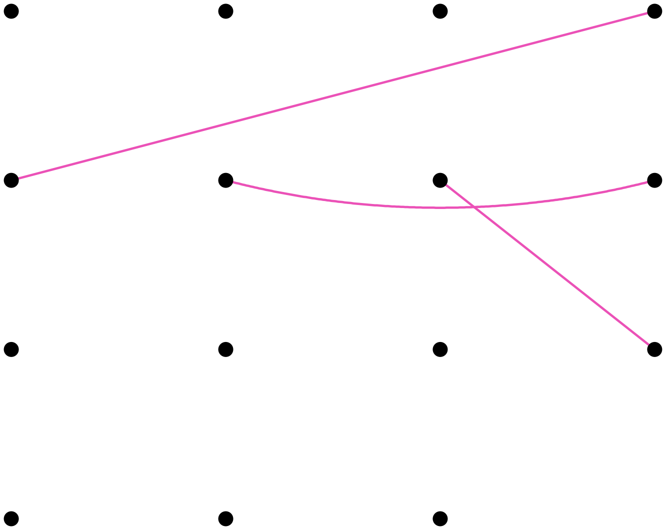

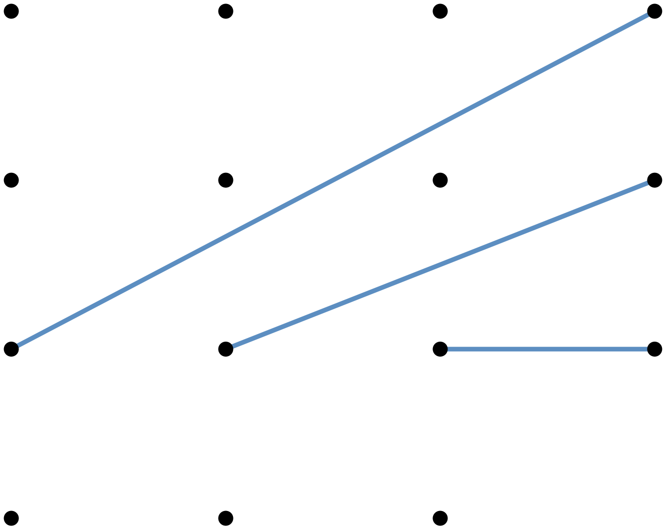

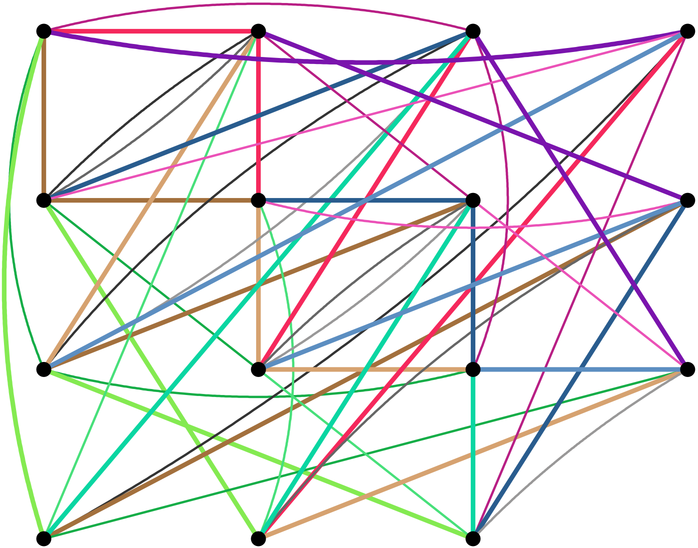

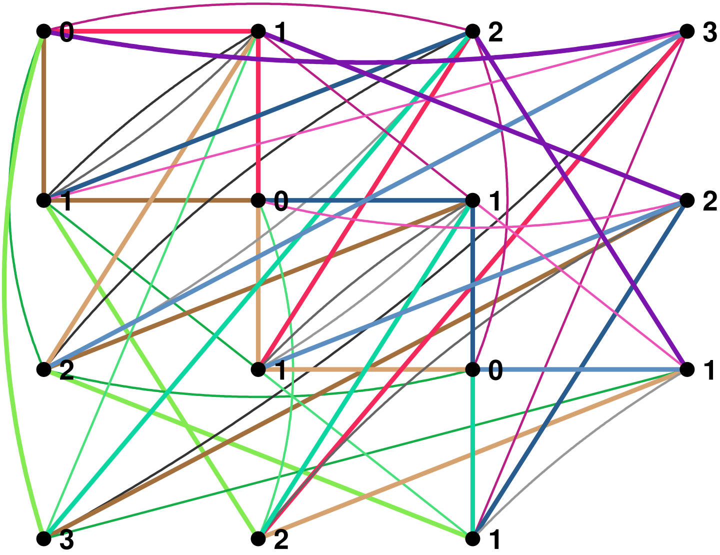
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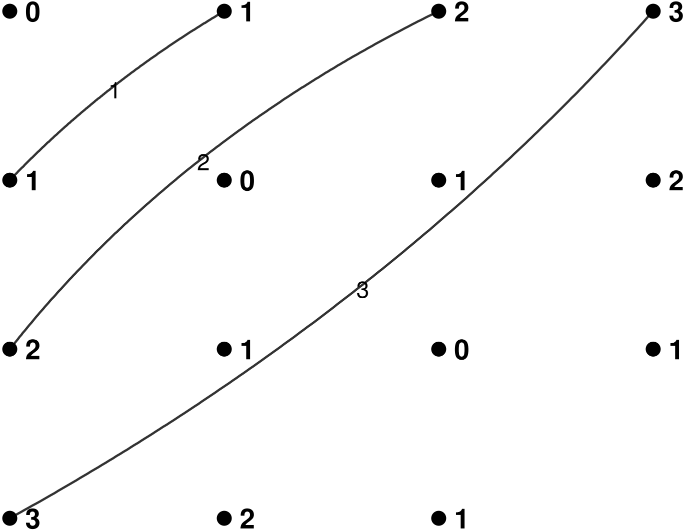
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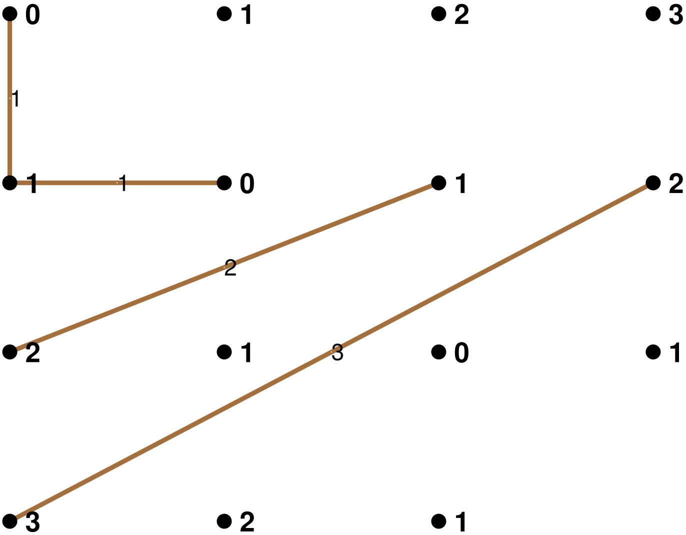
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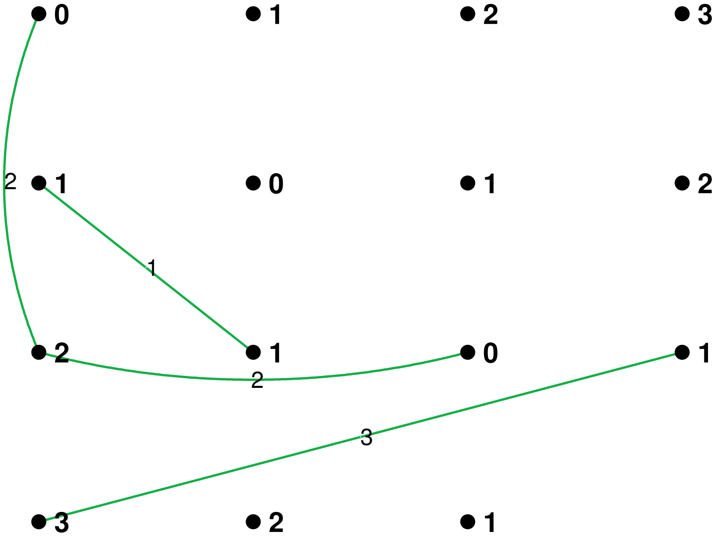
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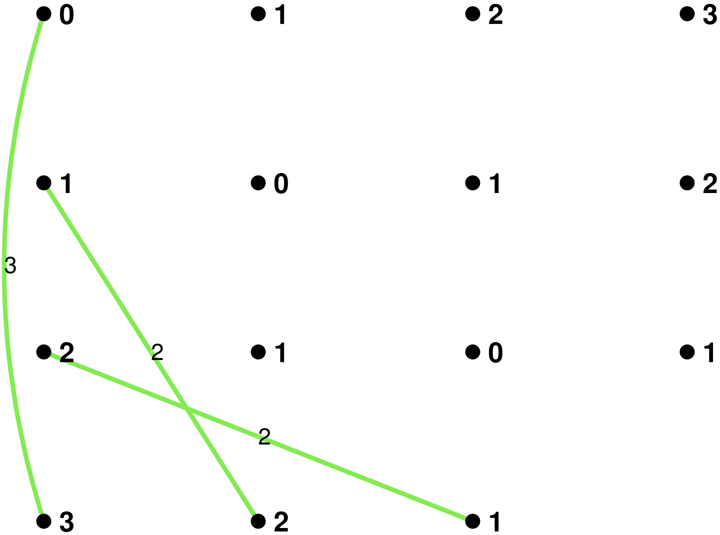
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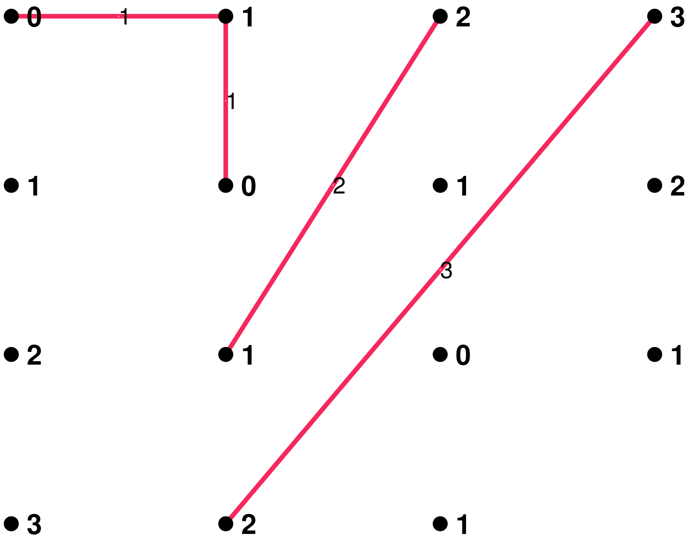

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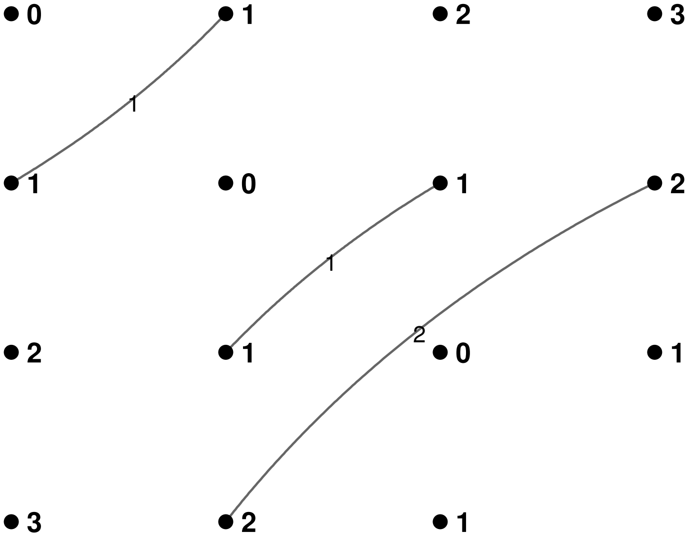
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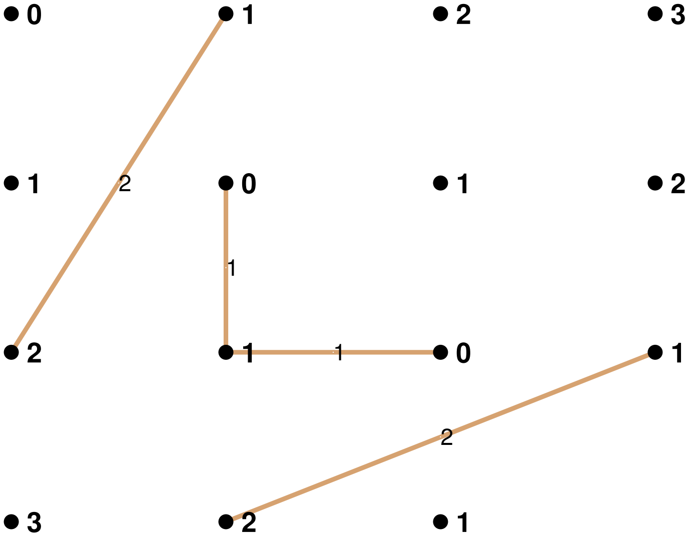
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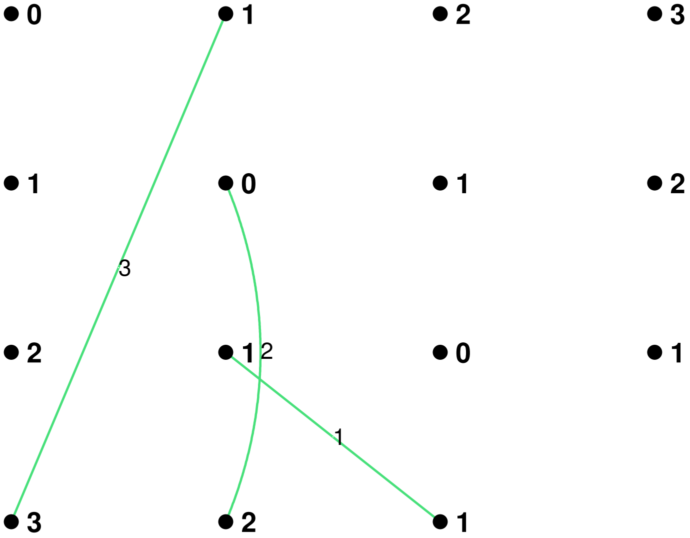

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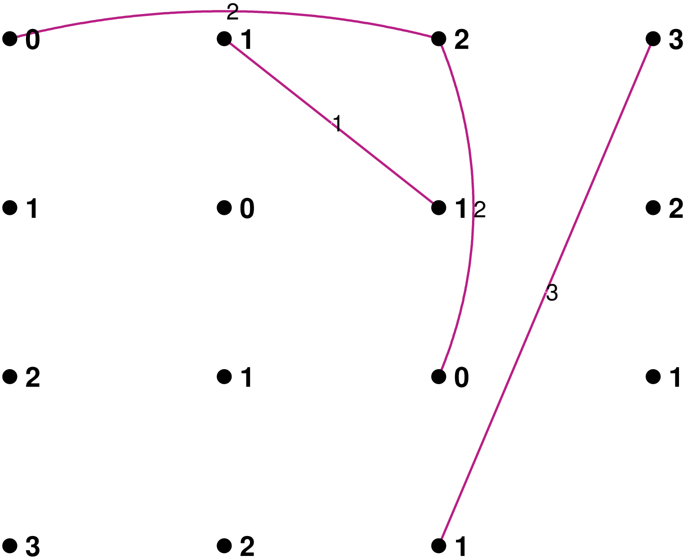

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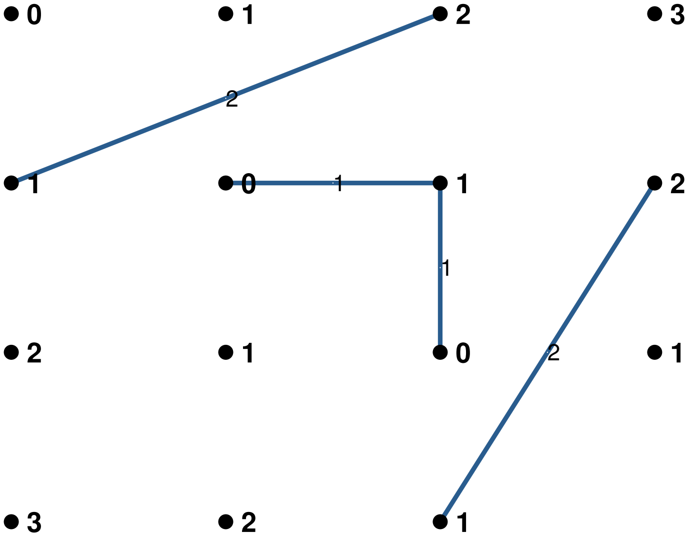
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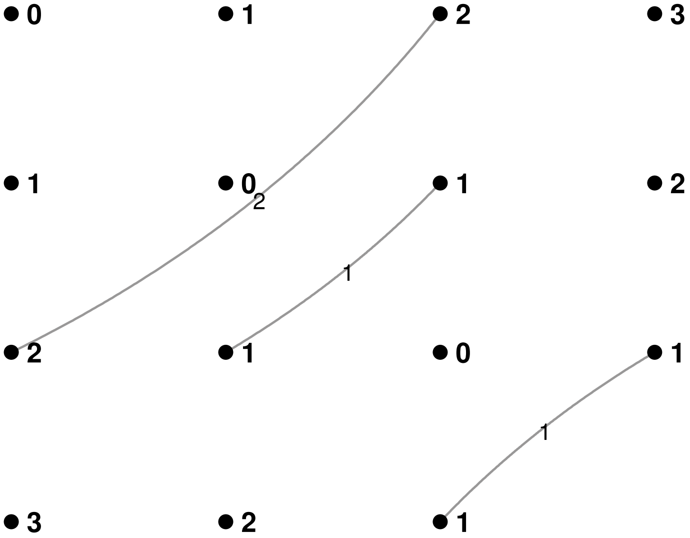
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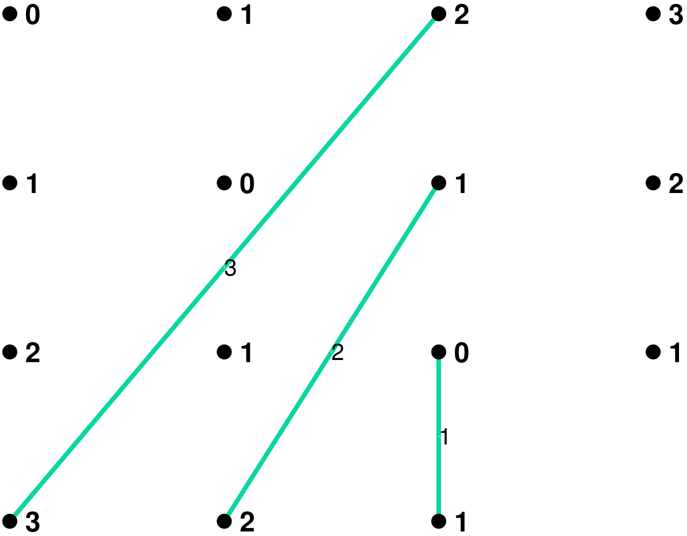

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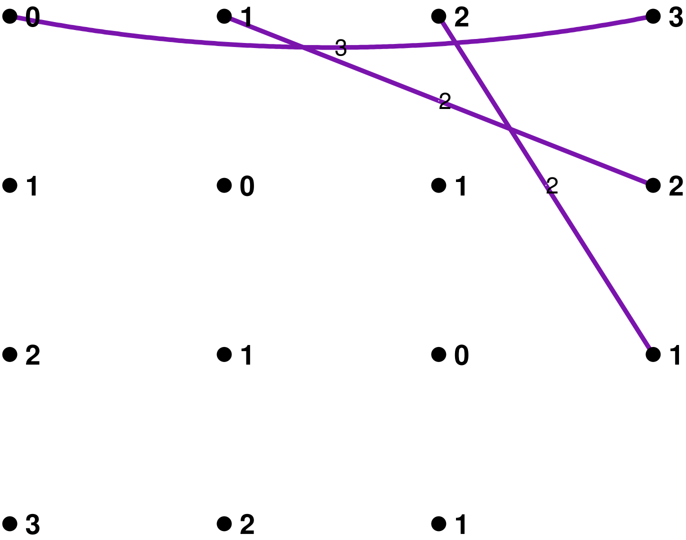
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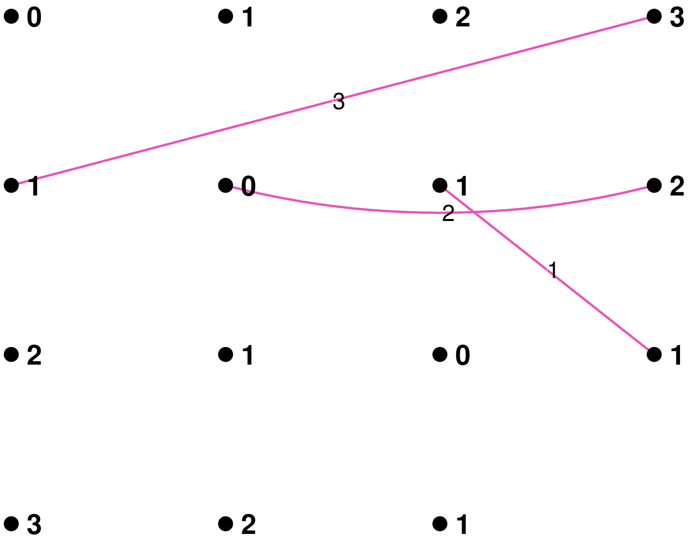
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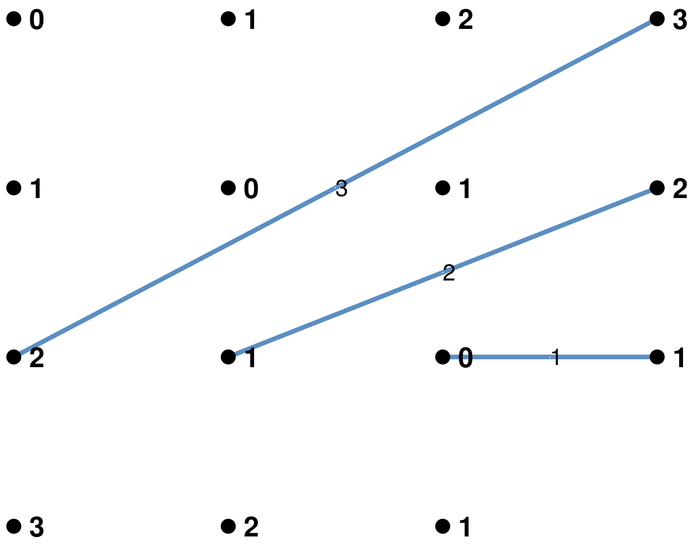
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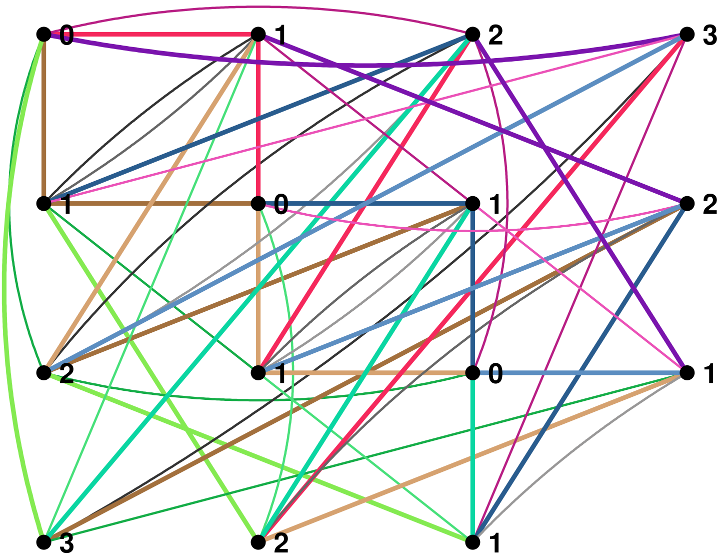
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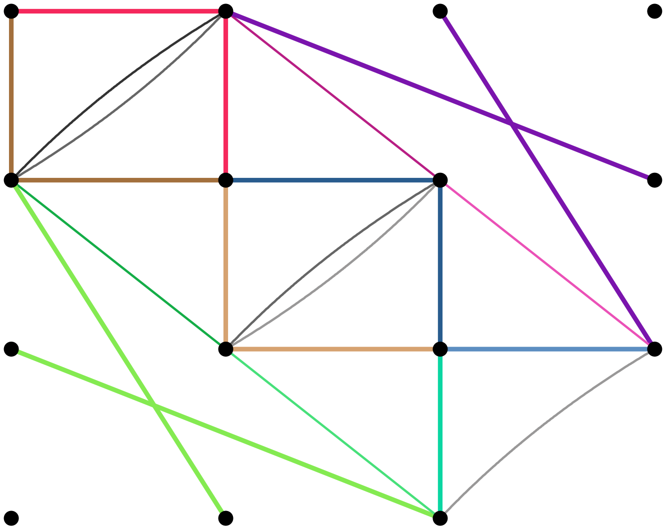

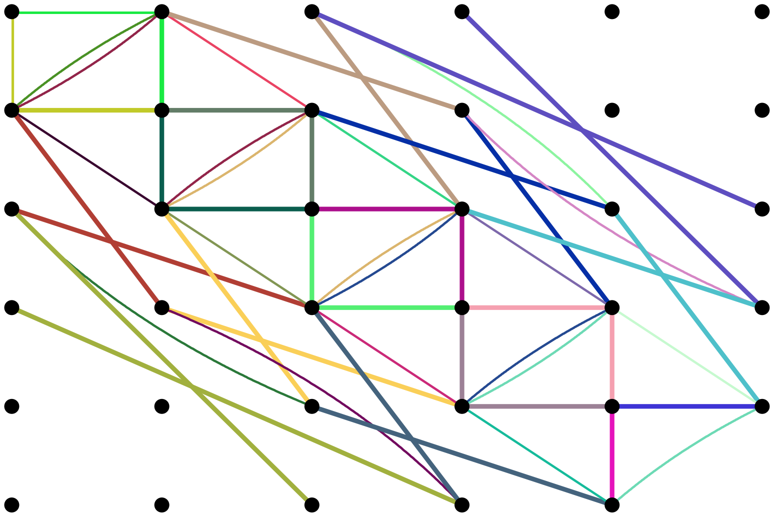

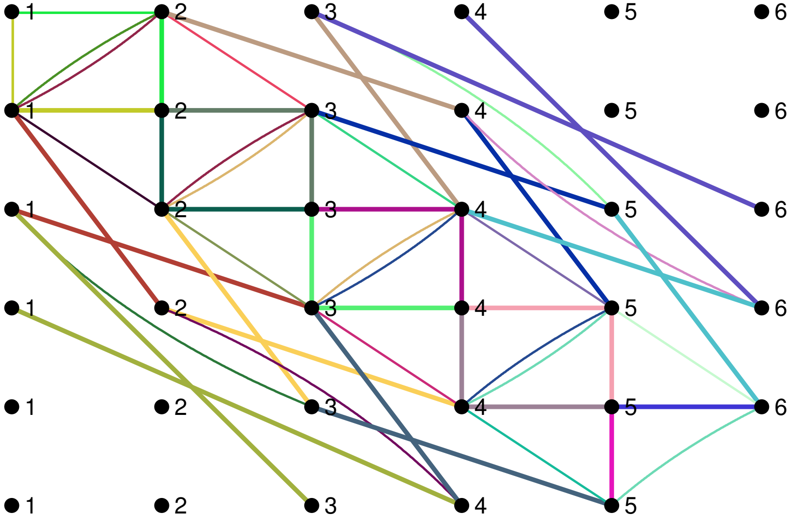
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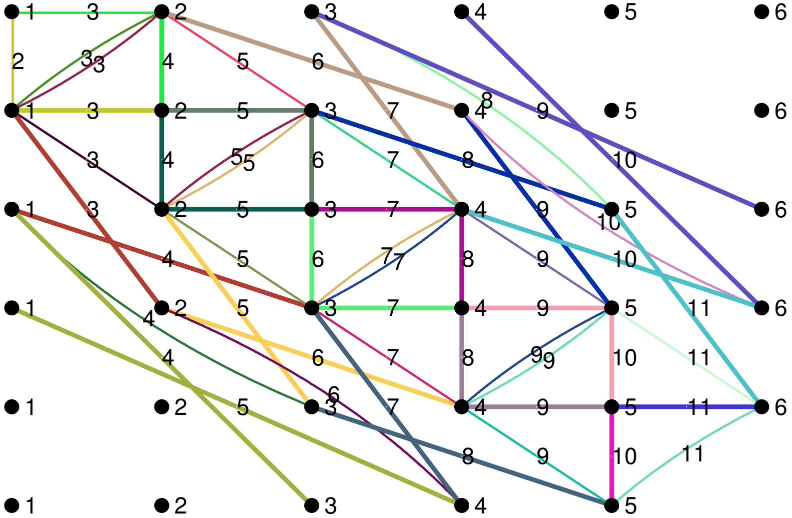

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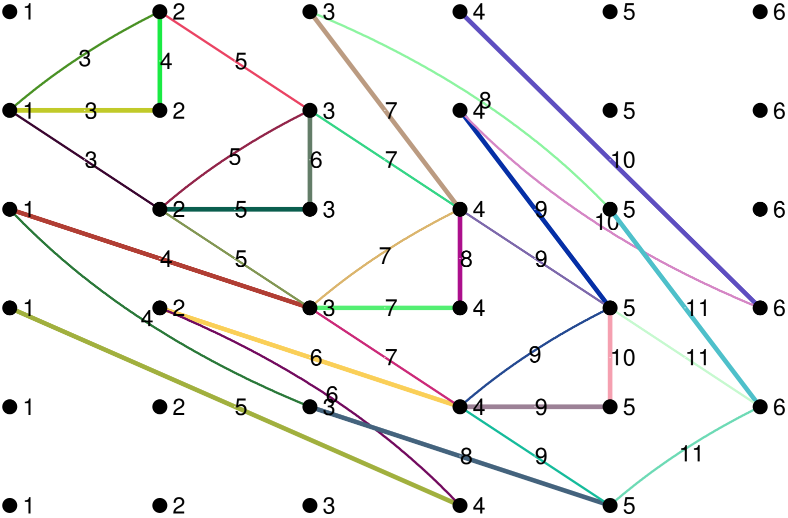
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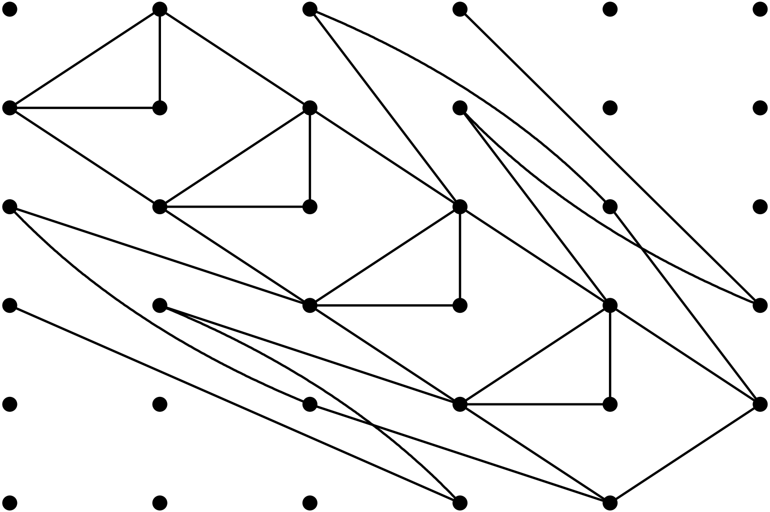

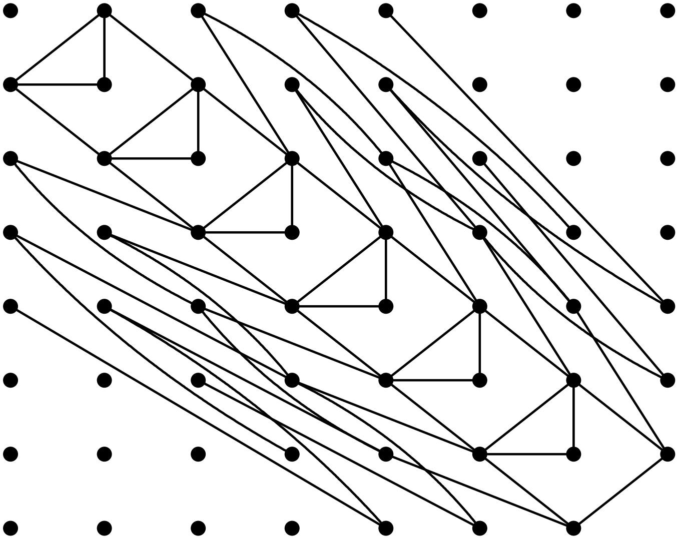

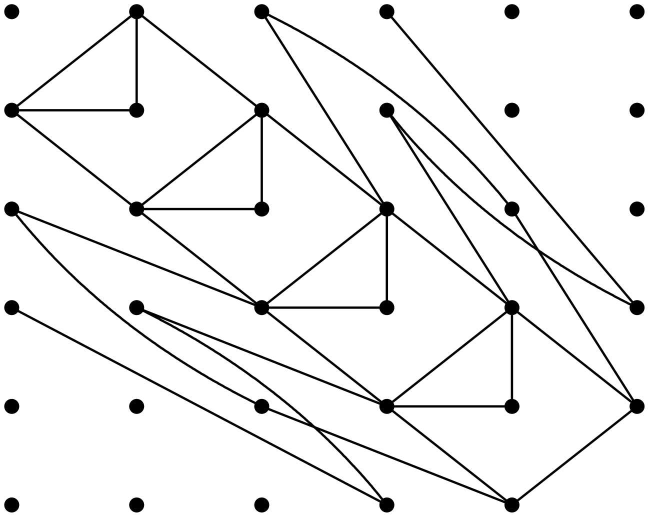

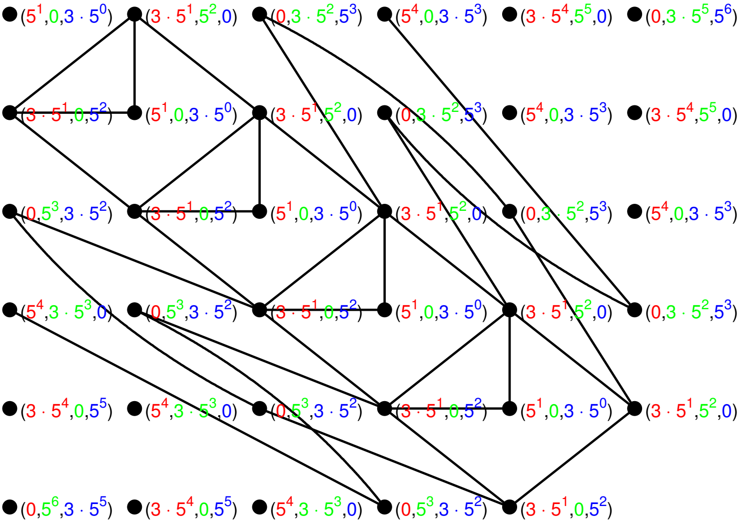

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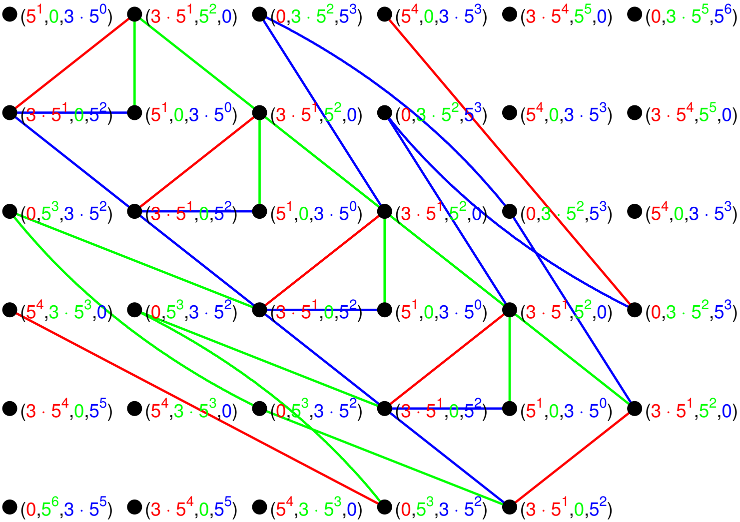
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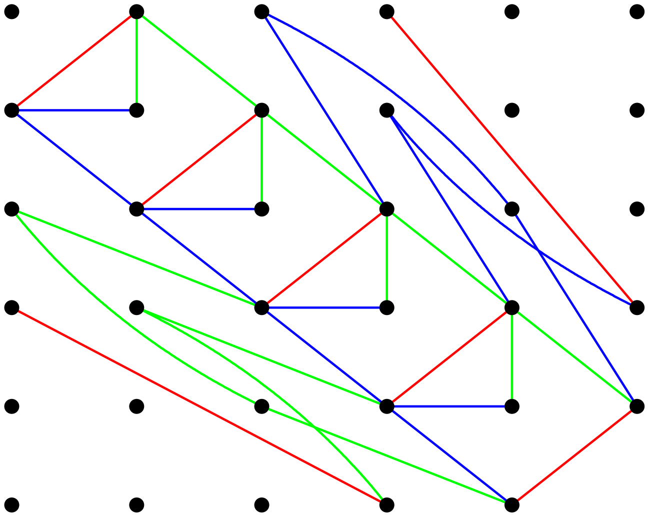

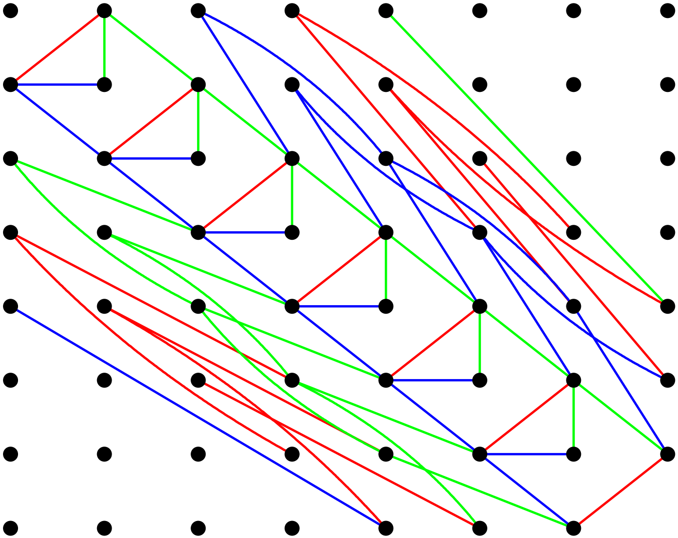

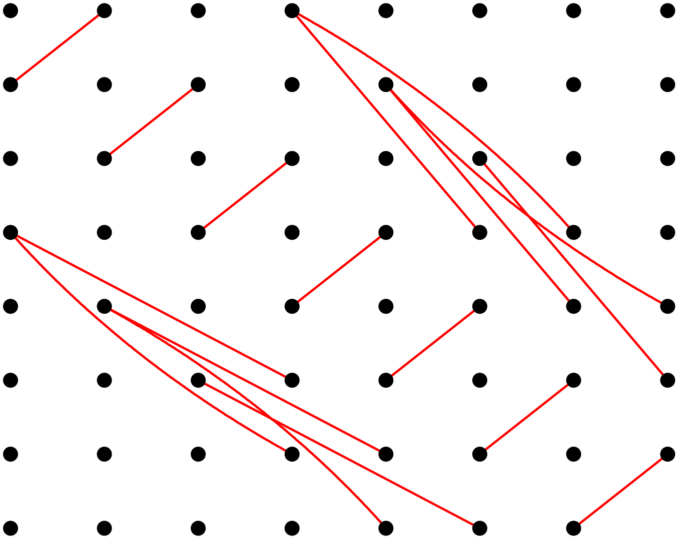

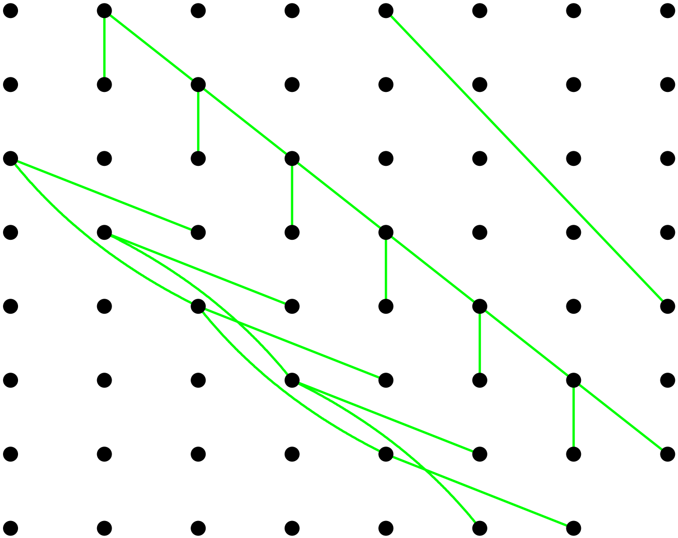

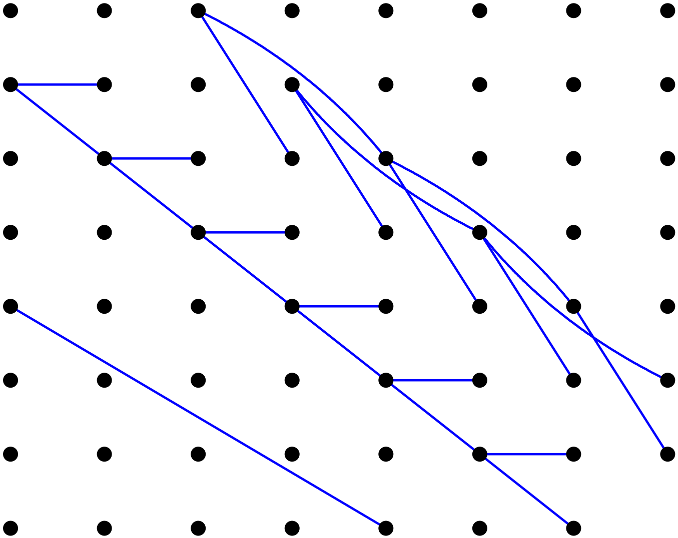

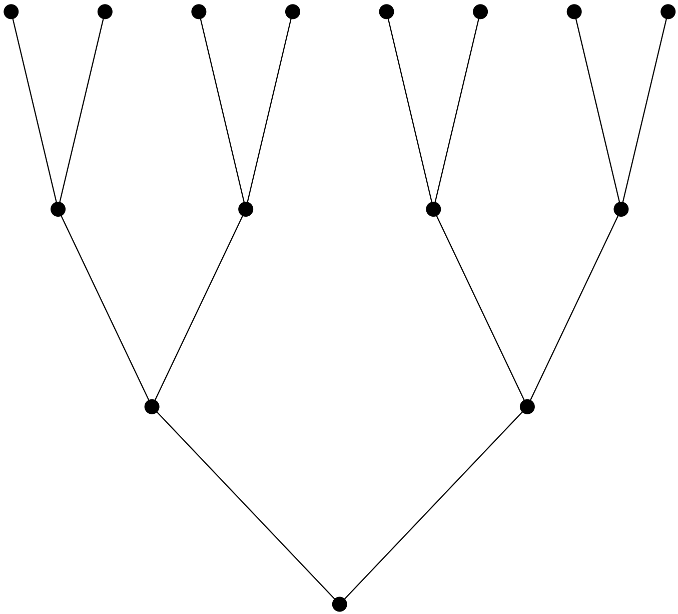

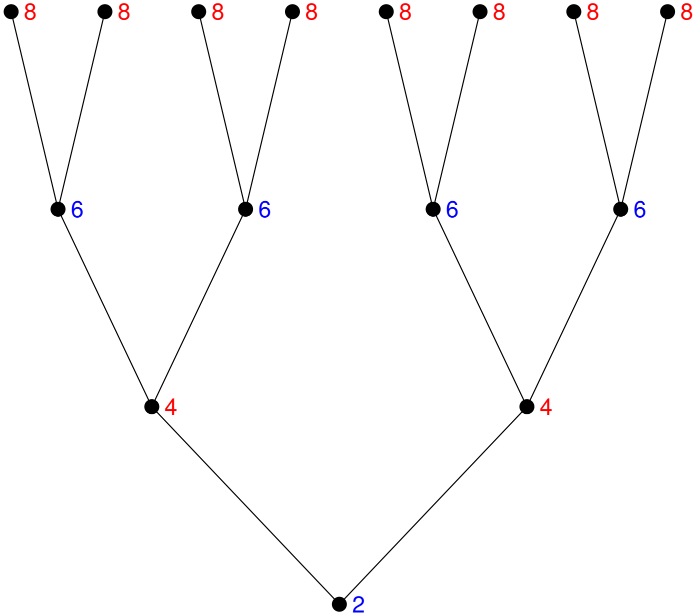
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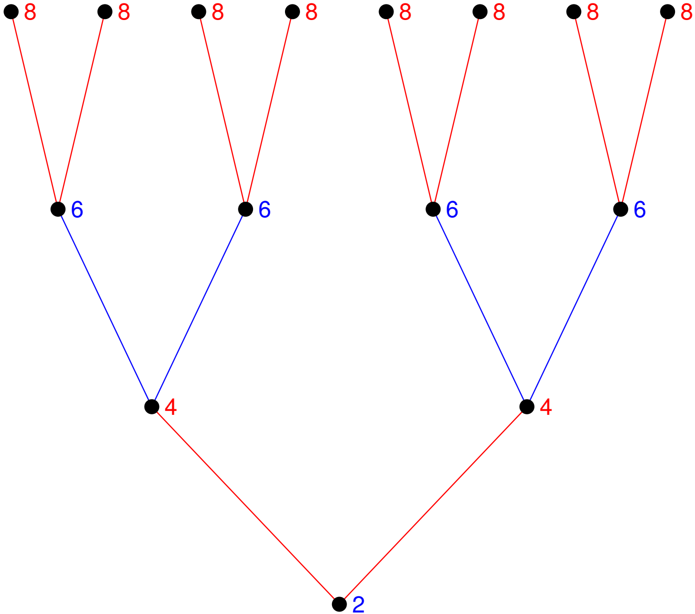
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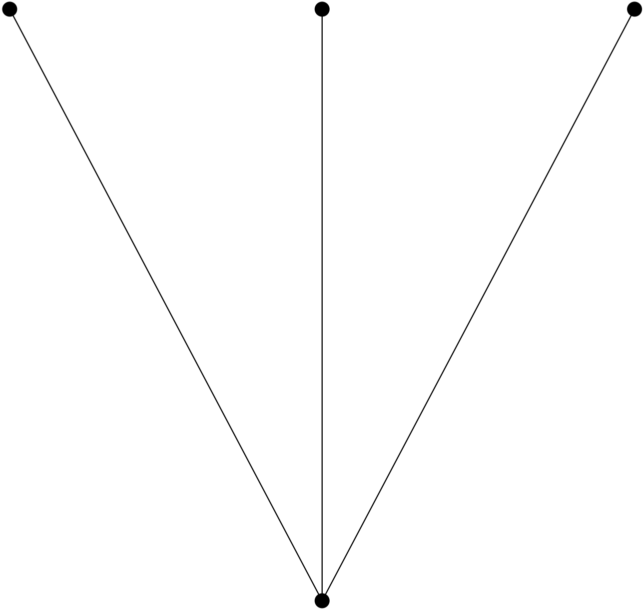

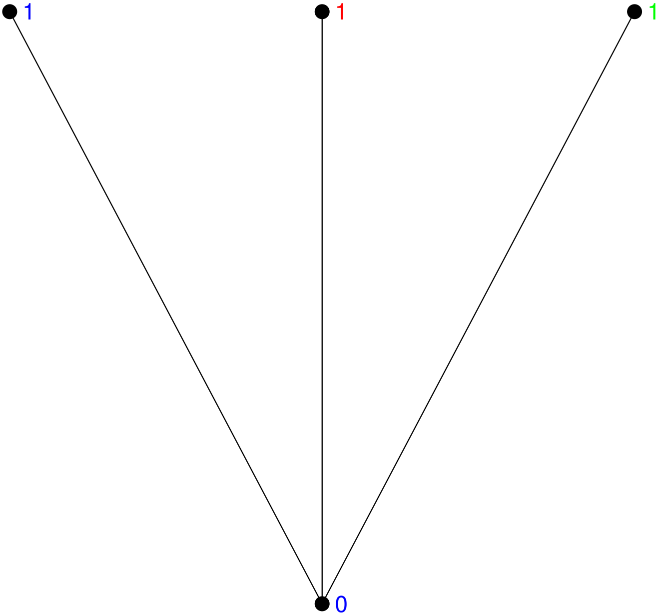
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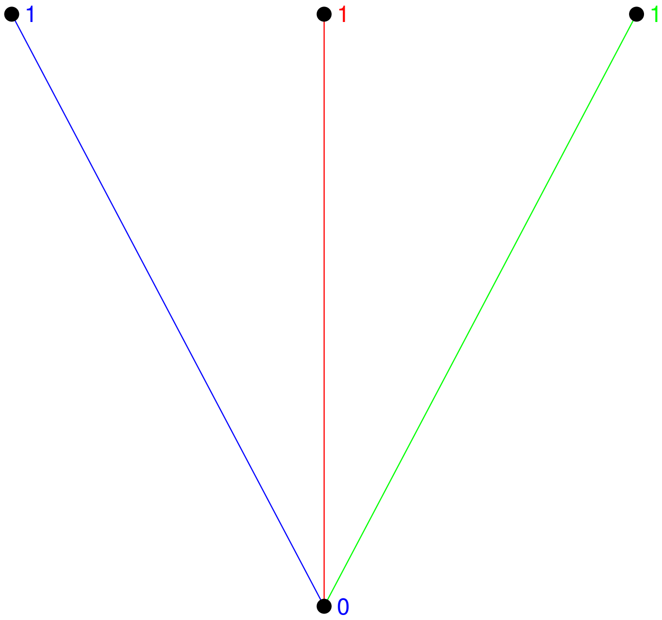
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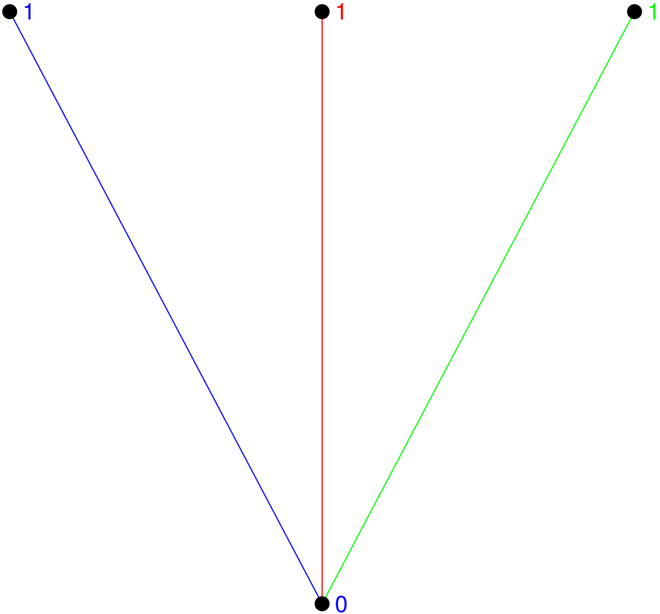
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