Algebraic Topology - Exercise 4

Solve the following questions:

0. Let X be path connected, and $\phi : \tilde{X} \to X$ a covering map, show that (If you didn't hand this one last week):
   (a) $\tilde{X}$ is path connected $\iff$ $\pi_1(X)$ acts transitively on the fiber at x_0 (Note that the action might not lift to a deck transformation if the lifted loop wasn't in the normalizer of $\phi_* \pi_1(\tilde{X}, \tilde{x}_0)$.).
   (b) $|(\phi^{-1}(x_1)| = |\phi^{-1}(x_2)|$ for any $x_1, x_2 \in X$.

1. Show that:
   (a) a covering map is always open.
   (b) If G act transitively on X, then $|G_x| = |G_y|$ for any $x, y \in X$.

2. Construct a non-normal cover of $S^1 \vee S^1$ (Note: in Exercise Session #5 we'll show what are the normal covers of $S^1 \vee S^1$, you can hand it in with Exercise 5 if you'd like).

3. Compute the fundamental group of the Klein bottle
   (a) Using its universal cover.
   (b) Using Van-Kampen's theorem.

4. Prove a version of the Fundamental Theorem of Algebra, a non-constant polynomial $p(z)$ with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$:
   (a) Define $f_r(s) = \frac{p(re^{2\pi is})}{|p(re^{2\pi is})|}$, for a polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ with no roots in $\mathbb{C}$. Show that $f_r(s)$ is a loop in $S^1 \subset \mathbb{C}$ for every $r \geq 0$, and compute $[f_r] \in \pi_1(S^1)$.
   (b) Take $r > \max(1, |a_0| + \cdots + |a_{n-1}|)$, and show that for $|z| = r$ and for every $t \in [0, 1]$, we get that $p_t(z) = z^n + t(p(z) - z^n)$ has no roots on the circle $|z| = r$.
   (c) Construct a homotopy between f_r and the loop $e^{2\pi ins}$, by recalling the non-triviality of $[e^{2\pi ins}] \in \pi_1(S^1)$ (why?), and the homotopy class of f_r, conclude that we must have that $n = 0$ and thus that $p(z)$ must be constant.

5. Let $p : X_H \to X$ be a path connected covering space, where $G = \pi_1(X)$ and $H = \pi_1(X_H)$, and also set $G(X_H)$ to be the group of deck transformations, show that:

   (a) $G(X_H)$ acts transitively on the fibers $\iff$ the stabilizer of a point in the fiber under the action of $\pi_1(X)$ is normal.

   (b) If H is normal then $G(X_H) \simeq G/H$.

   Extra exercises:

6. Construct a non-normal covering space of the Klein bottle by a Klein bottle and by a torus.

7. Compute the fundamental group of the Hawaiian earring, that is the bouquet of countably many circles of radius $\frac{1}{n}$ and center $(\frac{1}{n}, 0)$ at their common point.

8. For an example of a space with a very strange fundamental group, look up the the Harmonic Archipelago online.

9. Show that generally, for any $H \leq G$ we have that $G(X_H) \simeq N(H)/H$, where $N(H)$ is the normalizer of H.