Algebraic Topology - Exercise 8

Solve the following questions:

1. (a claim from class) Let S be a simplicial complex and $\alpha : S \to S$ a map from the realization of S to itself. Show that if the induced map $\pi_S(\alpha) : \pi(S, S) \to \pi(S, S)$ is identity (Recall that $\pi(T, S)$ is the set of all continuous maps from T to S up to homotopy.), then α is homotopic to Id_S.

2. Let $g \in \pi_1(X)$, prove that:

   (a) g is a commutator, i.e. $\exists a, b \in \pi_1(X)$ s.t. $g = [a, b] \iff$ there exists a map $\phi : (T^2 - D^2) \to X$ with $\phi|_{\partial(T^2-D^2)} = g$.

   (b) g is a product of commutators, that is $g = [a_1, b_1] \cdot \ldots \cdot [a_n, b_n]$ $\iff$ there exists a map $\phi : (T^2\#\ldots\#T^2 - D^2) \to X$ with $\phi|_{\partial(T^2\#\ldots\#T^2-D^2)} = g$.

3. Show that $\partial^2 = 0$.

4. Compute the Euler characteristic of the following spaces (using triangulations):

   (a) An orientable genus two surface.

   (b) The Mobius strip.

   (c) The Klein bottle.

   (d) The projective plane.

   (e) An n-dimensional sphere.

   Extra:

5. Finish the proof of Whitehead's theorem: let X be a topological space, construct a homotopy equivalence between the realization of the set (V,S) of maps from simplices to X we constructed in class to X.