1 Constructions

We want to show another construction for topological spaces.
Let X, Y be two topological spaces. Consider the space

$$cont(X,Y) = \{f : X \to Y | f \text{ is continuous}\} \tag{1}$$

We need to define a topology.

Definition 1. *The sub basis for the compact open topology is defined as follows:*

$$U_{K,V} = \{f \in cont(X,Y) | f(K) \subseteq V\} \tag{2}$$

s.t. $K \subseteq X$ compact and $V \subseteq Y$ is open.

This construction makes sense if X is compactly generated.

Definition 2. *X is compactly generated if*

$U \subseteq X$ *is open* $\iff \forall K \subseteq X$ *compact,* $U \cap K$ *is compact in* K*.*

What is it good for?

Theorem 1.
- *$\forall f \in cont(X \times Y, Z)$ there is a natural induced $f_* \in cont(X, cont(Y,Z))$ by $f_*(x)(y) = f(x,y)$. When X, Y, Z are all compactly generated, then f_* is a bijection.*
- *$\forall g \in cont(X \wedge Y, (Z, z_0))$ there is a natural induced $g_* \in cont((X, x_0), cont((Y, y_0), (Z, z_0)))$ by $g_*(x)(y) = g(x,y)$ and $g_*(x_0)(y_0) = z_0$. If X, Y, Z are compactly generated, then g_* is a bijection.*

Example 1. *Let $Y = S^1$. Then*

$$cont(X \wedge S^1, Z) \cong cont(X, cont(S^1, Z)), \tag{3}$$

which actually means

$$cont(\Sigma X, Z) \cong cont(X, \Omega(Z)). \tag{4}$$

If we take $Z = \Sigma X$ we get the following,

$$cont(\Sigma X, \Sigma X) \cong cont(X, \Omega\Sigma X) \tag{5}$$

and we denote by φ_1 the image of the id *mapping under that correspondence, which is the unit map.*

2 The Fundamental Group π_1

Definition 3. *Let (X, x_0) be a pointed topological space, $\Omega(X)$ is the set of all loops in X with x_0 as their base point, i.e. loops starting and ending at x_0.*

Definition 4. *If f is a path in X from x_0 to x_1 and if g is a path in X from x_1 to x_2 we define the concatenation $f * g$ of f and g to be the path h given by the equations:*

$$h(s) = \begin{cases} f(2s) & s \in [0, 1/2] \\ g(2s-1) & s \in [1/2, 1] \end{cases}$$

We think of h as the path whose first half is the path f and whose second half is the path g.

Notice however, that concatenation is NOT associative! That is, for f, g and h paths in X from x_0 to x_1, x_1 to x_2 and x_2 to x_3 respectively, it is almost never the case that $f * (g * h) = (f * g) * h$.

Now, we can consider $\Omega(X)/\sim$ (where $f \sim g \iff f$ is homotopic to g). We denote this quotient by $\pi_1(X, x_0)$. In addition, we define

$$[f] * [g] = [f * g], \tag{6}$$

and arrive at our first theorem:

Theorem 2. *$\big(\pi_1(X, x_0), *\big)$ is a group.*

Proof. It is easy to verify it using the following two lemmas:

Lemma 1. *If $k : X \to Y$ is continuous, F is a homotopy of paths between f and f', then $k \circ F$ is a homotopy of paths in Y of $k \circ f$ and $k \circ f'$.*

Lemma 2. *If $k : X \to Y$ is continuous, f, g are paths in X with $f(1) = g(0)$, then $k \circ (f * g) = (k \circ f) * (k \circ g)$.*

□

Lemma 3. *If $x_0, x_1 \in X$ are path-connected then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

So, for a path-connected space we can just omit the fixed point and write $\pi_1(X)$ as it is well-defined.

Now let us consider a continuous function $\varphi : X \to Y$. Naturally, φ induces a function $\varphi_* : \pi_1(X) \to \pi_1(Y)$ defined by $\gamma \mapsto \varphi \circ \gamma$. Set $\psi \in \mathit{cont}(X, Y)$, then it is easy to verify the following:

- φ_* is a group homomorphism.
- $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.
- $\varphi \sim \psi \Longrightarrow \varphi_* = \psi_*$

Corollary 1. *If $(X, x_0) \simeq (Y, y_0)$ then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.*

3 Covering Spaces

Definition 5. *Let X be topological space that is path-connected. A covering space of X is pair (E, ρ) s.t. E is a topological space, $\rho : E \to X$ continuous and surjective s.t. $\forall x \in X$ there is an open neighborhood U_x of X s.t. $\rho^{-1}(U_x) = \sqcup_{\alpha \in I} V_\alpha$ where $\rho\big|_{V_\alpha}$ is homeomorphism on U_x and the sets V_α are disjoint.*

Example 2.
- *$X \times D$ for any discrete set D with the identity on each copy of X.*
- *$\varphi : S^1 \to S^1$ by $z \mapsto z^n$.*
- *$\psi : \mathbb{R} \to S^1$ by $x \mapsto e^{ix}$.*

Theorem 3. *Let (X, x_0) a topological space, (E, ρ) a covering space and $\rho(e_0) = x_0$.*

- *if $\gamma : [0, 1] \to X$ is a path s.t. $\gamma(0) = x_0$ then there exists a unique path $\bar{\gamma} : [0, 1] \to E$ s.t. $\bar{\gamma}(0) = e_0$ and $\rho \circ \bar{\gamma} = \gamma$. $\bar{\gamma}$ is called the lift of γ to (E, ρ) at the point e_0.*
- *Let$F : [0, 1]^2 \to X$ be a continuous function, then there exists unique $\bar{F} : [0, 1]^2 \to E$ s.t. $\bar{F}(0, 0) = e_0$ and $\rho \circ \bar{F} = F$. Moreover, if F is a homotopy of paths, then so is $\bar{F}$.*

Proof. We will highlight key points in the proof for the first item. The second item is done in a similar way. The idea is to define $\bar{\gamma}$ in parts.
Note that $\bigcup\limits_{x \in X} \rho^{-1}(U_x)$ is an open cover of [0,1].
Since $[0, 1]$ is compact, we can find a finite sub-cover, and since ρ is homemorphism on each of these open sets, it has an inverse. Thus, we have only one way to define $\bar{\gamma}$:

$$\bar{\gamma}(t) = \rho^{-1} \circ \gamma(t) \tag{7}$$

We start with the set containing the start point e_0, and continue set by set. After a finite number of steps, we have uniquely defined $\bar{\gamma}$. □

4 The Fundamental Group of S^1

Theorem 4. *The fundamental group of S^1 is isomorphic to $(\mathbb{Z}, +)$.*

Proof. Again, we will only highlight key points in the proof.
Let b_0 be the point (1,0) of S^1. We shall construct an isomorphism,

$$\phi : (\pi_1(S^1, (1, 0)), *) \to (\mathbb{Z}, +).$$

For this purpose, consider the covering map $\rho : R \to S^1$ by $\rho(x) = (\cos 2\pi x, \sin 2\pi x)$.

If γ is a loop in S^1 based at b_0 let $\tilde{\gamma}$ be the lift of γ to to a path in R beginning at 0. The point $\tilde{\gamma}(1)$ must be a point of the set $\rho^{-1}(b_0)$; that is, $\tilde{\gamma}(1)$

must equal some integer n. We define $\phi([\gamma])$ to be that integer.
In the next lectures we will see why it is well-defined, and why is it an isomorphism. $\square$