Points of algebraic varieties over finite rings.

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Let R be a finite ring. Denote

$$h_X(R) := \frac{\#X(R)}{\#R^{\dim X_{\mathbb{Q}}}}.$$

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- Let $\mathbb{Q}_{p^n}$ be the unique unramified extension of $\mathbb{Q}_p$ of order n. Let $\mathbb{Z}_{p^n}$ be its ring of integers and π be its uniformizer. Consider $R := \mathbb{Z}_q/\pi^k$.

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Theorem (Cluckers-Loeser $\sim$ 2005)

There exists a finite set S of primes s.t. for any $q \in \mathcal{P}_S$

$$h_X(q,k) = h'_X(q,k)$$

Main Results

Theorem (A.-Avni 2014)

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4. *For almost any prime p: $h_X(p,k)$ is bounded.*
5. *For any $q \in \mathcal{P}$: $h_X(q,k)$ is bounded.*
6. *There exists a finite set of primes S s.t. $h_X(q,k) - h_X(q,1) = O(\frac{1}{q})$ for $q \in \mathcal{P}_S$.*

Jet schemes and rational singularities

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- *The jet schemes of X are irreducible.*

Lang Weil bounds and application of the Chebotarev theorem

Theorem (Lang-Weil)

Let X be an absolutely irreducible scheme.

$$\#X(\mathbb{F}_p) = p^{\dim X} + O(p^{\dim X - \frac{1}{2}}).$$

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$$h'_X(p,k) = h'_{Jet_k(X)}(p,1) = c_p(Jet_k(X)) + O(\frac{1}{\sqrt{p}}).$$

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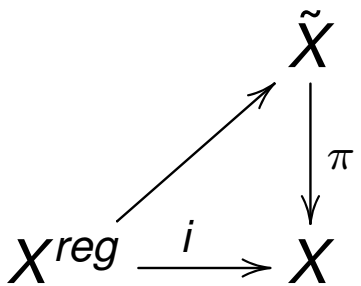

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Let:

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Then $\phi_*(m)$ *has continuous density.*

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$$h_X(q,k) = \frac{\mu(\phi^{-1}(\pi^k \mathbb{Z}_q^b))}{\nu(\pi^k \mathbb{Z}_q^b)}$$

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$$h_X(q,k) = \frac{\mu(\phi^{-1}(\pi^k \mathbb{Z}_q^b))}{\nu(\pi^k \mathbb{Z}_q^b)} = \frac{\phi_*(\mu)(\pi^k \mathbb{Z}_q^b)}{\nu(\pi^k \mathbb{Z}_q^b)}.$$

Theorem (Hrushovski-Kazhdan, et al.)

Let X be a definable set in an algebraic variety over a valued field, and ω be a definable top differential form on X depending on a parameter n in the valuation group. Let

$$f(q,n) := \int_{X(\mathbb{Q}_q)} |\omega|.$$

Then

$$f(q,n) = \sum f_i(q,n) \# X_i(\mathbb{F}_q),$$

where X_i are schemes and $f_i(q,n)$ are "simple-minded" non-negative functions.

2&4 $\Longrightarrow$ 3

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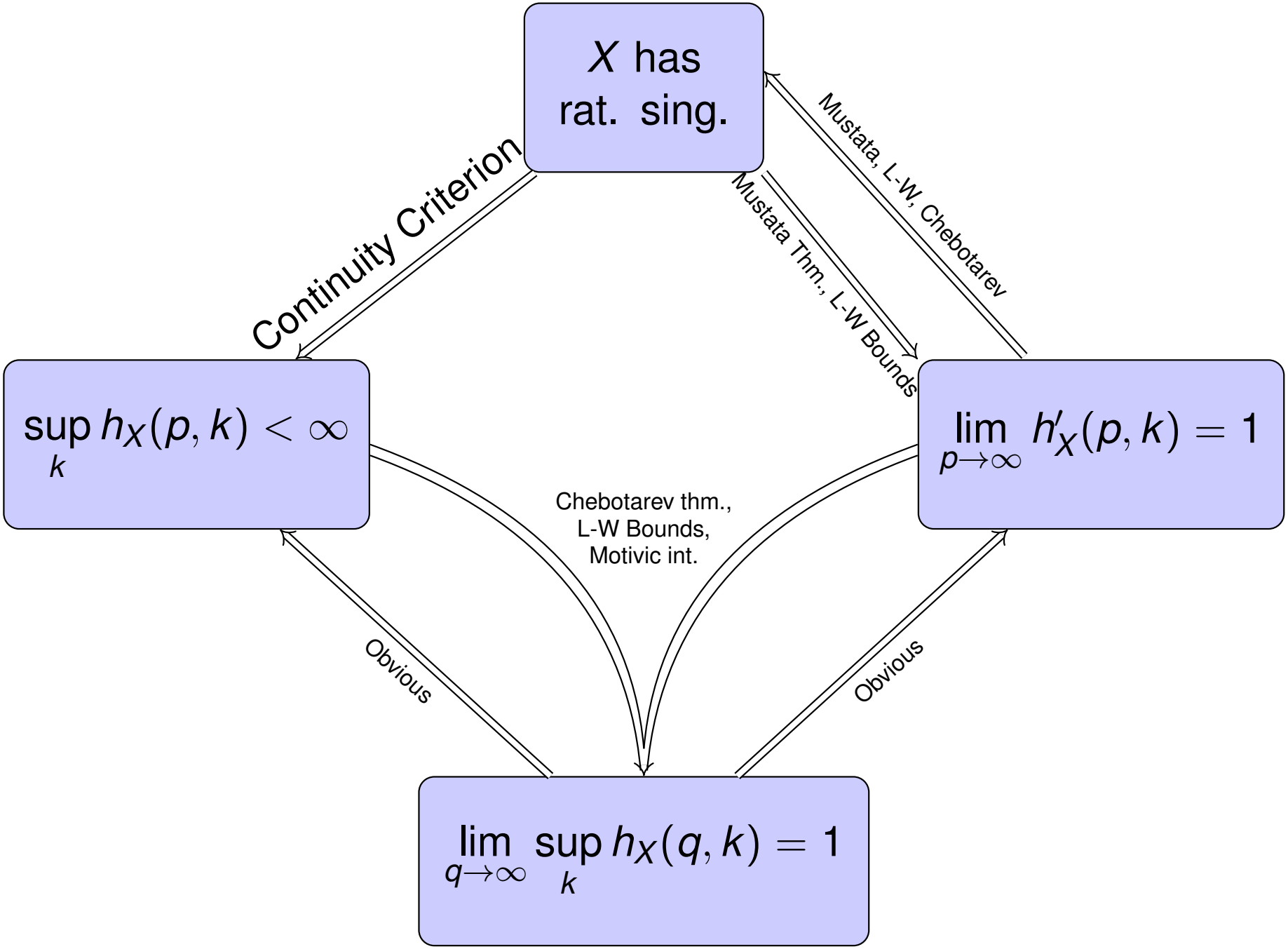
X has rat. sing.
Continuity Criterion
Mustata, L-W, Chebotarev
Mustata Thm., L-W Bounds
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Applications

Theorem (A.-Avni 2014)

Let $n > 500$ and G be a semisimple group defined over $\mathbb{Z}$. Then there exists a constant c s.t. for any integer k and $A \subset G(\mathbb{Z}/k\mathbb{Z})$:

$$\mathrm{Prob}([g_1, h_1] \cdots [g_n, h_n] \in A) < c \cdot \mathrm{Prob}(g \in A),$$

for random elements $g, g_1 \ldots g_n, h_1 \ldots h_n \in G(\mathbb{Z}/k\mathbb{Z})$

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$$\#\{\pi \in \mathrm{irr}(G(\mathbb{Z})) | \dim \pi < n\} < Cn^{1000}.$$