Restriction for general linear groups: the local non-tempered Gan-Gross-Prasad conjecture

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Classical results

Let $G_n = \mathrm{GL}_n(F)$, where F is a local field.

Theorem (AGRS 2010, SZ 2012)

Let π_1 and π_2 be irreducible (smooth) representations of G_{n+1} and G_n. Then

$$\dim \mathrm{Hom}_{G_n}(\pi_1, \pi_2) \leq 1.$$

But still need to know existence of quotients..

Theorem (JPSS 1983)

Let $\pi_1 \in \mathrm{Irr}(G_{n+1})$ and $\pi_2 \in \mathrm{Irr}(G_n)$ both generic. Then

$$\mathrm{Hom}_{G_n}(\pi_1, \pi_2) \neq 0$$

Ext-analogue version

Theorem (D. Prasad, Aizenbud-Sayag)

Let π_1 and π_2 be admissible representations of G_{n+1} and G_n respectively. Then

$$\dim \mathrm{Ext}^i_{G_n}(\pi_1, \pi_2) < \infty$$

Theorem (Conjectured by D. Prasad, Proved C.-Savin 2018 arXiv)

Let $\pi_1 \in \mathrm{Irr}(G_{n+1})$ and $\pi_2 \in \mathrm{Irr}(G_n)$ both generic. Then, for all $i \geq 1$,

$$\mathrm{Ext}^i_{G_n}(\pi_1, \pi_2) = 0$$

Today, our concern goes back to Hom-branching law on a special class of representations – Arthur type.

Arthur parameters

Let W_F be the Weil group of F. Let WD_F be the Weil-Deligne group. Let

$$WD_F = \begin{cases} W_F \times \mathrm{SL}_2(\mathbb{C}) & \text{if } F \text{ is non-Archimedean} \\ W_F & \text{if } F \text{ is Archimedean} \end{cases}$$

An Arthur parameter is the set of ${}^L G$-orbits of maps

$$\psi : WD_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G = \mathrm{GL}_n(\mathbb{C})$$

such that $\psi|_{WD_F}$ has bounded image i.e. has tempered Langlands parameter, and the restriction to $\mathrm{SL}_2(\mathbb{C})$-factor is algebraic.

Arthur type representations

- $\mathrm{Sym}^k(\mathbb{C}^2)$: $(k+1)$-diml irr. rep. of $\mathrm{SL}_2(\mathbb{C})$
- Arthur parameter, as a finite $WD_F \times \mathrm{SL}_2(\mathbb{C})$-representation ψ, takes the form

$$M_A = \sum_d M_d \otimes \mathrm{Sym}^d(\mathbb{C}^2), \tag{1}$$

where each M_d is a tempered representation of WD_F

- For each Arthur parameter ψ, one assigns a L-parameter given by

$$\phi_\psi(w) = \psi(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix})$$

- a Langlands parameter M as described above, and gives a G_n-representation of Arthur type denoted by π_M

Relevant Arthur parameters

Definition (Gan-Gross-Prasad)

Let M_A and N_A be Arthur parameters. Then M_A and N_A are **relevant** if there exist tempered *WD*-representations $M_0^+, \ldots, M_r^+$ and $M_0^-, \ldots, M_s^-$ such that

$$M_A = \sum_{d=0}^{r} M_d^+ \otimes \mathrm{Sym}^d(\mathbb{C}^2) \oplus \sum_{e=1}^{s} M_e^- \otimes \mathrm{Sym}^{e-1}(\mathbb{C}^2),$$

$$N_A = \sum_{d=1}^{r} M_d^+ \otimes \mathrm{Sym}^{d-1}(\mathbb{C}^2) \oplus \sum_{e=0}^{s} M_e^- \otimes \mathrm{Sym}^{e}(\mathbb{C}^2).$$

Remark: The notion of relevant is symmetric.

Gan-Gross-Prasad conjecture

Conjecture (Gan-Gross-Prasad $\sim$ 2019)

Let F be a local field. Let π_M and π_N be Arthur type representations of G_{n+1} and G_n respectively. Then

$$\mathrm{Hom}_{G_n}(\pi_M, \pi_N) \neq 0 \Leftrightarrow \ M_A \text{ and } N_A \text{ are relevant.}$$

Main result

Theorem (C. 2020 arXiv)

If F is non-Archimedean, then the conjecture is true.

Previous results:

- GGP proved when the Deligne $\mathrm{SL}_2(\mathbb{C})$ in WD_F acts trivially
- M. Gurevich proved the only if direction
- Gourevitch-Sayag proved some results towards Archimedean case

Arthur type representations

From now on, F is a non-Archimedean field.

- Zelevinsky segment $\Delta = [\nu^a\rho, \nu^b\rho]$, where $b - a \in \mathbb{Z}_{\geq 0}$ and ρ is irr. unitarizable cuspidal rep.
- (Zelevinsky) Square-integrable representations $\mathrm{St}([\nu^a\rho, \nu^b\rho])$: the unique irreducible quotient of

$$\nu^a\rho \times \ldots \times \nu^b\rho$$

- Speh representations: fix irreducible unitarizable representation ρ, and fix positive integers d, m: let $u_\rho(m, d)$ be the unique irreducible quotient of

$$\mathrm{St}(\nu^{(m-1)/2}\Delta_\rho(d)) \times \ldots \times \mathrm{St}(\nu^{-(m-1)/2}\Delta_\rho(d)),$$

where $\Delta_\rho(d) = [\nu^{-(d-1)/2}\rho, \nu^{(d-1)/2}\rho]$.

- Arthur type representations: product of (parabolically induced from) Speh representations

Arthur type representations

Key properties of Arthur type representations (Bernstein, Tadić):

- unitarizable
- independent of the ordering of Speh representations in the product $\Longrightarrow$ different ordering gives different filtration in Mackey theory

BZ Derivatives of Arthur type representations

- Derivative: Let $R_i = \left\{ \begin{pmatrix} I_{n-i} & x \\ & u \end{pmatrix} : u \in U_i \right\}$. The i-th derivative $\pi^{(i)}$, as G_{n-i}-repn. of π is defined as the (normalized) ψ-twisted Jacquet functor of R_i, where ψ is generic character on the part U_i.
- Level of π: largest integer such that $\pi^{(i)} \neq 0$.
- Shifted derivative: $\pi^{[i]} = \nu^{1/2} \cdot \pi^{(i)}$ e.g.

$$\mathrm{triv}_n^{[1]} = \mathrm{triv}_{n-1}$$

- For the level k^* of π, define $\pi^- = \pi^{[k^*]}$.

skip left derivatives

Derivatives

Imposing the Gelfand-Kazhdan involution

$$\theta(g) = g^{-t},$$

we have the left derivative of π:

$$^{(i)}\pi = \theta(\theta(\pi)^{(i)})$$

and the shifted derivatives

$$^{[i]}\pi = \nu^{-1/2} \cdot {}^{(i)}\pi$$

A consequence of Zelevinsky theory: When $k^* =$ level of π,

$$\pi^- = \pi^{[k^*]} \cong {}^{[k^*]}\pi$$

is irreducible.

Theorem (C. 2019 arXiv)

Let $\pi \in \mathrm{Irr}(G_n)$. If i is not the level of π, then $\pi^{[i]}$ and $^{[i]}\pi$ do not have isomorphic irreducible quotients and do not have isomorphic irreducible submodules.

Derivatives of Arthur representations

- Shifted highest derivative of a Speh representation (Zelevinsky):

$$u_\rho(m, d)^- \cong u_\rho(m, d-1)$$

  is still Speh

- Shifted highest derivative of an Arthur type representation:

$$\begin{aligned}&(u_{\rho_1}(m_1, d_1) \times \ldots \times u_{\rho_r}(m_r, d_r))^- \\ \cong &u_{\rho_1}(m_1, d_1 - 1) \times \ldots \times u_{\rho_r}(m_r, d_r - 1)\end{aligned}$$

  is still of Arthur type

- General derivative of a Speh representation is isomorphic to a ladder representation (Lapid-Mínguez)

Reformulate the conjecture..

Reformulated conjecture

Let F be non-Archimedean. Let π_M and π_N be G_{n+1} and G_n-representations of Arthur types respectively. Then $\mathrm{Hom}_{G_n}(\pi_M, \pi_N) \neq 0$ if and only if there exists Speh representations $\pi_{p,1}, \ldots, \pi_{p,r}$ and $\pi_{q,1}, \ldots, \pi_{q,s}$ such that

$$\pi_M \cong \pi_{p,1} \times \ldots \times \pi_{p,r} \times \pi^{-}_{q,1} \times \ldots \times \pi^{-}_{q,s}$$

$$\pi_N \cong \pi^{-}_{p,1} \times \ldots \times \pi^{-}_{p,r} \times \pi_{q,1} \times \ldots \times \pi_{q,s}.$$

Filtration from Mackey theory and BZ theory

Theorem

Let $\pi_1 \in \mathrm{Alg}(G_{n_1})$ and let $\pi_2 \in \mathrm{Alg}(G_{n_2})$. The parabolically induced module $(\pi_1 \times \pi_2)|_{G_{n_1+n_2-1}}$ admits a filtration with successive quotients: $k = 0, 1, \ldots,$

$$\pi_1^{[k]} \times \mathcal{RS}_k(\pi_2),$$

where, for $k \geq 2$,

$$\mathcal{RS}_k(\pi_2) = \mathrm{ind}_H^{G_{n_2+k}} \pi_2 \boxtimes \psi,$$

where

- $H = H^R_{k-2} = \left\{ \begin{pmatrix} g & & * \\ & 1 & v \\ & & u \end{pmatrix} \right\}, g \in G_{n_2}, u \in U_{k-2},$
- *ψ is a generic character on $U_{k-2} \ltimes F^{k-2}$,*
- *Still need to make sense for degenerate cases $k = 0, 1$*

Filtration

1. When $k = 0$,
$$(\nu^{1/2}\pi_1) \times (\pi_2|_{G_{n_2-1}})$$
2. When $k = 1$, (Fourier-Jacobi model)
$$\pi_1^{[1]} \times (\nu^{-1/2}\pi_2 \otimes S(F^{n_2}))$$
3. When $k \geq 2$,
$$\pi_1^{[k]} \times \mathcal{RS}_{k-2}(\pi_2),$$
where
$$\mathcal{RS}_{k-2}(\pi_2) = \mathrm{ind}_{H^R}^{G_n} \pi_2 \otimes \psi_{k-2},$$
where
$$H^R = \left\{ \begin{pmatrix} g & 0 & * \\ & 1 & x \\ & & u \end{pmatrix} \right\}$$

An example

Let $\pi_1 = \mathrm{triv}_3 \times \mathrm{triv}_1 \times \mathrm{triv}_1$ in $\mathrm{GL}_5(F)$.
We obtain a filtration on $\pi_1|_{G_4}$ with successive quotients:

$$(\nu^{1/2}\mathrm{triv}_3) \times ((\mathrm{triv}_1 \times \mathrm{triv}_1)|_{G_1}),$$

$$\mathrm{triv}_2 \times (\nu^{-1/2}(\mathrm{triv}_1 \times \mathrm{triv}_1) \otimes S(F^2)),$$

The filtration is coaser than the Bernstein-Zelevinsky filtration. For which the second copy will be further decomposed into three copies. However, one can use **induction** in this filtration.

First reduction to bottom piece of the filtration

Duality for restriction:

Proposition

Let $\pi_1 \in \mathrm{Alg}(G_{n+1})$ and $\pi_2 \in \mathrm{Alg}(G_n)$. For all i,

$$\mathrm{Ext}^i_{G_n}(\pi_1, \pi_2^\vee) \cong \mathrm{Ext}^i_{G_{n+1}}(\pi_2 \times \sigma, \pi_1^\vee)$$

for a suitable choice of cuspidal representation $\sigma \in \mathrm{Alg}(\mathrm{GL}_2)$.

Thus the two problems

$$\mathrm{Hom}_{G_n}(\pi_1, \pi_2) \neq 0, \quad \mathrm{Hom}_{G_{n+1}}(\pi_2^\vee \times \sigma, \pi_2^\vee) \neq 0$$

are equivalent.

Reductions

By using the duality, we may assume a special form on π_M:

$$u_{\rho_1}(m_1, d_1) \times \pi'_M$$

where $m_1 + d_1$ is largest among all the Speh representation factors in π_M and π_N. Let $u = u_{\rho_1}(m_1, d_1)$. This allows one to establish Ext vanishing of 'upper' layers (i.e. $k \neq$ level of u) via comparing cuspidal supports:

$$\mathrm{Ext}^i_{G_n}(u^{[k]} \times \mathcal{RS}_k(\pi'_M), \pi_N) = 0$$

and so it reduces to study the bottom layer:

$$\mathrm{Ext}^i_{G_n}(u^- \times \mathcal{RS}_{k^*}(\pi'_M), \pi_N) \cong \mathrm{Ext}^i_{G_n}(u \times \pi'_M, \pi_N)$$

Reduction to:

1. Problem (A): Study the Arthur type quotient of $\mathcal{RS}_k(\pi'_M)$;
2. Problem (B): Study the product $u^- \times \mathcal{RS}_k(\pi'_M)$

Gan-Gross-Prasad type reduction and Problem (A)

Problem (A): Study the Arthur type quotient of $\mathcal{RS}_k(\pi'_M)$
Solution: GGP-type reduction

Proposition

$$\mathrm{Hom}_G(\mathcal{RS}_{k^*}(\pi'_M), \pi'_N) \cong \mathrm{Hom}_G(\pi'_M \times \sigma, \pi'_N)$$

for some cuspidal σ

Upshot: σ is chosen to be unitarizable and so $\pi_1 \times \sigma$ is still Arthur type. The Arthur type quotient of $\mathcal{RS}(\pi_1)$ can be deduced from an inductive case. So roughly, we now know later Hom and solved Problem (A)!

Problem (B) and 'if direction'

Suppose (M_A, N_A) are relevant. Write

$$\pi_M = u \times \pi'_M, \quad \pi_N = u^- \times \pi'_N,$$

where $u = u_{\rho_1}(m_1, d_1)$.

$$(\pi'_M, \pi'_N) \text{ relevant}$$

$$\Updownarrow \text{ Induction}$$

$$\sigma \times \pi'_M \text{ has a quotient } \pi'_N$$

$$\Updownarrow \text{ GGP type reduction}$$

$$\mathcal{RS}(\pi'_M) \text{ has a quotient of } \pi'_N$$

$$\Downarrow \text{ exactness of product}$$

$$\text{Problem (B): } u^- \times \mathcal{RS}(\pi'_M) \text{ has a quotient of } u^- \times \pi'_N$$

$$\Updownarrow \text{ Filtration on product}$$

$$u \times \pi'_M \text{ has a quotient of } u^- \times \pi'_N$$

Only if direction

Two problems for going backwards for only if direction:

1. In prior, an Arthur quotient of $u^{-} \times \mathcal{RS}(\pi'_M)$ may not take the form $u^{-} \times \lambda$; and
2. λ may not come from an Arthur type quotient of $\mathcal{RS}(\pi'_M)$

The first problem is easier: Frobenius reciprocity and show irreducibility on product

The second problem:

Example

$1 \times \nu$ has only quotient $\langle [1, \nu] \rangle$, but $\nu \times (1 \times \nu)$ is semisimple with two composition factors $\nu \times \mathrm{St}([1, \nu])$ and $\nu \times \langle [1, \nu] \rangle$. Thus producting with ν on $1 \times \nu$ breaks the extension. (Note that if we mulitply ν on the right, the resulting repn. is still indecomposable.

Product functor

To study preserving extension, it is more convenient to view product as a functor.

Example

Let $F : \mathcal{C} \to \mathcal{D}$ be an additive exact functor preserving simple objects. Let X be an indecomposable object of length 2 in $\mathcal{C}$ with composition factors X_1, X_2. then

$$\mathrm{Hom}_{\mathcal{D}}(F(X), F(X_i)) \cong \mathrm{Hom}_{\mathcal{C}}(X, X_i)$$

if and only if $F(X)$ is still indecomposable.

Product functor

Example

Let σ be a cuspidal representation of G_n. Let $\mathcal{E}$ be a full subcategory of $\mathrm{Alg}(G_k)$ whose objects π are of finite length and the cuspidal support of π does not contain σ. Then for any $\pi_1, \pi_2 \in \mathcal{E}$,

$$\mathrm{Hom}_{G_{n+k}}(\sigma \times \pi_1, \sigma \times \pi_2) \cong \mathrm{Hom}_{G_k}(\pi_1, \pi_2)$$

from Frobenius reciprocity and geometric lemma.

Product Functor

For an irreducible $\pi \in \mathrm{Alg}(G_n)$, define a functor

$$\times_\pi : \mathrm{Alg}(G_k) \to \mathrm{Alg}(G_{n+k})$$

given by

$$\times_\pi(\pi') = \pi \times \pi'.$$

Lemma

$\times_\pi$ *is a faithful functor i.e. for* $\pi_1, \pi_2 \in \mathrm{Alg}(G_k)$,

$$\mathrm{Hom}(\pi_1', \pi_2') \hookrightarrow \mathrm{Hom}(\pi \times \pi_1', \pi \times \pi_2').$$

Proof.

This follows from that $\times_\pi$ is exact and sends non-zero objects to non-zero objects. $\square$

Preserving simple objects

Question: For $\pi_i \in \mathrm{Irr}(G_{n_i})$ ($i = 1, 2$), when $\pi_1 \times \pi_2$ is irreducible? (Work of Zelevinsky, Tadić, Lapid-Mínguez, Gurevich,..)

Proposition (Lapid-Mínguez 2016)

Let $\mathrm{Alg}_{\mathcal{C}}(G_n)$ be a full subcategory of $\mathrm{Alg}(G_n)$ whose objects π are of finite length and with composition factors whose cuspidal support has representations either

1. lying in cuspidal support $\mathrm{cupp}(u_\rho(m, d))$; or
2. not lying in the cuspidal line $\mathrm{cupp}_{\mathbb{Z}}(\nu^{(m+d)/2}\rho)$.

For any irreducible $\pi \in \mathrm{Alg}_{\mathcal{C}}(G_n)$, $u_\rho(m, d) \times \pi$ is irreducible.

Preserving extensions

Theorem (C. 2020 arXiv)

Let $\pi \in \mathrm{Alg}_{\mathcal{C}}(G_n)$ of length 2. Then

$$u_\rho(m,d) \times \pi \text{ is indecomposable } \Leftrightarrow \pi \text{ is indecomposable}$$

Theorem (C. 2020 arXiv)

Restrict $\times_{u_\rho(m,d)}$ to the full subcategory $\mathrm{Alg}_{\mathcal{C}}(G_n)$. Then $\times_{u_\rho(m,d)}$ is fully-faithful.

From second adjointness of Frobenius reciprocity and adjointness of tensor product, the product $\times_\pi$ has right functor, denoted R_π:

$$\mathrm{Hom}(\pi \times \pi_1, \pi_2) \cong \mathrm{Hom}(\pi_1, R_\pi((\pi_2)) \tag{2}$$

Corollary

Let π_2 be in $\mathrm{Alg}_{\mathcal{C}}(G_{n+k})$. Then

$$R_\pi(\pi \times \pi_2) \cong \pi_2.$$

Consequences on Jacquet modules

Example

Let

$$\pi = \langle \Delta \rangle \times \langle \Delta \rangle \in \mathrm{Irr}(G_{2n})$$

Let $N = \begin{pmatrix} I_n & * \\ & I_n \end{pmatrix}$. Frobenius reciprocity, we know that π_{N^-} is an indecomposable $G_n \times G_n$-representation with two factors both isomorphic to $\langle \Delta \rangle \boxtimes \langle \Delta \rangle$. However,

$$\mathrm{Ext}^1_{G_n \times G_n}(\langle \Delta \rangle \boxtimes \langle \Delta \rangle, \langle \Delta \rangle \boxtimes \langle \Delta \rangle) \cong \mathbb{C}^2$$

and so there are still several possibilities on the structure. The corollary rules out the following structure:

$$\langle \Delta \rangle \boxtimes \lambda,$$

where λ is length 2 with factors isomorphic to $\langle \Delta \rangle$.

Back to application on GGP conjecture

Recall we want to study an Arthur quotient π_N of

$$u_\rho(m, d-1) \times \mathcal{RS}(\pi'_M)$$

and by Frobenius reciprocity and Lapid-Mínguez irreducibility criteria, we have

$$\pi_N \cong u_\rho(m, d-1) \times \lambda$$

and the adjointness gives that λ is quotient of $\mathcal{RS}(\pi'_M)$.

skip to the end

Some other consequences

The Bernstein-Zelevinsky derivative approach leads to study:

$$\mathrm{Hom}_{G_n}(\pi_M^{[j+1]}, {}^{(j)}\pi_N) \neq 0$$

for all $j \geq 0$.

Derivative approach

The proof also shows the following:

Corollary

Suppose π_M and π_N are Arthur type representations of G_{n+1} and G_n respectively. Suppose their associated Arthur parameters M_A and N_A are relevant. Then there exists exactly one k^* such that

$$\mathrm{Hom}_{G_n}(\pi_M^{[k^*+1]}, {}^{(k^*)}\pi_N) \neq 0$$

and so

$$\mathrm{Hom}_{G_n}(\pi_M, \pi_N) \cong \oplus_k \mathrm{Hom}_{G_n}(\pi_M^{[k+1]}, {}^{(k)}\pi_N)$$

Conjecture on Ext-branching laws

Conjecture

Let π_M and π_N be Arthur type representations. Then, for all i,

$$\mathrm{Ext}^i_{G_n}(\pi_M|_{G_n}, \pi_N) \cong \oplus_k \mathrm{Ext}^i_{G_n}(\pi_M^{[k+1]}, {}^{(k)}\pi_N)$$

Some examples: Hom-branching law (i.e. $i = 0$), when either π_M or π_N is generic

Thank you!