A Meeting Point for Analysis, Arithmetic and Geometry

A. Aizenbud

Weizmann Institute of Science

Joint with Nir Avni

http://www.wisdom.weizmann.ac.il/~aizenr/

Analysis

Let f be a polynomial of n variables (with integer coefficients).

Analysis

Let f be a polynomial of n variables (with integer coefficients).

Question

To compute $Vol(\{x \in \mathbb{R}^n : |f(x)| < \varepsilon\})$

Analysis

Let f be a polynomial of n variables (with integer coefficients).

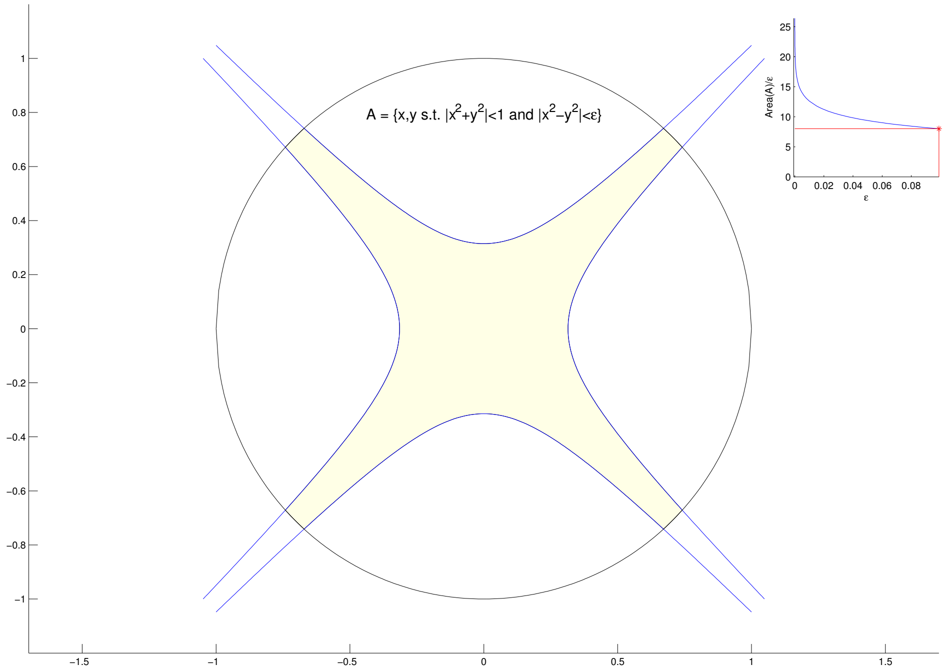
Question

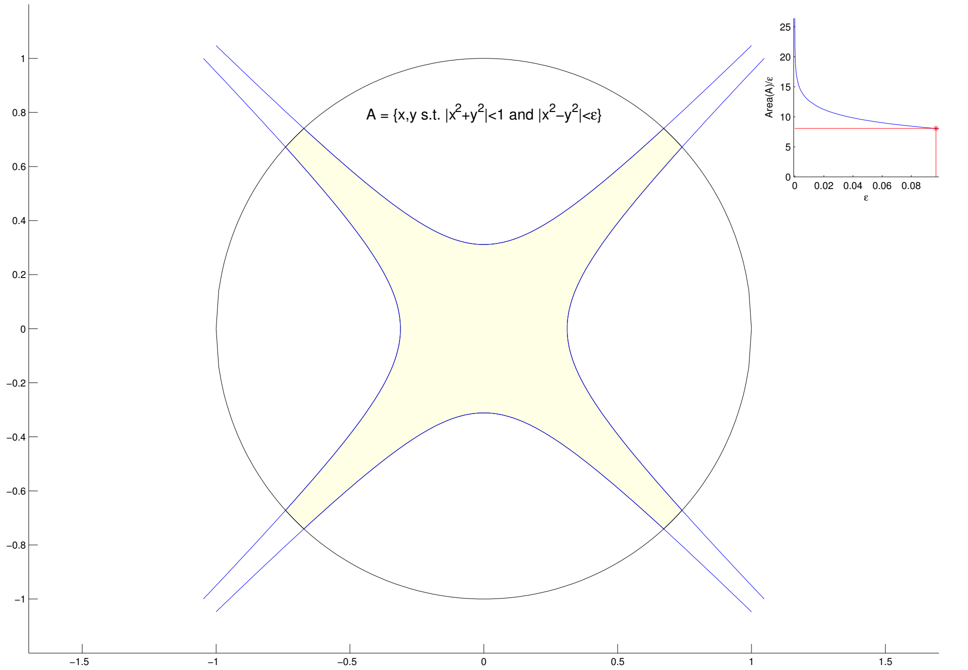
To compute $Vol\left(\{x \in \mathbb{R}^n : |f(x)| < \varepsilon\}\right)$

"Definition"

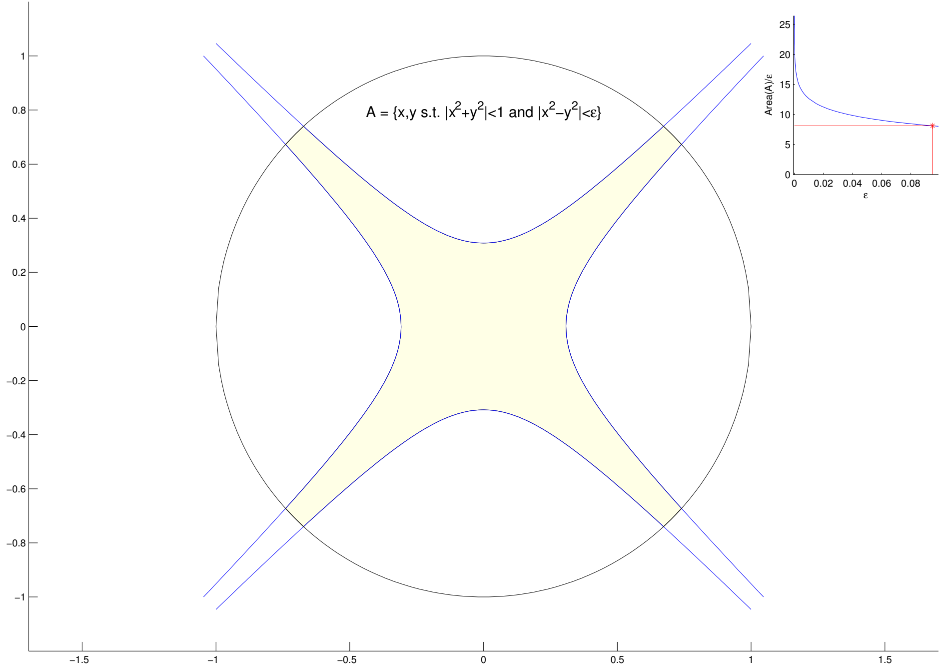
We say that f *is analytically good if*

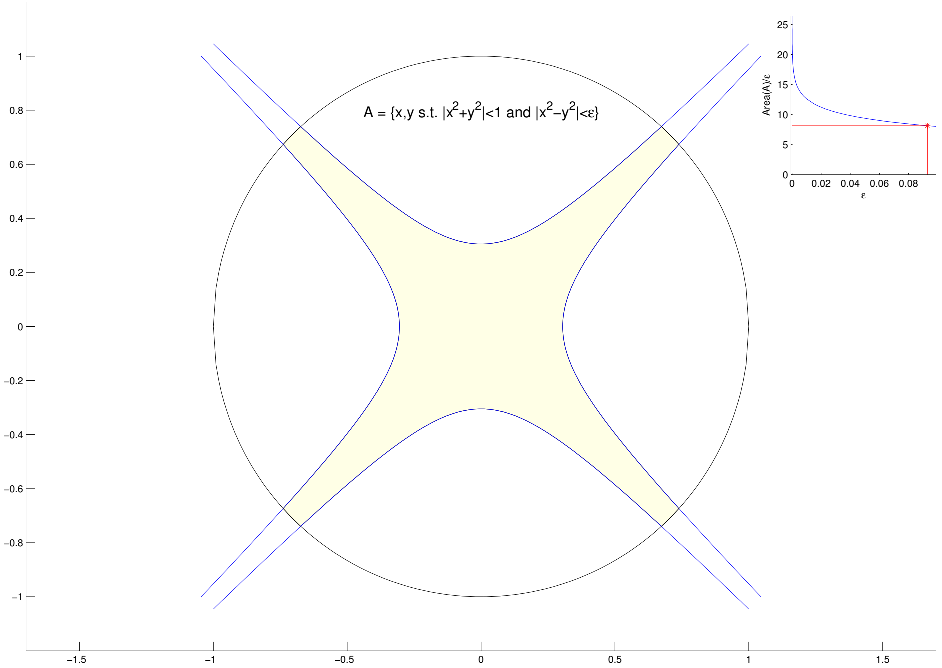
$$\lim_{\varepsilon \to 0} \frac{Vol\left(\{x \in \mathbb{R}^n : |f(x)| < \varepsilon; ||x|| < 1\}\right)}{\varepsilon} < \infty$$


A = {x,y s.t. |x2+y2|<1 and |x2−y2|<ε}
Area(A)/ε
ε
0
0.02
0.04
0.06
0.08
25
20
15
10
5
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−0.5
0.5
1.5

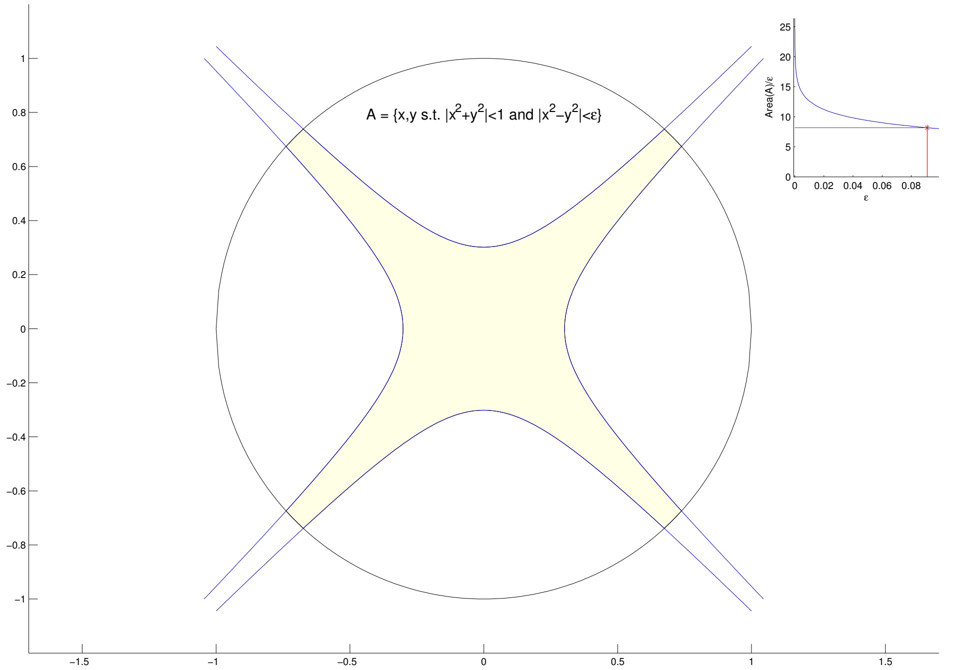


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

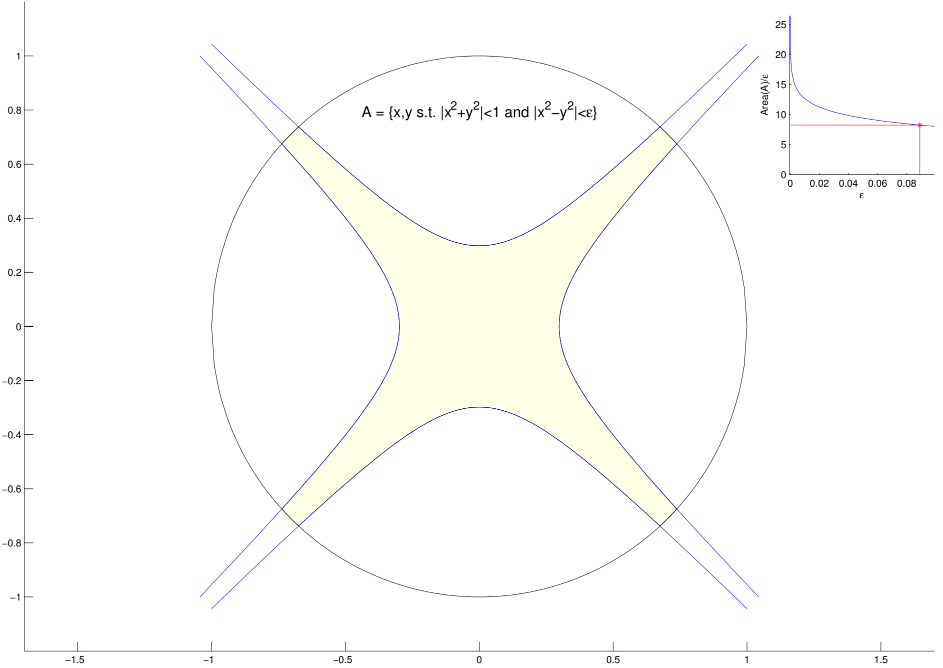

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5



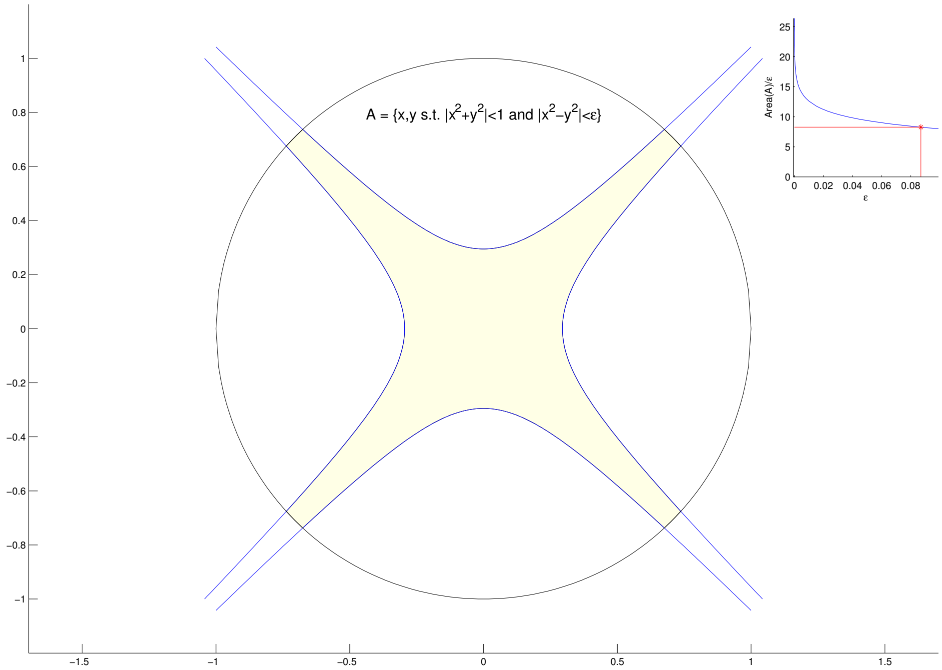
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5

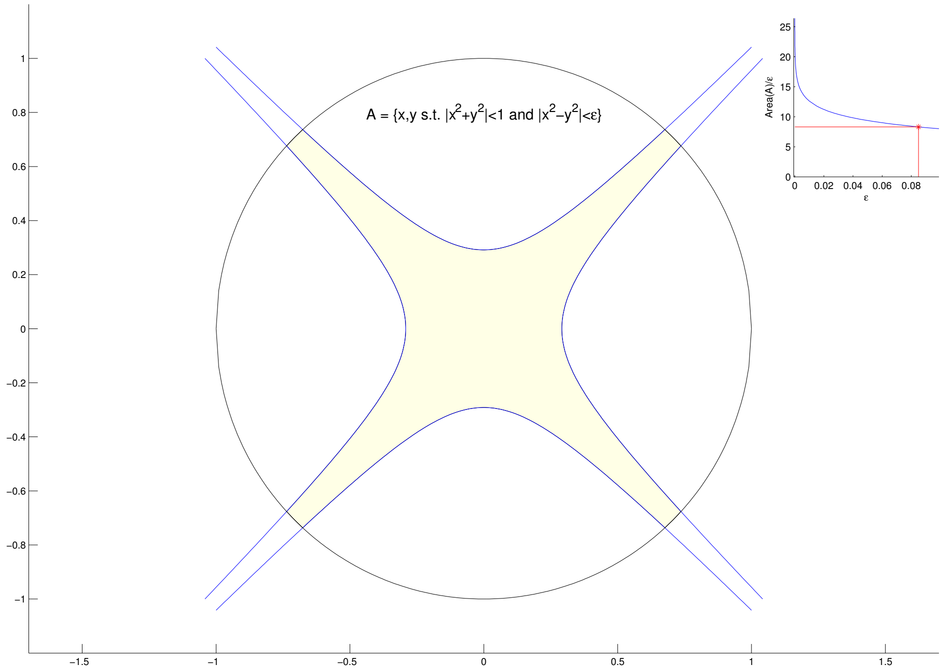


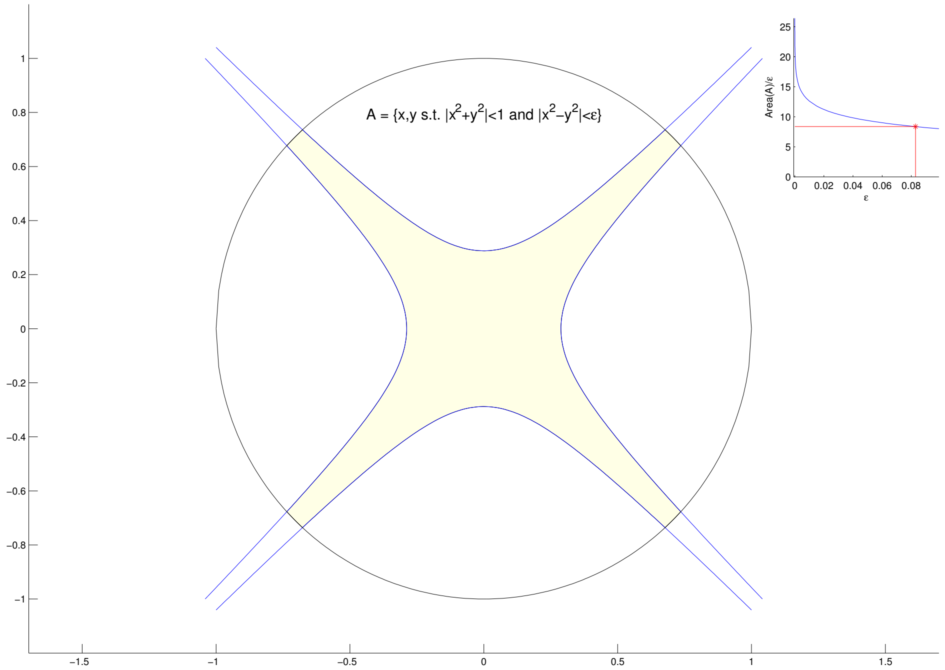
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε



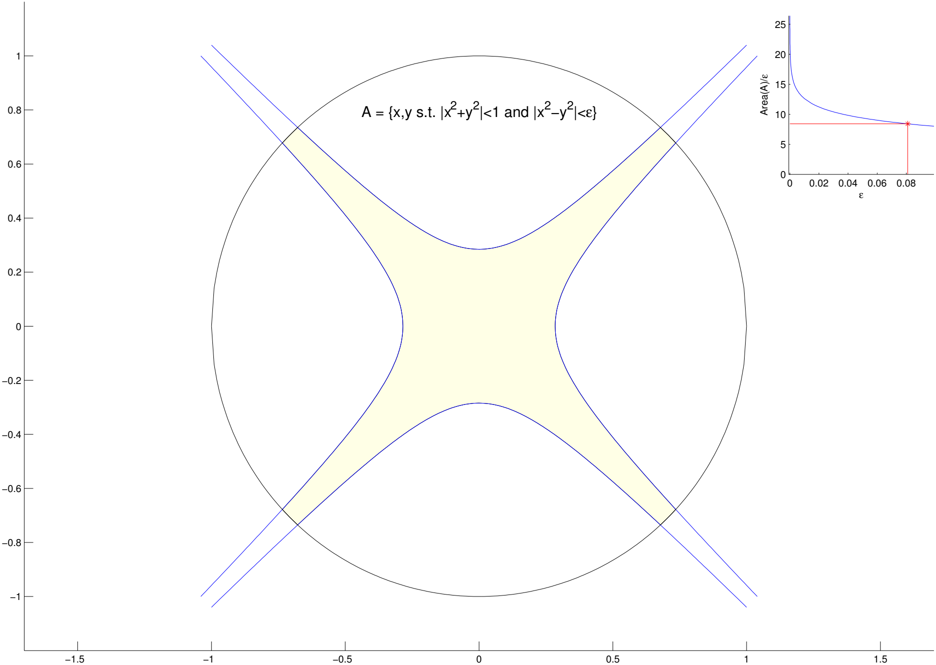
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−0.5
0.5
1.5

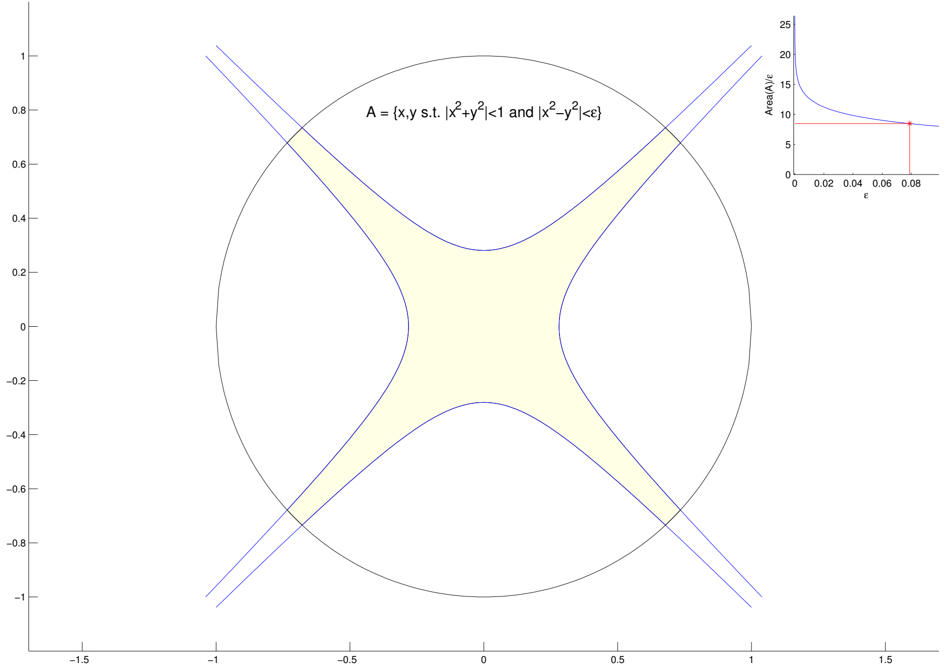

A = {x,y s.t. |x²+y²|<1 and |x²−y²|<ε}
Area(A)/ε
ε



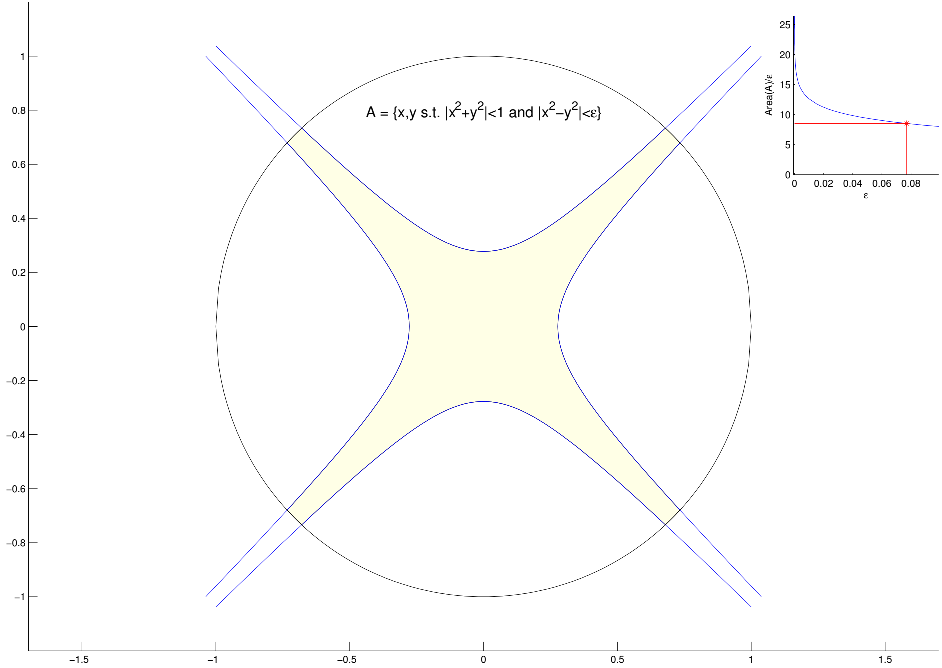
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0
0.02
0.04
0.06
0.08
5
10
15
20
25
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−0.5
0.5
1.5



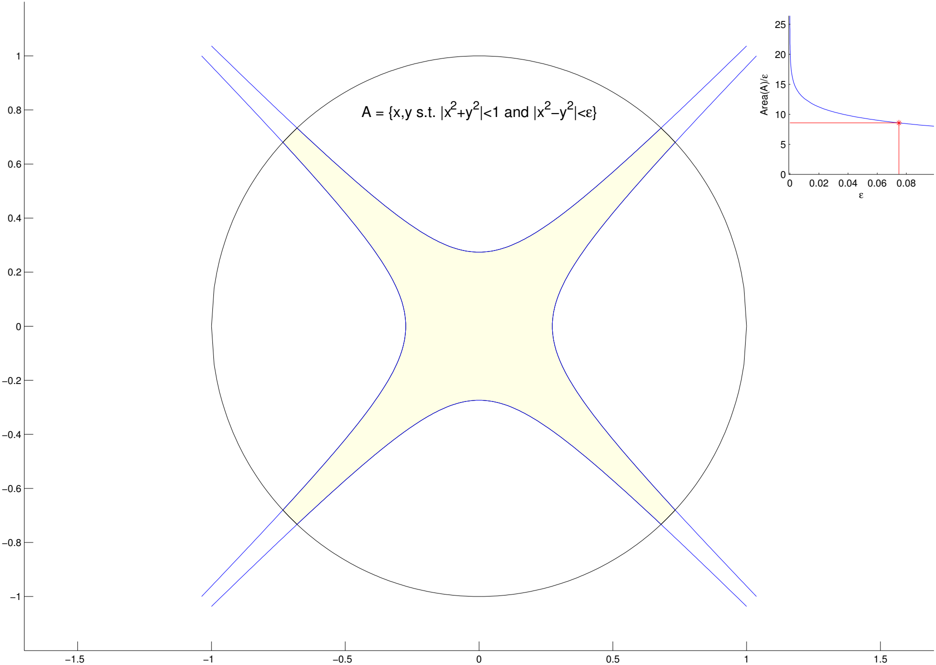
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε



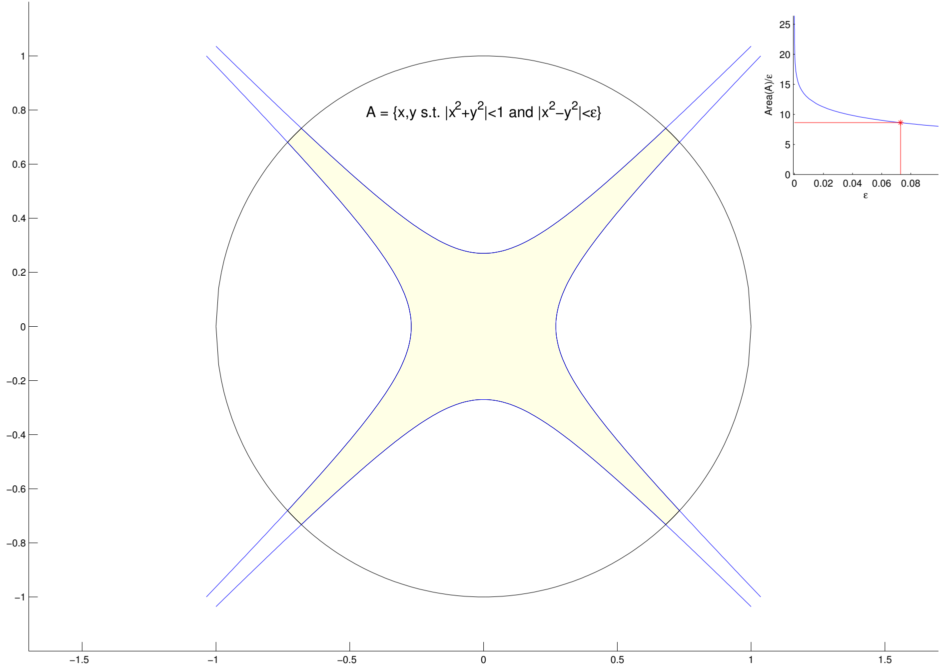
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5
Area(A)/ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
ε

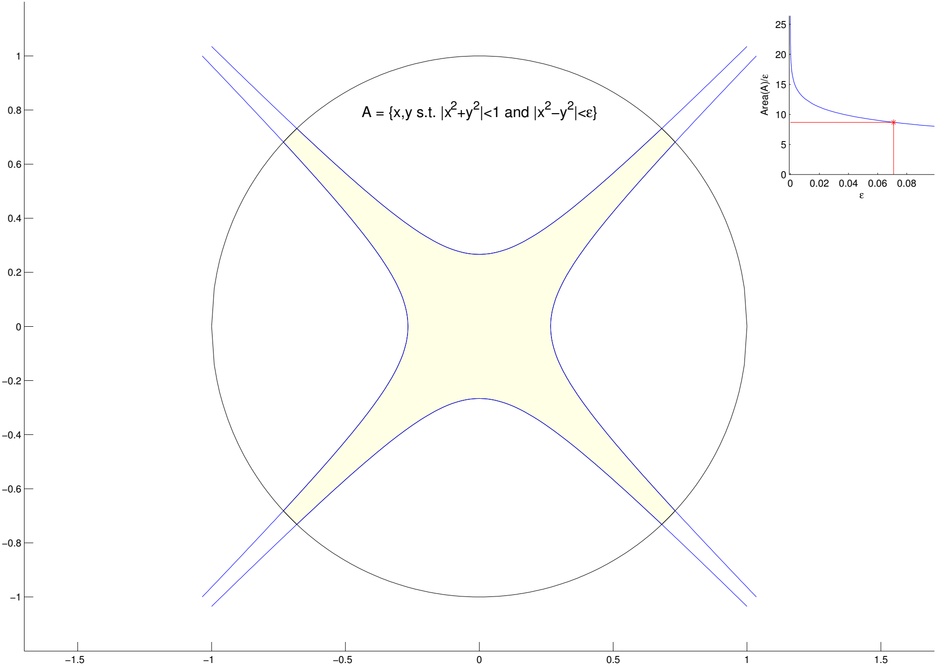


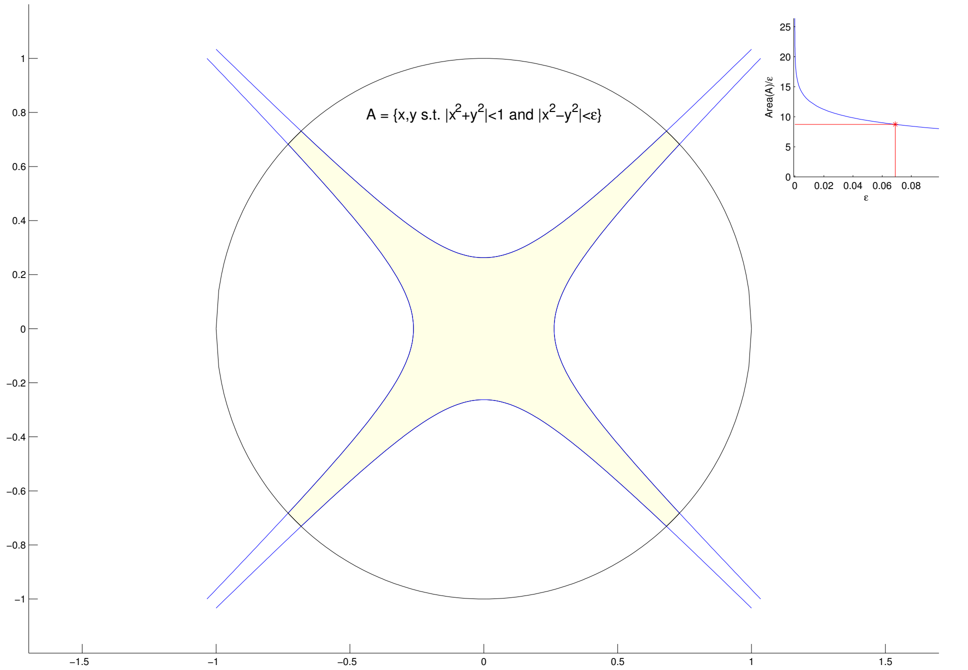
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε



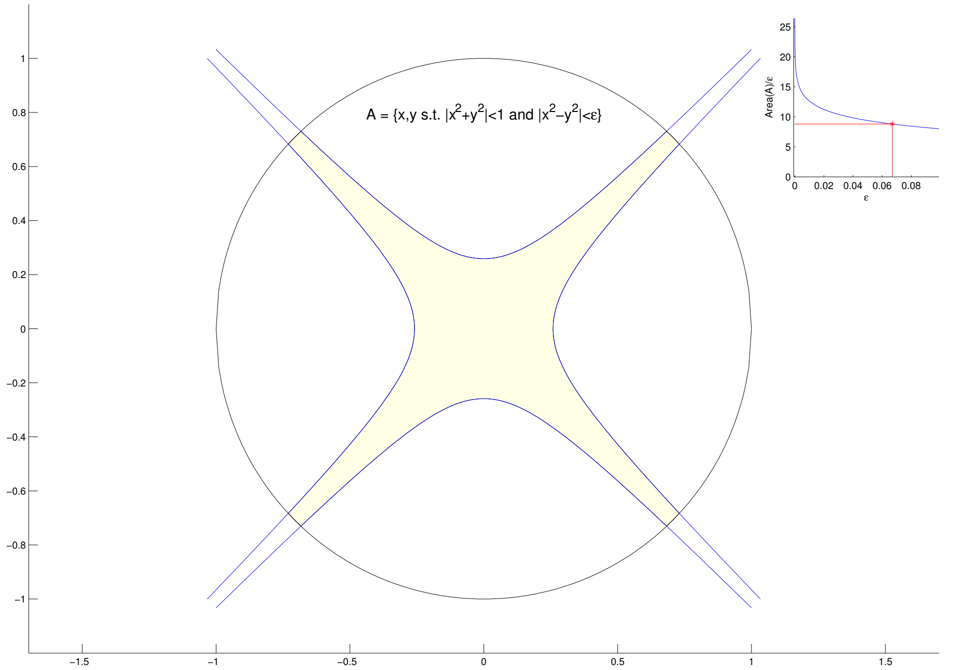
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5
Area(A)/ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
ε

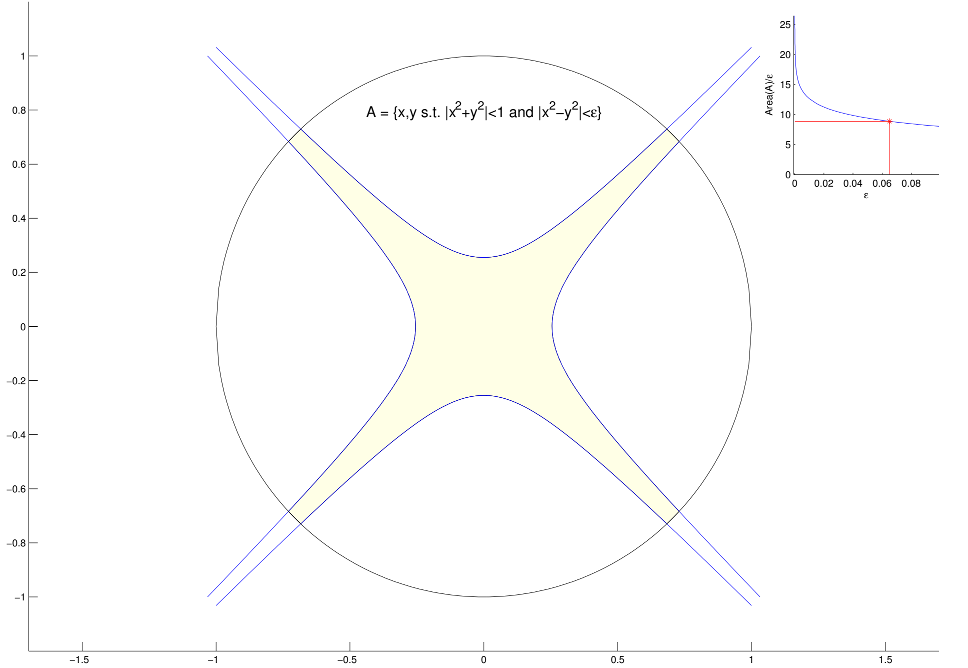

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0
0.02
0.04
0.06
0.08
5
10
15
20
25
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−0.5
0.5
1.5

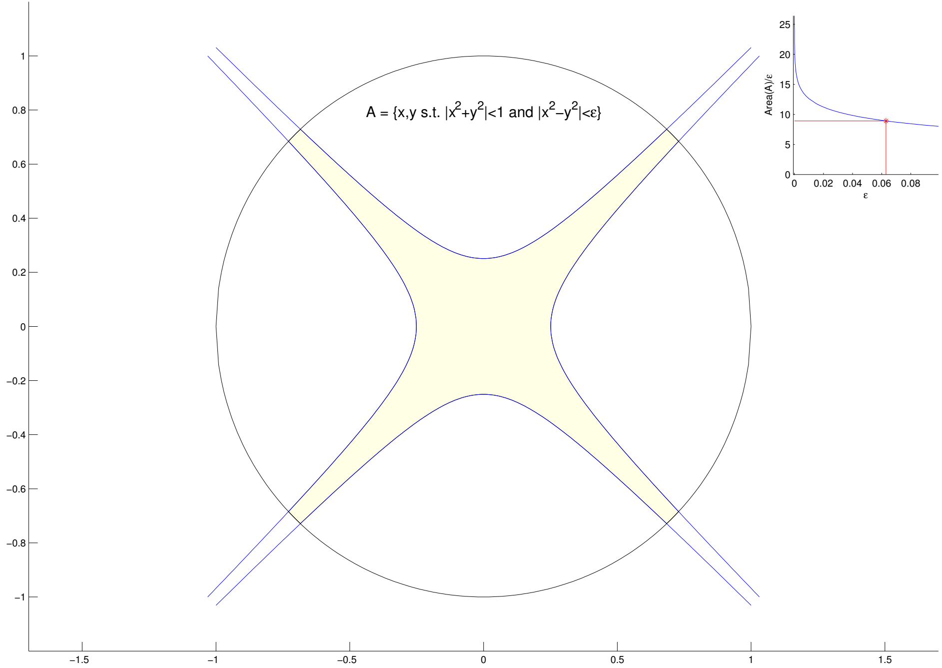


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

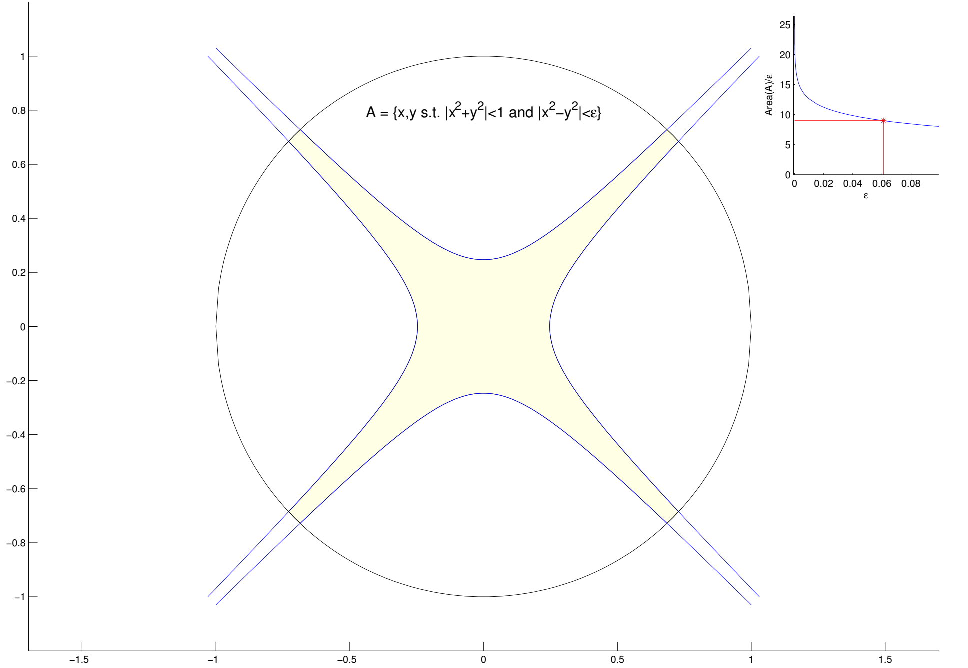


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

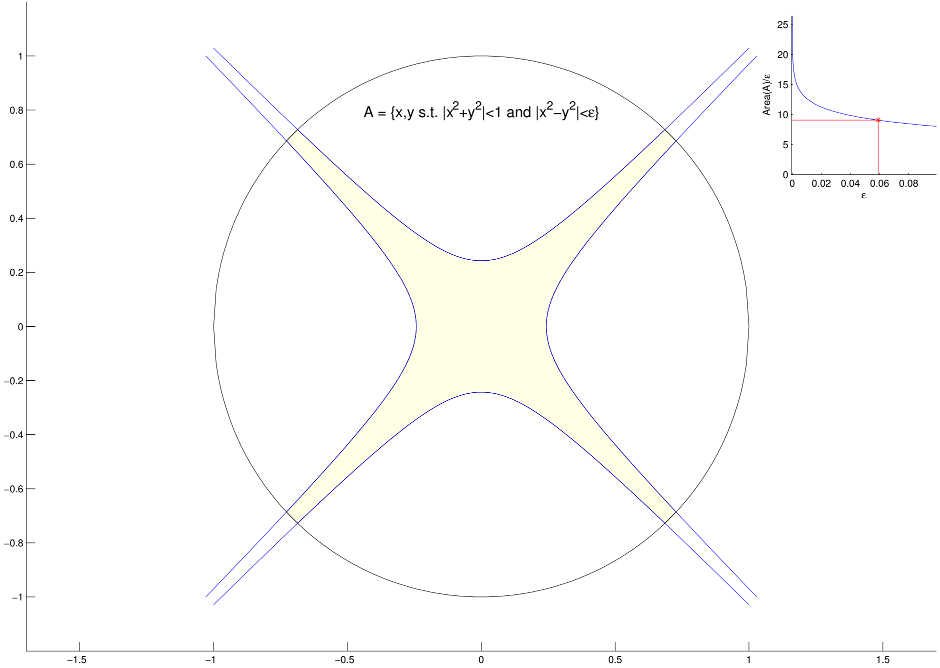

A = {x,y s.t. |x²+y²|<1 and |x²−y²|<ε}
Area(A)/ε
ε

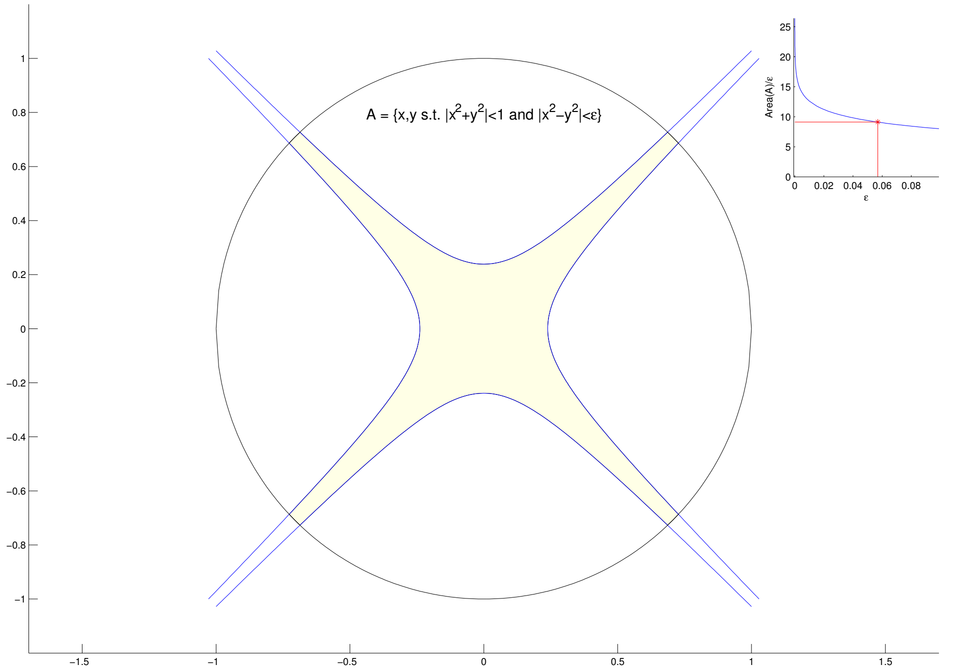


A = {x,y s.t. |x²+y²|<1 and |x²−y²|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5

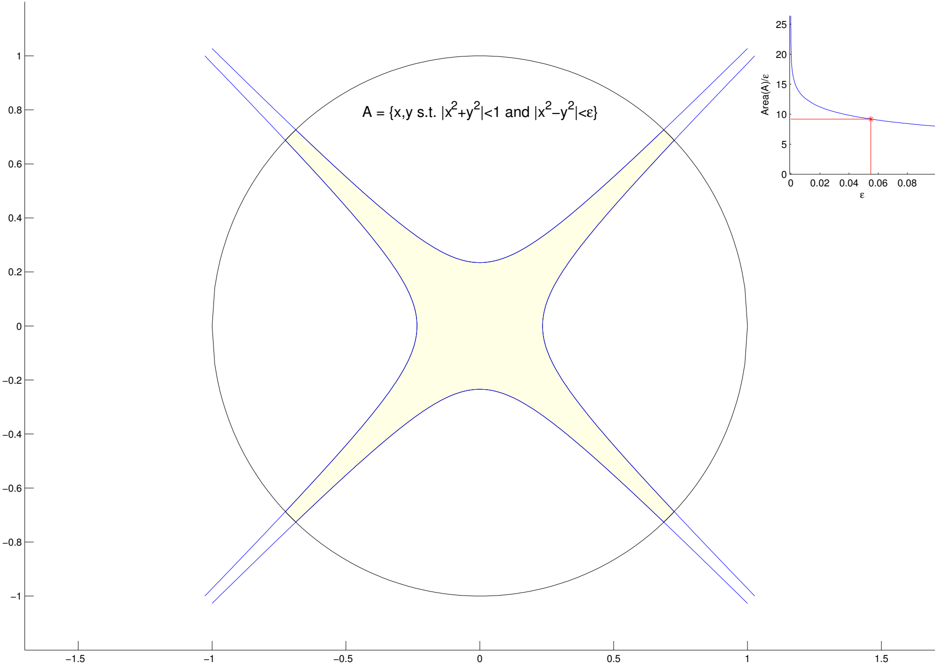


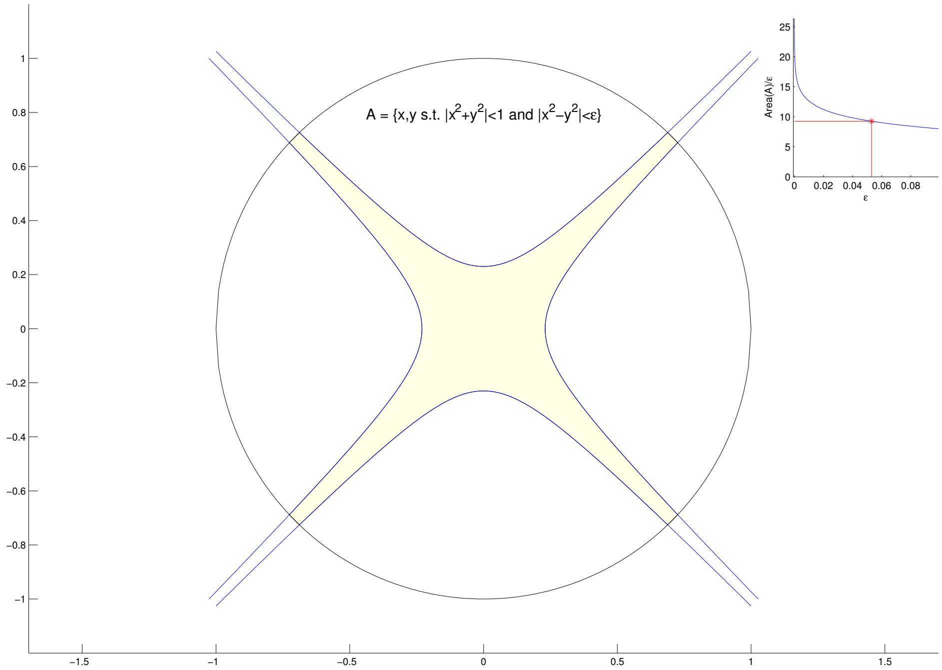
A = {x,y s.t. |x2+y2|<1 and |x2−y2|<ε}
Area(A)/ε
ε


A = {x,y s.t. |x2+y2|<1 and |x2−y2|<ε}
Area(A)/ε
ε
0
0.02
0.04
0.06
0.08
5
10
15
20
25
−1.5
−1
−0.5
0.5
1
1.5
−0.8
−0.6
−0.4
−0.2
0.2
0.4
0.6
0.8

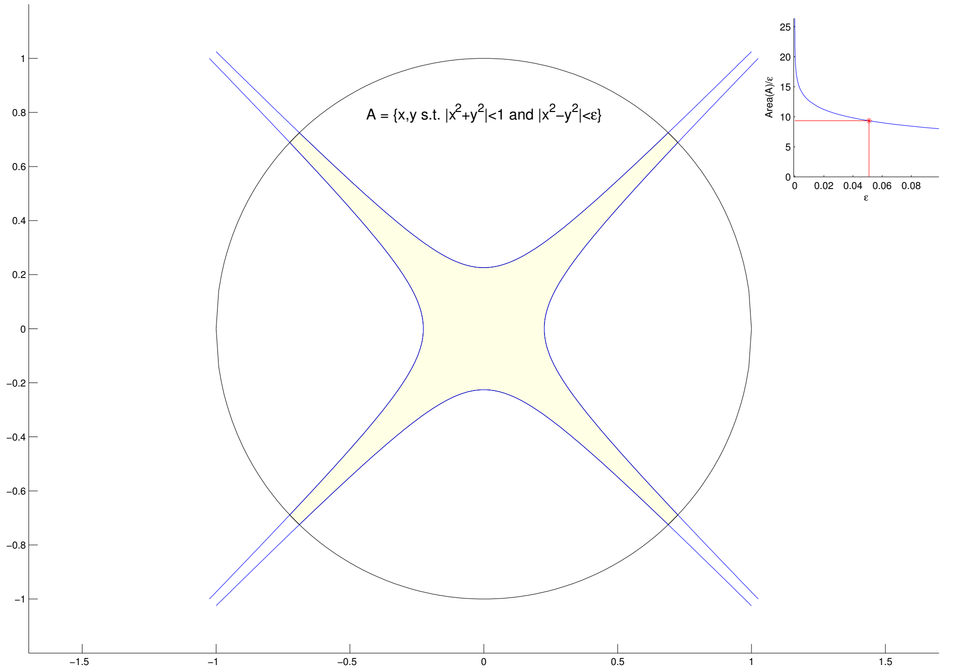


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

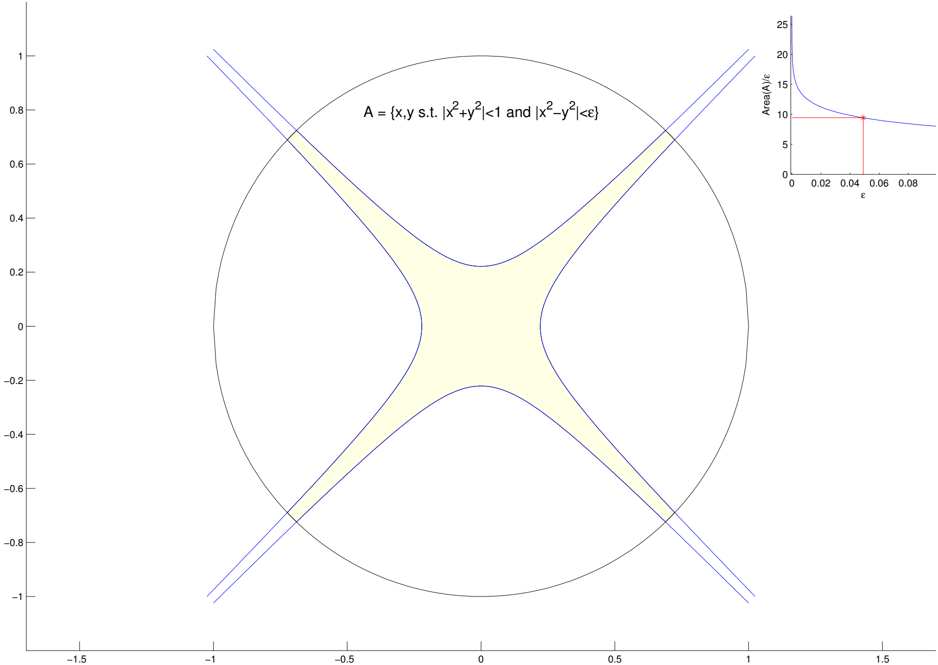

A = {x,y s.t. |x2+y2|<1 and |x2−y2|<ε}
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5
Area(A)/ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
ε

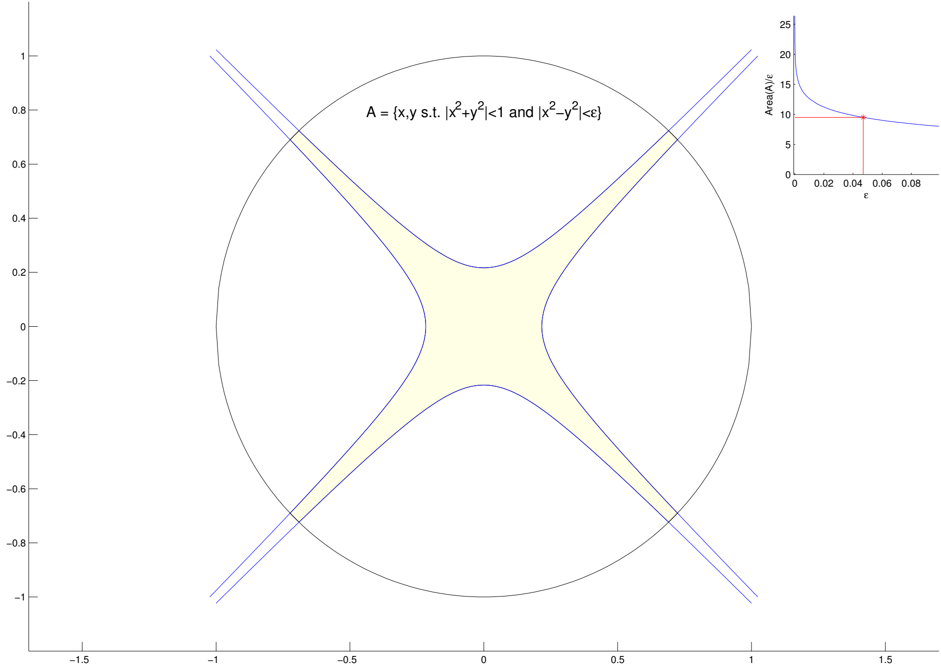


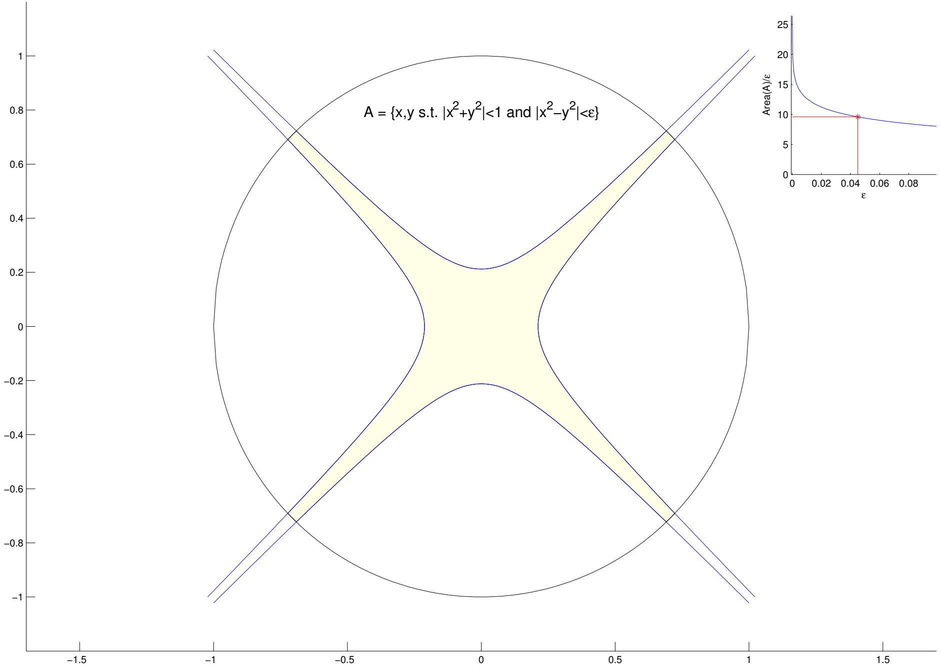
A = {x,y s.t. |x2+y2|<1 and |x2−y2|<ε}
Area(A)/ε
ε

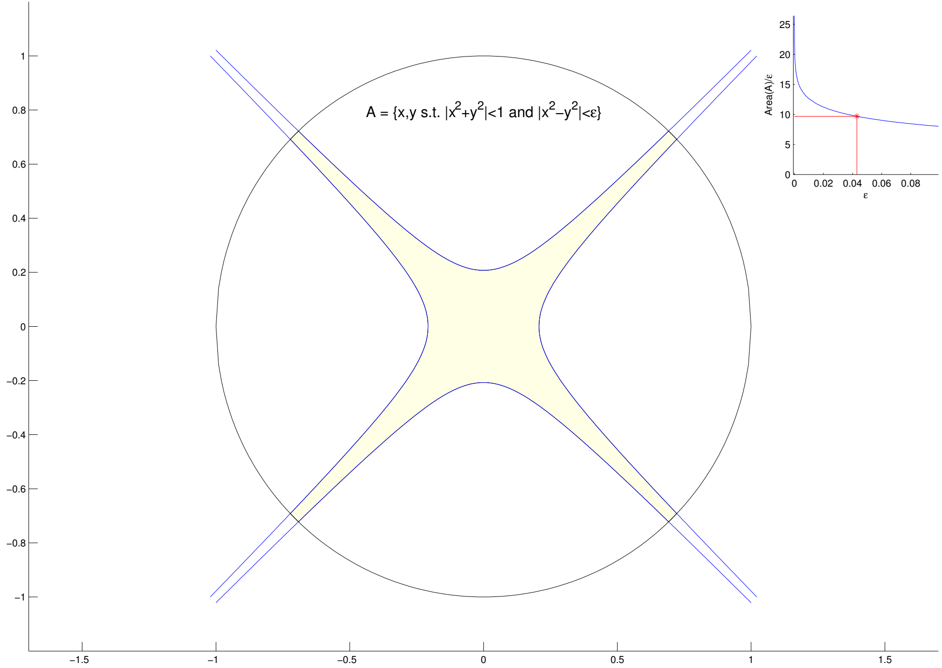


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

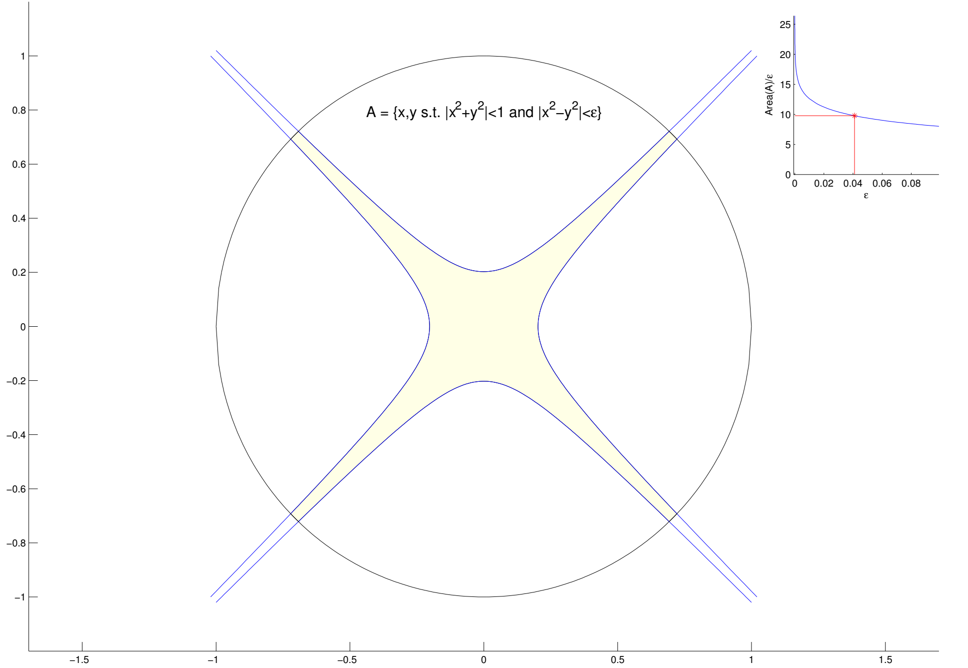

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0 0.02 0.04 0.06 0.08
0 5 10 15 20 25
−1.5 −1 −0.5 0 0.5 1 1.5
−1 −0.8 −0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8 1


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

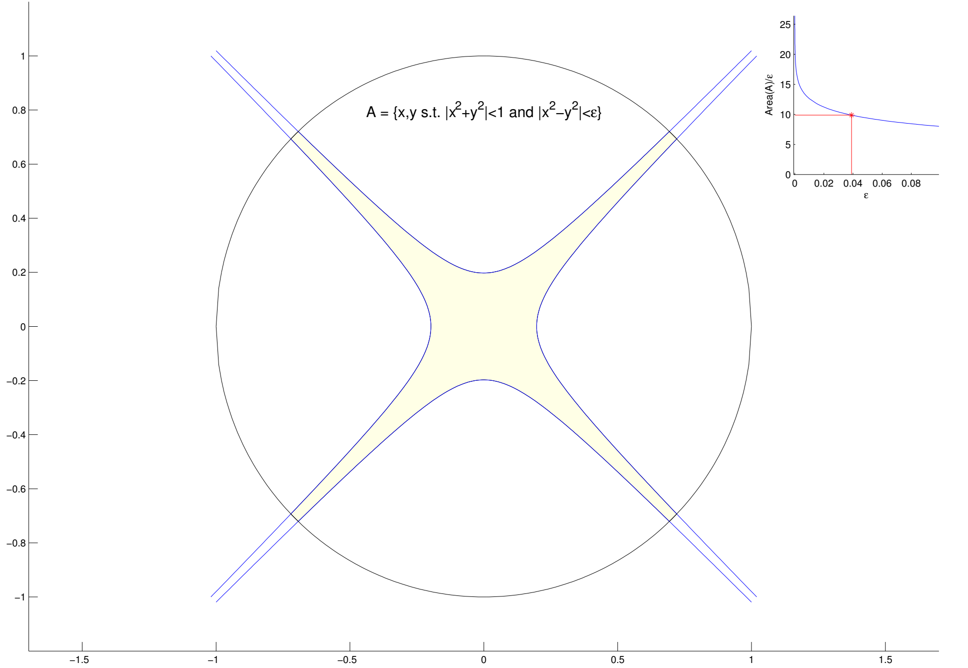

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−0.5
0.5
1.5



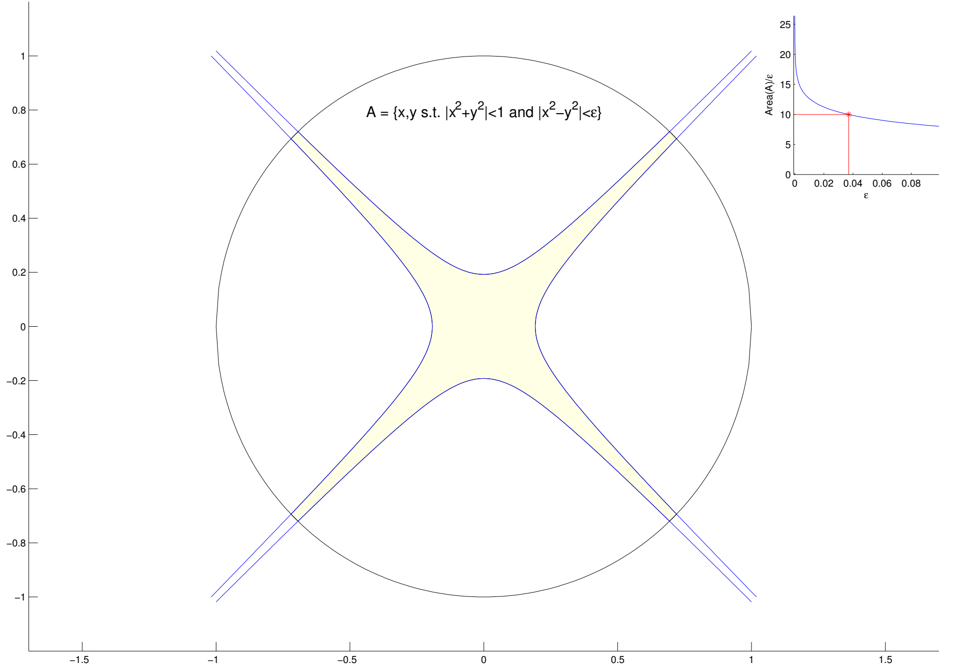
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

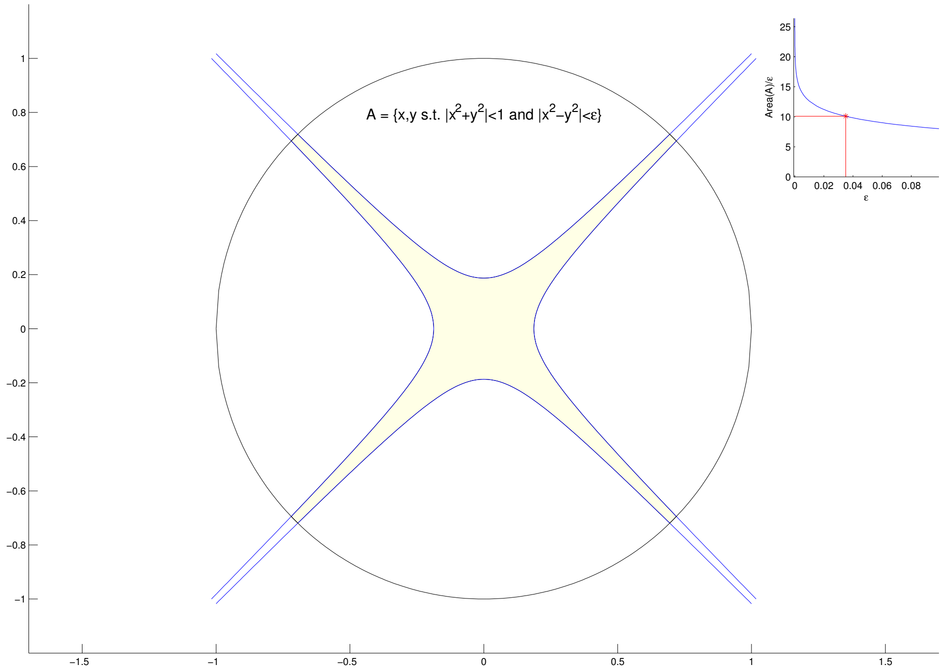


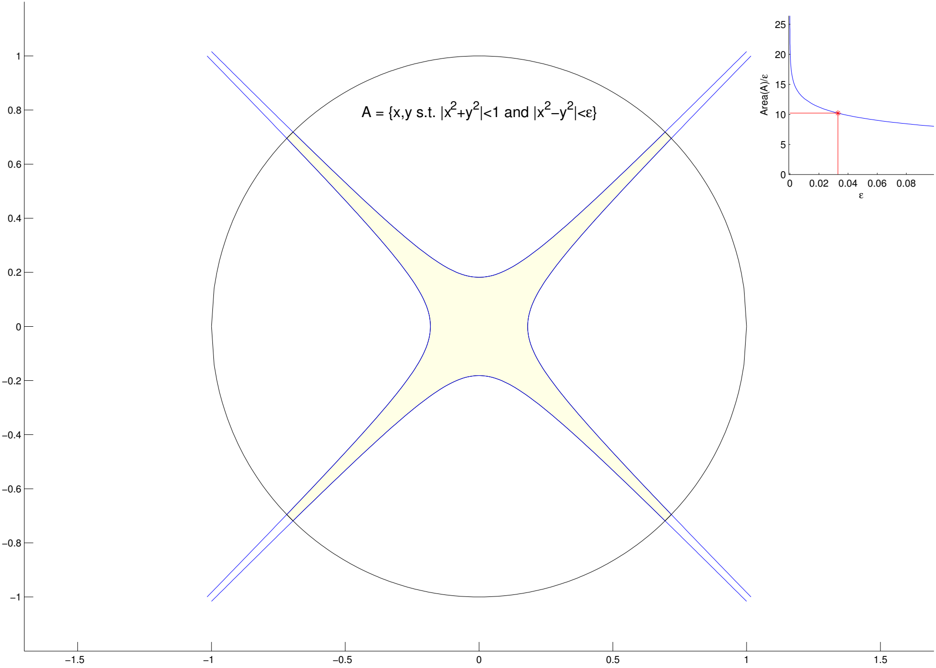
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0
0.02
0.04
0.06
0.08
5
10
15
20
25
−1.5
−1
−0.5
0.5
1
1.5
−0.2
−0.4
−0.6
−0.8
0.2
0.4
0.6
0.8



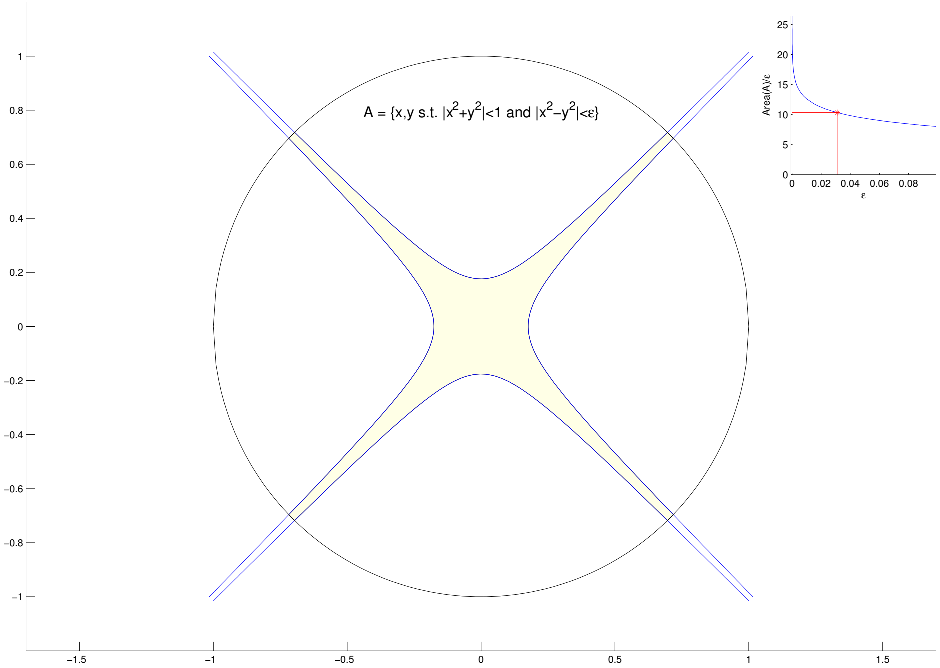
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

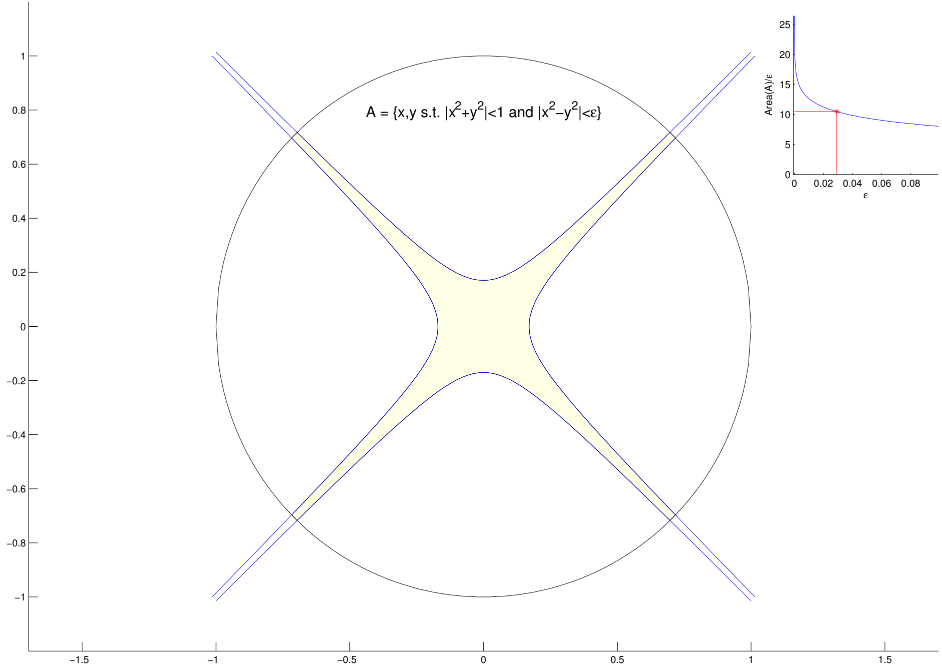

A = {x,y s.t. |x2+y2|<1 and |x2−y2|<ε}
Area(A)/ε
ε
−1.5
−1
−0.5
0
0.5
1
1.5
0.02
0.04
0.06
0.08
5
10
15
20
25


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

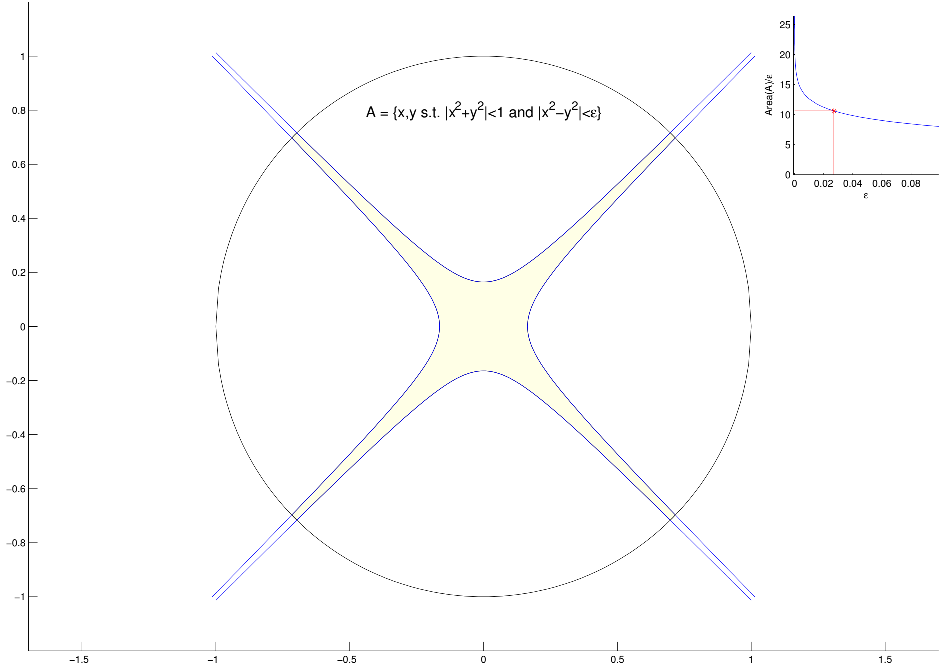


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

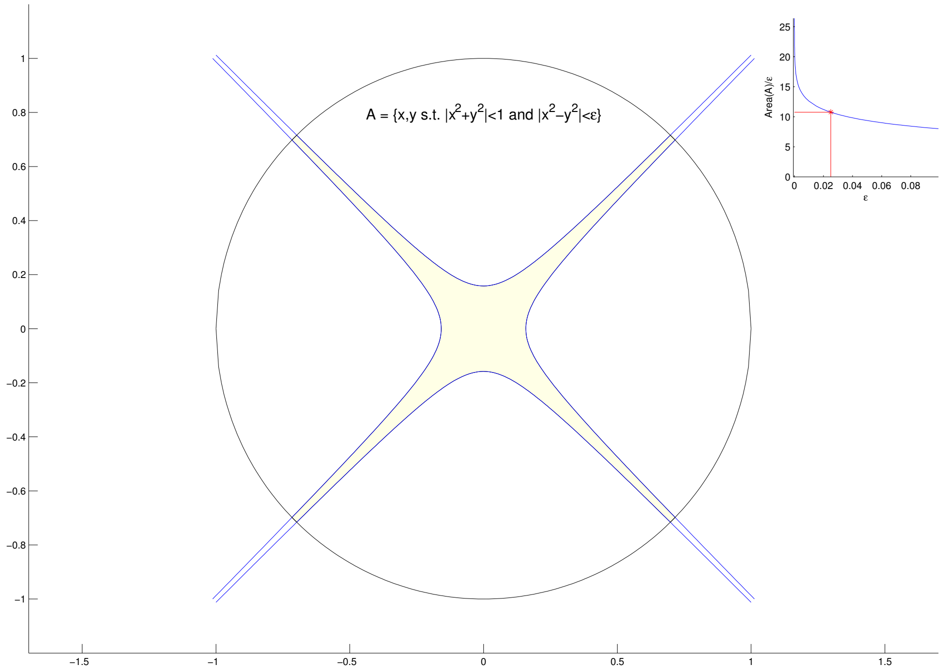

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε



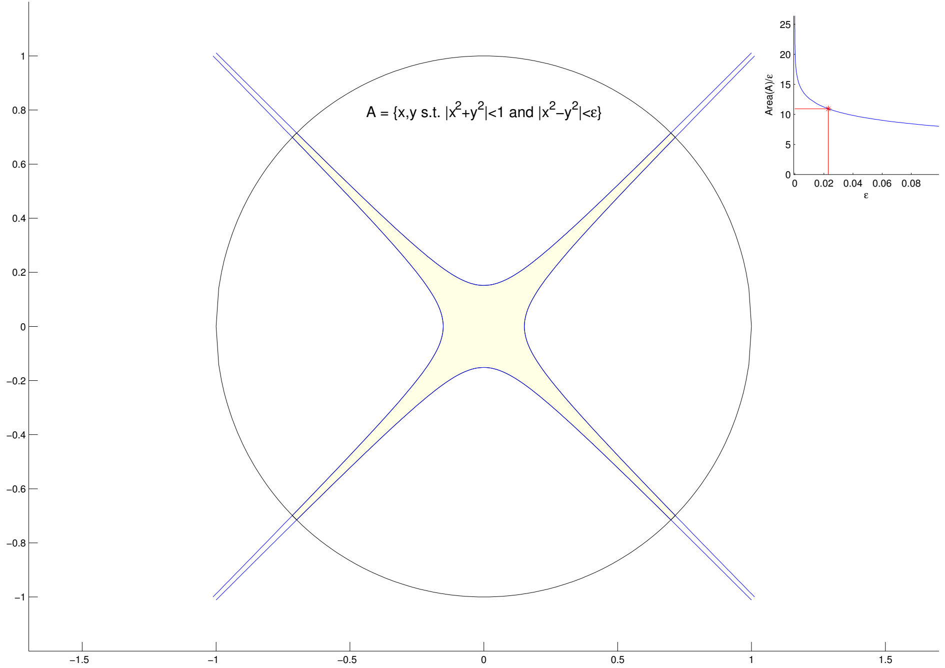
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

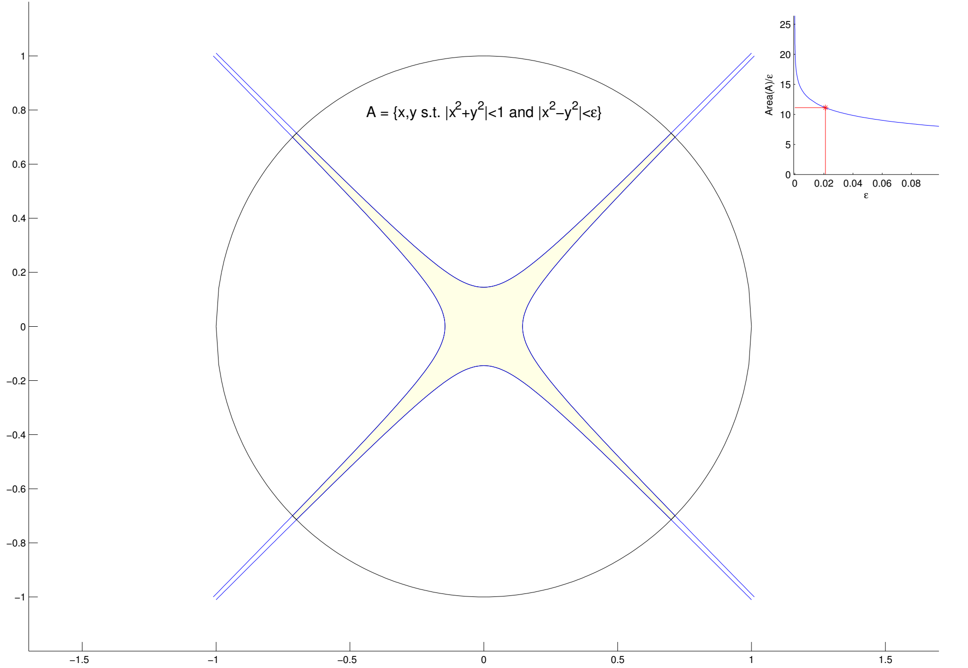


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

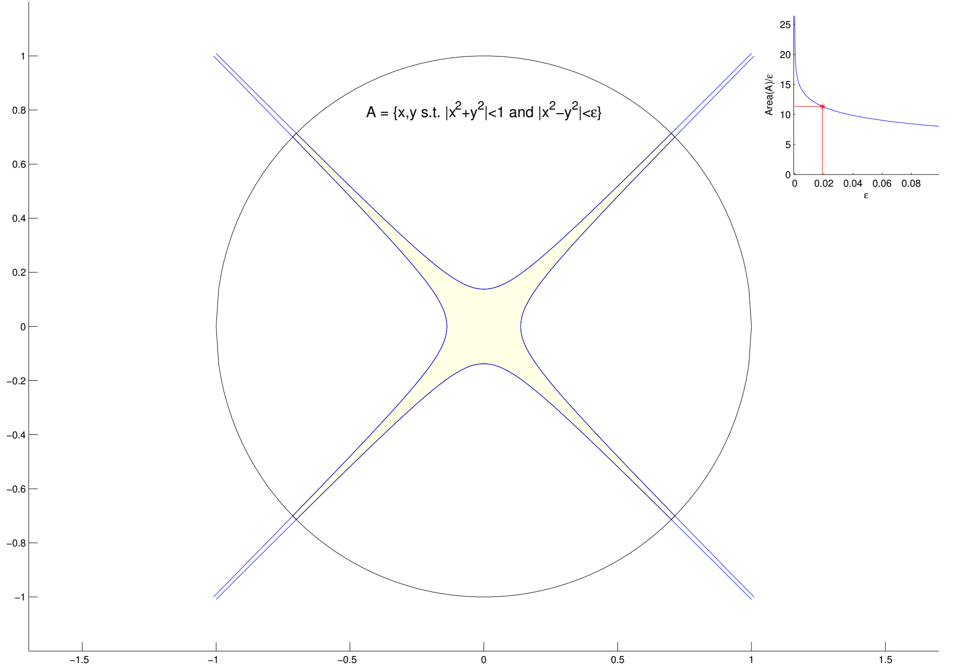


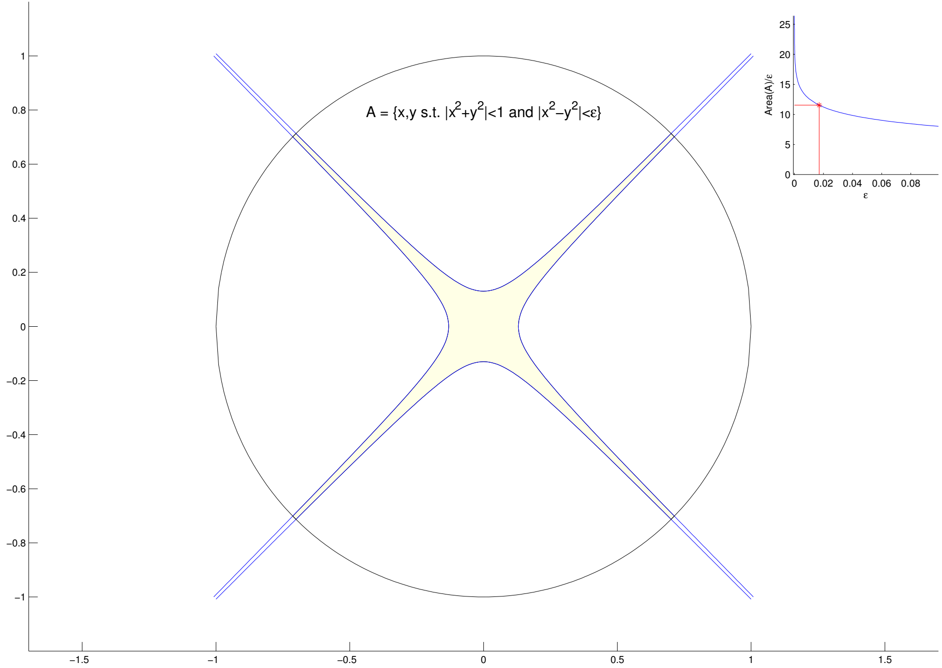
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

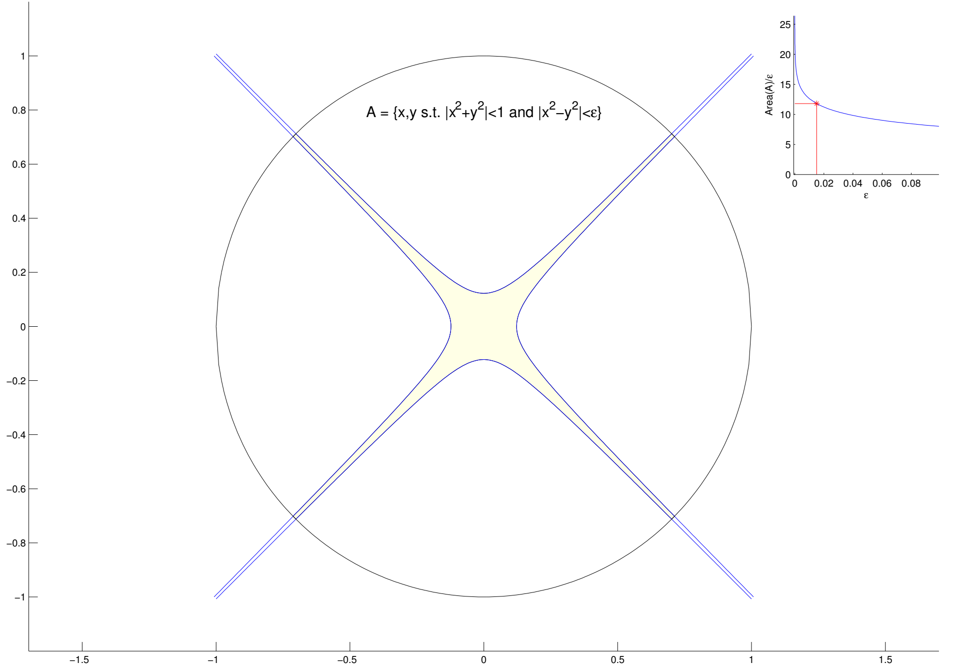

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε



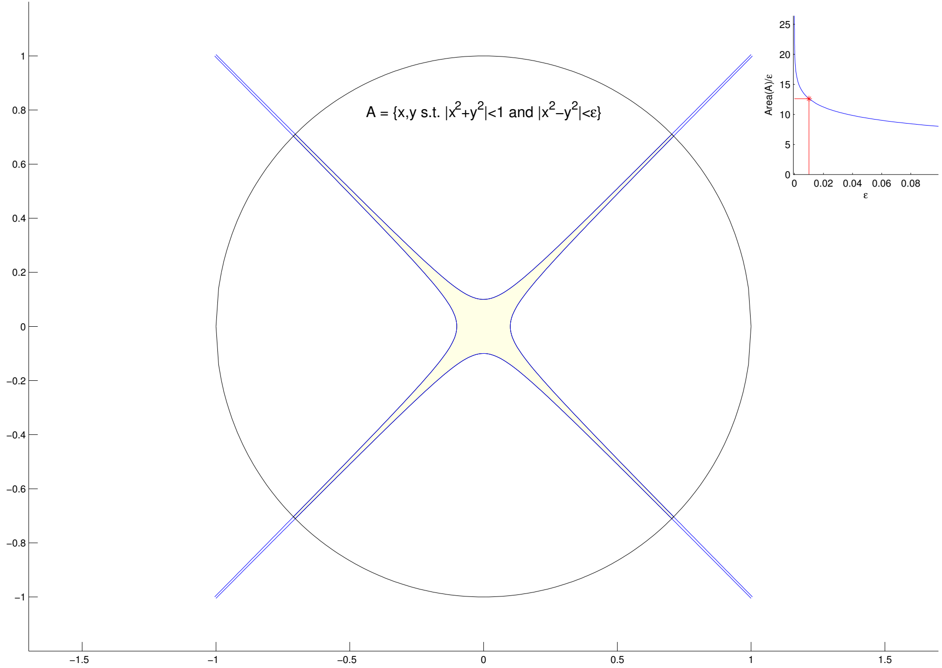
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5


A = {x,y s.t. |x2+y2|<1 and |x2−y2|<ε}
Area(A)/ε
ε

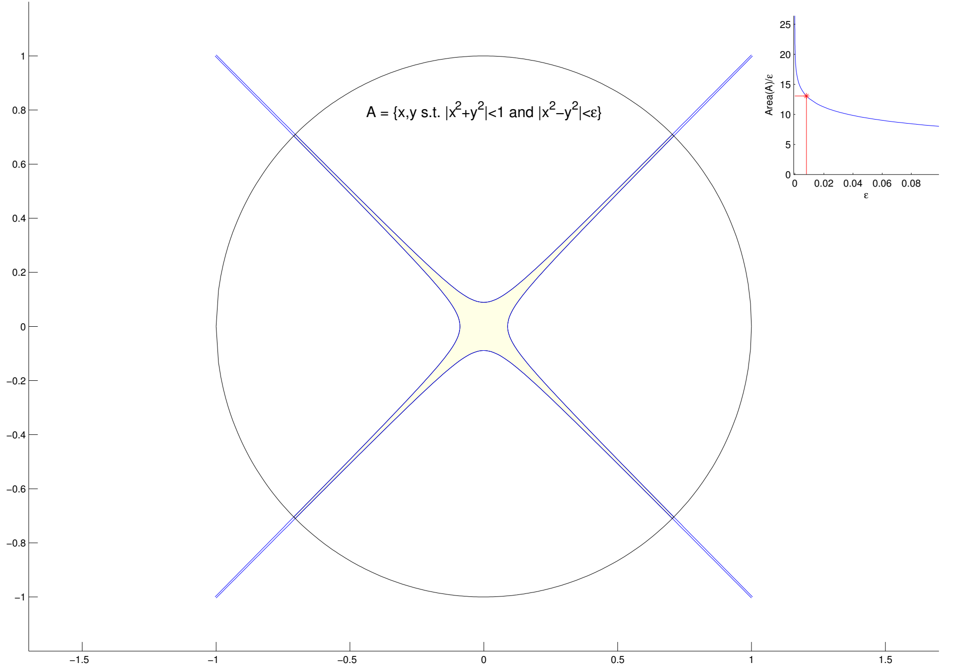

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5

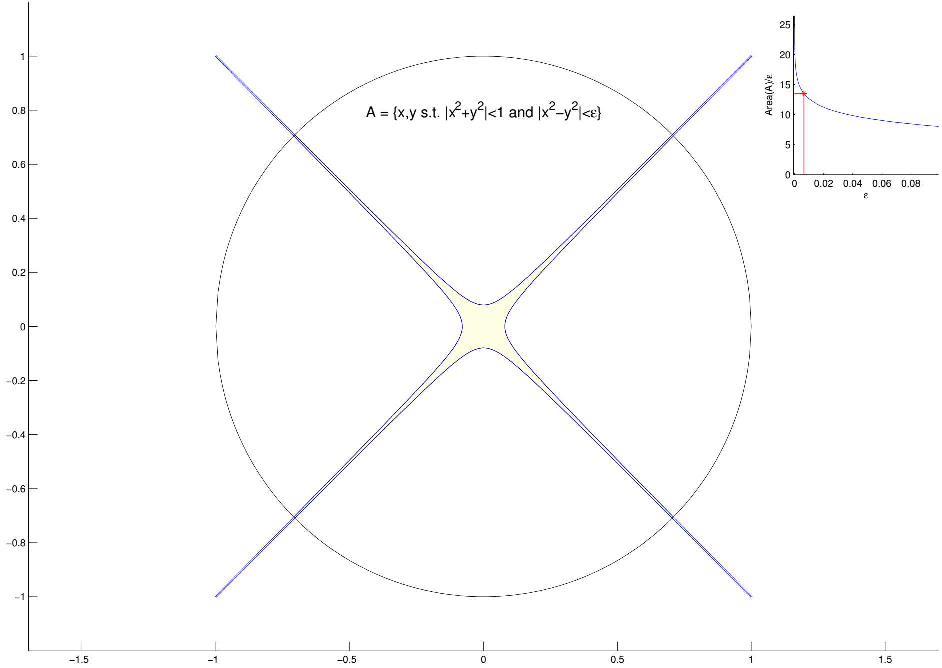


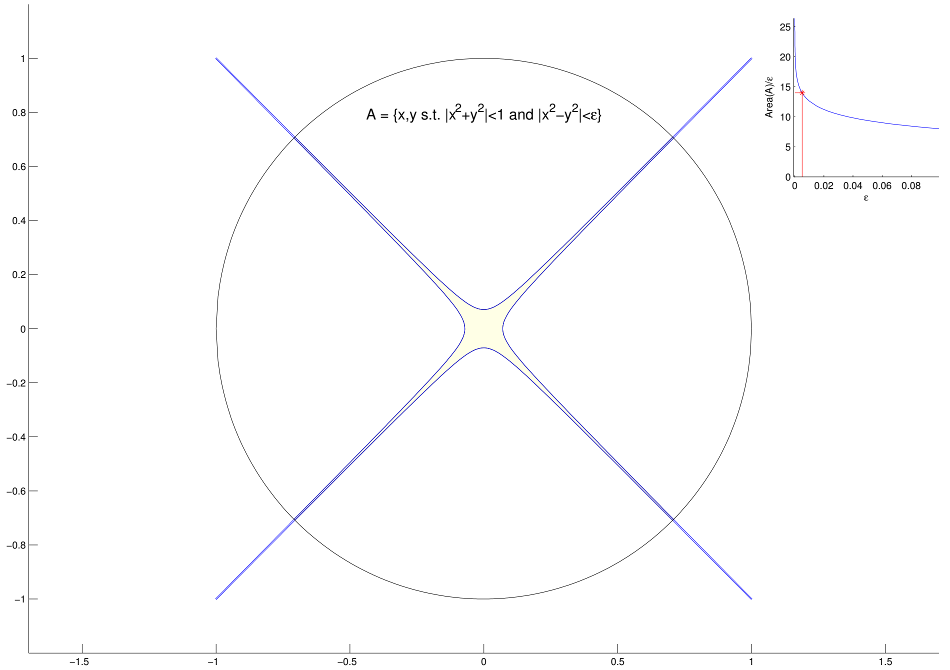
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5

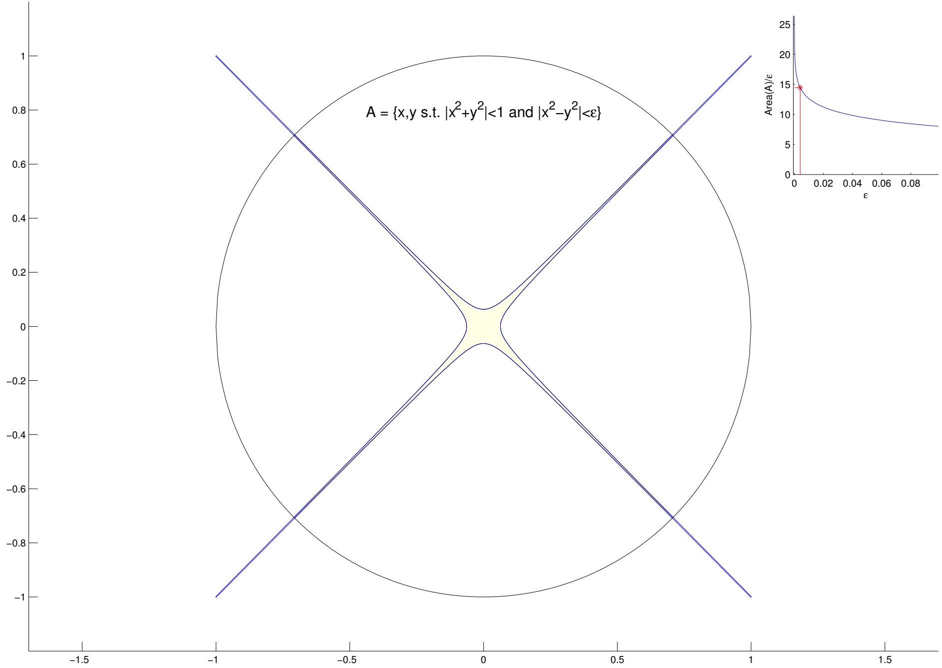


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

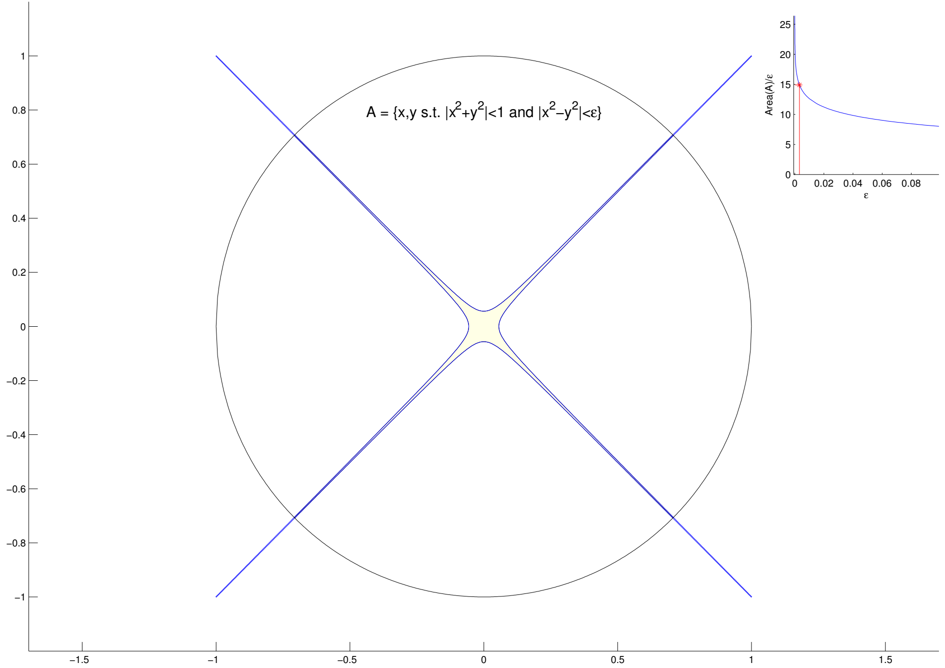

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0
0.02
0.04
0.06
0.08
5
10
15
20
25
−1.5
−1
−0.5
0.5
1
1.5
−0.2
−0.4
−0.6
−0.8
0.2
0.4
0.6
0.8


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

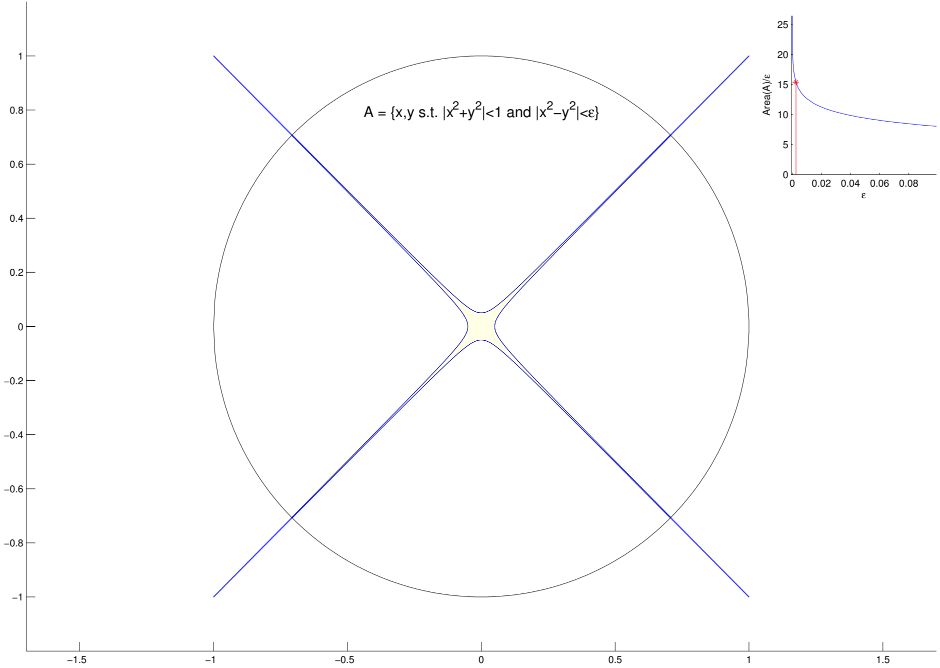

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5



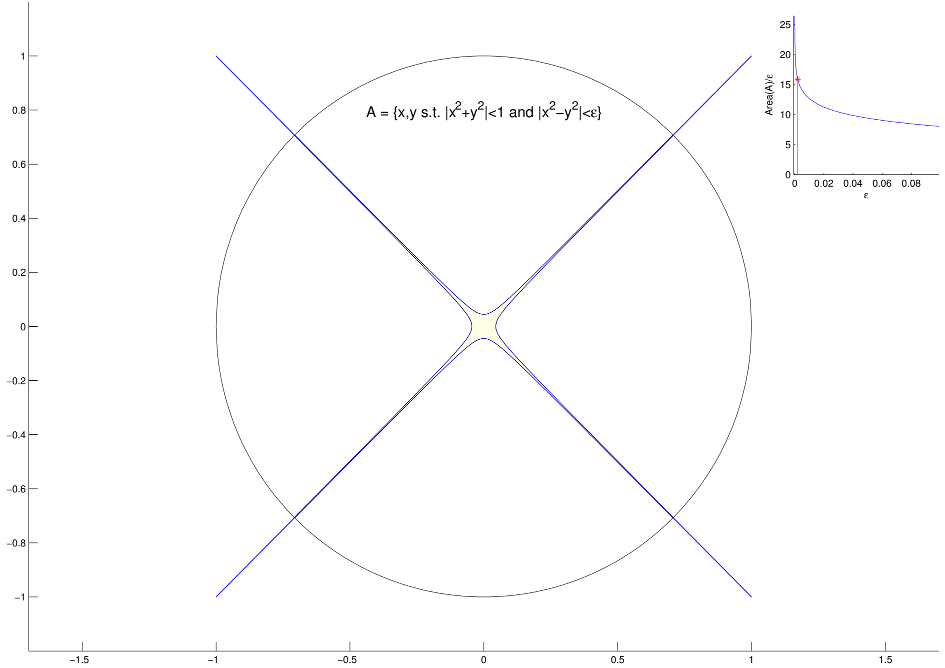
A = {x,y s.t. |x²+y²|<1 and |x²−y²|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−0.5
0.5
1.5



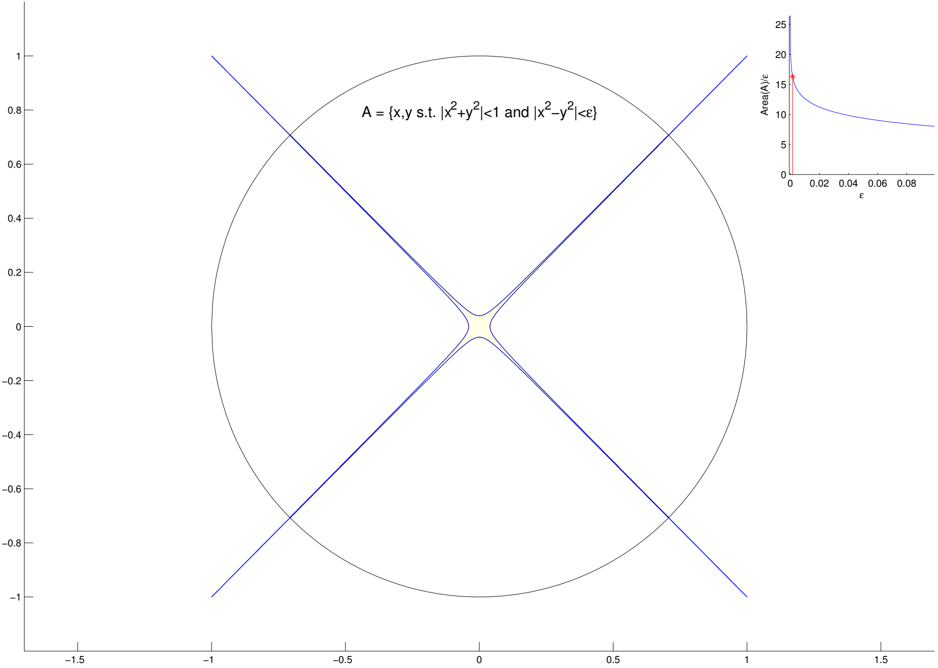
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

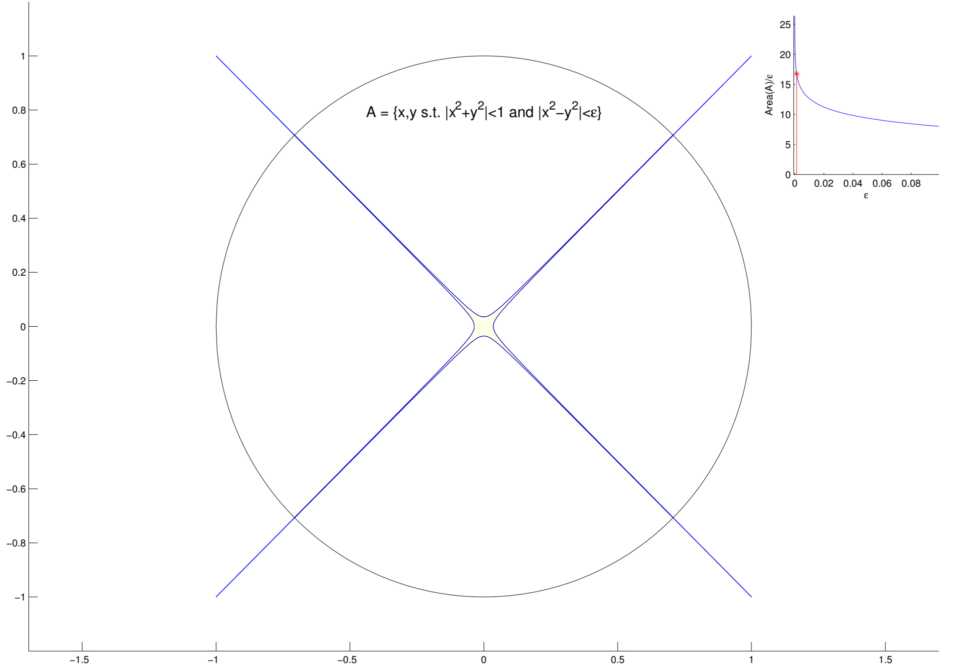


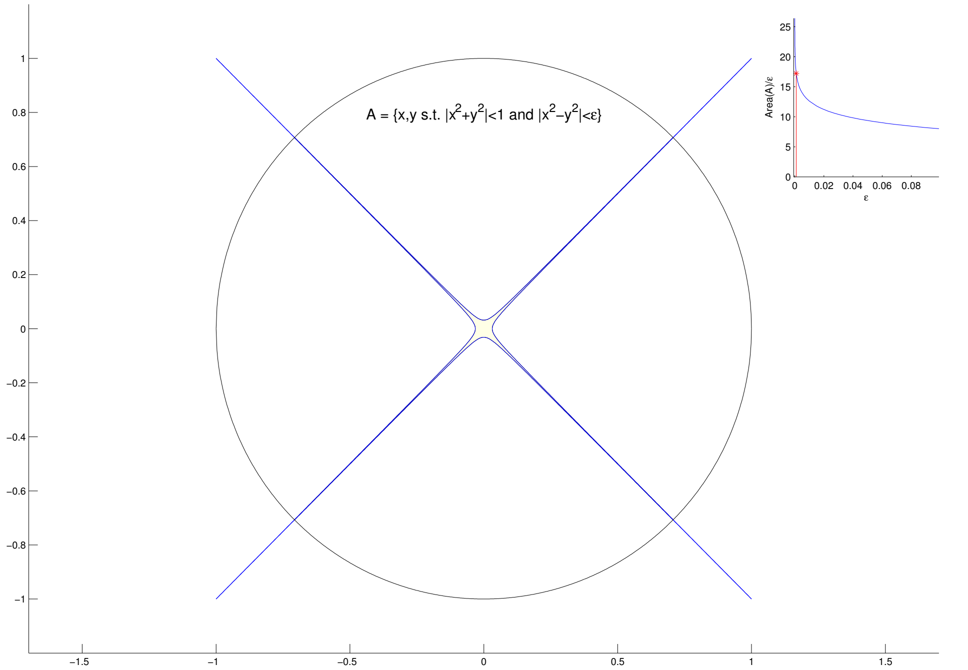
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε



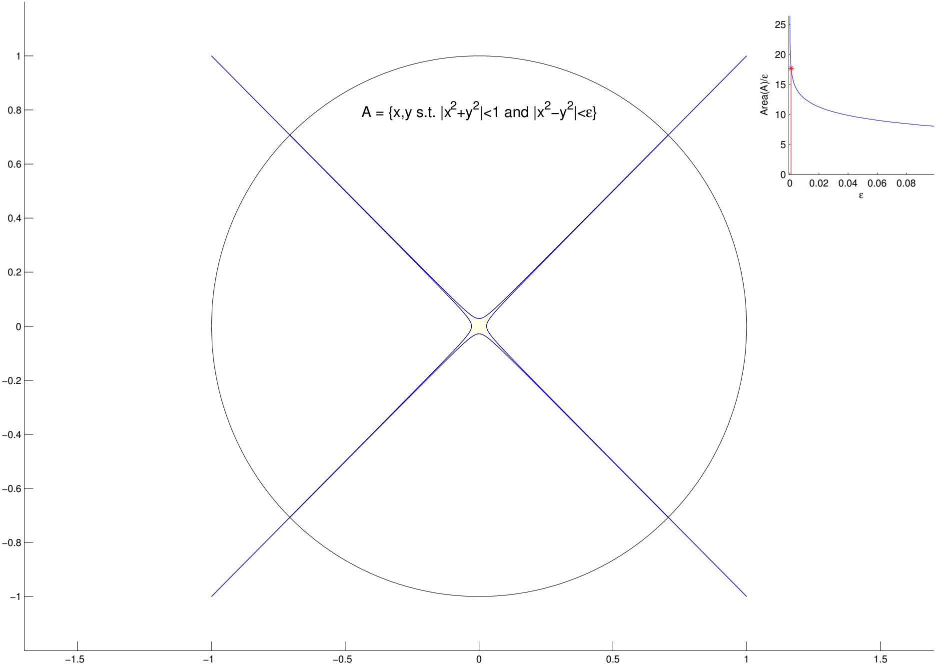
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5

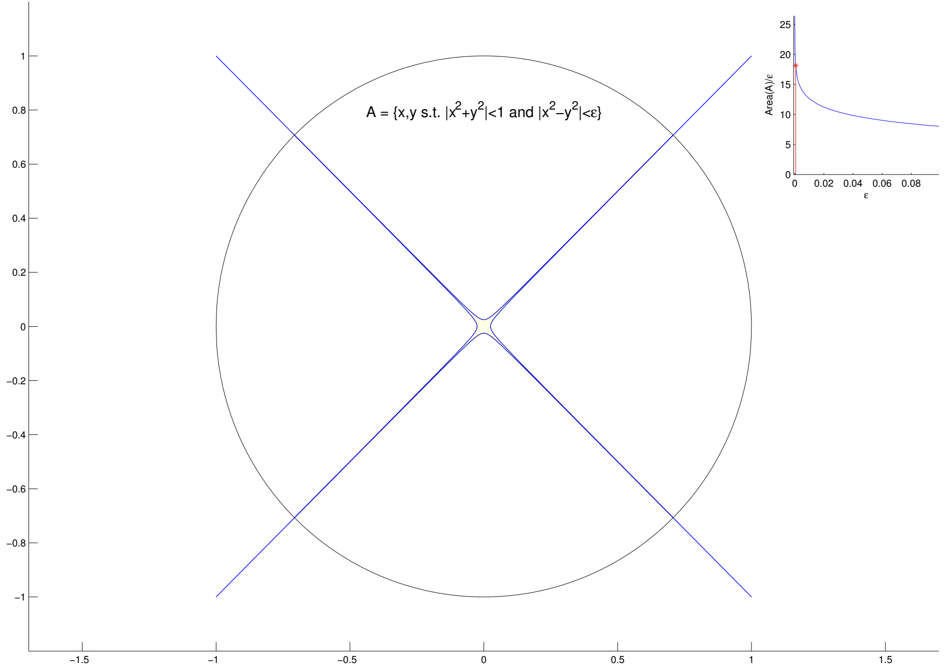

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5

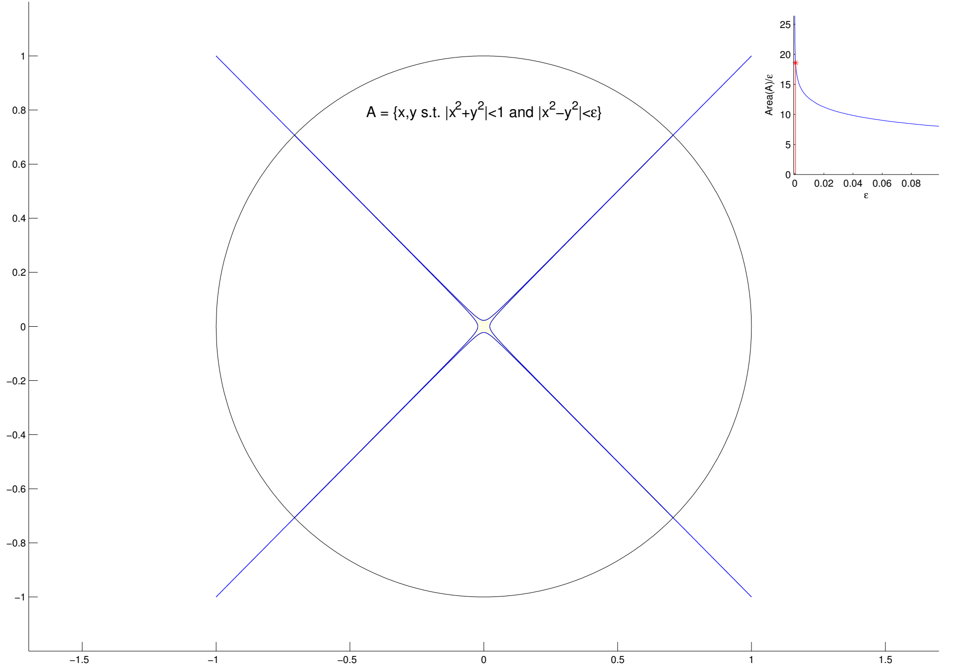


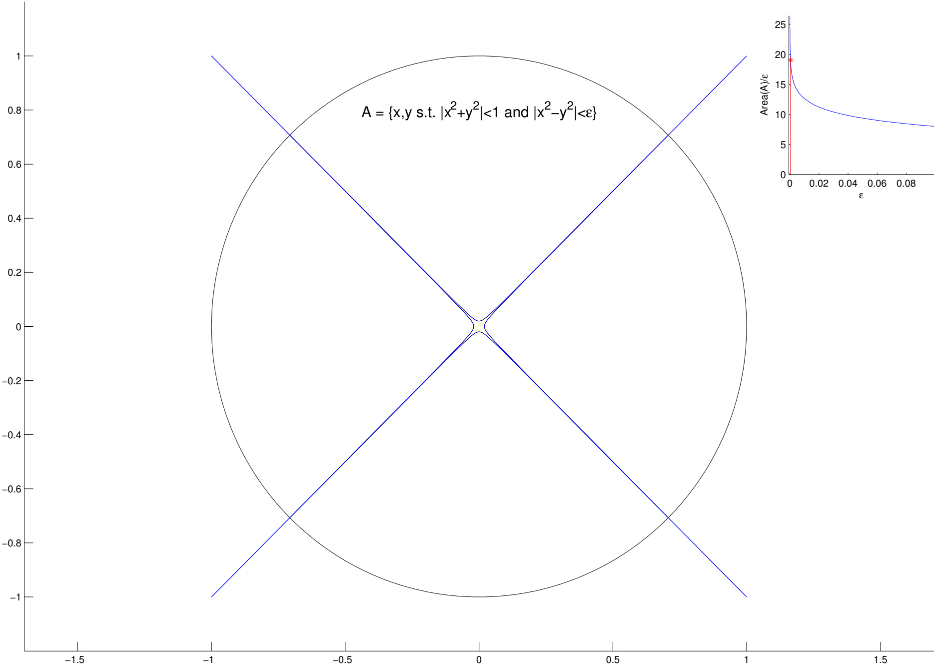
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

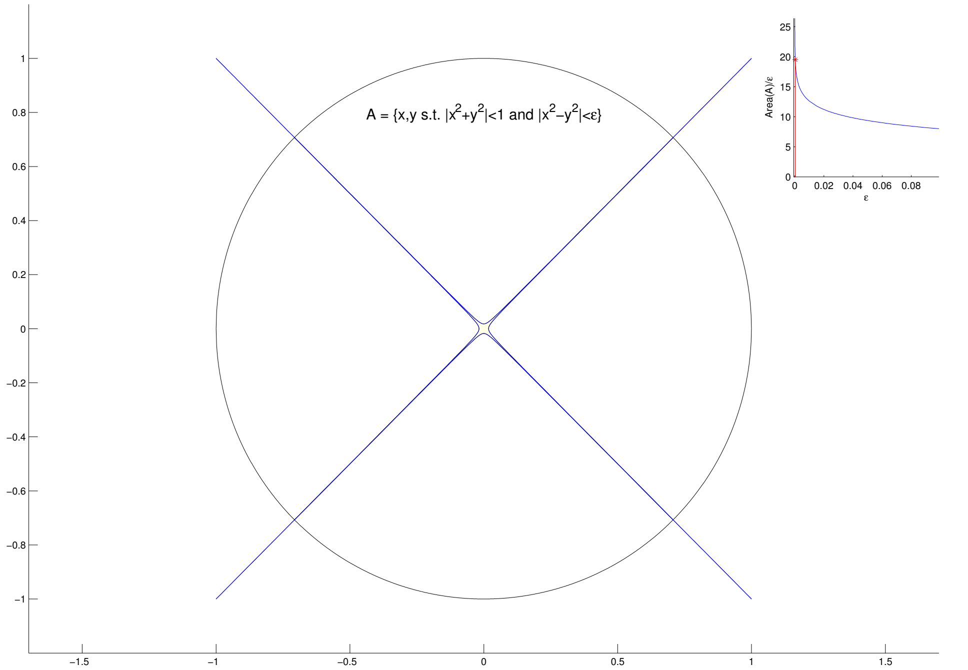

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5



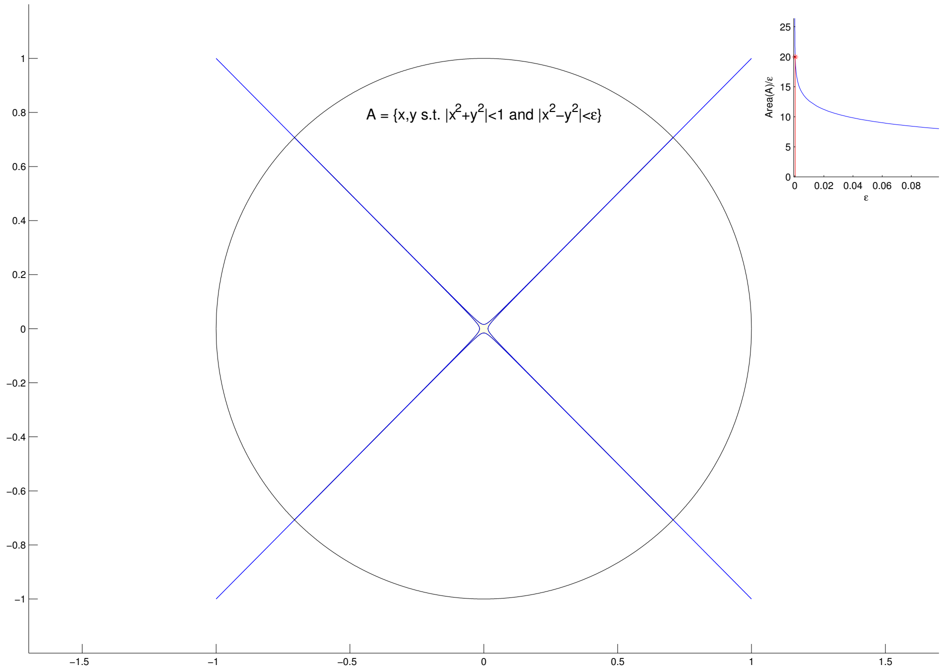
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5

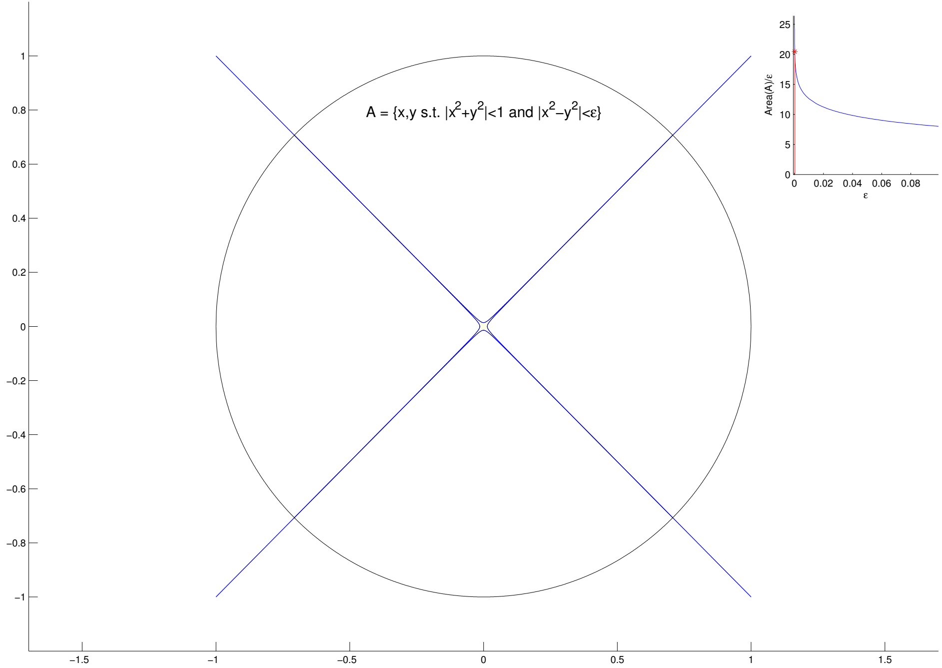

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0
5
10
15
20
25
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−0.5
0.5
1.5

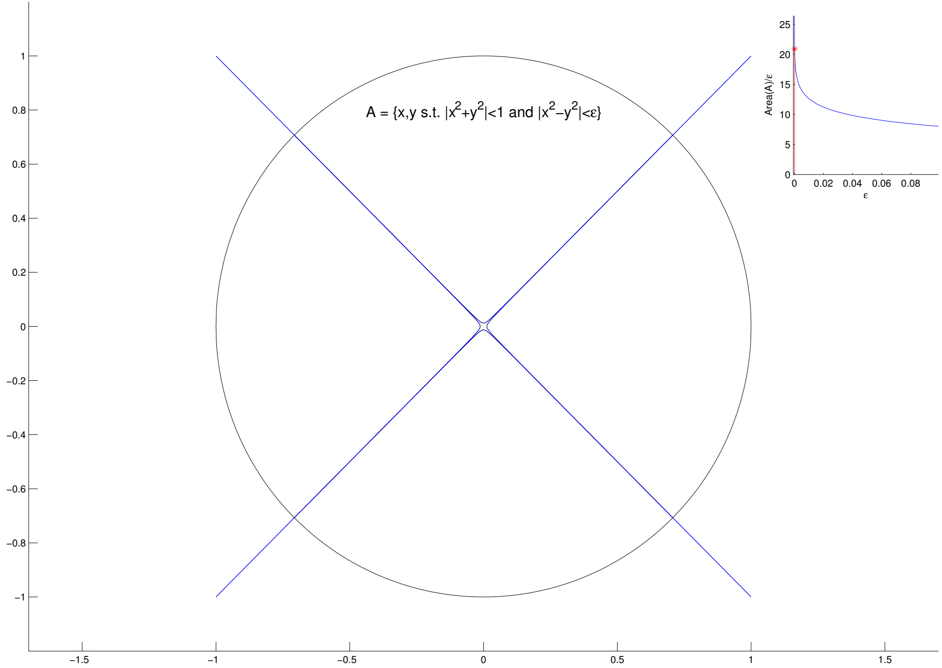


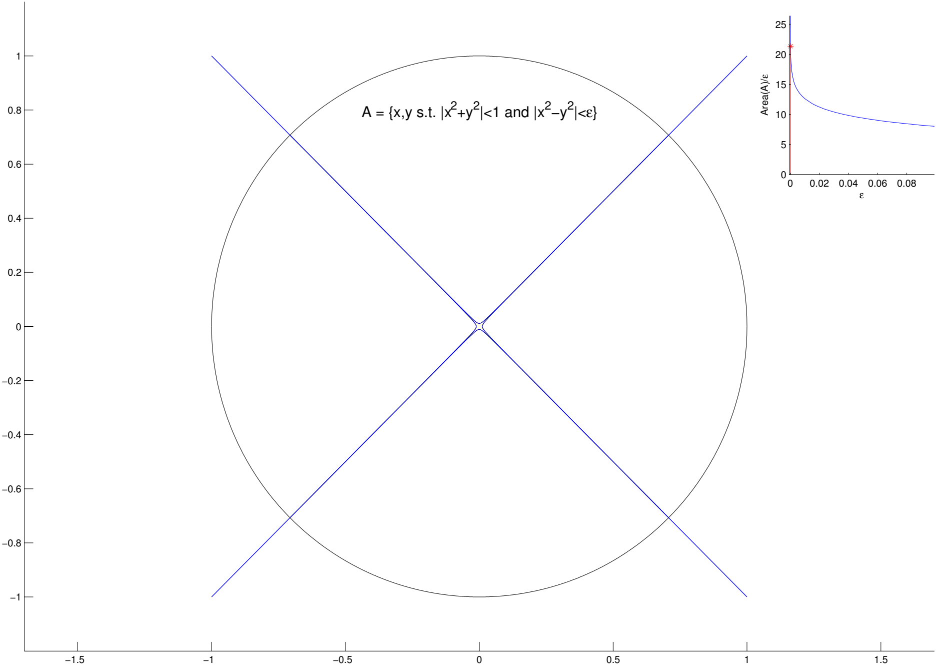
A = {x,y s.t. $|x^2+y^2|<1$ and $|x^2-y^2|<\varepsilon$}
Area(A)/ε
ε

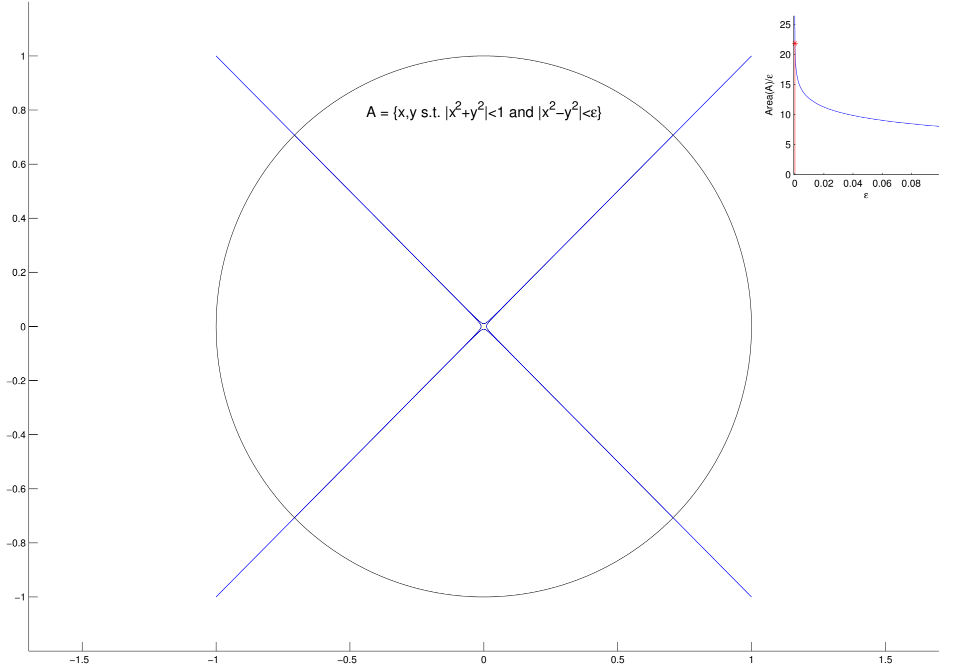


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

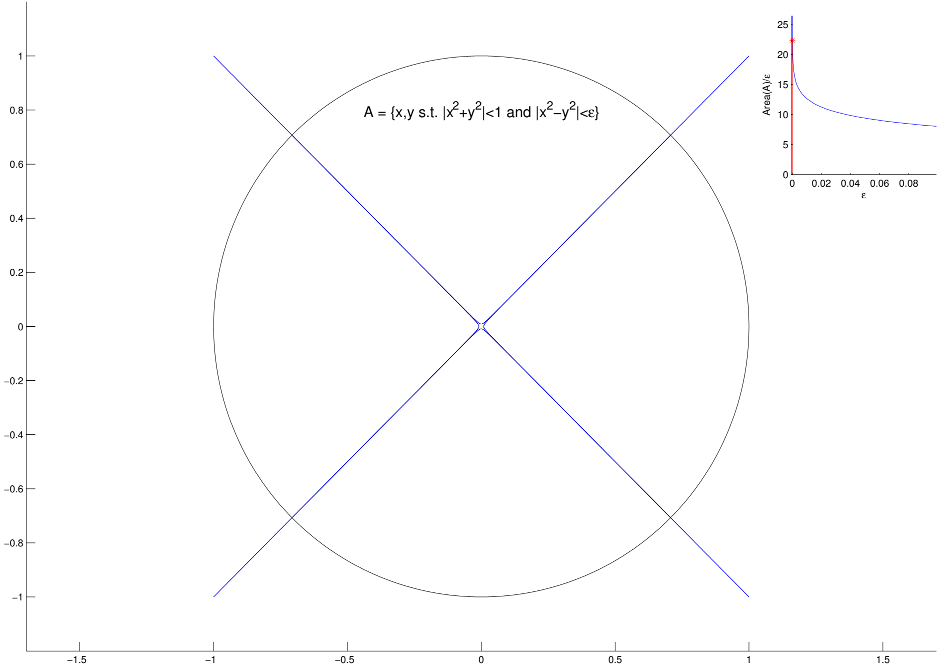

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−0.5
0.5
1.5


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−1
−0.5
0
0.5
1
1.5

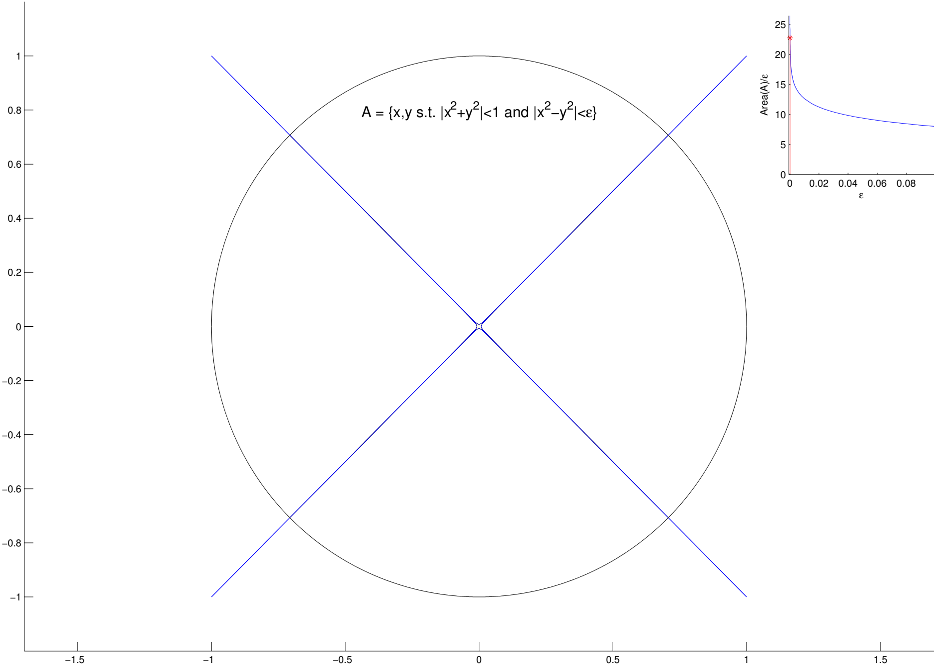

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

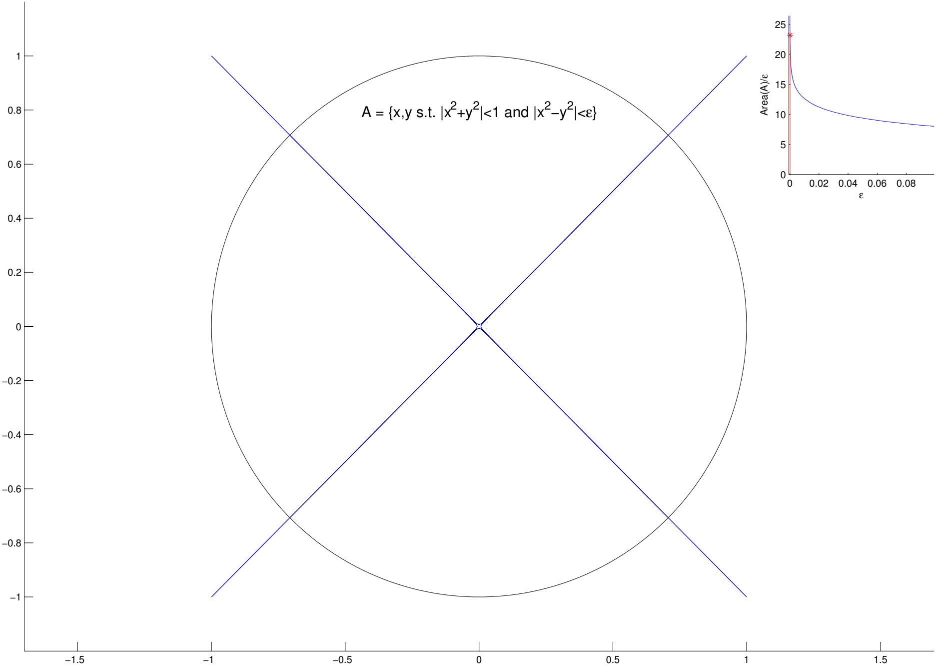


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−0.5
0.5
1.5

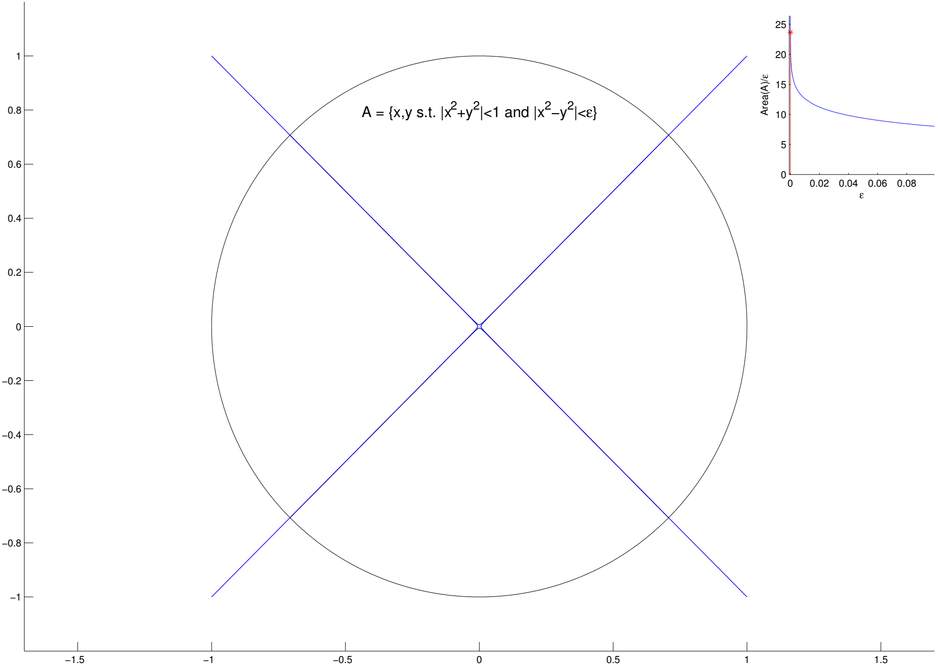


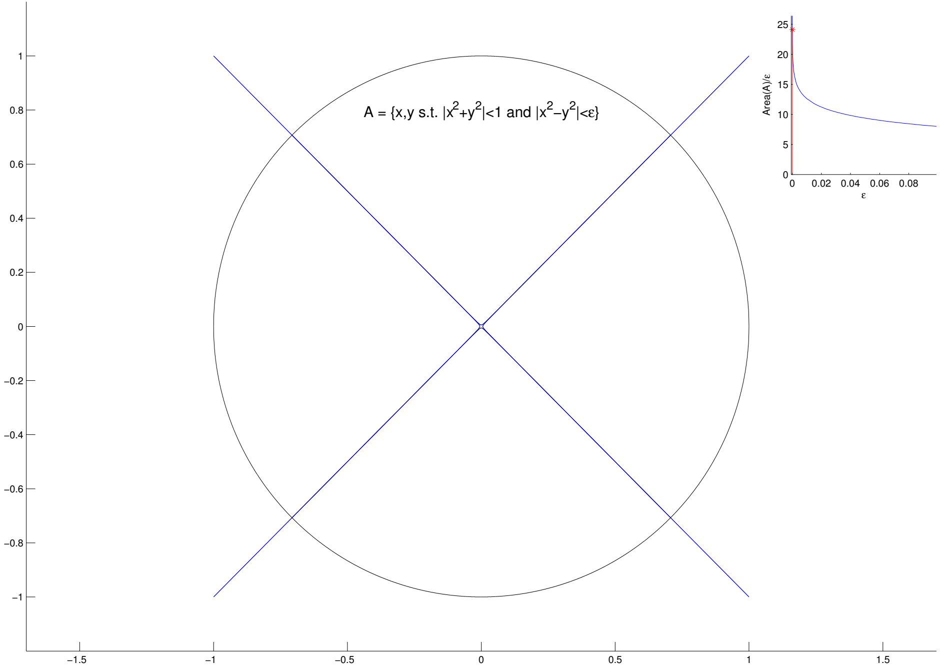
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0 0.02 0.04 0.06 0.08
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.5 −1 −0.5 0 0.5 1 1.5


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−0.5
0.5
1.5

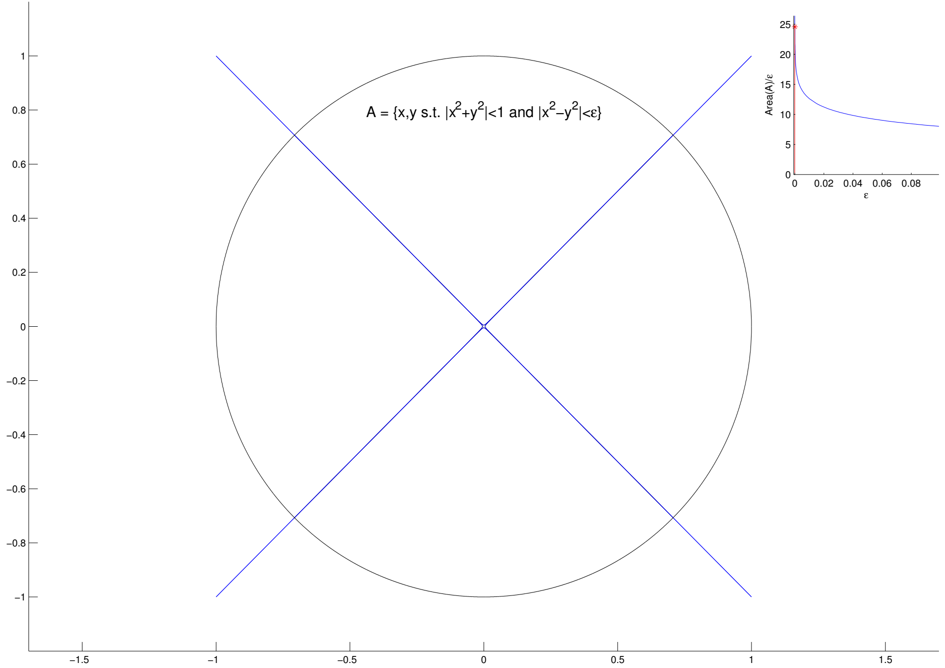


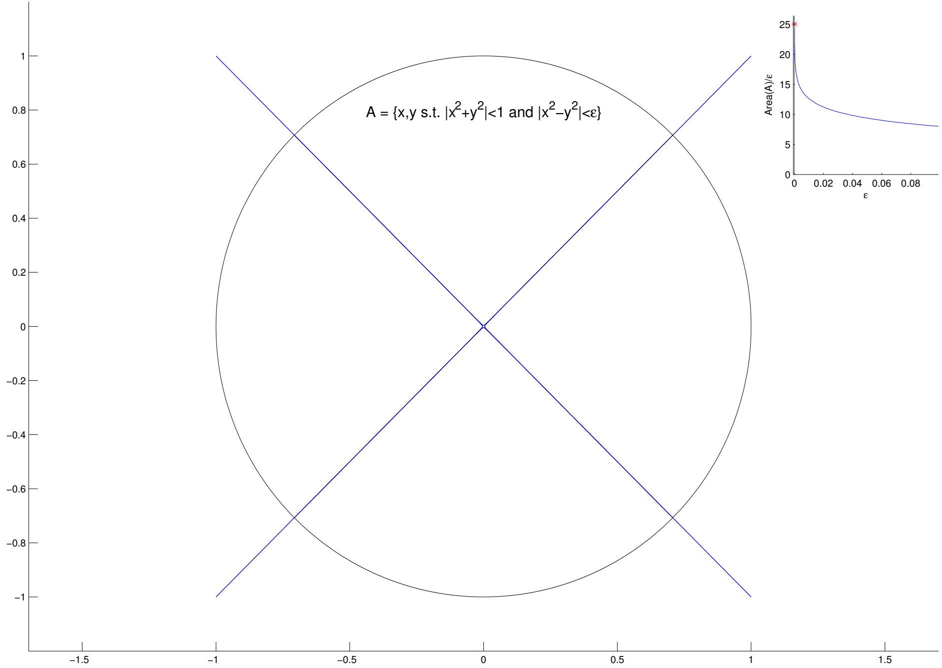
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0
0.02
0.04
0.06
0.08
5
10
15
20
25
−1.5
−1
−0.5
0.5
1
1.5
−0.2
−0.4
−0.6
−0.8
0.2
0.4
0.6
0.8


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−0.5
0.5
1.5

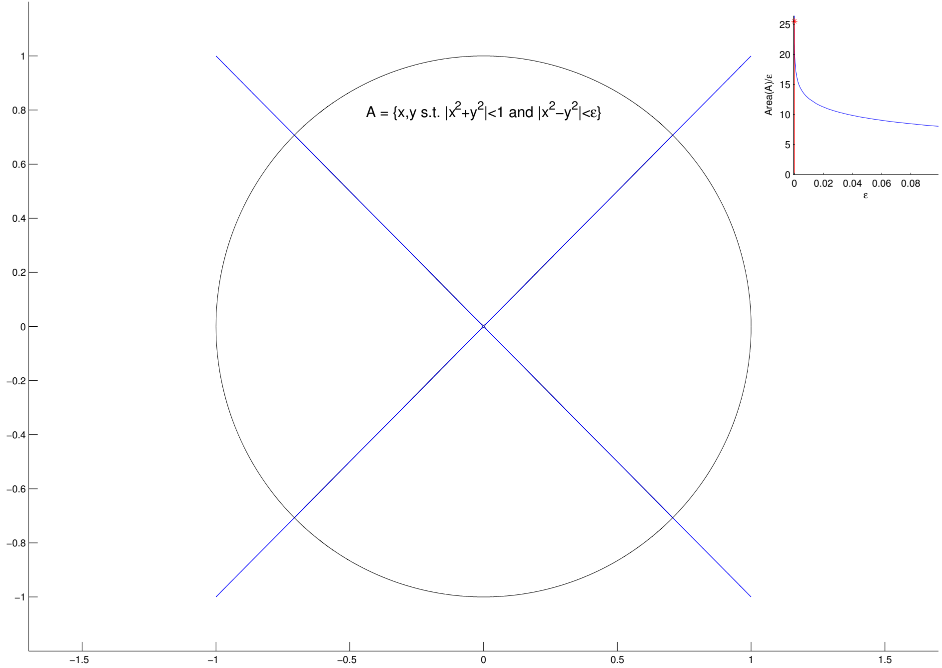


A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
25
20
15
10
5
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1.5
−0.5
0.5
1.5

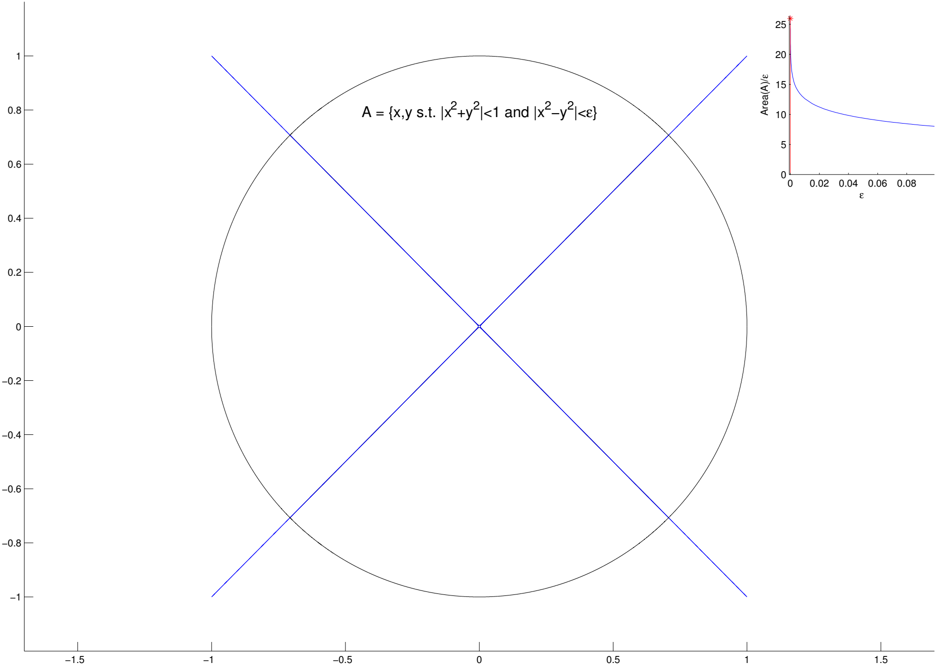

A = {x,y s.t. |x²+y²|<1 and |x²−y²|<ε}
Area(A)/ε
ε

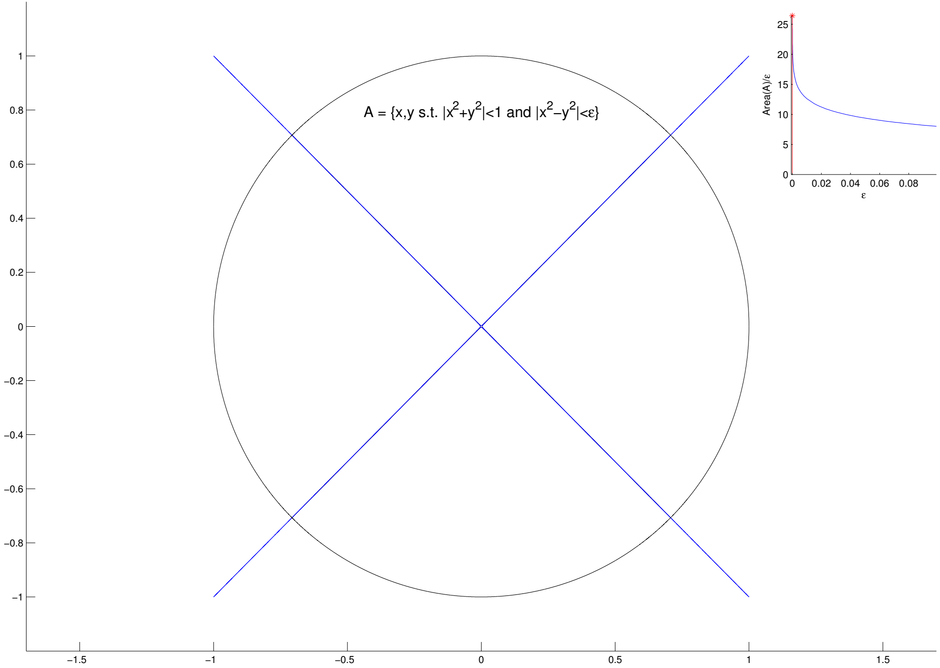


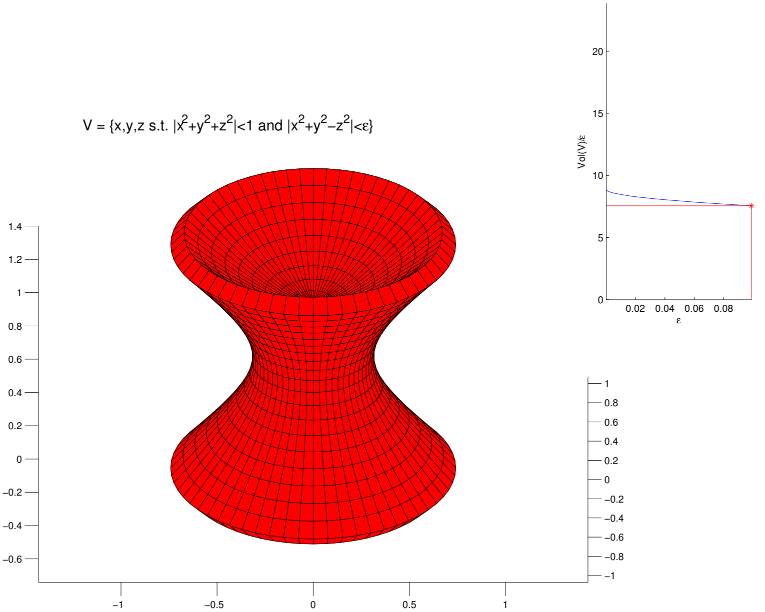
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε



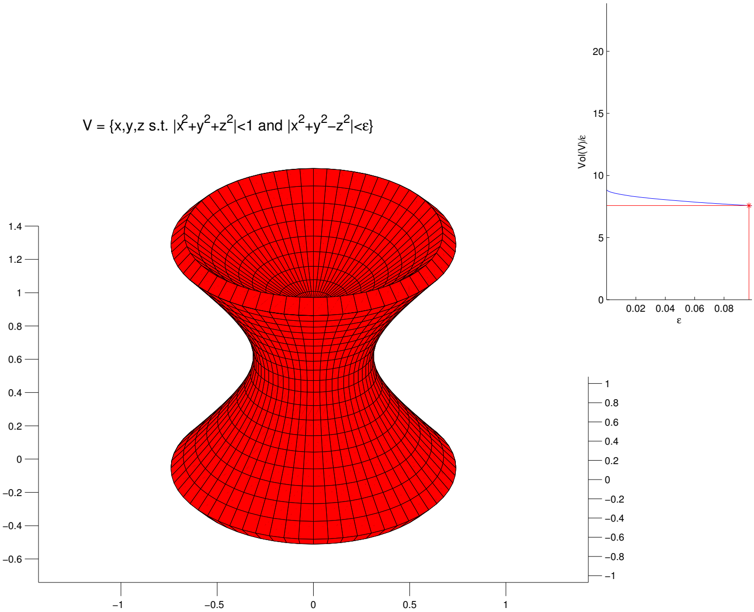
A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε
0
0.02
0.04
0.06
0.08
5
10
15
20
25
−1.5
−1
−0.5
0.5
1
1.5
−0.2
−0.4
−0.6
−0.8
0.2
0.4
0.6
0.8


A = {x,y s.t. |x²+y²|<1 and |x²−y²|<ε}
Area(A)/ε
ε

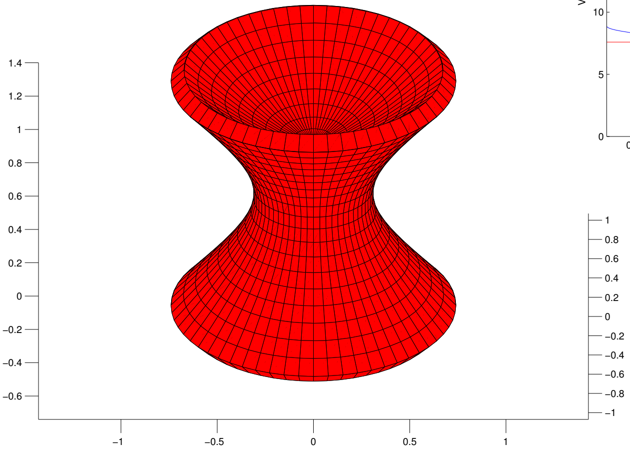

A = {x,y s.t. |x^2+y^2|<1 and |x^2−y^2|<ε}
Area(A)/ε
ε

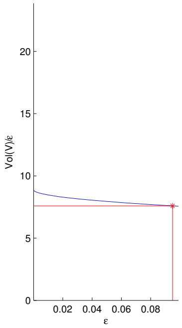


V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

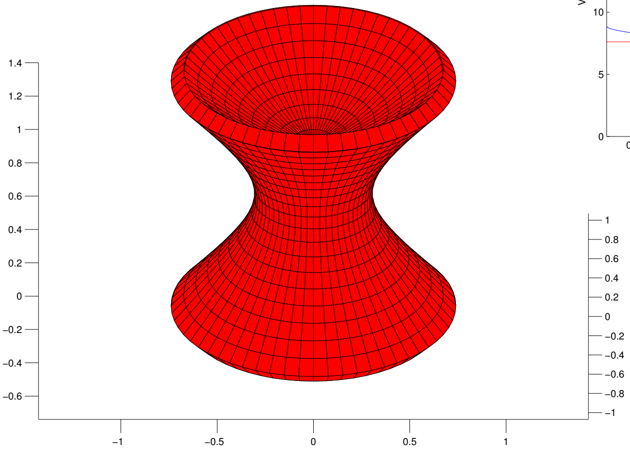

V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

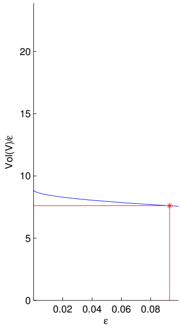
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

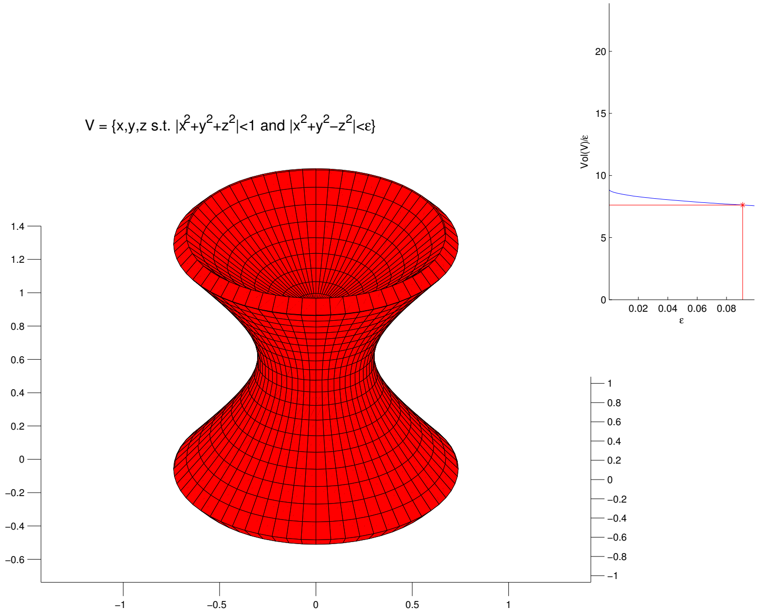


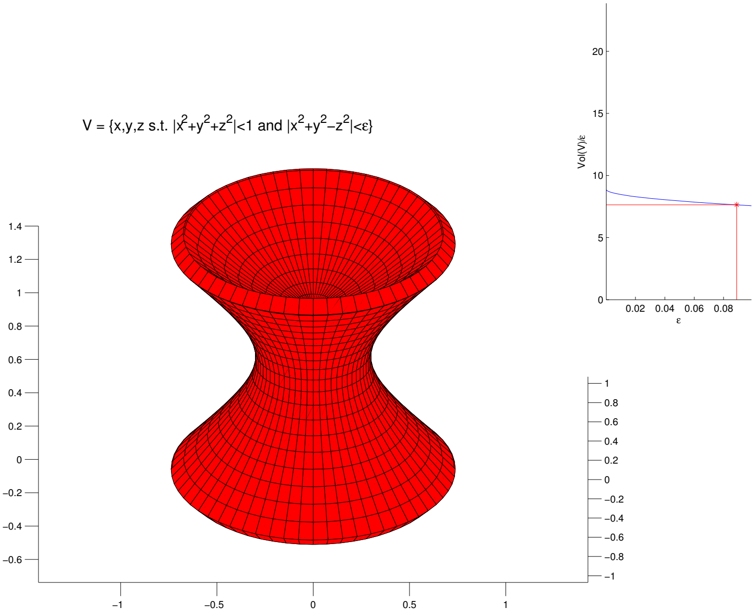


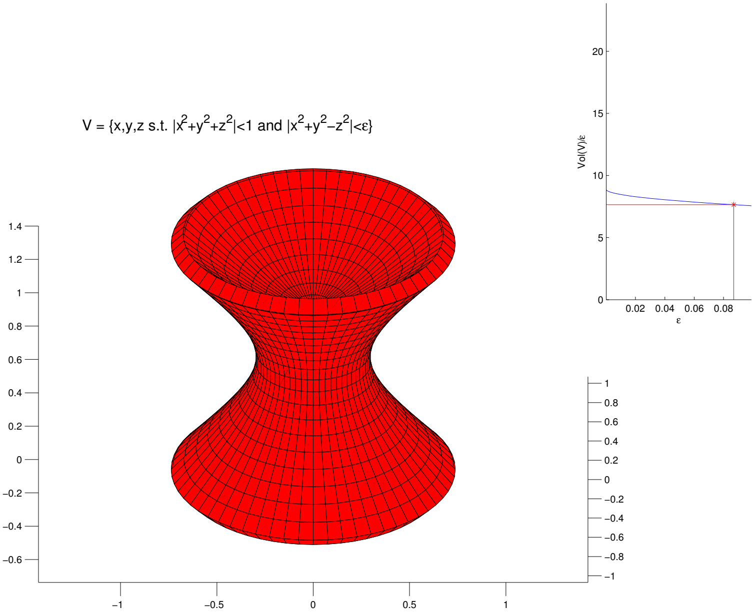
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



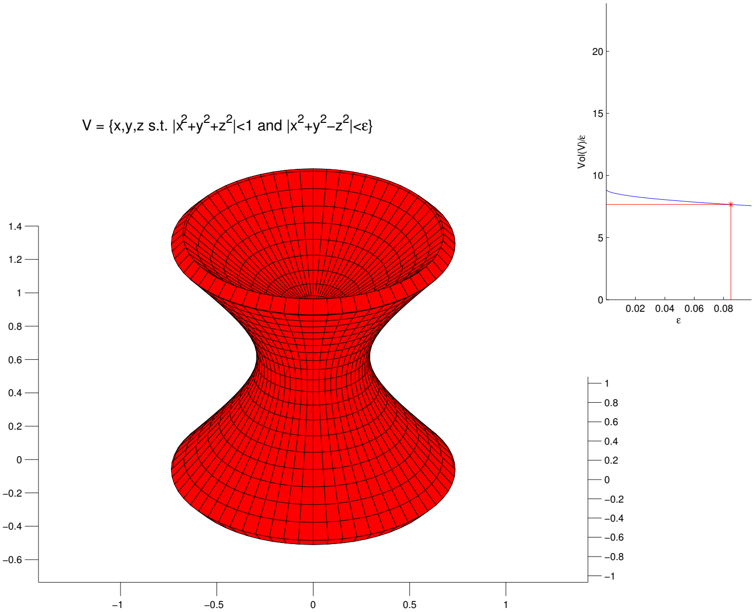



V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε

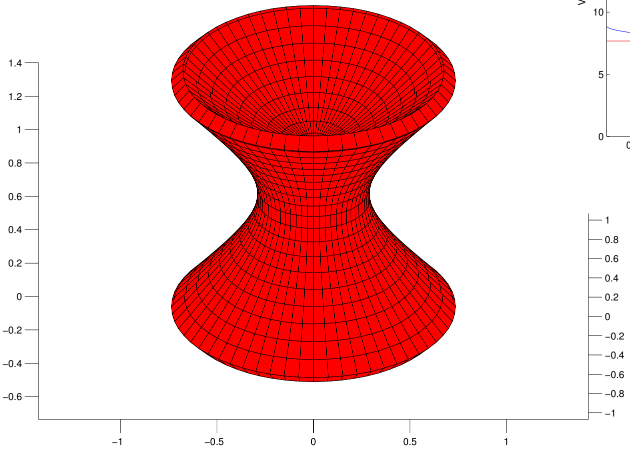

V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε

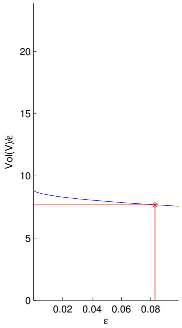


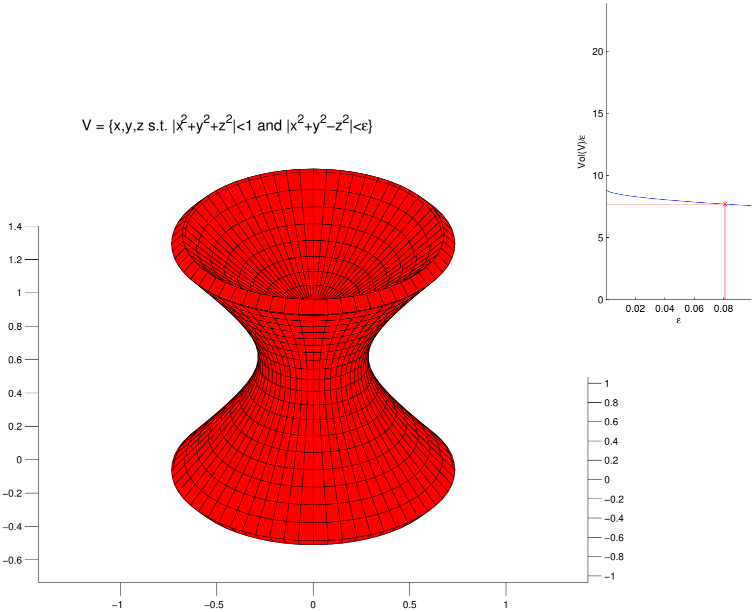
V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0 0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

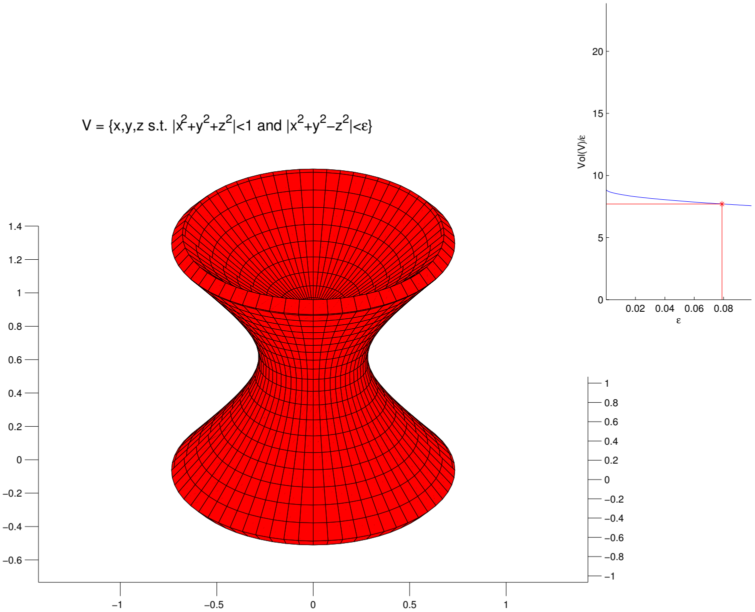

V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0.02
0.04
0.06
0.08
0
5
10
15
20
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
0.8
0.6
0.4
0.2
−0.8

$$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$$

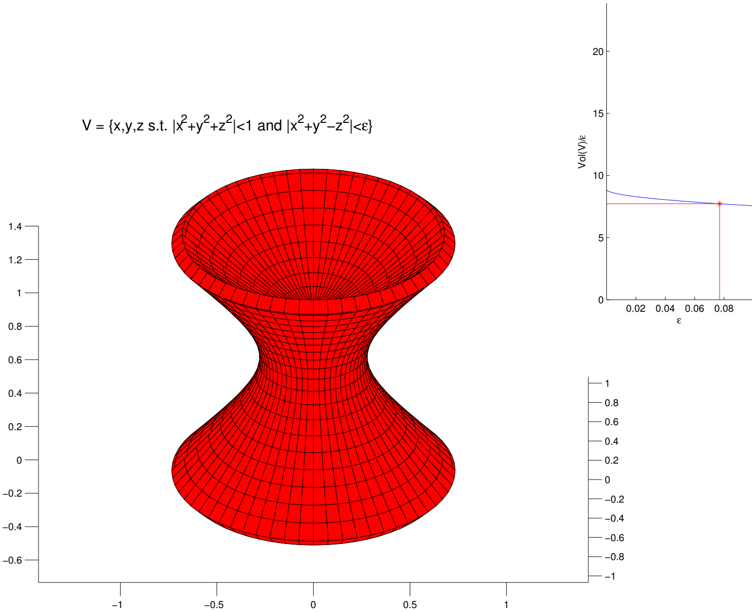


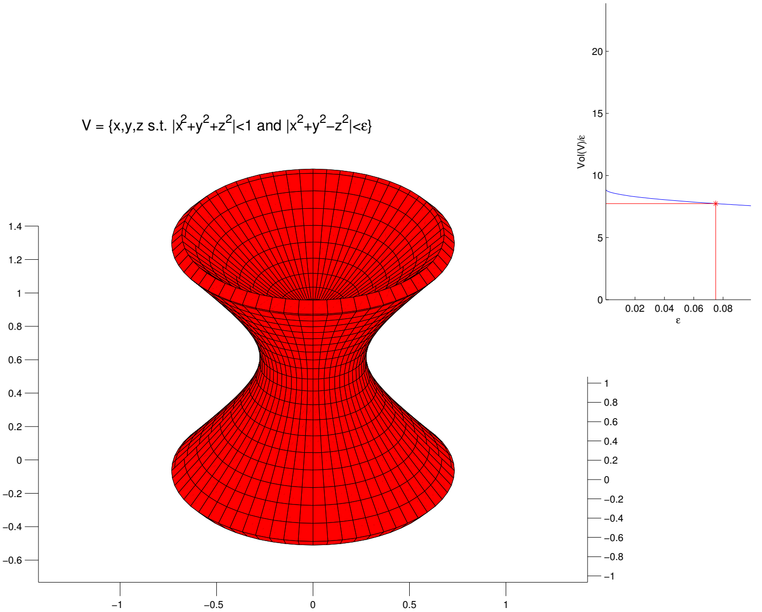



V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
−1 −0.5 0 0.5 1
−0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8 1 1.2 1.4
−1 −0.8 −0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8 1



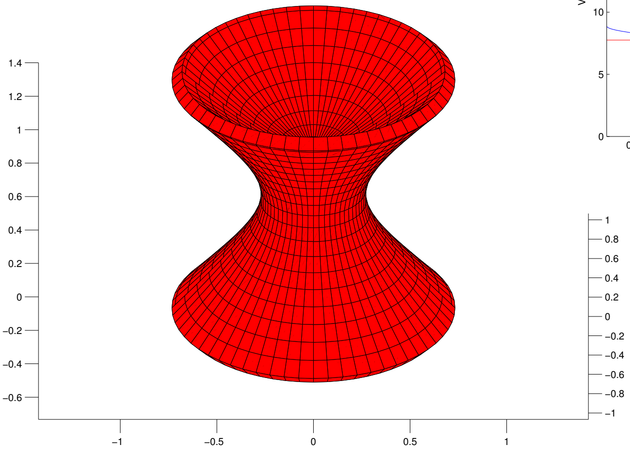
V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε

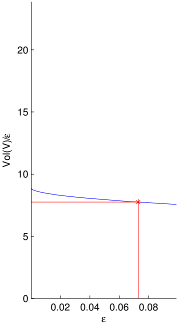

V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0.02
0.04
0.06
0.08
0
5
10
15
20
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1



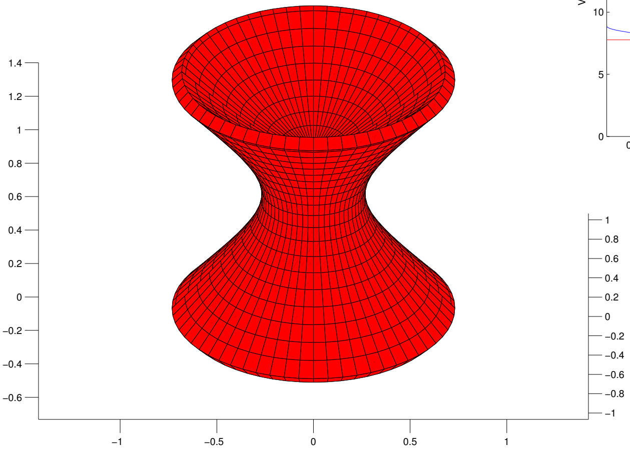
V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0
5
10
15
20
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

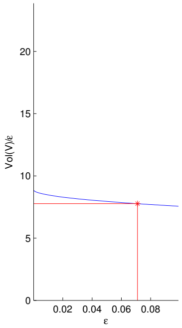
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

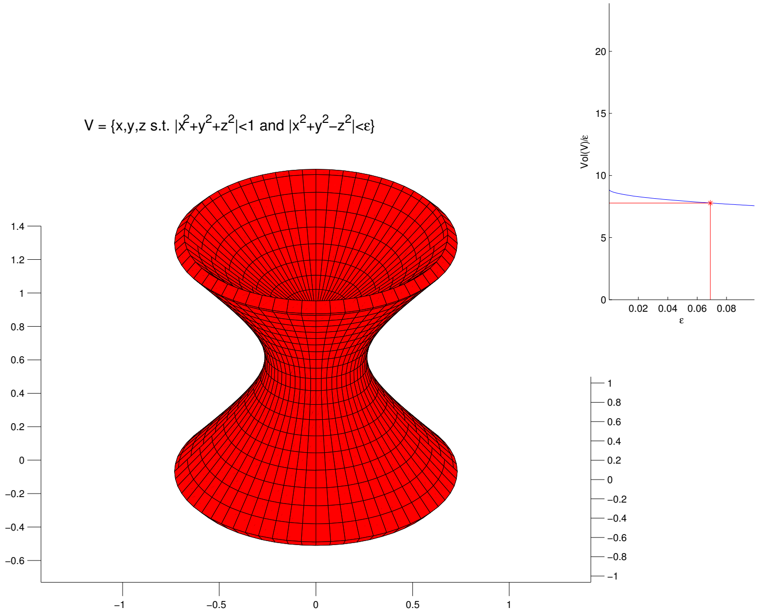




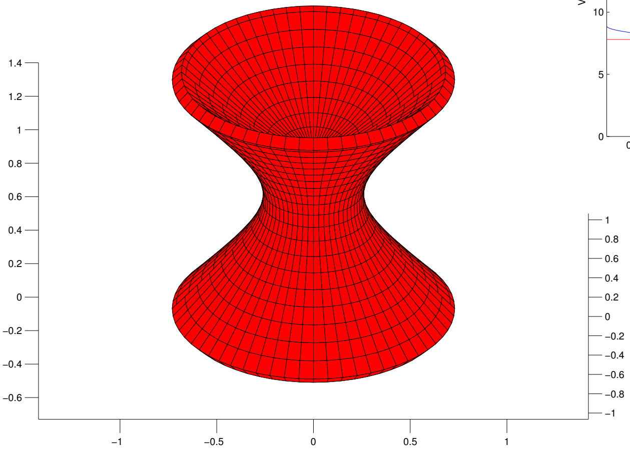
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

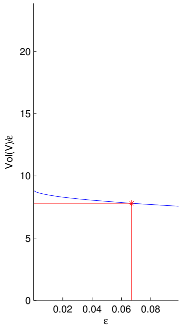


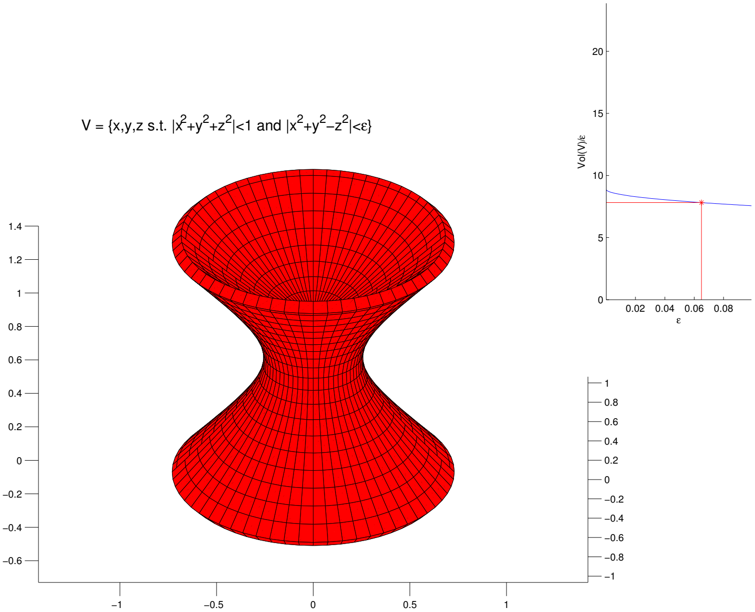



V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
0.02
0.04
0.06
0.08
0
5
10
15
20
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0.5
−0.8
−1

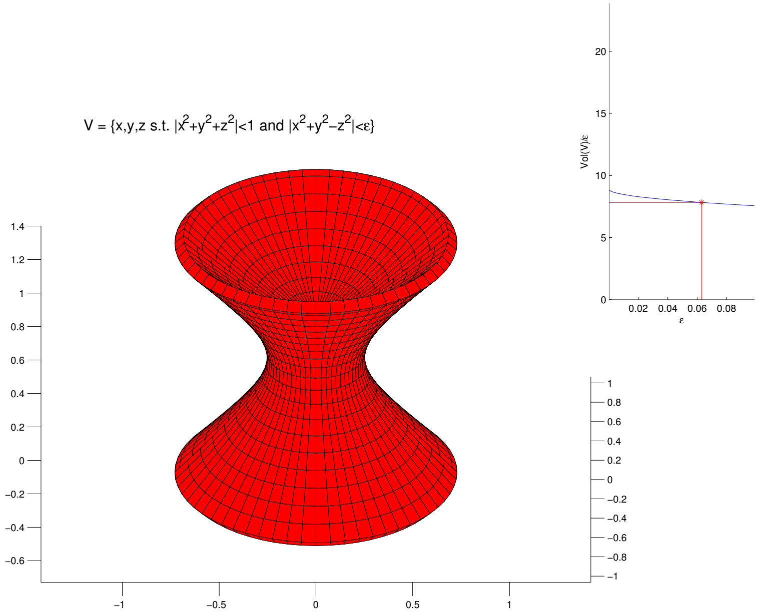
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

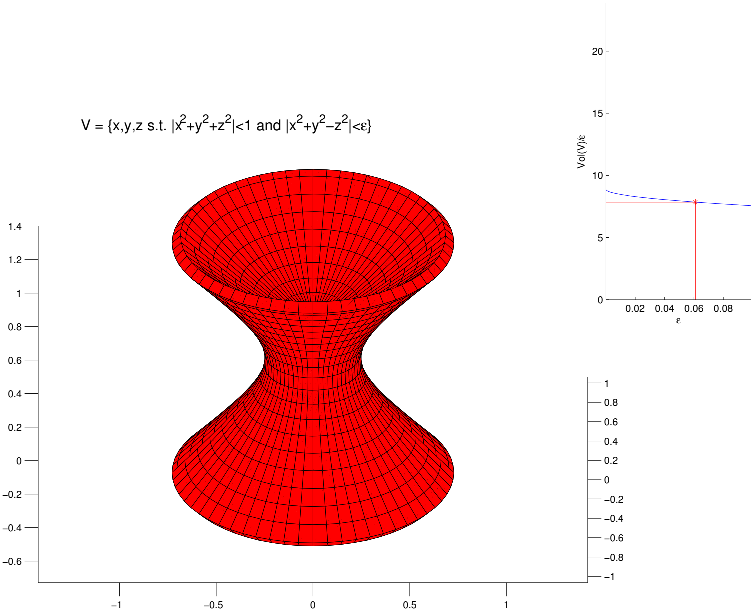






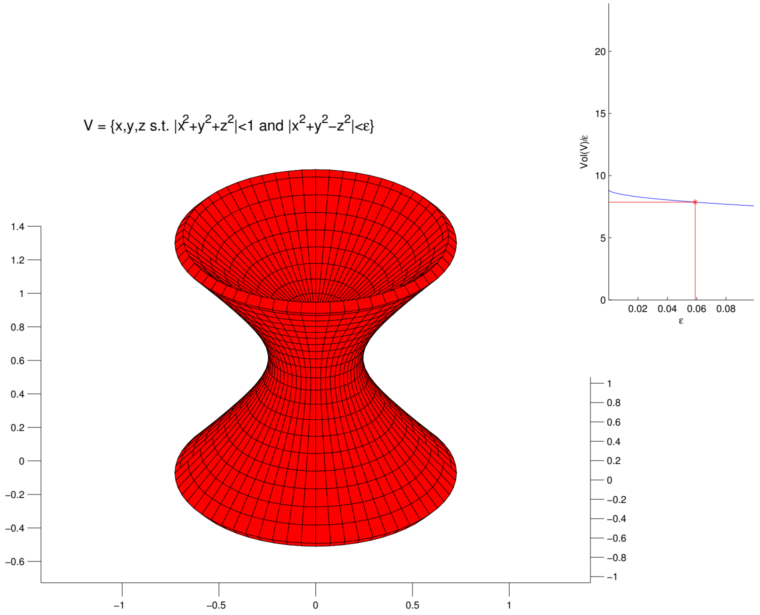
V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0.02
0.04
0.06
0.08
0
5
10
15
20
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

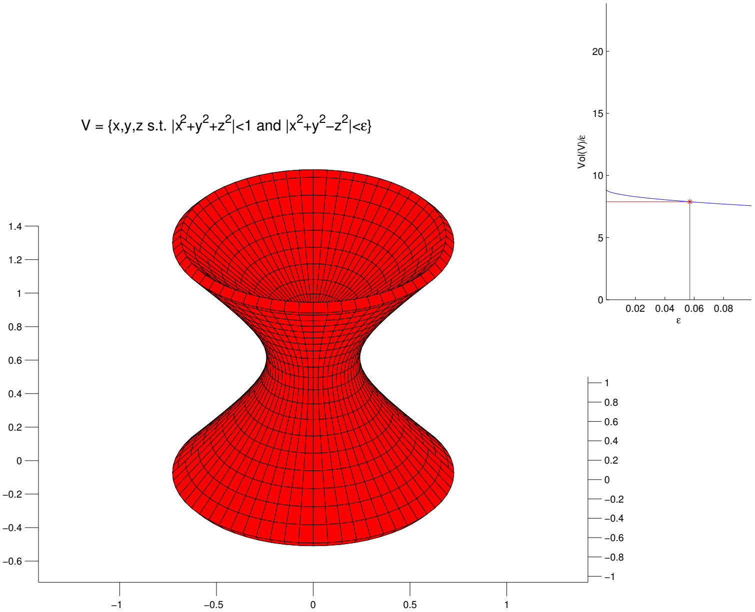

V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε

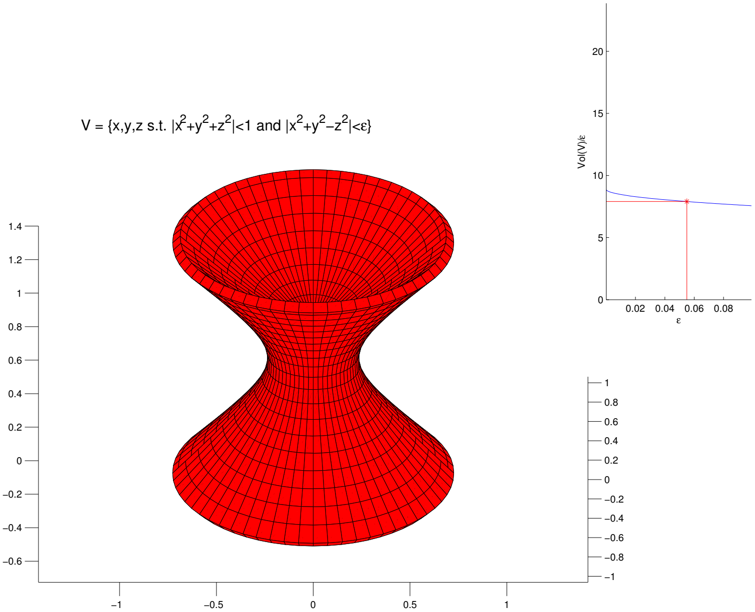


V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
20
15
10
5
0
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

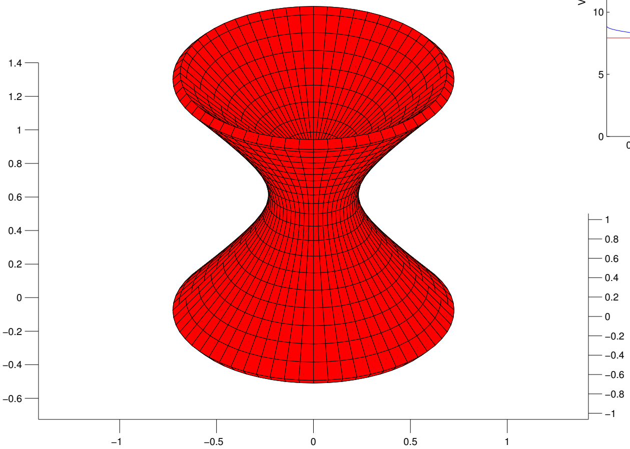
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

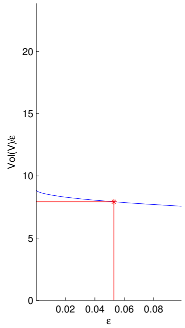



V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
20
15
10
5
0
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
−0.8
−1

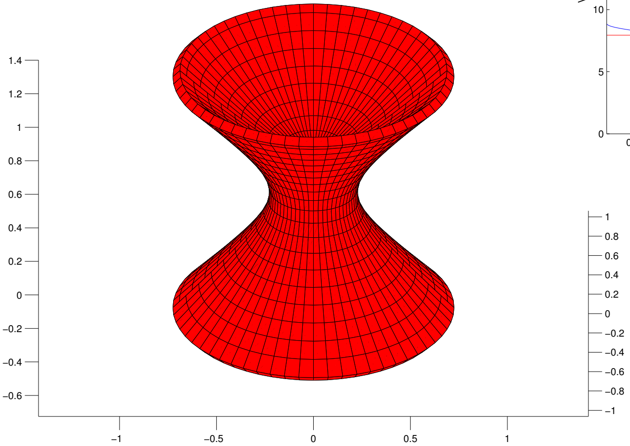

V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
20
15
10
5
0
0.02
0.04
0.06
0.08
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

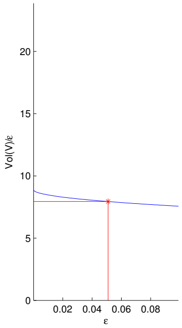
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

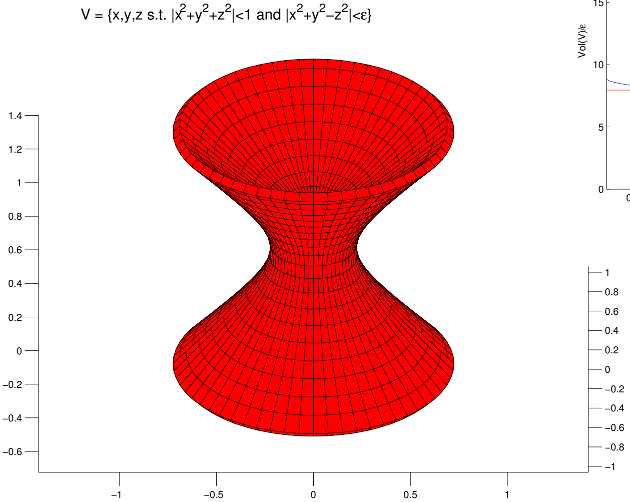




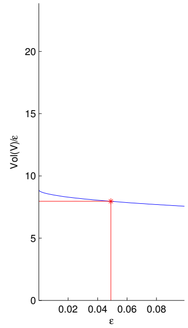
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



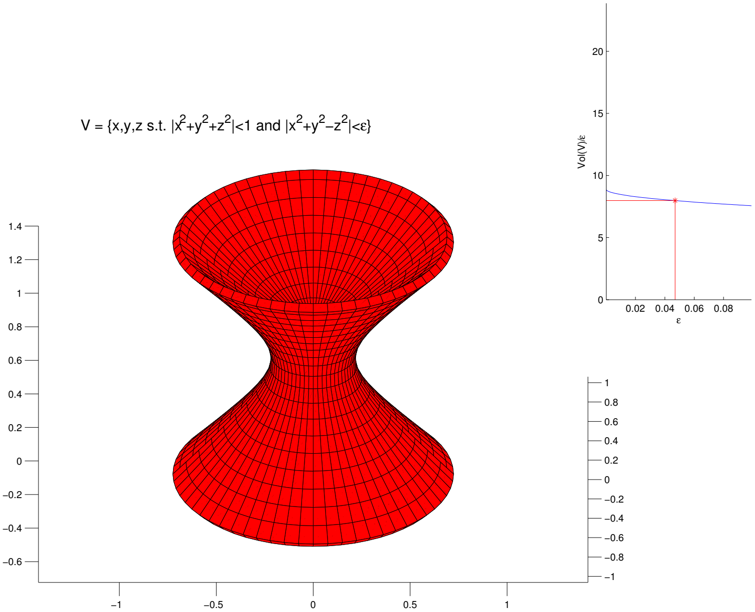


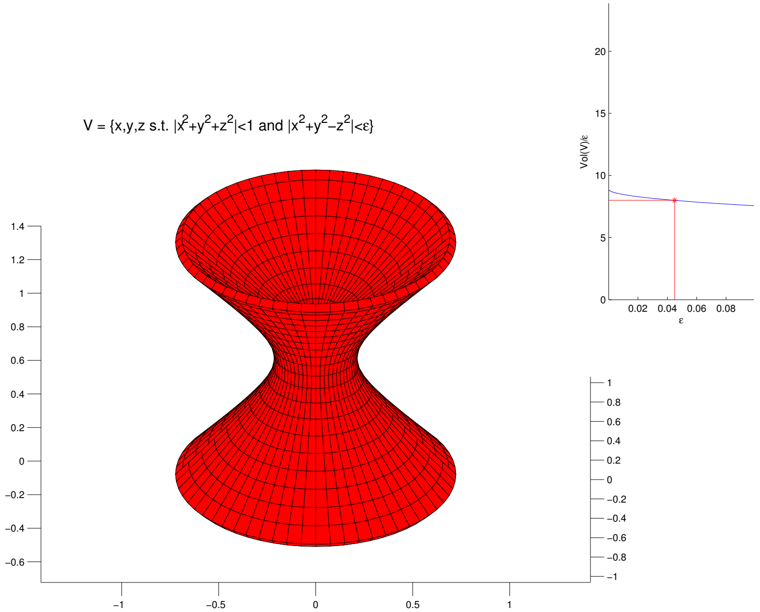


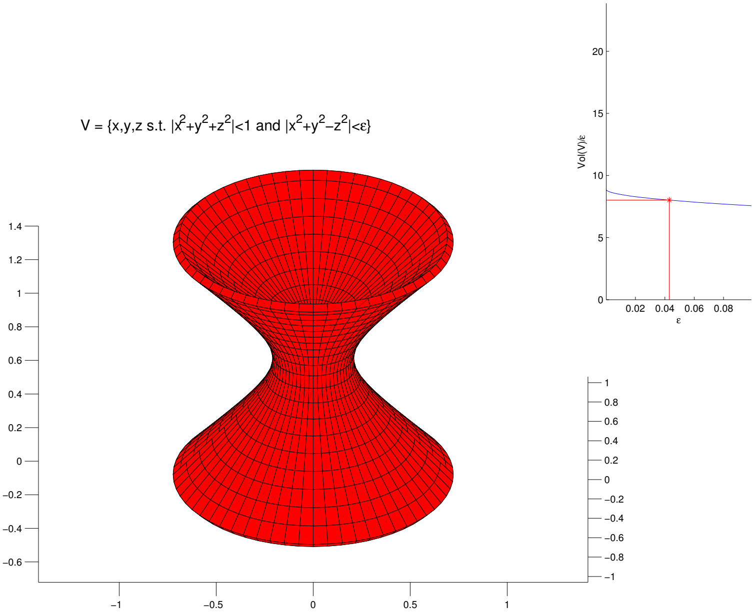
V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

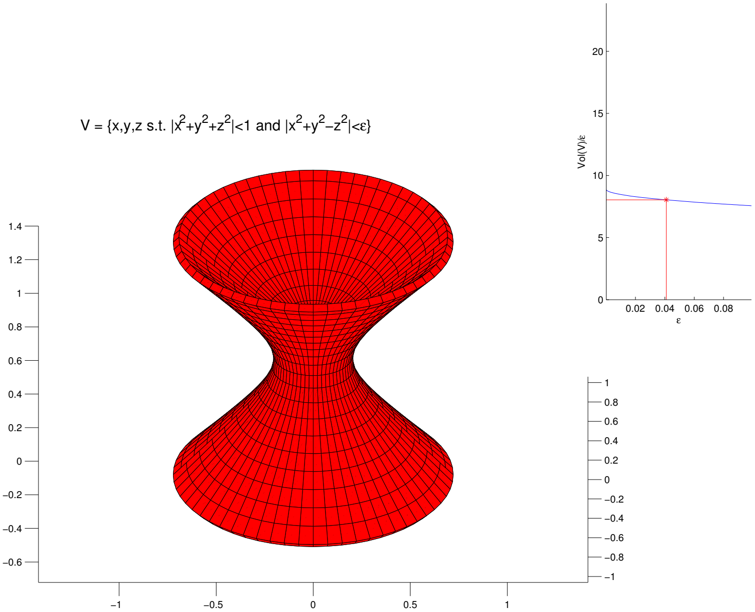


Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε

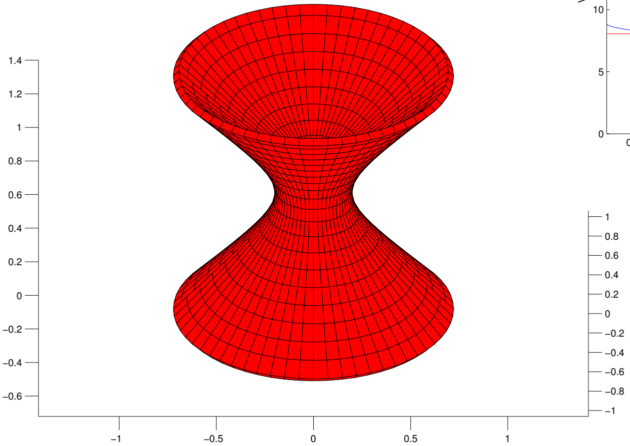

V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

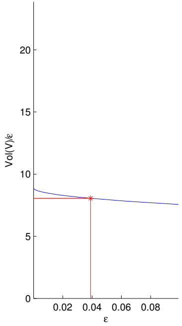

V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

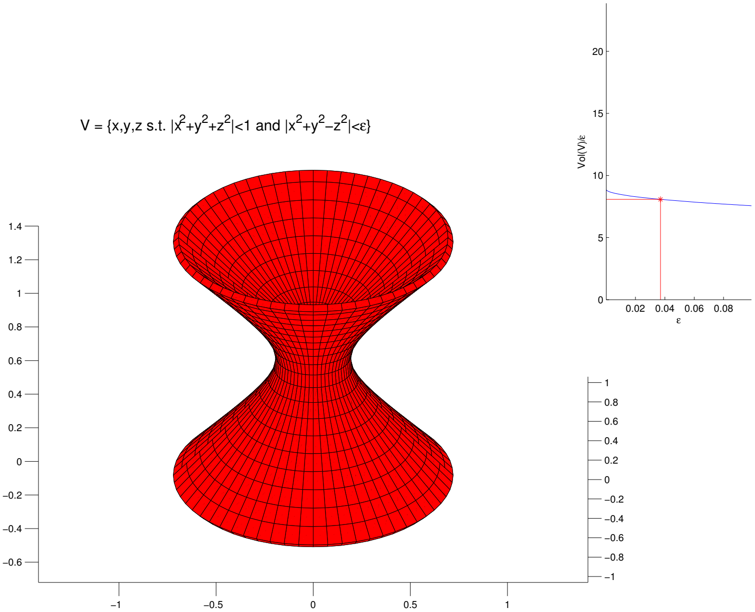


V = {x,y,z s.t. $|x^2+y^2+z^2|<1$ and $|x^2+y^2-z^2|<\varepsilon$}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

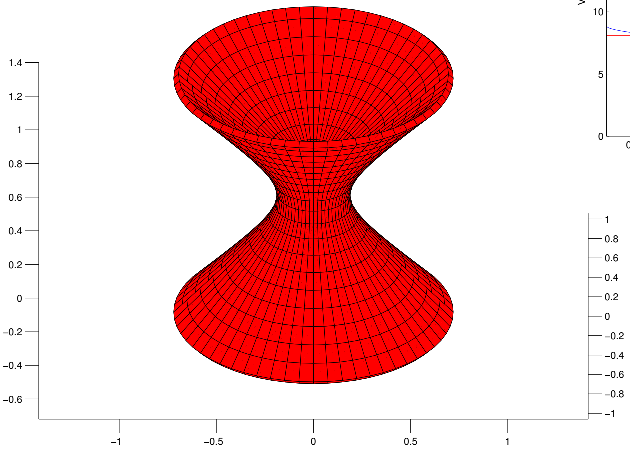
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

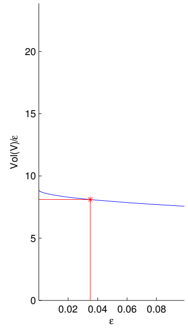


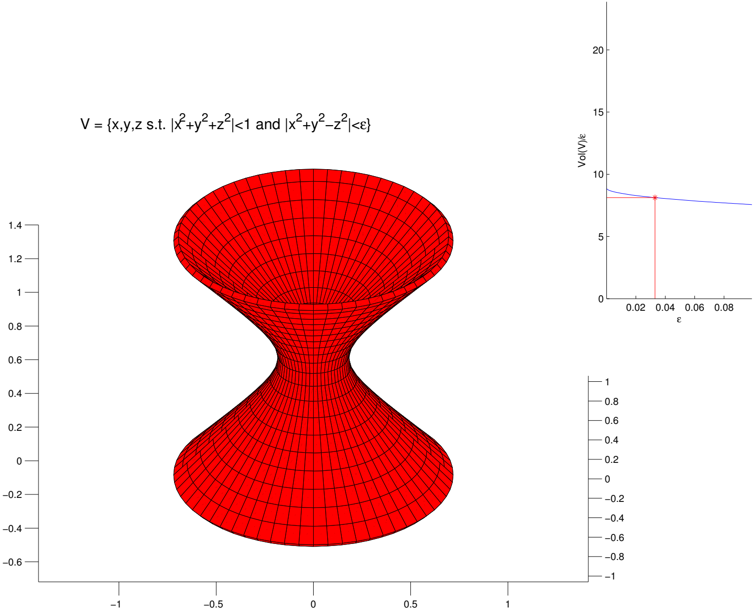



V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε

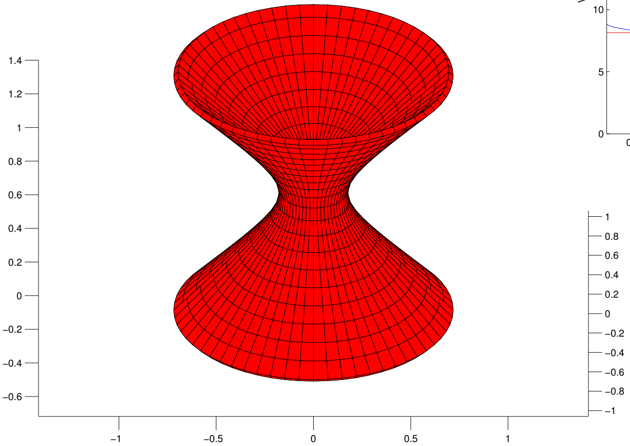
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

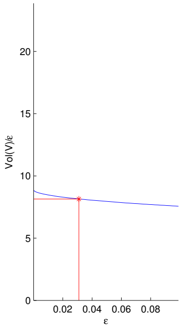




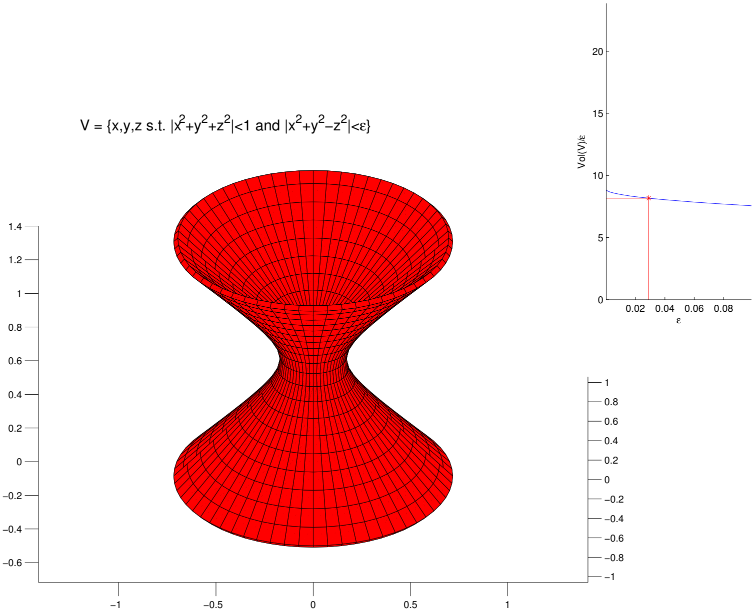

V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
0.02
0.04
0.06
0.08
0
5
10
15
20
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0.5
−0.8
−1

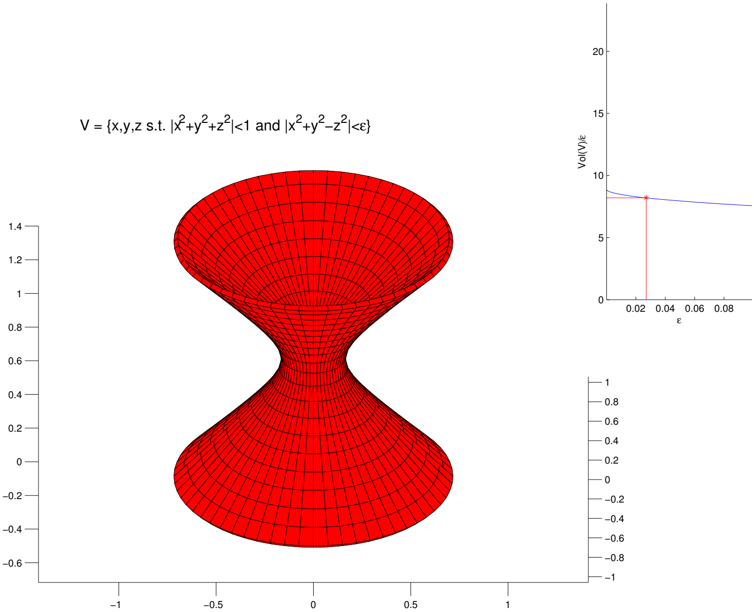
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



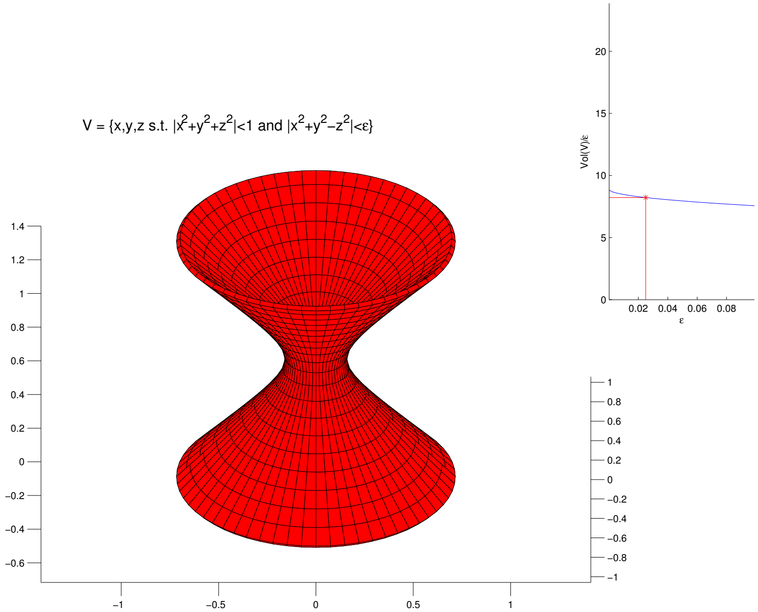


$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

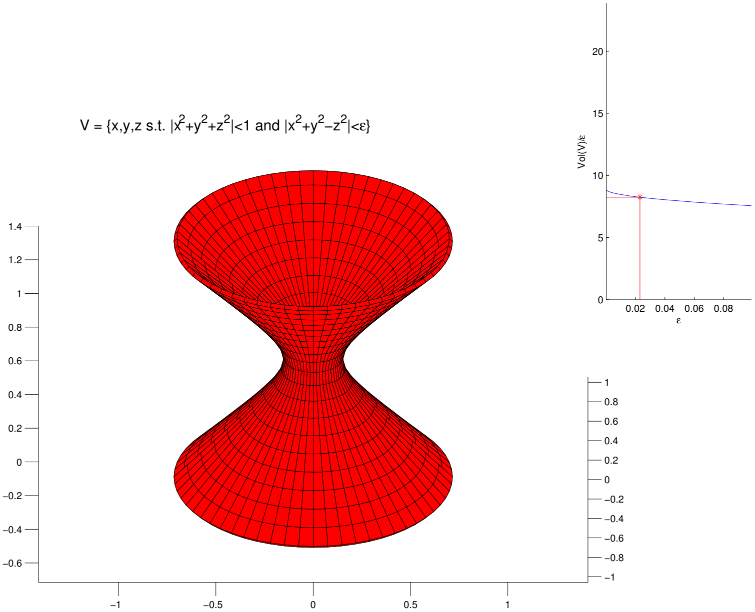


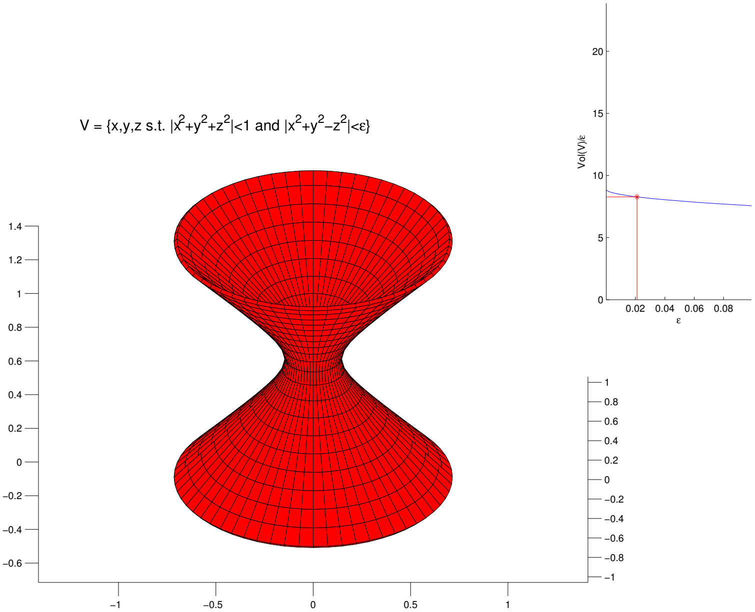


V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
20
15
10
5
0
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

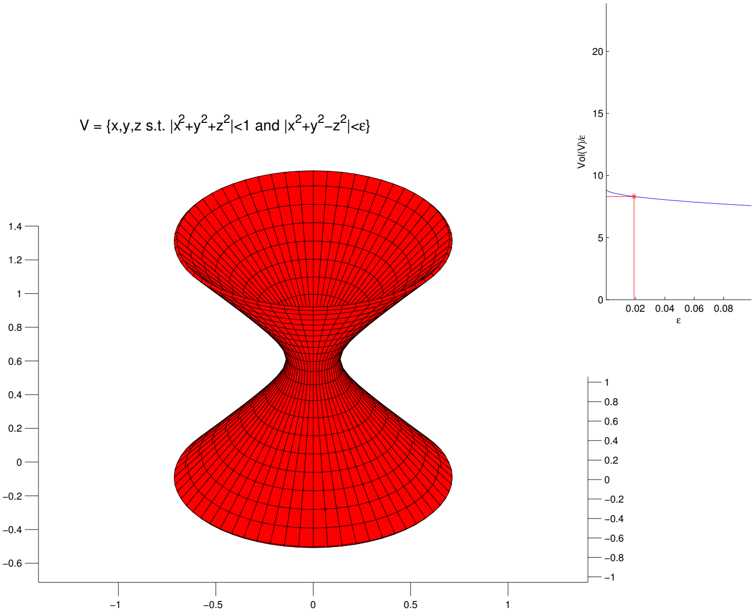

V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

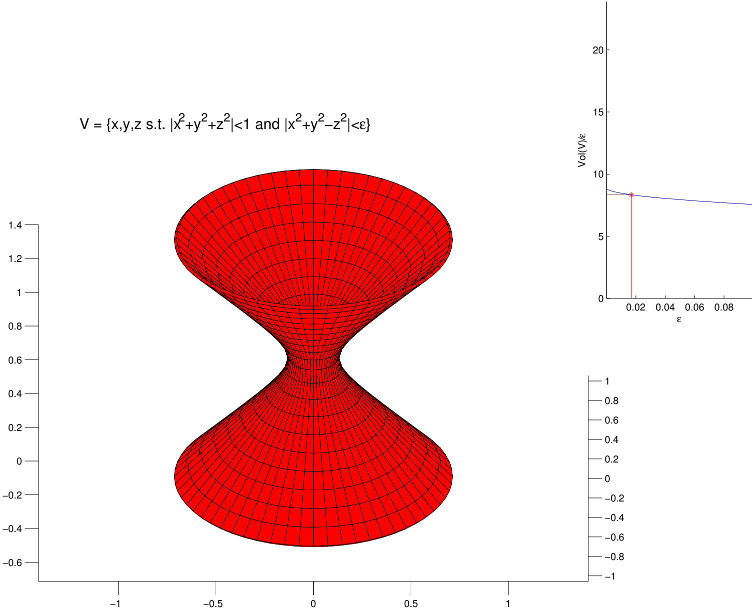
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$





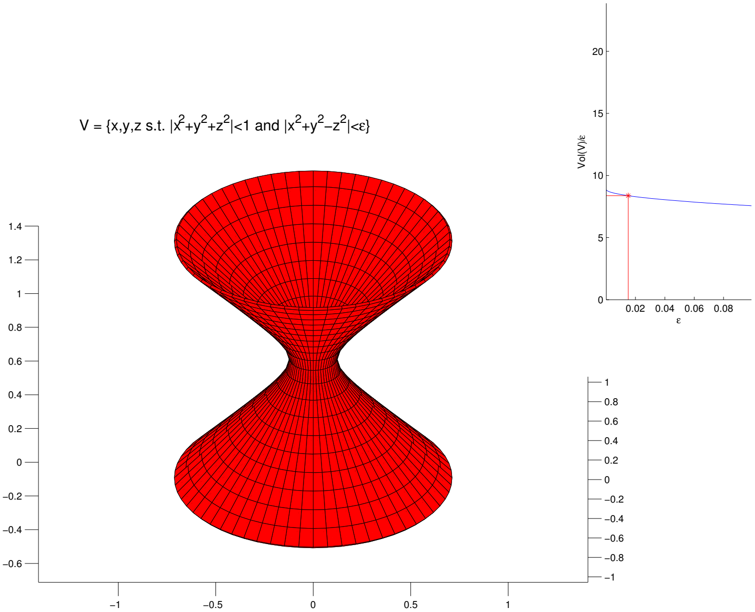
V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
20
15
10
5
0
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
−1 −0.5 0 0.5 1
−0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8 1 1.2 1.4
−1 −0.8 −0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8 1

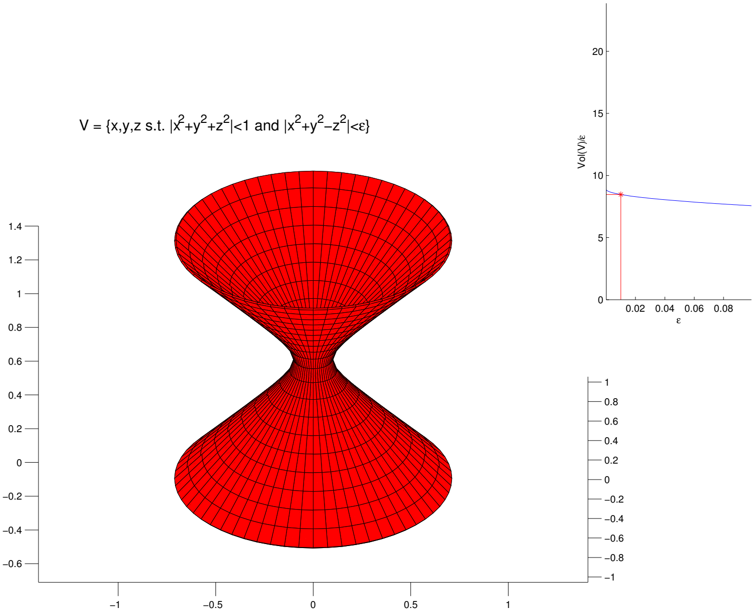


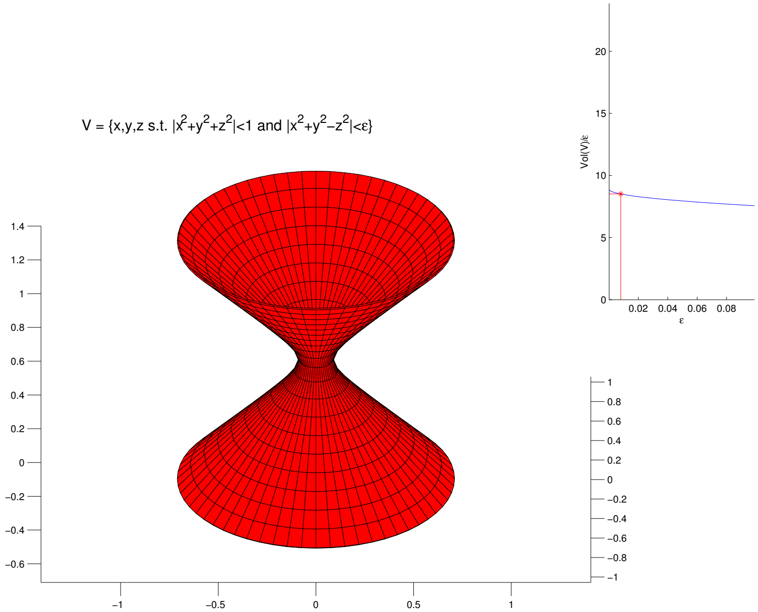
V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

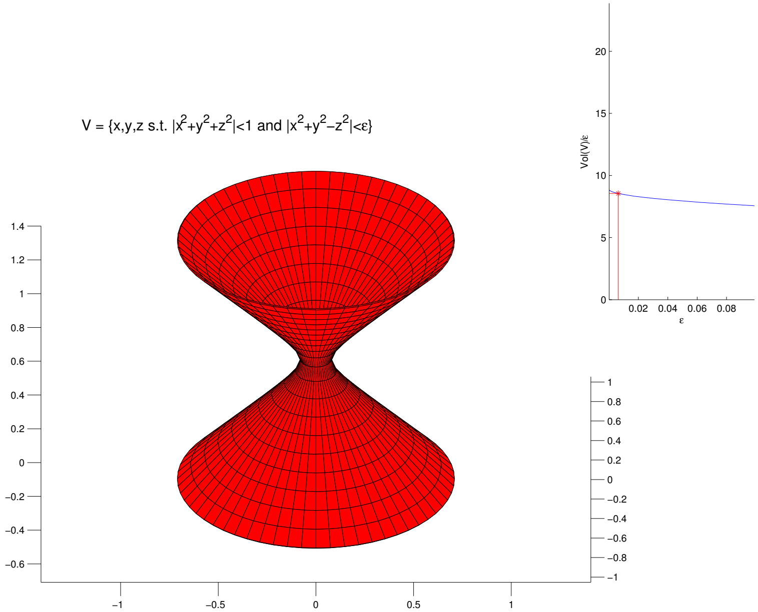
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



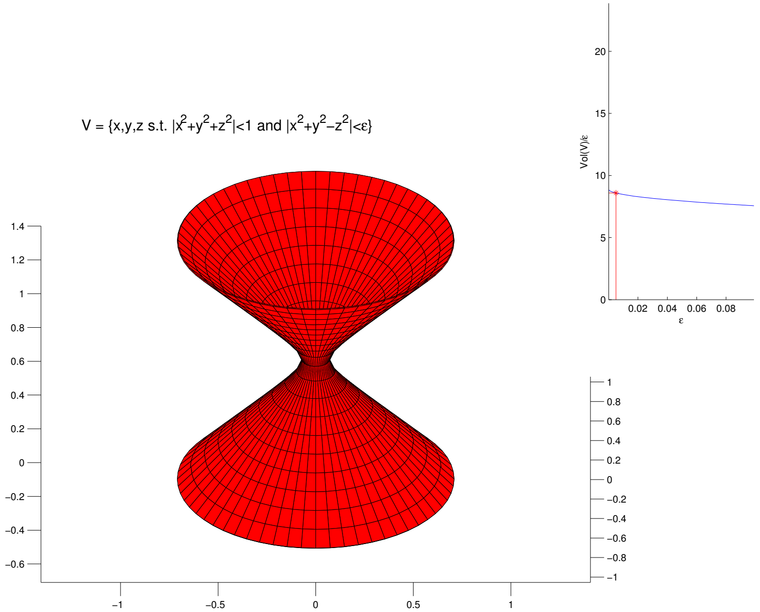
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

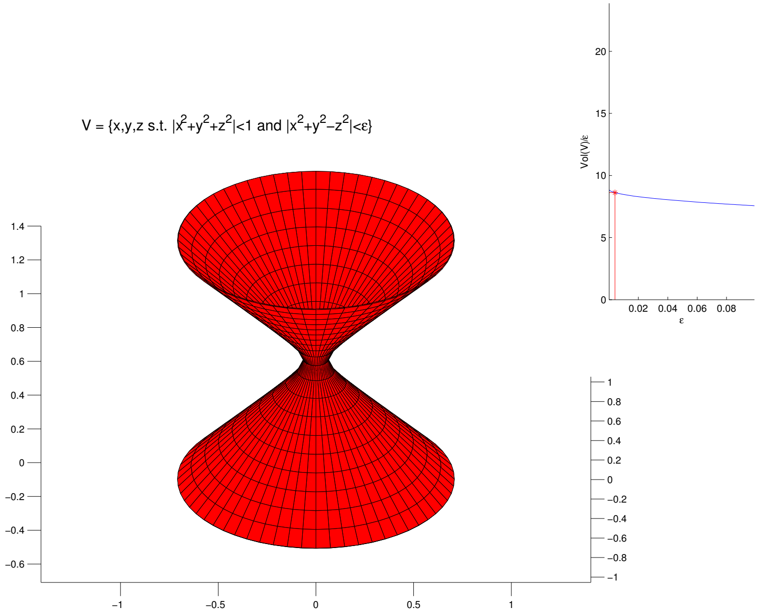


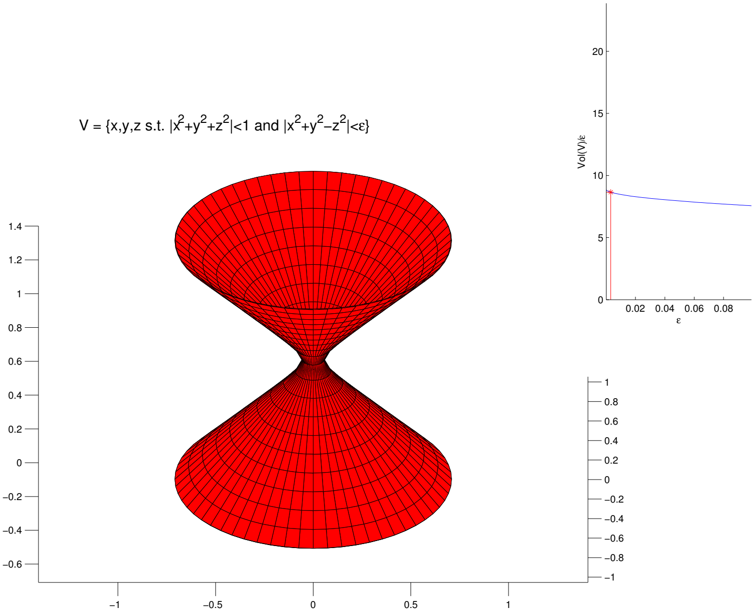

V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20



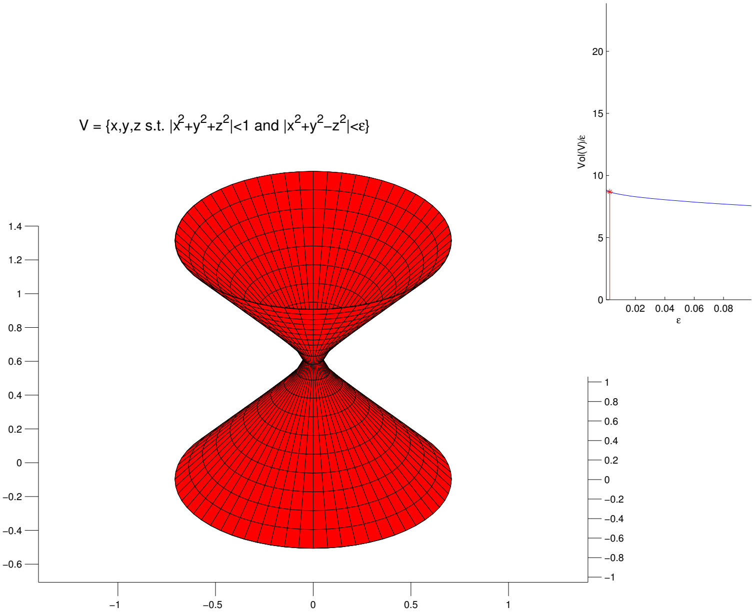
V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
20
15
10
5
0
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

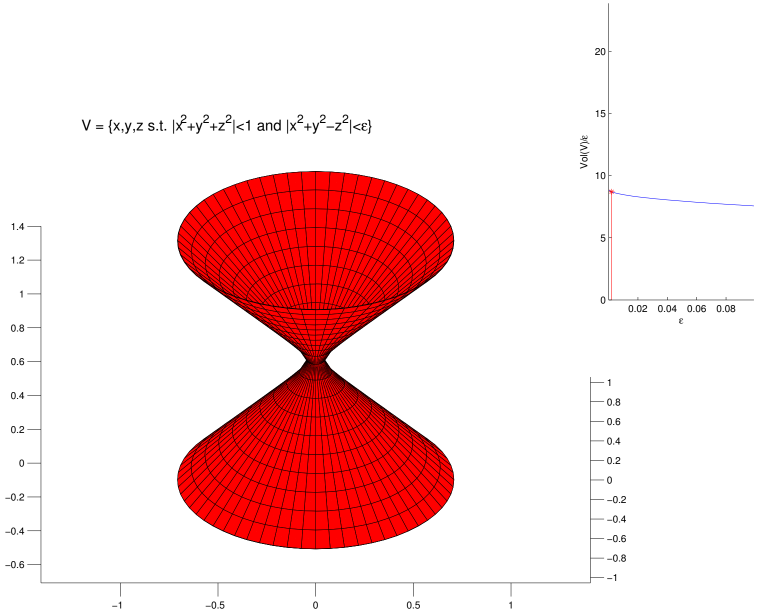

V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

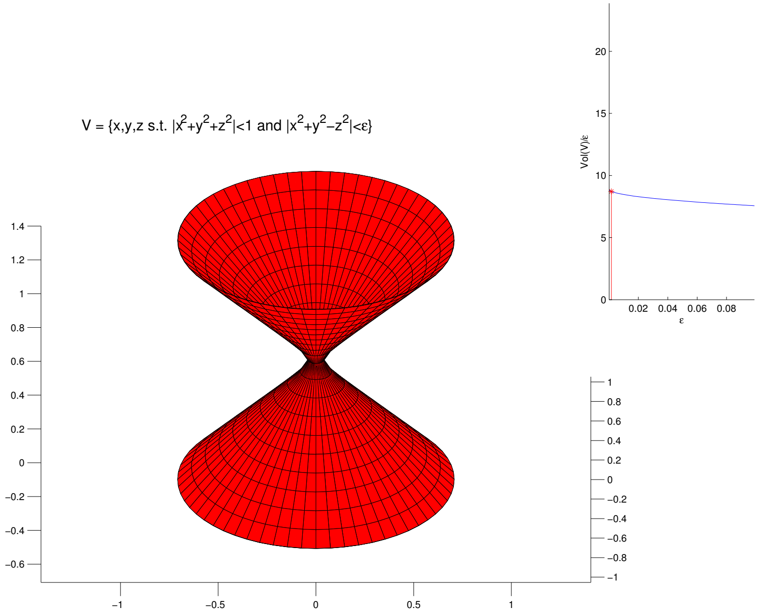

V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0.02 0.04 0.06 0.08
0 5 10 15 20
1.4 1.2 1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6
−1 −0.5 0 0.5 1
1 0.8 0.6 0.4 0.2 0 −0.2 −0.4 −0.6 −0.8 −1

$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

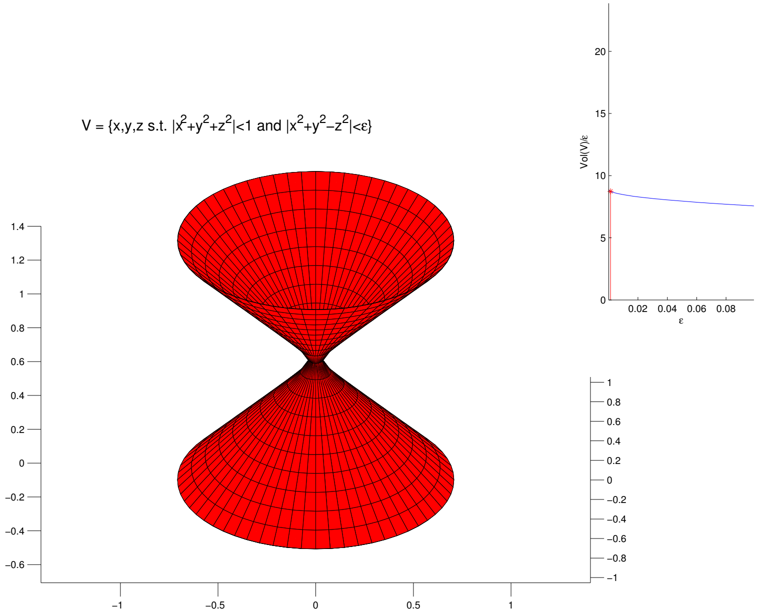




V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε

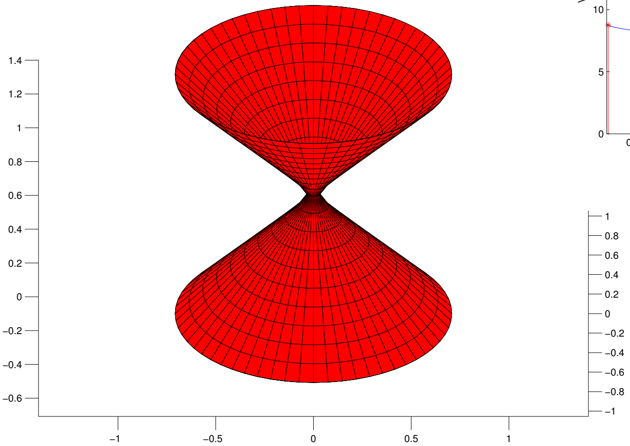


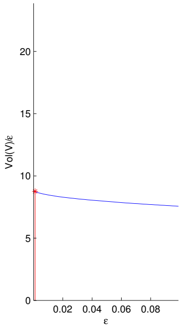
V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
20
15
10
5
0
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1



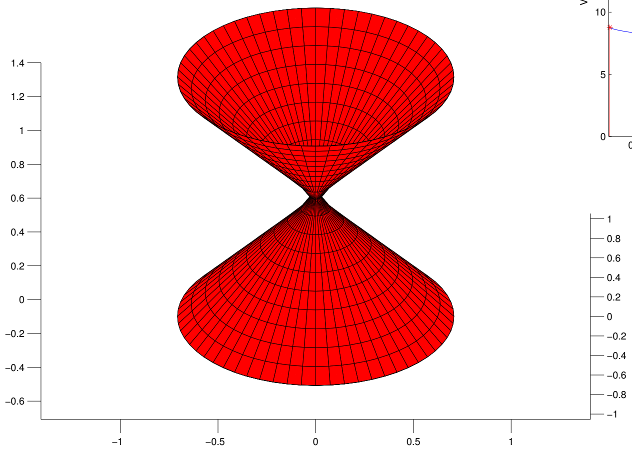
V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε

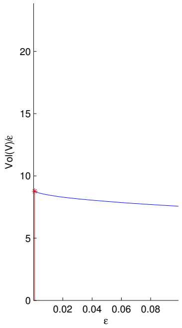
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



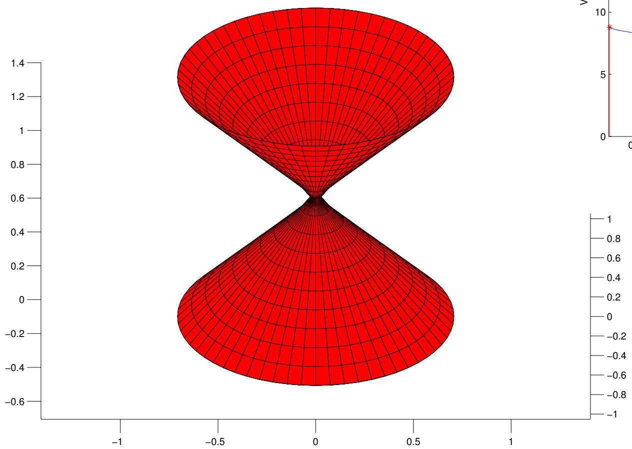


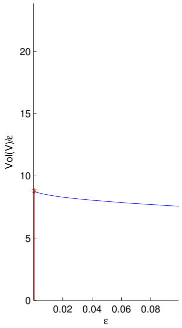
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

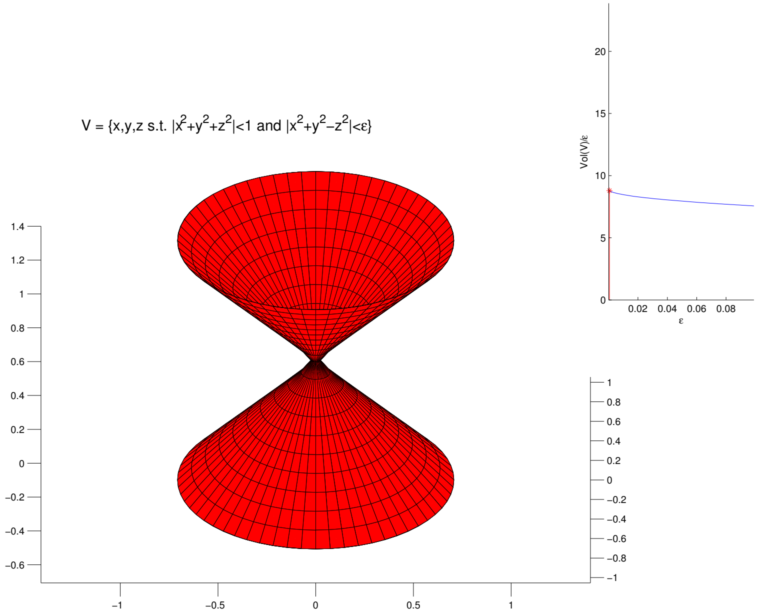




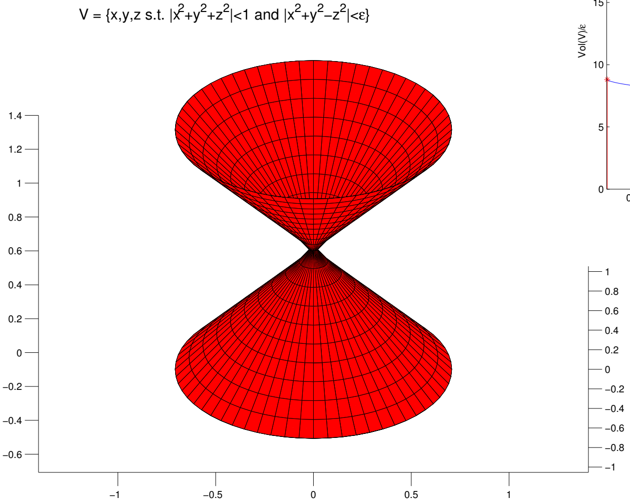
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

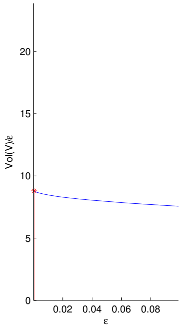


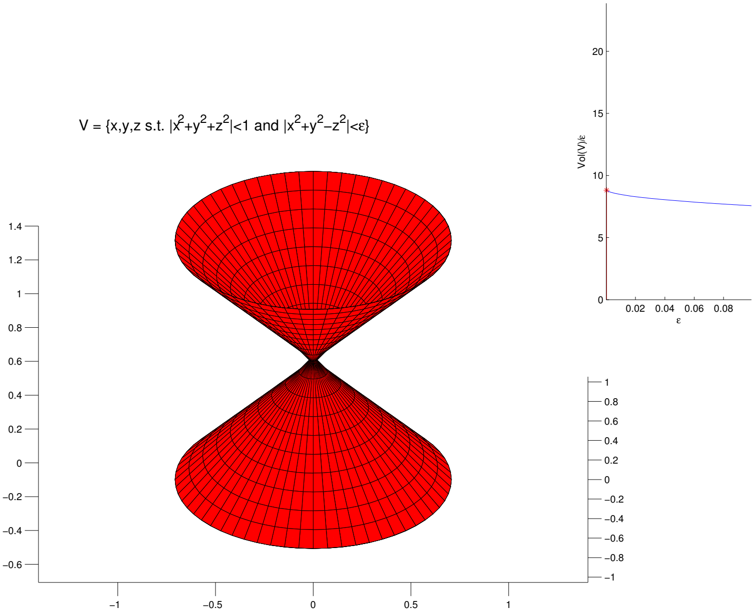




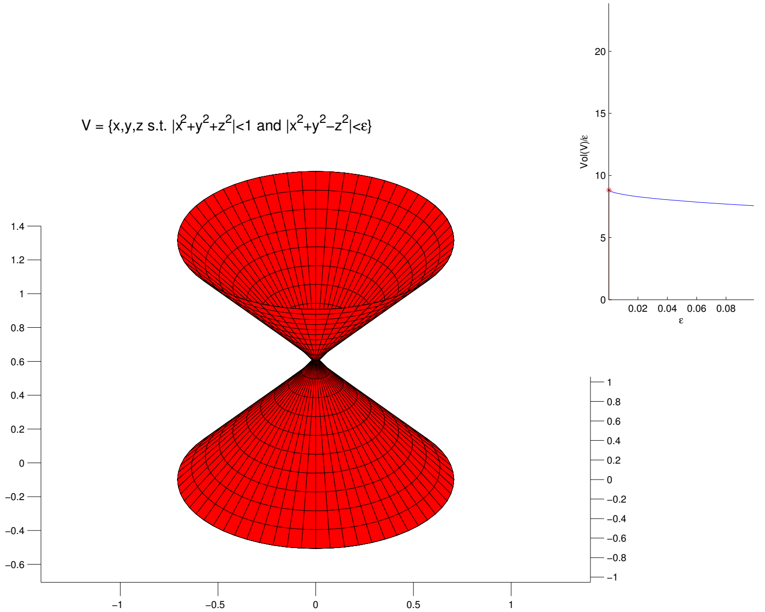
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$
Vol(V)/ε
ε

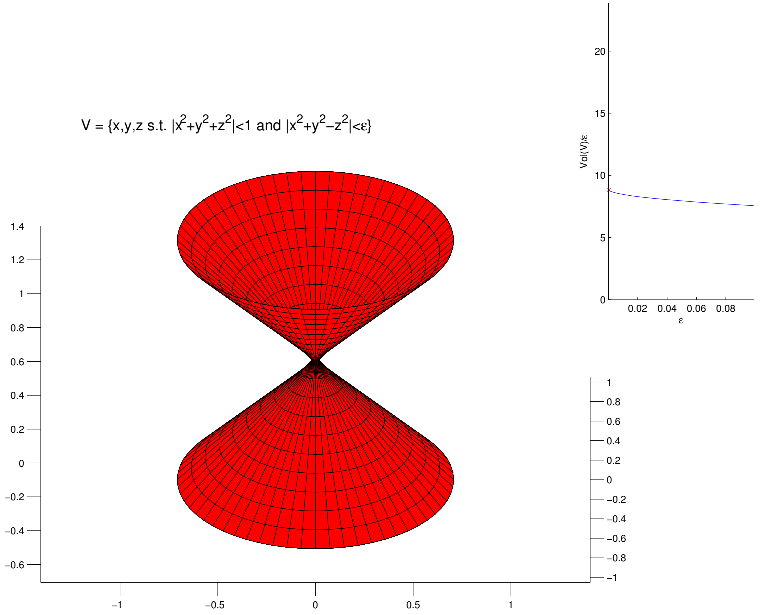

V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

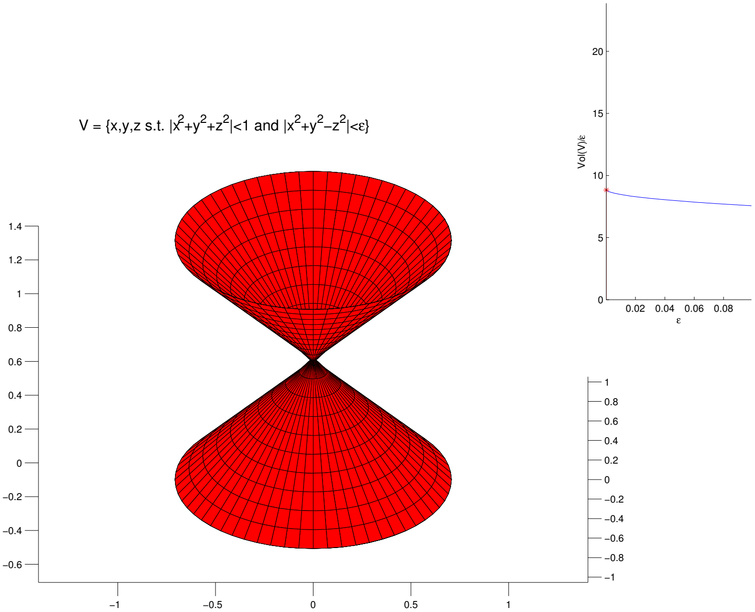

Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0
5
10
15
20
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

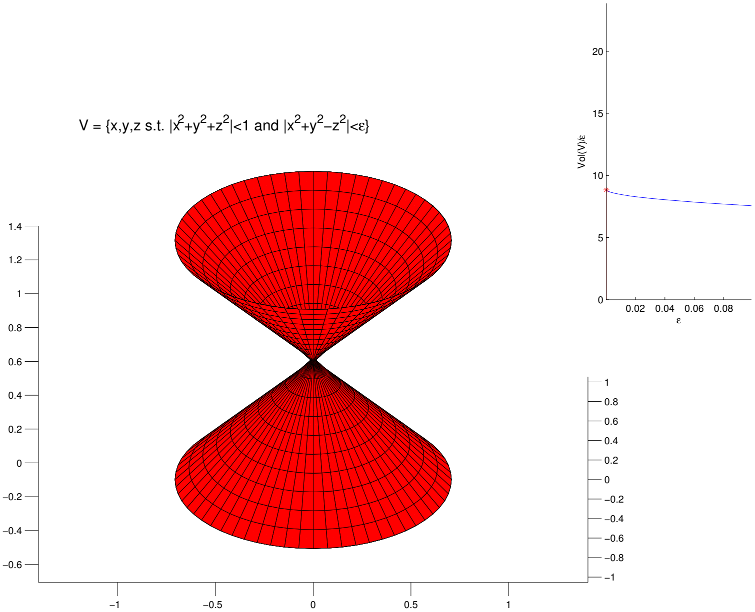
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



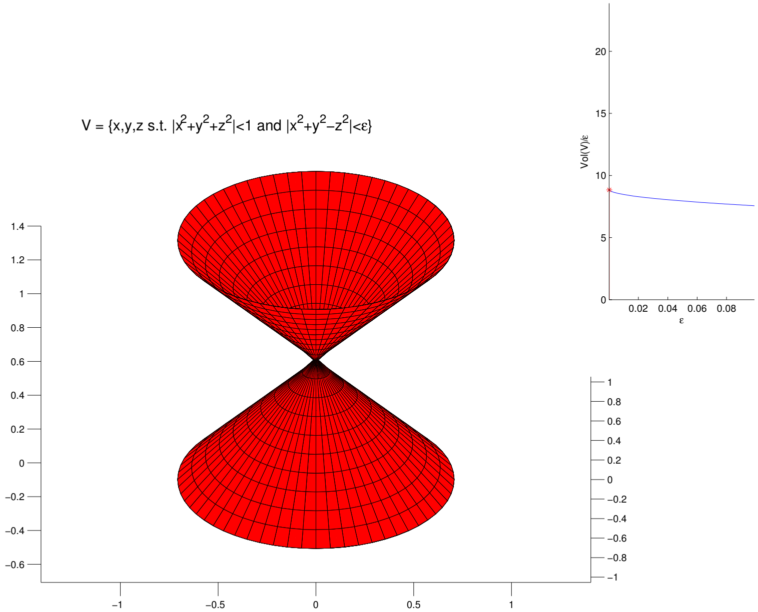

V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε



V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε

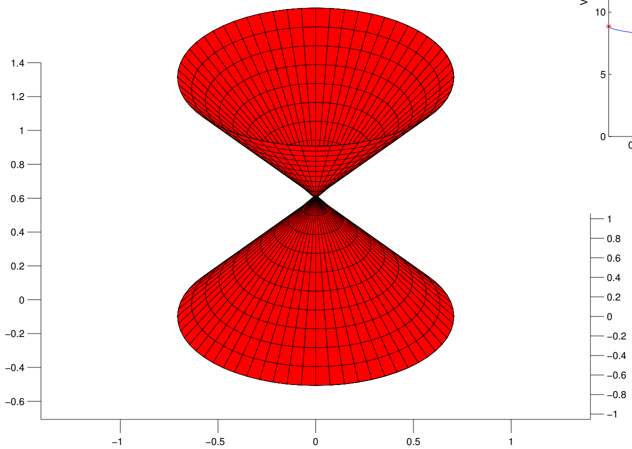


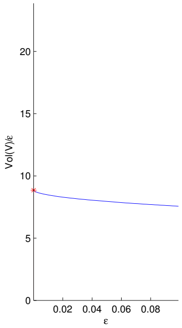
V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε



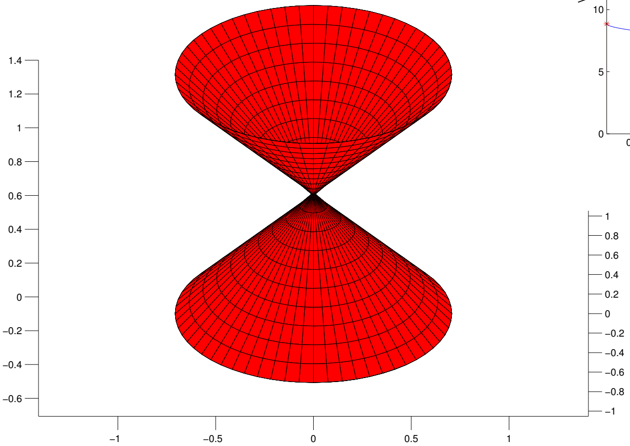
V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε

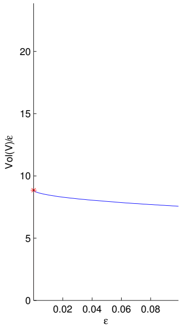
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



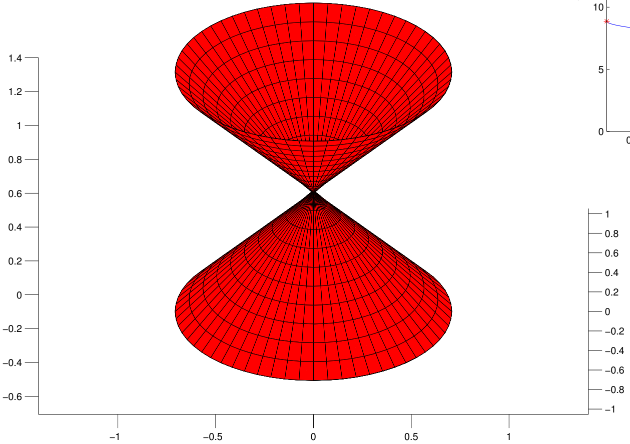


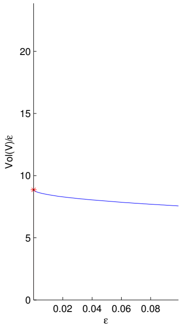
$$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$$



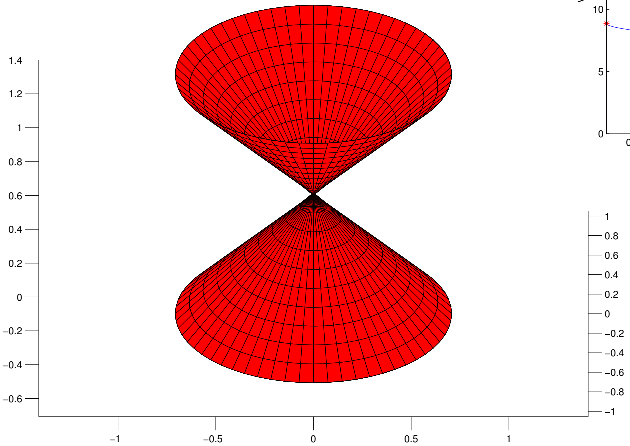


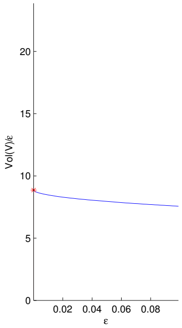
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$

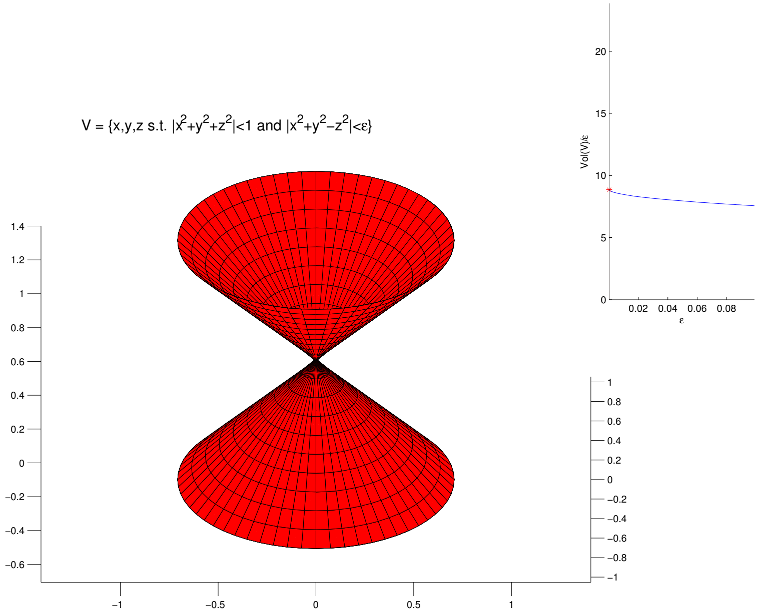




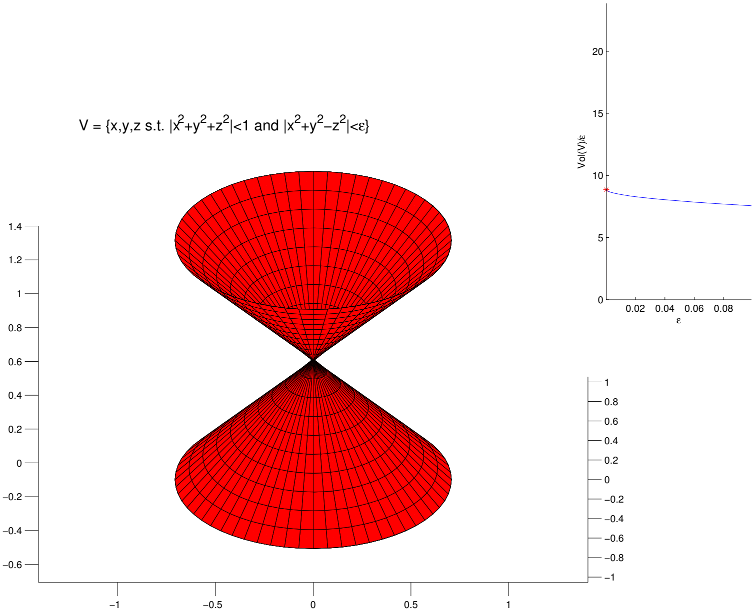
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$



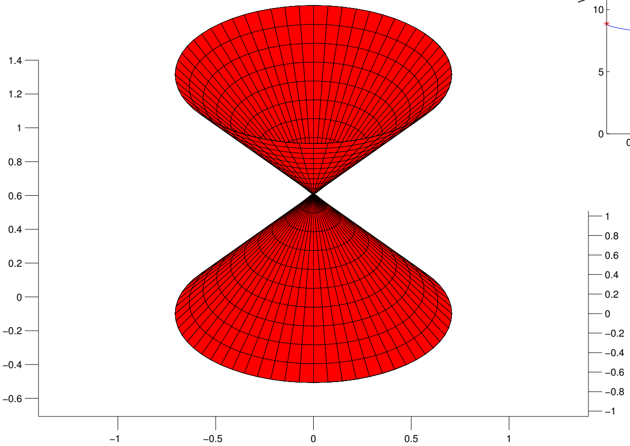


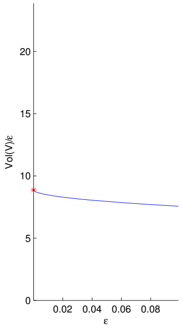


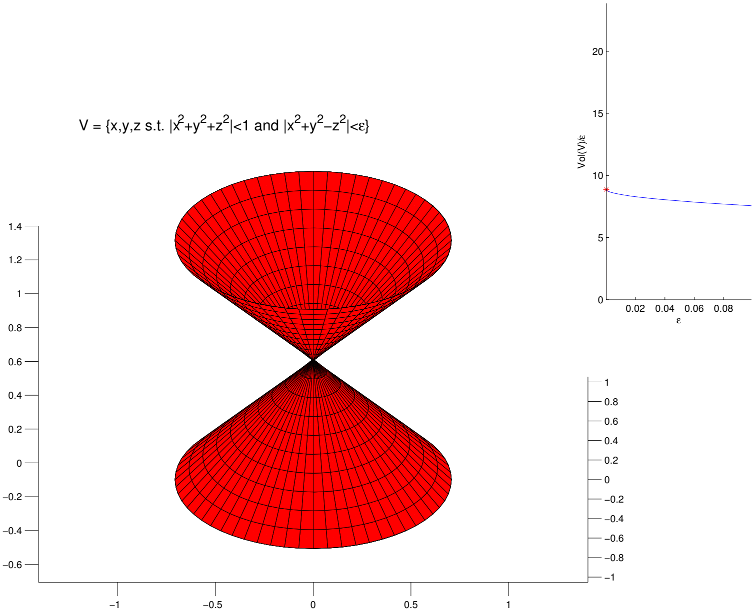
V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε

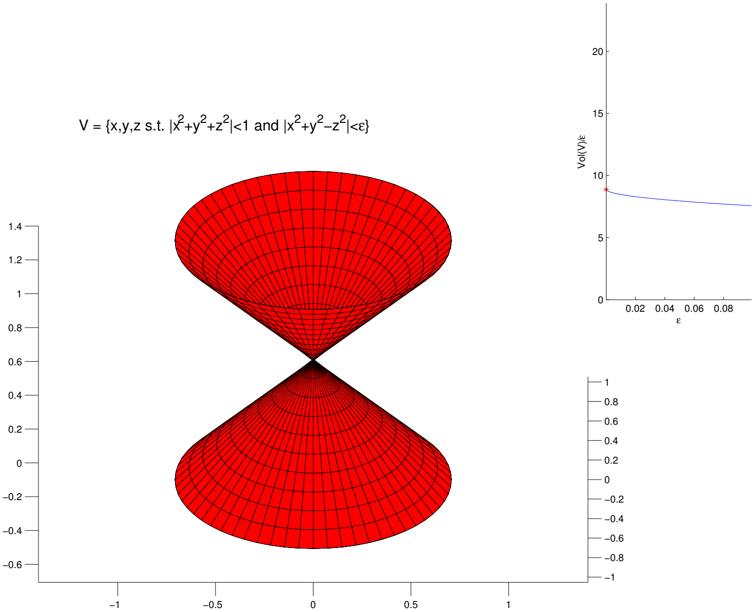

V = {x,y,z s.t. |x2+y2+z2|<1 and |x2+y2−z2|<ε}
Vol(V)/ε
ε
0.02
0.04
0.06
0.08
0
5
10
15
20
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

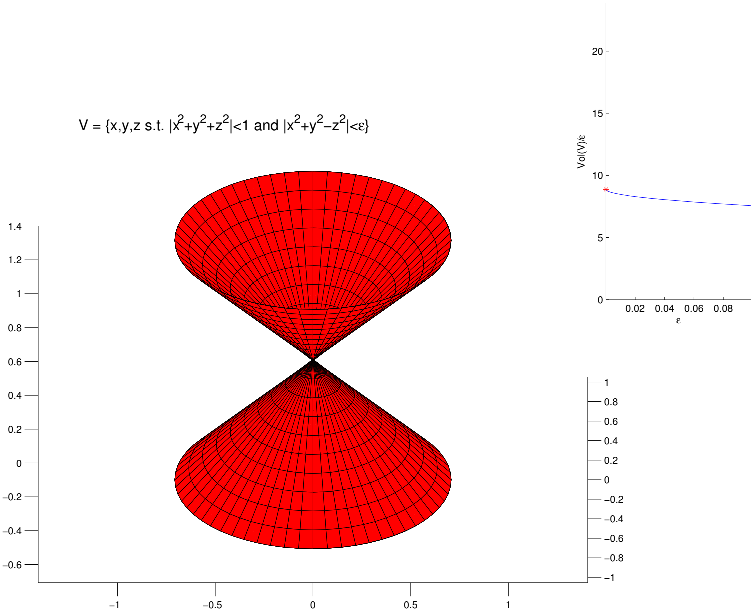
$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$






V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
Vol(V)/ε
20
15
10
5
0
0.02
0.04
0.06
0.08
ε


V = {x,y,z s.t. |x²+y²+z²|<1 and |x²+y²−z²|<ε}
Vol(V)/ε
ε
0
5
10
15
20
0.02
0.04
0.06
0.08
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1


V = {x,y,z s.t. |x^2+y^2+z^2|<1 and |x^2+y^2−z^2|<ε}
Vol(V)/ε
ε
0.02
0.04
0.06
0.08
0
5
10
15
20
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−1
−0.5
0
0.5
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1

Arithmetic

Questions

Arithmetic

Questions

- *How many integer solutions* $f(x_1, \ldots x_n) = 0$ *have?*

Arithmetic

Questions

- *How many integer solutions* $f(x_1, \dots x_n) = 0$ *have?*
- *How many solutions* $f(x_1, \dots x_n) = 0 \mod k$ *have?*

Arithmetic

Questions

- *How many integer solutions* $f(x_1, \ldots x_n) = 0$ *have?*
- *How many solutions* $f(x_1, \ldots x_n) = 0 \mod k$ *have?*
- *How many solutions* $f(x_1, \ldots x_n) = 0 \mod p^m$ *have?*

Arithmetic

Questions

- *How many integer solutions* $f(x_1, \dots x_n) = 0$ *have?*
- *How many solutions* $f(x_1, \dots x_n) = 0 \mod k$ *have?*
- *How many solutions* $f(x_1, \dots x_n) = 0 \mod p^m$ *have?*

"Definition"

We say that f is arithmetically good if $\frac{\#\{x \in (\mathbb{Z}/k)^n : f(x) = 0\}}{k^{(n-1)}}$ *is bounded.*

Geometry

Geometry

For a set $X \subset \mathbb{R}^n$ Let

$$J_k(X) := \{\gamma : [0,1] \to X\}/ \sim_k,$$

Geometry

For a set $X \subset \mathbb{R}^n$ Let

$$J_k(X) := \{\gamma : [0,1] \to X\} / \sim_k,$$

where $\gamma_1 \sim_k \gamma_1$ iff

$$\lim_{t \to 0} \frac{\gamma_1(t) - \gamma_2(t)}{t^k} = 0.$$

For a set $X \subset \mathbb{R}^n$ Let

$$J_k(X) := \{\gamma : [0,1] \to X\}/\sim_k,$$

where $\gamma_1 \sim_k \gamma_1$ iff

$$\lim_{t \to 0} \frac{\gamma_1(t) - \gamma_2(t)}{t^k} = 0.$$

"Definition"

We say that f is geometrically good if for any k, the variety $J_k\left(\{x \in \mathbb{R}^n | f(x) = 0\}\right)$ is dense in $\{x \in J_k(\mathbb{R}^n) | f(x) = 0\}$

Relation

"Theorem"

The following are equivalent:

Relation

"Theorem"

The following are equivalent:

- f *is analytically good*
- f *is arithmetically good*
- f *is geometrically good*

Relation

"Theorem"

The following are equivalent:

- f *is analytically good*
- f *is arithmetically good*
- f *is geometrically good*

Reason:

Relation

"Theorem"

The following are equivalent:

- *f is analytically good*
- *f is arithmetically good*
- *f is geometrically good*

Reason:
Consider the fields

Relation

"Theorem"

The following are equivalent:

- f *is analytically good*
- f *is arithmetically good*
- f *is geometrically good*

Reason:
Consider the fields

- $\mathbb{R}$

Relation

"Theorem"

The following are equivalent:

- *f is analytically good*
- *f is arithmetically good*
- *f is geometrically good*

Reason:
Consider the fields

- $\mathbb{R}$
- $\mathbb{Q}_p$

Relation

"Theorem"

The following are equivalent:

- *f is analytically good*
- *f is arithmetically good*
- *f is geometrically good*

Reason:
Consider the fields

- $\mathbb{R}$
- $\mathbb{Q}_p$
- $\mathbb{F}_p((t))$

Relation

"Theorem"

The following are equivalent:

- *f is analytically good*
- *f is arithmetically good*
- *f is geometrically good*

Reason:
Consider the fields

- $\mathbb{R}$
- $\mathbb{Q}_p$
- $\mathbb{F}_p((t))$
- $\mathbb{C}((t))$

Examples

Example ($f = z^2 - x^2 + y^2$)

Change to spherical coordinates

$$f = r^2(\cos^2(\phi) - \sin^2(\phi)); \quad dV = r^2 \sin(\phi) dr d\theta d\phi.$$

"canceling" r^2 we get

$$f' = \cos(2\phi); \quad dV' = \sin(\phi) dr d\theta d\phi.$$

Examples

Example ($f = z^2 - x^2 + y^2$)

Change to spherical coordinates

$$f = r^2(\cos^2(\phi) - \sin^2(\phi)); \quad dV = r^2 \sin(\phi) dr d\theta d\phi.$$

"canceling" r^2 we get

$$f' = \cos(2\phi); \quad dV' = \sin(\phi) dr d\theta d\phi.$$

Example ($f = x^2 - y^2$)

Change to polar coordinates

$$f = r^2(\cos^2(\phi) - \sin^2(\phi)); \quad dV = r dr d\phi.$$

"canceling" r we get

$$f' = r\cos(2\phi); \quad dV' = dr d\phi.$$