Transfer of Characters under the Howe Duality Correspondence

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Dual Pair

- F local field (nonarchimedean).
- W symplectic vector space of dimension $2n$, with isometry group $\mathrm{Sp}(W)$.
- $V = V^{\pm}$ the two $2m+1$ dimensional quadratic spaces of discriminant 1, with isometry group $\mathrm{O}(V^{\pm})$.

Have a dual pair

$$\mathrm{Sp}(W) \times \mathrm{O}(V) \longrightarrow \mathrm{Sp}(W \otimes V).$$

Metaplectic Group and Weil Representation

Let $\mathrm{Mp}(W)$ be the unique two-fold nonlinear cover of $\mathrm{Sp}(W)$; it is called the metaplectic group.

Have:

$$\mathrm{Mp}(W) \times \mathrm{O}(V) \longrightarrow \mathrm{Mp}(W \otimes V).$$

For a fixed nontrivial character ψ of F, let

$$\omega_\psi = \text{Weil rep. of } \mathrm{Mp}(W \otimes V).$$

Pulling back gives a representation $\omega_{V,W,\psi}$ of $\mathrm{Mp}(W) \times \mathrm{O}(V)$.

Theta correspondence

For $\pi \in \mathrm{Irr}(\mathrm{O}(V))$, define a smooth rep. of $\mathrm{Mp}(W)$ by

$$\Theta(\pi) = (\omega_{V,W,\psi} \otimes \pi^{\vee})_{\mathrm{O}(V)}.$$

This is the big theta lift of π and $\pi \boxtimes \Theta(\pi)$ is the maximal π-isotypic quotient of $\omega_{V,W}$.

Likewise, for $\tilde{\pi} \in \mathrm{Irr}(\mathrm{Mp}(W))$, have smooth rep. $\Theta(\tilde{\pi})$ of $\mathrm{Mp}(W)$.

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Theorem (Howe Duality)

(i) $\Theta(\pi)$ *has finite length and a unique irreducible quotient* $\theta(\pi)$.

(ii) If $\pi_1 \neq \pi_2$, *then* $\theta(\pi_1) \neq \theta(\pi_2)$ *(if both nonzero).*

The analogous theorem holds if we start with $\tilde{\pi} \in \mathrm{Irr}(\mathrm{Mp}(W))$.

Equal Rank Case

We shall focus on the special case $m = n$, so that
$\dim V^{\pm} = \dim W + 1 = 2n + 1$.

Theorem (Local Shimura Correspondence)

The theta correspondence, together with the restriction from $O(V)$ to $SO(V)$, gives a bijection

$$\mathrm{Irr}(Mp(W)) \longleftrightarrow \mathrm{Irr}(SO(V^+)) \sqcup \mathrm{Irr}(SO(V^-)).$$

Moreover, under this bijection, discrete series representations correspond, and so do tempered representations.

$$\theta : \mathrm{Irr}_{temp}(\mathrm{SO}(V^+)) \sqcup \mathrm{Irr}_{temp}(\mathrm{SO}(V^-)) \longleftrightarrow \mathrm{Irr}_{temp}(\mathrm{Mp}(W)),$$

Characters

If $\pi \in \mathrm{Irr}(G(F))$, set

$$\Theta_\pi = \text{ Harish-Chandra character of } \pi.$$

It is a conjugacy-invariant distribution on $G(F)$, which is given by a locally L^1 smooth function on the regular semisimple locus:

$$\Theta_\pi : C_c^\infty(G(F)) \twoheadrightarrow C_c^\infty(G(F))_{G(F)^\Delta} \to \mathbb{C}.$$

If π is unitary and $\{e_i\}$ is an orthonormal basis of π, then

$$\Theta_\pi(f) = \sum_i \langle \pi(f) e_i, e_i \rangle.$$

If π is tempered, then Θ_π is a tempered distribution: it extends to a linear form on the Harish-Chandra-Schwarz space $\mathcal{C}(G(F)) \subset L^2(G(F))$. Indeed, for $f \in \mathcal{C}(G(F))$, the operator $\pi(f)$ is defined.

The Question

Suppose $\tilde{\pi} \in \mathrm{Irr}(\mathsf{Mp}(W))$ and $\pi \in \mathrm{Irr}(\mathsf{SO}(V^\epsilon))$ satisfy

$$\tilde{\pi} = \theta(\pi).$$

Question: How are the characters of π and $\tilde{\pi}$ related?

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- This question has been studied by T. Prezbinda when $F = \mathbb{R}$ and when one of the spaces is much smaller than the other (stable range).
- The relation is given by some formula, using what he calls the Cauchy-Harish-Chandra integral.
- The analytic difficulties in working with this integral is one obstacle in extending his results beyond the stable range.

We would like to propose a more conceptual approach.

An Approach to Character Relations

We shall:

- introduce spaces of test functions on $\mathrm{Mp}(W)$ and $\mathrm{SO}(V)$;
- define a notion of transfer of test functions from one group to another;
- show that this transfer descends to a well-defined map on the level of orbital integrals, thus inducing a transfer of invariant distributions.
- show that the transfer of Θ_{π} is equal to $\Theta_{\tilde{\pi}}$.
- describe the transfer in geometric terms (using a moment map).

Spaces of Test Functions

Consider the diagram

$$\Omega^\epsilon = \omega_{W,V^\epsilon} \otimes \overline{\omega_{W,V^\epsilon}}$$

$$C^\infty(\mathrm{Mp}(W)) \xleftarrow{\;p^\epsilon\;} \Omega^\epsilon \xrightarrow{\;q^\epsilon\;} C^\infty(\mathrm{SO}(V^\epsilon))$$

The two maps are defined by matrix coefficients:

$$p^\epsilon(\phi_1 \otimes \phi_2)(g) = \langle \phi_1, g\phi_2 \rangle \quad \text{and} \quad q^\epsilon(\phi_1 \otimes \phi_2)(h) = \langle \phi_1, h\phi_2 \rangle.$$

for $\phi_1 \otimes \overline{\phi_2} \in \Omega^\epsilon$. Set

$$\mathcal{S}^\epsilon(\mathrm{Mp}(W)) = \mathrm{Image}(p^\epsilon) \quad \text{and} \quad \mathcal{S}(\mathrm{SO}(V^\epsilon)) = \mathrm{Image}(q^\epsilon).$$

These are the spaces of test functions.

Transfer of Test Functions

We say that

$$f^{\epsilon} \in \mathcal{S}(\mathrm{SO}(V^{\epsilon})) \quad \text{and} \quad \tilde{f}^{\epsilon} \in \mathcal{S}^{\epsilon}(\mathrm{Mp}(W))$$

are transfer of each other if there exists $\Phi \in \Omega^{\epsilon}$ such that

$$p^{\epsilon}(\Phi) = f^{\epsilon} \quad \text{and} \quad q^{\epsilon}(\Phi) = \tilde{f}^{\epsilon}.$$

More generally, say that

$$f = (f^+, f^-) \in \mathcal{S}(\mathrm{SO}(V^{\pm})) := \mathcal{S}(\mathrm{SO}(V^+)) \oplus \mathcal{S}(\mathrm{SO}(V^-))$$

and

$$\tilde{f} = (\tilde{f}^+, \tilde{f}^-) \in \mathcal{S}(\mathrm{Mp}(W)) := \mathcal{S}^+(\mathrm{Mp}(W)) \oplus \mathcal{S}^-(\mathrm{Mp}(W))$$

are in correspondence if the $\pm$-components correspond.
Transfers always exist, by definition.

Properties of Test Functions

Lemma

$$C_c^\infty(Mp(W)) \subset \mathcal{S}(Mp(W)) \subset \mathcal{C}(Mp(W))$$

and

$$C_c^\infty(SO(V^\epsilon)) \subset \mathcal{S}(SO(V^\epsilon)) \subset \mathcal{C}(SO(V^\epsilon)).$$

Corollary

For $\pi \in \mathrm{Irr}_{temp}(SO(V^\epsilon))$ *and* $f \in \mathcal{S}(SO(V^\epsilon))$ *the operator* $\pi(f)$ *is defined and so is its trace*

$$\Theta_\pi(f) = \sum_{v \in ONB(\pi)} \langle \pi(f)v, v \rangle$$

Equivariance Properties

Lemma

(i) The map

$$p^{\epsilon} : \Omega^{\epsilon} \longrightarrow \mathcal{C}(Mp(W))$$

is $Mp(W) \times Mp(W)$-equivariant and $O(V^{\epsilon})^{\Delta}$-invariant. Thus $\mathcal{S}^{\epsilon}(Mp(W))$ is an $Mp(W) \times Mp(W)$-submodule of $\mathcal{C}(Mp(W))$.

Indeed, $p = p^{+} \oplus p^{-}$ induces an isomorphism

$$\bigoplus_{\epsilon} \Omega^{\epsilon}_{O(V^{\epsilon})^{\Delta}} \cong \mathcal{S}^{\epsilon}(Mp(W)).$$

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(ii) The map

$$q^{\epsilon} : \Omega^{\epsilon} \longrightarrow \mathcal{C}(SO(V^{\epsilon}))$$

is $SO(V^{\epsilon}) \times SO(V^{\epsilon})$-equivariant and $Mp(W)^{\Delta}$-invariant; it induces an isomorphism

$$(\Omega^{\epsilon})_{Mp(W)^{\Delta}} \cong \mathcal{S}(SO(V^{\epsilon})).$$

Isomorphism of Spaces of Orbital Integrals

Consider the composite:

$$\Omega = \oplus_\epsilon \Omega^\epsilon \to \mathcal{S}(\mathrm{Mp}(W)) \to \mathcal{S}(\mathrm{Mp}(W))_{\mathrm{Mp}(W)^\Delta}.$$

This map is $\mathrm{Mp}(W)^\Delta$-invariant and thus factors as:

$$\Omega \to \Omega_{\mathrm{Mp}(W)^\Delta} \cong \mathcal{S}(\mathrm{SO}(V^\pm)) \to \mathcal{S}(\mathrm{Mp}(W))_{\mathrm{Mp}(W)^\Delta}$$

Since the last arrow is also $\mathrm{SO}(V^\epsilon)^\Delta$-invariant, it further descends to

$$\mathcal{S}(\mathrm{SO}(V^\epsilon))_{\mathrm{SO}(V^\epsilon)^\Delta} \longrightarrow \mathcal{S}(\mathrm{Mp}(W))_{\mathrm{Mp}(W)^\Delta}.$$

Lemma

This construction gives an isomorphism

$$\mathcal{S}(SO(V^+))_{SO(V^+)^\Delta} \oplus \mathcal{S}(SO(V^-))_{SO(V^-)^\Delta} \cong \mathcal{S}(Mp(W))_{Mp(W)^\Delta}.$$

A Character Identity

The previous lemma allows one to transfer invariant distributions between $\mathcal{S}(\mathrm{Mp}(W))$ and $\mathcal{S}(\mathrm{SO}(V^{\pm}))$.

Theorem

Suppose that $\pi \in \mathrm{Irr}_{temp}(SO(V^{\epsilon}))$, so that $\tilde{\pi} = \theta(\pi) \in \mathrm{Irr}_{temp}(Mp(W))$. Then for f and $\tilde{f}$ in correspondence,

$$\Theta_{\pi}(f) = \Theta_{\tilde{\pi}}(\tilde{f}).$$

While this theorem relates the characters of π and $\tilde{\pi}$, the relation is only as explicit as one understands the transfer of test functions.

What we will discuss in the rest of this talk

- a sketch proof of the character identity.
- a geometric description of the transfer of test functions.

Sketch Proof via the Plancherel Theroem

For $\Phi = \phi_1 \otimes \phi_2 \in \Omega^\epsilon$, observe that

$$p(\Phi)(1) = \langle \phi_1, \phi_2 \rangle = q(\Phi)(1).$$

Since $p(\Phi) \in \mathcal{C}(\mathrm{Mp}(W))$, the Harish-Chandra-Plancherel theorem gives:

$$p(\Phi)(1) = \int_{\widehat{\mathrm{Mp}(W)}} \Theta_{\tilde{\pi}}(p(\Phi))\, d\mu_{\mathrm{Mp}(W)}(\tilde{\pi}).$$

Likewise,

$$q(\Phi)(1) = \int_{\widehat{\mathrm{SO}(V)}} \Theta_{\pi}(q(\Phi))\, d\mu_{\mathrm{SO}(V)}(\pi).$$

So we get the equality of both RHS's.

We have shown:

$$\int_{\widehat{\mathrm{Mp}(W)}} \Theta_{\tilde{\pi}}(p(\Phi))\, d\mu_{\mathrm{Mp}(W)}(\tilde{\pi}) = \int_{\widehat{\mathrm{SO}(V)}} \Theta_{\pi}(q(\Phi))\, d\mu_{\mathrm{SO}(V)}(\pi).$$

On the other hand, under the map

$$\theta : \mathrm{Irr}_{temp}(\mathrm{SO}(V^+)) \cup \mathrm{Irr}_{temp}(\mathrm{SO}(V^-)) \longleftrightarrow \mathrm{Irr}_{temp}(\mathrm{Mp}(W)),$$

one has (by G.-Ichino)

$$\theta_*(d\mu_{\mathrm{SO}(V^+)}) + \theta_*(d\mu_{\mathrm{SO}(V^-)}) = d\mu_{\mathrm{Mp}(W)}$$

This gives

$$\int_{\widehat{\mathrm{SO}(V)}} \Theta_{\theta(\pi)}(p(\Phi))\, d\mu_{\mathrm{SO}(V)}(\pi) = \int_{\widehat{\mathrm{SO}(V)}} \Theta_{\pi}(q(\Phi))\, d\mu_{\mathrm{SO}(V)}(\pi).$$

One can peel of the integrals on both sides using a Bernstein center type argument.

Doubling

The key for understanding the transfer is to interpret everything in terms of the doubling see-saw.

Let

$$\mathbb{W} = W \oplus (-W)$$

be the doubled symplectic space. This contains

$$W^{\Delta} = \{(w,w) : w \in W\} \quad \text{and} \quad W^{\nabla} = \{(w,-w) : w \in W\}$$

as maximal isotropic subspaces.

Likewise,

$$\mathbb{V} = V \oplus (-V)$$

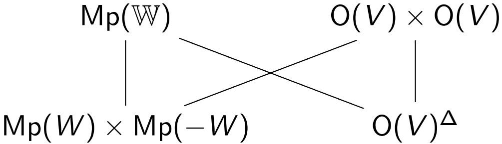
which contains V^{Δ} and V^{∇} as maximal isotropic subspaces.

Observe that one has isomorphisms of symplectic spaces:

$$V \otimes \mathbb{W} = (V \otimes W) \oplus (V \otimes (-W)) \cong (V \otimes W) \oplus ((-V) \otimes W) = \mathbb{V} \otimes W.$$

Doubling See-Saw

In $\mathrm{Sp}(V \otimes \mathbb{W}) = \mathrm{Sp}(\mathbb{V} \otimes W)$, there are 2 dual pairs fitting in a see-saw:



Moreover, if $\Omega_{\mathbb{W},V,\psi}$ denotes the Weil rep. for $\mathrm{Mp}(\mathbb{W}) \times \mathrm{O}(V)^{\Delta}$, then

$$\Omega_{\mathbb{W},V,\psi} \cong \omega_{W,V,\psi} \otimes \overline{\omega_{W,V,\psi}}$$

when restricted to $\mathrm{Mp}(W) \times \mathrm{Mp}(W) \times \mathrm{O}(V)^{\Delta}$.

$$\Omega_{\mathbb{W},V,\psi} \cong \omega_{W,V,\psi} \otimes \overline{\omega_{W,V,\psi}}$$

RHS is the domain Ω of the maps p and q in the definition of transfer; we see now that it is a Weil rep. for $\mathrm{Mp}(\mathbb{W}) \times \mathrm{O}(V)^{\Delta}$.

We want to interpret the map

$$p : \Omega \longrightarrow \mathcal{S}(\mathrm{Mp}(W))$$

through the lens of the other dual pair $\mathrm{Mp}(\mathbb{W}) \times \mathrm{O}(V)^{\Delta}$.

Since p is $\mathrm{O}(V)^{\Delta}$-invariant, it descends to

$$p : \Omega_{\mathrm{O}(V)^{\Delta}} \longrightarrow \mathcal{S}(\mathrm{Mp}(W))$$

Now the LHS is a rep. of $\mathrm{Mp}(\mathbb{W})$ which has been described by Rallis.

A Result of Rallis

Theorem

There is a natural morphism of $Mp(\mathbb{W})$-modules

$$\iota : \oplus_\epsilon \Omega^\epsilon \twoheadrightarrow \oplus_\epsilon \Omega^\epsilon_{O(V^\epsilon)} \cong I_{P(W^\Delta)}(0)$$

where the RHS is a degenerate principal series rep. of $Mp(\mathbb{W})$ induced from the Siegel parabolic stabilizing W^Δ.

The morphism ι is described as follows:

- The rep. Ω^ϵ can be realized on $\mathcal{S}(V^\epsilon \otimes W^\nabla)$.
- For $\Phi \in \mathcal{S}(V^\epsilon \otimes W)$,

$$\iota(\Phi)(g) = (g \cdot \Phi)(0) \quad \text{for } g \in \mathrm{Mp}(\mathbb{W}).$$

Degenerate P.S.

Now let's examine the degenerate p.s. $I_{P(W^\Delta)}(0)$ from other points of view. By definition, elements of $I_{P(W^\Delta)}(0)$ are smooth sections of a line bundle on the partial flag variety

$$P\backslash \mathrm{Sp}(\mathbb{W}) = P(W^\Delta)\backslash \mathrm{Sp}(\mathbb{W}),$$

parametrizing maximal isotropic subspaces of $\mathbb{W}$. Such sections are determined by their restrictions to open dense subsets.

There are 2 such open dense subsets we will use:

- the open Bruhat cell:

$$X_1 = P\backslash Pw_0N \subset P\backslash \mathrm{Sp}(\mathbb{W})$$

  with w_0 a Weyl element and

$$N \cong \mathrm{Sym}^2(W^\Delta) \cong \mathfrak{sp}(W).$$

- $\mathrm{Sp}(W) \times \mathrm{Sp}(W)$ has an open dense orbit (orbit of W^Δ):

$$X_2 = \mathrm{Sp}(W)^\Delta\backslash \mathrm{Sp}(W) \times \mathrm{Sp}(W) \subset P\backslash \mathrm{Sp}(\mathbb{W}).$$

So we have two injective restriction maps

$$\mathrm{rest}_{\mathrm{Mp}(W)} : I_P(0) \hookrightarrow C^\infty(\mathrm{Mp}(W))$$

and

$$\mathrm{rest}_{w_0 N} : I_P(0) \hookrightarrow C^\infty(N) = C^\infty(\mathfrak{sp}(W)).$$

Moreover the images of these maps contain $C_c^\infty(\mathrm{Mp}(W))$ and $C_c^\infty(\mathfrak{sp}(W))$.

Lemma

The map $p : \Omega \longrightarrow \mathcal{S}(Mp(W))$ is given by:

$$p = \mathrm{rest}_{Mp(W)} \circ \iota.$$

Hence $\mathrm{rest}_{Mp(W)}$ gives an isomorphism of $Mp(W) \times Mp(W)$-modules

$$\mathrm{rest}_{Mp(W)} : I_P(0) \cong \mathcal{S}(Mp(W)).$$

Denoting the image of $\mathrm{rest}_{w_0 N}$ by $\mathcal{S}(\mathfrak{sp}(W))$, we also have isomorphisms

$$\mathcal{S}(\mathrm{Mp}(W)) \cong I_P(0) \cong \mathcal{S}(\mathfrak{sp}(W)).$$

What is this isomorphism?

Since the intersection $X_1 \cap X_2$ is open dense, we have a birational map

$$c : \mathfrak{sp}(W) \longrightarrow \mathrm{Mp}(W)$$

which is essentially given by the Cayley transform

$$c(x) = (x-1)(x+1)^{-1}$$

(when projected to $\mathrm{Sp}(W)$).

Lemma

The above isomorphism sends $f \in \mathcal{S}(Mp(W))$ to the function

$$\phi(x) = f(c(x)) \cdot |\det(1-c(x))|^{\frac{\dim W+1}{2}}.$$

Summary

At this point, we have the following diagram:

$$\begin{array}{ccccc} & & \Omega^{\epsilon} = \mathcal{S}(V^{\epsilon} \otimes W) & & \\ & \swarrow^{p^{\epsilon}} & & \searrow^{q^{\epsilon}} & \\ \mathcal{S}(\mathrm{Mp}(W)) & & & & \mathcal{S}(\mathrm{SO}(V^{\epsilon})) \\ \Big\downarrow{\scriptstyle Cayley} & & & & \Big\downarrow{\scriptstyle Cayley} \\ \mathcal{S}(\mathfrak{sp}(W)) & & & & \mathcal{S}(\mathfrak{so}(V^{\epsilon})) \end{array}$$

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$$
\begin{array}{ccccc}
 & & \Omega^{\epsilon} = \mathcal{S}(V^{\epsilon} \otimes W) & & \\
 & \swarrow^{p^{\epsilon}} & & \searrow^{q^{\epsilon}} & \\
\mathcal{S}(\mathrm{Mp}(W)) & & & & \mathcal{S}(\mathrm{SO}(V^{\epsilon})) \\
\Big\downarrow \scriptstyle{Cayley} & & & & \Big\downarrow \scriptstyle{Cayley} \\
\mathcal{S}(\mathfrak{sp}(W)) & & & & \mathcal{S}(\mathfrak{so}(V^{\epsilon}))
\end{array}
$$

This diagram arises in another context: the moment map associated to the Hamiltonian $\mathrm{O}(V) \times \mathrm{Sp}(W)$-variety $V \otimes W$.

Moment Map

The moment map is a double fibration

$$\begin{array}{ccc} & V \otimes W & \\ {}^{p'}\swarrow & & \searrow^{q'} \\ \mathfrak{sp}(W)^* \cong \mathrm{Sym}^2 W^* & & \mathfrak{so}(V)^* \cong \wedge^2 V^* \end{array}$$

The maps are given by:

$$p'(T) = T \circ T^* \quad \text{and} \quad q'(T) = T^* \circ T.$$

- The map p' is $\mathrm{Sp}(W)$-equivariant and $\mathrm{SO}(V)$-invariant, whereas q' is $\mathrm{SO}(V)$-equivariant and $\mathrm{Sp}(W)$-invariant.
- It induces a correspondence of orbits between $\mathfrak{so}(V)$ and $\mathfrak{sp}(W)$.
- Consider invariant subsets $\mathfrak{so}(V)^{\heartsuit}$ and $\mathfrak{sp}(W)^{\heartsuit}$ corresponding to maximally nondegenerate maps. Then this correspondence descends to a bijection of closed orbits

$$\mathfrak{so}(V)^{\heartsuit}//\mathrm{SO}(V) \longleftrightarrow \mathfrak{sp}(W)^{\heartsuit}//\mathrm{Sp}(W).$$

Moment Map Correspondence

The moment map diagram

$$\begin{array}{ccc} & V \otimes W & \\ {}^{p'}\swarrow & & \searrow^{q'} \\ \mathfrak{sp}(W) & & \mathfrak{so}(V) \end{array}$$

induces by integration along the fibers:

$$\begin{array}{ccc} & \mathcal{S}(V \otimes W) & \\ {}^{p'_*}\swarrow & & \searrow^{q'_*} \\ \mathcal{S}(\mathfrak{sp}(W)^{\heartsuit}) & & \mathcal{S}(\mathfrak{so}(V)^{\heartsuit}) \end{array}$$

This defines a "moment map correspondence" of the two spaces of test functions, which descends to give an isomorphism

$$\mathcal{S}(\mathfrak{sp}(W)^{\heartsuit})_{\mathrm{Sp}(W)^{\Delta}} \cong \mathcal{S}(\mathfrak{so}(V)^{\heartsuit})_{\mathrm{SO}(V^{\pm})}$$

Transfer and Moment Map

We may ask if the maps p and p'_* are the same? Here we identify $\mathcal{S}(\mathrm{Mp}(W))$ with the space $\mathcal{S}(\mathfrak{sp}(W))$ (via normalized Cayley transform)

Proposition

$$p = \mathcal{F} \circ p'_*.$$

where

$$\mathcal{F} : \mathcal{S}(\mathfrak{sp}(W)^{\heartsuit}) \longrightarrow \mathcal{S}(\mathfrak{sp}(W))$$

is the Fourier transform (of distributions).

So have commutative diagram:

$$\begin{array}{ccc}
\mathcal{S}(V \otimes W) & \xrightarrow{p'_*} & \mathcal{S}(\mathfrak{sp}(W)^{\heartsuit}) \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathcal{F}} \\
\mathcal{S}(\mathrm{Mp}(W)) & \xrightarrow[\text{Cayley}]{} & \mathcal{S}(\mathfrak{sp}(W))
\end{array}$$

Geometric Description of Transfer

So our final geometric description of the transfer of test functions can be summarized as

- given $\tilde{f} \in \mathcal{S}(\mathrm{Mp}(W))$ and $f \in \mathcal{S}(\mathrm{SO}(V))$, we may regard

$$\tilde{f} \in \mathcal{S}(\mathfrak{sp}(W)) \quad \text{and} \quad f \in \mathcal{S}(\mathfrak{so}(V))$$

  via normalized Cayley transform.

- Then $\tilde{f}$ and f correspond if their Fourier transform $\mathcal{F}(\tilde{f})$ and $\mathcal{F}(f)$ have equal orbital integrals (under the moment map correspondence), i.e. are equal in

$$\mathcal{S}(\mathfrak{sp}(W)^{\heartsuit})_{\mathrm{Sp}(W)^{\Delta}} \cong \mathcal{S}(\mathfrak{so}(V)^{\heartsuit})_{\mathrm{SO}(V^{\pm})}$$

Corollary

If $\tilde{\mathcal{O}}$ is an $Sp(W)$-orbit in $\mathfrak{sp}(W)^{\heartsuit}$ with corresponding $SO(V)$-orbit $\mathcal{O}$ in $\mathfrak{so}(V)^{\heartsuit}$, then the transfer map identifies the Fourier transform of the orbital integrals $\mu_{\tilde{\mathcal{O}}}$ and $\mu_{\mathcal{O}}$.

Character Identity and Doubling Zeta Integrals

Suppose that $\Phi = \phi_1 \otimes \phi_2 \in \Omega$ satisfies

$$\tilde{f} = p(\Phi) \quad \text{and} \quad f = q(\Phi),$$

so that $\tilde{f}$ and f correspond. Then

$$\Theta_\pi(f) = \sum_{v \in ONB(\pi)} \int_{\mathrm{SO}(V)} f(h) \cdot \langle hv, v \rangle \, dh = \sum_v \overline{Z(0, \phi_1, \phi_2, v, v)}$$

where $Z(s, \phi_1, \phi_2, v_1, v_2)$ is the doubling zeta integral. Likewise.

$$\Theta_{\tilde{\pi}}(\tilde{f}) = \sum_{w \in ONB(\tilde{\pi}} \int_{\mathrm{Mp}(W)} \tilde{f}(g) \cdot \langle gv, v \rangle \, dg = \sum_w \overline{Z(0, \phi_!, \phi_2, w, w)}.$$

So the main character identity is saying

$$\sum_v Z(0, \phi_1, \phi_2, v, v) = \sum_w Z(0, \phi_1, \phi_2, w, w).$$

This identity was shown in a 2019 Duke paper of Hang Xue!

Periods and Theta Correspondence

The techniques for deriving the main character identity can be applied in the setting of the relative Langlands program to give relative character identities.

In theta correspondence, given a dual pair $G \times H$, it is typical for one to relate a period $\mathcal{P}$ on G with another period $\mathcal{P}'$ on H. Given a period $\mathcal{P}$, one can associate a relative character. Thus, in the above setting, one may ask if there is an identity relating the relative character for $\mathcal{P}$ and that for $\mathcal{P}'$.

For example:

- The theory discussed before corresponds to the group case, where $G = G_0 \times G_0$ and the period $\mathcal{P}$ is the G_0^{Δ}-period.
- For $\mathrm{O}_{n-1}\backslash\mathrm{O}_n$ vs. $(N,\psi)\backslash\mathrm{SL}_2$, see paper with Xiaolei Wan.
- Wan's thesis deal with $\mathrm{U}_2\backslash\mathrm{SO}_5$ v.s. $(N,\psi)\backslash\mathrm{PGL}_2 \times T\backslash\mathrm{PGL}_2$, using the theta correspondence for $\mathrm{PGSp}_4 \times \mathrm{PGSO}_4$.

THANK YOU FOR YOUR ATTENTION!