Isolation of the cuspidal spectrum and application to the Gan-Gross-Prasad conjecture for unitary groups

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- Lindenstrauss-Venkatesh ('05) : soft proof of Weyl's law for spherical cusp forms on congruence locally symmetric spaces.
- Set $\Gamma = \mathrm{PGL}_2(\mathbb{Z}) \curvearrowright \mathbb{H} = \mathrm{G}/\mathrm{K}$: upper half plane with $\mathrm{G} = \mathrm{PGL}_2(\mathbb{R})$, $\mathrm{K} = \mathrm{PO}_2(\mathbb{R})$.
- $\mathrm{C}_c^\infty(\mathrm{K}\backslash\mathrm{G}/\mathrm{K})$ acts on $\mathrm{L}^2(\Gamma\backslash\mathbb{H})$ (by right convolution) and we have the spectral decomposition :

$$\mathrm{L}^2(\Gamma\backslash\mathbb{H}) = \mathrm{L}^2_{cusp} \oplus \mathbb{C}\mathbf{1} \oplus \int_{\mathbb{R}}^{\oplus} \mathbb{C}\mathrm{E}_{1/2+it}\,dt.$$

- For $k \in \mathrm{C}_c^\infty(\mathrm{K}\backslash\mathrm{G}/\mathrm{K})$, $\mathrm{R}(\mathrm{k})\mathrm{E}_{1/2+\lambda} = \widehat{\mathrm{k}}(\lambda)\mathrm{E}_{1/2+\lambda}$ where $\widehat{\mathrm{k}}(\lambda) = \int_{\mathrm{G}} \mathrm{k}(g)\Xi_\lambda(g)dg$ spherical transform.
- Helgason : $\mathrm{k} \mapsto \widehat{\mathrm{k}}$ induces $\mathrm{C}_c^\infty(\mathrm{K}\backslash\mathrm{G}/\mathrm{K}) \simeq \mathrm{PW}_{\mathrm{even}}(\mathbb{C})(:= \mathcal{F}\,\mathrm{C}^\infty_{c,\mathrm{even}}(\mathbb{R}))$.
- On the other hand, $\mathrm{T}_p\mathrm{E}_{1/2+\lambda} = (p^\lambda + p^{-\lambda})\mathrm{E}_{1/2+\lambda}$.
- Set $\mathrm{U}_p : \mathrm{C}_c^\infty(\mathrm{K}\backslash\mathrm{G}/\mathrm{K}) \to \mathrm{C}_c^\infty(\mathrm{K}\backslash\mathrm{G}/\mathrm{K})$, $(\mathrm{U}_p\mathrm{k})^\wedge(\lambda) = (p^\lambda + p^{-\lambda})\widehat{\mathrm{k}}(\lambda)$.
- We have : $\mathrm{R}_{p,\mathrm{k}} := \mathrm{T}_p\mathrm{R}(\mathrm{k}) - \mathrm{R}(\mathrm{U}_p\mathrm{k})$ kills $(\mathrm{L}^2_{cusp})^\perp$.
- "High in the cusp" $\mathrm{R}(k)$ & $\mathrm{R}(\mathrm{U}_p k)$ commute with horizontal translations whereas T_p doesn't.
- $\Rightarrow$ We can arrange $\mathrm{R}_{p,k} \neq 0 \Rightarrow$ existence of even Maass forms.

- More generally, L & V construct "many" operators on $\mathrm{L}^2(\Gamma\backslash \mathrm{G}(\mathbb{R})/\mathrm{K}_\infty)$ with cuspidal image (where : $\mathrm{G}/\mathbb{Q}$ split adjoint, $\Gamma \subset \mathrm{G}(\mathbb{Q})$ congruence subgroup, $\mathrm{K}_\infty \subset \mathrm{G}(\mathbb{R})$ maxl compact) $\rightsquigarrow$ Weyl's law.

Remarks :

- L & V operators always kill some interesting automorphic forms like $\mathsf{Sym}^2\,\varphi$ for φ a form on GL_2 ;
- It only works for forms that are spherical at the Archimedean place.

Schwartz spaces

- $G/\mathbb{Q}$ conn. reductive, $\mathbb{A} = \mathbb{R} \times \prod'_p \mathbb{Q}_p = \mathbb{R} \times \mathbb{A}_f$, $K = K_S \times \prod_{p \notin S} K_p \subset G(\mathbb{A}_f)$ "level" with K_p hyperspecial for $p \notin S$.
- Schwartz space :

$$\mathcal{S}(G(\mathbb{R})) = \{f \in C^\infty(G(\mathbb{R})) \mid \forall D: \text{ polyn. differential op.}, |Df| \ll 1\}.$$

It is a space of "very rapidly decreasing" functions (together with all their derivatives) analog to

$$\mathcal{S}_{\exp}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \mid \forall n \geqslant 0, R > 0, |f^{(n)}(x)| \ll e^{-R|x|}\}.$$

The space $\mathcal{S}(G(\mathbb{R}))$ is an algebra under convolution $*$.

- Global Schwartz space :

$$\mathcal{S}(G(\mathbb{A}))_K = \mathcal{S}(G(\mathbb{R})) \otimes \bigotimes_p{}' C_c(K_p \backslash G(\mathbb{Q}_p)/K_p)$$

that is the space generated by functions of the form $f_\infty \times \prod_p f_p$ where $f_\infty \in \mathcal{S}(G(\mathbb{R}))$, $f_p \in C_c(K_p \backslash G(\mathbb{Q}_p)/K_p)$ and $f = \mathbf{1}_{K_p}$ for a.a. p.

Multipliers

- For $p \notin S$, $\mathcal{M}_p(\mathrm{G}) = \mathrm{C}_c(\mathrm{K}_p \backslash \mathrm{G}(\mathbb{Q}_p)/\mathrm{K}_p)$ (spherical Hecke alg) acts on itself by convolution.
- Multipliers at ∞ : let

$$\mathcal{M}_\infty(\mathrm{G}) = \mathrm{End}_{cont,\mathcal{S}(\mathrm{G}(\mathbb{R}))-bimod}(\mathcal{S}(\mathrm{G}(\mathbb{R})))$$

  be the space of continuous bimodule endomorphisms of $\mathcal{S}(\mathrm{G}(\mathbb{R}))$. It can be identified with the space of "rapidly decreasing invariant distributions on $\mathrm{G}(\mathbb{R})$" acting on $\mathcal{S}(\mathrm{G}(\mathbb{R}))$ by $*$.
- S-multipliers : $\mathcal{M}^{\mathrm{S}}(\mathrm{G}) = \mathcal{M}_\infty(\mathrm{G}) \otimes {\bigotimes_{p\notin \mathrm{S}}}' \mathcal{M}_p(\mathrm{G}) \overset{*}{\curvearrowright} \mathcal{S}(\mathrm{G}(\mathbb{A}))$.

Quasi-cuspidal convolution operators

- Let $\pi = \pi_\infty \otimes \bigotimes'_p \pi_p$ be an irred. cuspidal representation of $\mathrm{G}(\mathbb{A})$ st $\pi^{\mathrm{K}} \neq 0$.
- For every $p \notin \mathrm{S}$, $\mathcal{M}_p(\mathrm{G})$ acts on $\pi_p^{\mathrm{K}_p}$ by a character $\lambda_p(\pi)$ (Satake parameter).
- We say that π is S-CAP if there exists an Eisenstein series E on $\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A})/\mathrm{K}$ such that $\lambda_p(\pi) = \lambda_p(\mathrm{E})$ for all $p \notin \mathrm{S}$.

Theorem A (R.B.P., Y. Liu, W. Zhang, X. Zhu)

Assume that π is not S-CAP. Then, there exists $\mu_\pi \in \mathcal{M}^{\mathrm{S}}(\mathrm{G})$ such that for every $f \in \mathcal{S}(\mathrm{G}(\mathbb{A}))$ we have :

1. $\mathrm{R}(\mu_\pi * f)$ *acts by zero on* $\mathrm{L}^2_{cusp}(\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A}))^\perp$ *;*
2. $\pi(\mu_\pi * f) = \pi(f)$.

Remarks :

- If $\mathrm{G} = \mathrm{GL}_n$, every cuspidal representation is not S-CAP (Jacquet-Shalika) ;
- The proof is robust and allows for many variants (e.g. isolation of a cuspidal datum).

Spectral description of multipliers

- For simplicity assume G split, fix a Borel $\mathrm{B} \subset \mathrm{G}$ and let $\mathrm{A} \twoheadleftarrow \mathrm{B}$ be the universal Cartan, $\mathrm{W} = \mathrm{W}(\mathrm{G}, \mathrm{A})$ the Weyl group and $\widehat{\mathrm{A}} = \mathrm{X}^*(\mathrm{A}) \otimes \mathbb{C}^\times$ the dual torus.
- For every p, $\widehat{\mathrm{A}} \simeq \widehat{\mathrm{A}}_p$ the gp of unramified chars of $\mathrm{A}(\mathbb{Q}_p)$ by $\chi \otimes p^s \mapsto |\chi|_p^s$.
- Satake isomorphism :

$$\mathcal{M}_p(\mathrm{G}) \simeq \mathbb{C}[\widehat{\mathrm{A}}_p]^W, \mu \mapsto \widehat{\mu}$$

  st if $\lambda \in \widehat{\mathrm{A}}_p$ and $v_\lambda \in \mathrm{I}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{G}(\mathbb{Q}_p)}(\lambda)^{\mathrm{K}_p}$ (normalized induction), $\mathrm{I}(\mu) v_\lambda = \widehat{\mu}(\lambda) v_\lambda$.
- Similarly, for $\mu \in \mathcal{M}_\infty(\mathrm{G})$ and π_∞ : irred adm. repn of $\mathrm{G}(\mathbb{R})$ we have

$$\pi_\infty(\mu) = \widehat{\mu}(\pi_\infty)\,\mathrm{Id} \quad \text{(Schur)}$$

  and μ is characterized by $\pi_\infty \mapsto \widehat{\mu}(\pi_\infty)$.
- Harish-Chandra isom :

$$\mathcal{Z}(\mathfrak{g}(\mathbb{C})) \simeq \mathbb{C}[\mathrm{Lie}(\mathrm{A})^*_{\mathbb{C}}]^W \simeq \mathbb{C}[\mathrm{Lie}(\widehat{\mathrm{A}})]^W$$

  where $\mathcal{Z}(\mathfrak{g}(\mathbb{C}))$ denotes the center of the enveloping algebra of $\mathfrak{g}(\mathbb{C})$.
- Every π_∞ has an infinitesimal character $\lambda_\infty(\pi_\infty) \in \mathcal{Z}(\mathfrak{g}(\mathbb{C}))^\wedge \simeq \mathrm{Lie}(\widehat{\mathrm{A}})/\mathrm{W}$.
- We only look for multipliers $\mu \in \mathcal{M}_\infty(\mathrm{G})$ such that $\widehat{\mu}$ factorizes through $\pi_\infty \mapsto \lambda_\infty(\pi_\infty)$. Notation : $\mathcal{M}_\infty^{\mathrm{inf}}(\mathrm{G})$ (*infinitesimal multipliers).*

Archimedean multipliers

- $\widehat{G(\mathbb{R})}^{temp}$: set of all tempered irred. repns of $G(\mathbb{R})$. Put
$$\mathrm{Inf}^{temp} = \lambda_\infty\left(\widehat{G(\mathbb{R})}^{temp}\right) \subset \mathrm{Lie}(\widehat{A})/W.$$
- Harish-Chandra : π_∞ tempered iff $\exists\, B \subset P \subset G$ with $P \twoheadrightarrow M \twoheadrightarrow A_M$, σ discrete series of $M(\mathbb{R})$ and $\lambda \in \sqrt{-1}\,\mathrm{Lie}(A_M)^*_{\mathbb{R}} \subset \mathrm{Lie}(A)^*_{\mathbb{C}} = \mathrm{Lie}(\widehat{A})$ st
$$\pi_\infty \hookrightarrow I_{P(\mathbb{R})}^{G(\mathbb{R})}(\sigma \otimes \lambda).$$
- "Tubular neighborhood" ($C > 0$) : $\pi_\infty \in \widehat{G(\mathbb{R})}^{temp}_{<C}$ iff $\exists\, B \subset P \subset G$, σ d.s. of $M(\mathbb{R})$ and $\lambda \in \mathrm{Lie}(A_M)^*_{\mathbb{C}}$ st $\|\Re(\lambda)\| < C$ and
$$\pi_\infty \hookrightarrow I_{P(\mathbb{R})}^{G(\mathbb{R})}(\sigma \otimes \lambda).$$
- We set
$$\mathrm{Inf}^{temp}_{<C} = \lambda_\infty\left(\widehat{G(\mathbb{R})}^{temp}_{<C}\right) \subset \mathrm{Lie}(\widehat{A})/W.$$
- Example : if $G = SL_2$, $\mathrm{Lie}(\widehat{A})/W = \mathbb{C}/\{\pm 1\}$ and we have
$$\mathrm{Inf}^{temp} = i\mathbb{R}/\{\pm 1\} \cup \mathbb{Z}/\{\pm 1\}, \quad \mathrm{Inf}^{temp}_{<C} = V_C/\{\pm 1\} \cup \mathbb{Z}/\{\pm 1\}$$
where V_C is the vertical band $|\Re(z)| < C$.

Archimedean multipliers cont'd

Theorem B (R.B.P., Y. Liu, W. Zhang, X. Zhu)

Let $\widehat{\nu} : \mathrm{Lie}(\widehat{A}) \to \mathbb{C}$ *be holomorphic and such that*

- $\widehat{\nu}$ *is* W*-invariant ;*
- *For every* $C > 0$, $\widehat{\nu}$ *is bounded by a polynomial on* $\mathrm{Inf}^{\mathrm{temp}}_{<C}$*.*

Then, there exists $\mu \in \mathcal{M}^{\mathrm{inf}}_{\infty}(G)$ *such that* $\widehat{\mu}(\pi_\infty) = \widehat{\nu}(\lambda_\infty(\pi_\infty))$ *for every irred adm repn* π_∞ *of* $G(\mathbb{R})$*.*

- Arthur's multipliers : let $K_\infty \subset G(\mathbb{R})$ maxl compact then

$$C^\infty_c(G(\mathbb{R}))_{(K_\infty)} \curvearrowleft \mathcal{M}^A_\infty = \mathrm{PW}(\mathrm{Lie}(\widehat{A}))^W := \mathcal{F}\, C^\infty_c(\mathrm{Lie}(A)_{\mathbb{R}})^W$$

  where (K_∞) means K_∞-finite (on both sides) and $\mathcal{F}$ is the Fourier transform. Not sufficient for our purpose (ess. b/c PW functions are bounded by an exponential).
- Delorme's multipliers :

$$\mathcal{S}(G(\mathbb{R}))_{(K_\infty)} \curvearrowleft \mathcal{M}^D_\infty = \mathcal{F}\, \mathcal{S}_{\exp}(\mathrm{Lie}(A)_{\mathbb{R}})^W$$

  is more or less what we want (fns of rapid decay in vertical strips but no condition of growth in the real direction). However, need to show that this extends by continuity to $\mathcal{S}(G(\mathbb{R})) \rightsquigarrow L^2$-argument (Plancherel formula)+translation by fin. diml repns.

On the proof of Theorem A

First recall the statement :

Theorem A

Let π be a cuspidal repn of $\mathrm{G}(\mathbb{A})$ that is not S-CAP. Then, there exists $\mu_\pi \in \mathcal{M}^{\mathrm{S}}(\mathrm{G})$ such that for every $f \in \mathcal{S}(\mathrm{G}(\mathbb{A}))$ we have :

1. $\mathrm{R}(\mu_\pi * f)$ *acts by zero on* $\mathrm{L}^2_{cusp}(\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A}))^\perp$ *;*
2. $\pi(\mu_\pi * f) = \pi(f)$.

- Set $\mathfrak{X}^S = \underbrace{\mathrm{Lie}(\widehat{\mathrm{A}})/\mathrm{W}}_{\text{inf. chars.}} \times \prod_{p \notin S} \underbrace{\widehat{\mathrm{A}}_p/\mathrm{W}}_{\text{Sat. param. at } p}$ and
  $\lambda^S(\pi) = (\lambda_\infty(\pi), (\lambda_p(\pi))_{p\notin \mathrm{S}}) \in \mathfrak{X}^S$.
- We define similarly $\lambda^{\mathrm{S}}(\mathrm{E})$ for an Eisenstein series E on $\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A})/K$ and we set
  $$\mathfrak{X}^S_{\mathrm{Eis}} = \left\{\lambda^{\mathrm{S}}(\mathrm{E}) \mid \mathrm{E} \text{ Eis. series}\right\} \subset \mathfrak{X}^S.$$
- Note that $\mathcal{M}^{\mathrm{S,inf}}(\mathrm{G}) = \mathcal{M}^{\mathrm{inf}}_\infty(\mathrm{G}) \bigotimes'_{p\notin \mathrm{S}} \mathcal{M}_p(\mathrm{G})$ can be seen as a space of functions on $\mathfrak{X}^{\mathrm{S}}$ by $\mu \mapsto \widehat{\mu}$.
- We are looking for $\mu_\pi \in \mathcal{M}^{\mathrm{S,inf}}(\mathrm{G})$ such that $\widehat{\mu}_\pi$ vanishes on $\mathfrak{X}^S_{\mathrm{Eis}}$ but not on $\lambda^S(\pi)$.

- Eisenstein series come in a countable number of families $\mathcal{F} = \{\mathrm{E}(\varphi_\lambda)\}_{\varphi,\lambda}$ where $\mathrm{B} \subset \mathrm{P} \subsetneq \mathrm{G}$, $\mathrm{P} \twoheadrightarrow \mathrm{M} \twoheadrightarrow \mathrm{A}_M$, $\varphi \in \sigma \subset \mathcal{A}(\mathrm{M}(\mathbb{Q})\mathrm{N}_\mathrm{P}(\mathbb{A})\backslash \mathrm{G}(\mathbb{A}))$ and $\lambda \in \mathrm{Lie}(\mathrm{A}_\mathrm{M})^*_\mathbb{C} \subset \mathrm{Lie}(\widehat{\mathrm{A}})$.
- $\lambda^S(\mathcal{F})$ is then the image in $\mathfrak{X}^\mathrm{S}$ of a coset for

$$\left\{(\lambda, (p^\lambda)_{p\notin \mathrm{S}}) \mid \lambda \in \mathrm{Lie}(\mathrm{A}_\mathrm{M})^*_\mathbb{C}\right\} \subset \mathrm{Lie}(\widehat{\mathrm{A}}) \times \prod_{p\notin \mathrm{S}} \widehat{\mathrm{A}}_p.$$

- Harish-Chandra finiteness : $\lambda_\infty(\pi) \notin \lambda_\infty(\mathcal{F})$ for all except a finite number of $\mathcal{F}$'s. Moreover, the $\lambda_\infty(\mathcal{F})$ are images of affine subspaces of $\mathrm{Lie}(\widehat{\mathrm{A}})$ that are quite "sparse" (Donnelly) $\Rightarrow$ we can find $\mu_\infty \in \mathcal{M}_\infty^{\mathrm{inf}}(\mathrm{G})$ st $\widehat{\mu}_\infty$ vanishes on all of them except finitely many and $\widehat{\mu}_\infty(\lambda_\infty(\pi)) \neq 0$. (Here it is crucial not to work with Arthur's multipliers).
- For the remaining $\mathcal{F}$'s, we can separate $\lambda^\mathrm{S}(\pi)$ from $\lambda^\mathrm{S}(\mathcal{F})$ by using product of W-translates of functions of the form

$$(\lambda_\infty, (\lambda_p)_{p\notin \mathrm{S}}) \mapsto \alpha(\frac{p^{\lambda_\infty}}{\lambda_p}) - c_{\alpha,p}$$

where $\alpha : \widehat{\mathrm{A}} \to \mathbb{C}^\times$ is a character and $c_{\alpha,p} \in \mathbb{C}$.

Application to the Gan-Gross-Prasad conjecture

- Let E/F be a quad. ext. of number fields. $W \subset V$ Hermitian spaces over E of dim n, $n+1$.
- Set $\mathrm{G} = \mathrm{U}(V) \times \mathrm{U}(W) \hookleftarrow \mathrm{H} = \mathrm{U}(W)$ and let $\pi = \pi_V \boxtimes \pi_W \hookrightarrow \mathcal{A}_{\text{cusp}}(\mathrm{G}(F) \backslash \mathrm{G}(\mathbb{A}_F))$.
- Automorphic period : $\mathcal{P}_{\mathrm{H}} : \varphi \in \pi \mapsto \int_{\mathrm{H}(F)\backslash \mathrm{H}(\mathbb{A})} \varphi(h)dh$.
- Let $\pi_{\mathrm{E}} = \pi_{V,\mathrm{E}} \boxtimes \pi_{W,\mathrm{E}}$ be the base-change to $\mathrm{G}_E = \mathrm{GL}_{n+1,\mathrm{E}} \times \mathrm{GL}_{n,\mathrm{E}}$ and $\mathrm{L}(s,\pi_{\mathrm{E}}) = \mathrm{L}(s, \pi_{V,\mathrm{E}} \times \pi_{W,\mathrm{E}})$ be the Rankin-Selberg L-function.

Conjecture (Gan-Gross-Prasad)

Assume that π_{E} is generic. TFAE :

1. $\mathrm{L}(1/2, \pi_{\mathrm{E}}) \neq 0$ *;*
2. $\exists\, \pi'$ *st* $\pi'_{\mathrm{E}} = \pi_{\mathrm{E}}$ *and* $\mathcal{P}_{\mathrm{H}} \mid_{\pi'} \neq 0$.

Remark : Actually, we should allow π' to live on some inner forms of G.

- Refinement (Ichino-Ikeda conjecture) : $|\mathcal{P}_H(\varphi)|^2 \sim \mathrm{L}(1/2, \pi_E)$ (generalizing Waldspurger's formula on toric periods for GL_2).

Theorem C (R.B.P., Y. Liu, W. Zhang, X. Zhu)

Assume that π_E is cuspidal. Then, both conjectures hold for π.

Sketch of proof :

- Comparison of two relative trace formulas proposed by Jacquet-Rallis : RTF_G and $\mathrm{RTF}_{G'}$ where $G' = \mathrm{GL}_{n+1,E} \times \mathrm{GL}_{n,E}$;
- Formally, these can be written

$$\sum_{\varphi \in \mathcal{A}(G)} \mathcal{P}_H(R(f)\varphi)\overline{\mathcal{P}_H(\varphi)} = \mathrm{RTF}_G(f) = \sum \text{relative orbital int.}, \; f \in \mathcal{S}(G(\mathbb{A})),$$

$$\sum_{\varphi \in \mathcal{A}(G')} \mathcal{P}_{H_1}(R(f')\varphi)\overline{\mathcal{P}_{H_2}(\varphi)} = \mathrm{RTF}_{G'}(f') = \sum \text{R.O.I.}, \; f' \in \mathcal{S}(G'(\mathbb{A})),$$

  where $H_1 = \mathrm{GL}_{n,E} \hookrightarrow G'$ (R-S period) and $H_2 = \mathrm{GL}_{n+1,F} \times \mathrm{GL}_{n,F} \hookrightarrow G'$ (Flicker-Rallis period).
- Neither expression converge in general. However, if $R(f) = R_{\mathrm{cusp}}(f)$ and $R(f') = R_{\mathrm{cusp}}(f')$ then the left hand sides are convergent.(The RHS have been regularized by M. Zydor).
- There is a transfer $f \leftrightarrow f'$ defined by certain equalities of R.O.I. and it is known that we can transfer functions in both directions (W. Zhang, Z. Yun, J. Gordon, H. Xue).

$$\sum_{\varphi\in\mathcal{A}(G)} \mathcal{P}_H(R(f)\varphi)\overline{\mathcal{P}_H(\varphi)} = \mathrm{RTF}_G(f) = \sum \text{relative orbital int.},$$

$$\sum_{\varphi\in\mathcal{A}(G')} \mathcal{P}_{H_1}(R(f')\varphi)\overline{\mathcal{P}_{H_2}(\varphi)} = \mathrm{RTF}_{G'}(f') = \sum \text{R.O.I.}.$$

- As π and π_E are both cuspidal, we can construct $\mu_\pi \in \mathcal{M}^S(G)$ and $\mu_{\pi_E} \in \mathcal{M}^S(G')$ as in Theorem A (killing the non-cuspidal spectrum but not π and π_E) ;
- Actually, can arrange μ_π and μ_{π_E} st if $f \leftrightarrow f'$ then $\mu_\pi * f \leftrightarrow \mu_{\pi_E} * f'$ (requires spectral char. of transfer).
- So for $f \leftrightarrow f'$, $\mathrm{RTF}_G(\mu_\pi * f)$ and $\mathrm{RTF}_{G'}(\mu_{\pi_E} * f')$ make sense and have the same geometric expansions. Comparing spectral sides, we get :

$$\sum_{\substack{\pi' \\ \pi'_E=\pi_E}} \sum_{\varphi\in\pi'} \mathcal{P}_H(R(f)\varphi)\overline{\mathcal{P}_H(\varphi)} = \sum_{\varphi\in\pi_E} \mathcal{P}_{H_1}(R(f')\varphi)\overline{\mathcal{P}_{H_2}(\varphi)}.$$

- Hence,

$$\exists \pi' \text{ st. } \pi'_E = \pi_E \ \&\ \mathcal{P}_H\,|_{\pi'} \neq 0 \Leftrightarrow \begin{cases} \mathcal{P}_{H_1}\,|_{\pi_E} \neq 0 \Leftrightarrow L(1/2,\pi_E) \neq 0 \text{ (J-P-S-S)} \\ \& \\ \mathcal{P}_{H_2}\,|_{\pi_E} \neq 0 \text{ (automatic, F-R)} \end{cases}$$

Some open questions

Let $G/\mathbb{R}$ be a connected reductive group.

- We have constructed many elements in $\mathcal{M}_\infty^{\mathrm{inf}}(G)$ (multipliers only depending on the infinitesimal character) : did we get all of them ?
- Are there other elements in $\mathcal{M}_\infty(G)$? Besides infinitesimal multipliers we can also act by the center of $G(\mathbb{R})$ but e.g. when $G = PGL_2$ we don't have any multiplier separating the p.s.
$$I_{B(\mathbb{R})}^{G(\mathbb{R})}(\lambda) \text{ and } I_{B(\mathbb{R})}^{G(\mathbb{R})}(\mathrm{sgn} \otimes \lambda)$$
when $\lambda : \mathbb{R}_+^\times \to \mathbb{C}^\times$ is in generic position.
- What about a Paley-Wiener theorem for $\mathcal{S}(G(\mathbb{R}))$?

Thank you !