Category Theory Spring 2015 Exercise 6

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1. **[S]** Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Prove that the subcategory of objects in $\mathcal{A}$ which map to 0 in $\mathcal{B}$ is a Serre subcategory.

2. Let $\mathcal{C}$ be a Serre subcategory of $\mathcal{A}$, and $f : X \to Y$ in $W_{\mathcal{C}}$ (as defined in class). Show that for any $g : Z \to Y$ there is a commutative square

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

   with $f' \in W_{\mathcal{C}}$.

3. **[S]** Let $G : \mathcal{A} \to \mathcal{B}$ be an exact functor, and $\mathcal{C}$ the kernel of G as defined in class. Show that the induced functor $\mathcal{A}/\mathcal{C} \to \mathcal{B}$ is faithful.

4. **[S]** Let A be an Noetherian integral domain. Let $\mathcal{A}$ be the (abelian) category of finitely generated A modules, and $\mathcal{C}$ be the subcategory of torsion modules in $\mathcal{A}$. Show that $\mathcal{C}$ is a Serre subcategory and that $\mathcal{A}/\mathcal{C}$ is equivalent to the category of finite dimensional $Frac(A)$ vector spaces.

5. Let X be a topological space and $Z \subset X$ closed. Let $\mathrm{Sh}(X)$ be the category of sheaves of abelian groups on X and $\mathrm{Sh}_Z(X)$ be the subcategory of sheaves supported on Z. Show that $\mathrm{Sh}_Z(X)$ is a Serre subcategory, and that $\mathrm{Sh}(X)/\mathrm{Sh}_Z(X) \cong \mathrm{Sh}(X \setminus Z)$