BEILINSON–BERNSTEIN LOCALIZATION VIA WONDERFUL ASYMPTOTICS

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SECTIONS

- Introduction and motivation
- Filtrations on $\mathcal{O}_G$ and $\mathcal{D}_G$
- The horocycle space, Vinberg semigroup, and wonderful compactification
- Results and relations to other work

This talk is based on joint work with D. Ben-Zvi: arXiv:1901.01226 (new version on the way).

LOCALIZATION AND D-MODULES

Let G be a connected reductive group over $\mathbb{C}$

- Suppose X is a smooth G-variety
- The infinitesimal action of G:

$$\mathfrak{g} = \text{Lie}(G) \to \Gamma(X, \mathcal{T}_X) = \{\text{vector fields on } X\}$$

- Algebra homomorphism:

$$U\mathfrak{g} \to D_X = \Gamma(X, \mathcal{D}_X) = \{\text{differential operators on } X\}$$

- Localization functor:

$$\text{Loc} : U\mathfrak{g}\text{-mod} \to D(X) = \{\text{D-modules on } X\}$$

$$M \mapsto \mathcal{D}_X \otimes_{U(\mathfrak{g})} M$$

MORE ON LOCALIZATION

Take $X = G/B$ to be the flag variety. Beilinson-Bernstein Localization:

$$\begin{array}{ccc} U\mathfrak{g}\text{-mod} & \xrightarrow{\text{Loc}} & D(G/B) \\ \uparrow & \nearrow_{\simeq} & \\ U\mathfrak{g}\text{-mod}_0 & & \end{array}$$

where $U\mathfrak{g}\text{-mod}_0$ is the subcategory of modules with trivial central character.

Application: Proof of the Kazhdan-Lusztig conjecture on multiplicities of simples in standards.

Remark: Let $f : X \to Y$ be a G-equivariant map. Localization does not commute with pullback or pushforward in general:

$$\begin{array}{ccccc} & & U\mathfrak{g}\text{-mod} & & \\ & \swarrow^{\text{Loc}} & & \searrow^{\text{Loc}} & \\ D(X) & & ? \longleftrightarrow ? & & D(Y) \end{array}$$

IN THIS TALK ...

We'll be interested in the action of $G \times G$ on:

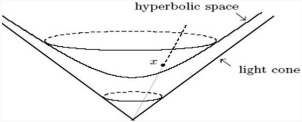
- G itself by right and left multiplication
- the horocycle space $\mathcal{Y}$ for G
- the Vinberg semigroup $\mathbb{V}_G$

$$\begin{array}{ccccc}
\mathcal{Y} & \hookrightarrow & \mathbb{V}_G & \hookleftarrow & G \\
\downarrow & & \downarrow \pi & & \downarrow \\
\{0\} & \hookrightarrow & \mathbb{A}^r & \hookleftarrow & \{\text{generic}\}
\end{array}$$

INSPIRATION FROM HARMONIC ANALYSIS ON HYPERBOLIC SPACE

- Let $\mathbb{H} = \mathsf{PSL}_2(\mathbb{R})/\mathsf{SO}_2(\mathbb{R})$ be hyperbolic space.
- Let $\mathsf{Y} = \mathsf{PSL}_2(\mathbb{R})/\{\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}\}$ be the light cone minus $\{0\}$. This can be identified with the space of horocycles in $\mathbb{H}$.
- Let V be the closure of the interior of the light cone, minus $\{0\}$.

$$\begin{array}{ccccc} \mathsf{Y} & \hookrightarrow & \mathsf{V} & \hookleftarrow & \mathbb{H} \times \mathbb{R}_{>0} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \hookrightarrow & \mathbb{R}_{\geq 0} & \hookleftarrow & \mathbb{R}_{>0} \end{array}$$



- $\mathsf{V}/\mathbb{R}_{>0} = \overline{\mathbb{H}}$ is the usual compactification of hyperbolic space.
- *Very* roughly:

$$\begin{array}{ccc} \mathsf{Fun}(\mathbb{H}) & \xrightarrow{\text{horocycle transform}} & \mathsf{Fun}(\mathsf{Y}) \\ \text{wave equation} \downarrow & & \uparrow \text{restrict} \\ \mathsf{Fun}(\mathbb{H} \times \mathbb{R}_{\geq 0}) & \xrightarrow{\text{“scattering theory”}} & \mathsf{Fun}(\mathsf{V}) \end{array}$$

NOTATION

We work over $\mathbb{C}$.

- Fix $T \subseteq B \subseteq G$.
- Let $\Lambda = X^*(T)$ be the character lattice of T, i.e. weight lattice of G.
- Let $\Delta = \{\alpha_1, \ldots, \alpha_r\} \subseteq \Lambda$ be the positive simple roots. We have the following partial order on Λ:

$$\mu \leq \lambda \qquad \Leftrightarrow \qquad \lambda - \mu = \sum_i n_i \alpha_i, \qquad n_i \geq 0$$

- Let $\Lambda^+ \subseteq \Lambda$ be the cone of dominant weights.

We have a bijection:

$$\left\{ \begin{matrix} \text{finite-dimensional irreducible} \\ \text{representations of } G \end{matrix} \right\} \longleftrightarrow \Lambda^+.$$

$$V_\lambda \longleftrightarrow \lambda$$

Let $\mathcal{O}_G$ denote the algebra of functions on G.

Peter-Weyl Theorem

There is an isomorphism of $G \times G$ representations

$$\bigoplus_{\lambda \in \Lambda^+} V_\lambda^* \otimes V_\lambda \longrightarrow \mathcal{O}_G$$

given by matrix coefficients: $f \otimes v \mapsto [g \mapsto \langle f, g \cdot v \rangle]$.

Lemma. There is a Λ-filtration on $\mathcal{O}_G$ given by:

$$(\mathcal{O}_G)_{\leq \lambda} = \bigoplus_{\mu \leq \lambda} V_\mu^* \otimes V_\mu.$$

Reason: If $V_\nu \subseteq V_\lambda \otimes V_\mu$, then $\nu \leq \lambda + \mu$.

Consider $G \times G$ acting on G by right and left multiplication. We have an algebra homomorphism:

$$\mu : U\mathfrak{g} \otimes U\mathfrak{g} \to D_G = \Gamma(G, \mathcal{D}_G)$$

Proposition

The image of μ can be identified with $U\mathfrak{g} \otimes_{Z(U\mathfrak{g})} U\mathfrak{g}$. Moreover, there is a Λ-filtration on D_G given by

$$(D_G)_{\leq \lambda} = \mathrm{Image}(\mu) \cdot (\mathcal{O}_G)_{\leq \lambda}.$$

Another way to think about this filtration:

$$\mathrm{Derv}(\mathcal{O}_G)_{\leq \lambda} = \{\theta \in \mathrm{Derv}(\mathcal{O}(G)) \mid \theta(\mathcal{O}(G)_{\leq \mu}) \subseteq \mathcal{O}(G)_{\leq \lambda + \mu}\}$$

EXAMPLE OF SL_2

Let $G = SL_2$. Then $\Lambda = \mathbb{Z}$ and $\Delta = \{2\}$, so the partial order is:

$$\cdots \leq -4 \leq -2 \leq 0 \leq 2 \leq 4 \leq \dots$$

$$\cdots \leq -3 \leq -1 \leq 1 \leq 3 \leq 5 \leq \dots$$

We have $\mathcal{O}_{SL_2} = \mathbb{C}[a, b, c, d]/(ad - bc = 1)$.

Trivializing D_{SL_2} using right-invariant vector fields we obtain

$$\mu : U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2) \to D_{SL_2} \simeq \mathcal{O}(SL_2) \star U(\mathfrak{sl}_2)$$

$$\begin{aligned} X \otimes 1 &\mapsto X \\ 1 \otimes E &\mapsto -a^2E + c^2F + acH \\ 1 \otimes F &\mapsto b^2E - d^2F + bdH \\ 1 \otimes H &\mapsto 2abE - 2cdF - (ad + bc)H \end{aligned}$$

Note: $\mu(1 \otimes \mathrm{Casimir}) = \mu(\mathrm{Casimir} \otimes 1)$, where
$\mathrm{Casimir} = H^2 + 2EF + 2FE$ is a generator of the center of $U(\mathfrak{sl}_2)$.

Notation: Let B^- denote the opposite Borel, with unipotent radical $N^- = R_{\mathrm{unip}}(B^-)$. We have $B \cap B^- = T$.

Definition

The horocycle space for G is $\mathcal{Y} = \frac{G/N \times N^- \backslash G}{T}$.

- **Fact**: The space $\mathcal{Y}$ is a degeneration of G, in the sense that

$$\mathrm{Spec}(\mathrm{gr}(\mathcal{O}_G)) \text{ is the affine closure of } \mathcal{Y}.$$

- There is a $G \times G$-equivariant principle T-bundle

$$q : \mathcal{Y} \to G/B \times B^- \backslash G$$

- **Example**: For $G = SL_2$:

$$\mathcal{Y}_{SL_2} = \frac{\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}^2 \setminus \{0\}}{\mathbb{C}^\times} = \{\text{rank one 2 by 2 matrices}\} \xrightarrow{q} \mathbb{P}^1 \times \mathbb{P}^1$$

$$M \mapsto (\mathrm{kernel}(M), \mathrm{image}(M))$$

Let

$$\mathrm{Rees}(\mathcal{O}_G) = \bigoplus_{\lambda \in \Lambda} \mathcal{O}(G)_{\leq \lambda} z^{\lambda} \subseteq \mathcal{O}(G) \otimes \mathbb{C}[\Lambda]$$

be the Rees algebra associated to $\mathcal{O}_G$ with the Peter-Weyl filtration.

Definition

The Vinberg semigroup for G is $\mathbb{V}_G = \mathrm{Spec}(\mathrm{Rees}(\mathcal{O}_G))$.

- $\mathrm{Rees}(\mathcal{O}(G))$ is naturally a bialgebra (not Hopf!).
- $\Lambda = X^*(T)$ grading on $\mathrm{Rees}(\mathcal{O}(G))$ $\Rightarrow$ T-action on $\mathbb{V}_G$.
- For $\lambda \in \Lambda^+$ regular, we denote the λ-semistable locus of $\mathbb{V}_G$ for the action of T by $\mathbb{V}_G^\circ$ (= S^{pr} in Vinberg[1]).

Remark. This is not Vinberg's original definition; see work of Brion[2].

[1] Vinberg, On reductive algebraic semigroups, Trans. of AMS-Series 2 169 (1995), 145–182

[2] Brion, The total coordinate ring of a wonderful variety, J. of Algebra 313 (2007), no. 1, 61–99.

THE WONDERFUL COMPACTIFICATION

The inclusion $\mathbb{C}[z^\alpha \mid \alpha \in \Delta\,] \hookrightarrow \mathcal{O}(\mathbb{V}_G) = \mathrm{Rees}(\mathcal{O}(G))$ induces a $G \times G \times T$-equivariant map

$$\begin{array}{ccccc} \mathcal{Y} & \hookrightarrow & \mathbb{V}_G^\circ & \hookleftarrow & \frac{G \times T}{Z(G)} \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow \\ \{0\} & \hookrightarrow & \mathbb{A}^r & \hookleftarrow & \frac{T}{Z(G)} \end{array}$$

The map π is flat with smooth fibers. The torus T acts on $\mathbb{V}_G^\circ$ freely.

Theorem (Martens-Thaddeus)

For $\lambda \in \Lambda^+$ regular, the GIT quotient of $\mathbb{V}_G$ at λ is the de Concini – Procesi ‘wonderful’ compactification of the adjoint group $G_{adj} = G/Z(G)$, i.e. $\overline{G_{adj}} = \mathbb{V}_G /\!\!/_\lambda T$.

$\to$ Martens and Thaddeus, Compactifications of reductive groups as moduli stacks of bundles, Compositio Math. 152 (2016), no. 1, 62–98.

EXAMPLE OF SL_2

Example: For $\mathrm{G} = \mathrm{SL}_2$, we have that $\mathbb{V}_{\mathrm{SL}_2} = \mathrm{Mat}_{2\times 2}$. The diagram on the previous slide becomes:

$$
\begin{array}{ccccc}
\mathcal{Y} = \{\mathrm{rk}\ 1\} & \hookrightarrow & \mathbb{V}_{\mathrm{G}}^{\circ} = \mathrm{Mat}_{2\times 2} \setminus \{0\} & \hookleftarrow & \frac{\mathrm{SL}_2 \times \mathbb{C}^{\times}}{\mathbb{Z}/2\mathbb{Z}} = \mathrm{GL}_2 \\
\downarrow & & \downarrow{\scriptstyle \pi = \mathrm{det}} & & \downarrow \\
\{0\} & \hookrightarrow & \mathbb{A}^1 & \hookleftarrow & \mathbb{C}^{\times}
\end{array}
$$

The wonderful compactification in this case is $\overline{\mathrm{PSL}_2} = \mathbb{P}^3$.

RELATIVE DIFFERENTIAL OPERATORS

Recall the matrix coefficients filtration on $\mathcal{D}_G$.

Question: What is $\mathrm{Rees}(\mathcal{D}_G)$?

$$\begin{array}{ccccc}
\mathcal{Y} & \hookrightarrow & \mathbb{V}_G^\circ & \hookleftarrow & \frac{G \times T}{Z(G)} \\
\downarrow & & \downarrow \pi & & \downarrow \\
\{0\} & \hookrightarrow & \mathbb{A}^r & \hookleftarrow & \frac{T}{Z(G)}
\end{array}$$

Definition. Let $\mathcal{D}_\pi \subseteq \mathcal{D}_{\mathbb{V}_G^\circ}$ be the subsheaf relative differential operators on $\mathbb{V}_G^\circ$ (generated by vector fields that preserve the fibers of π).

Proposition. There is a natural identification $\mathrm{Rees}(\mathcal{D}_G) \simeq \mathcal{D}_\pi$, and a functor

$$\mathrm{Rees} : D(G)^{\mathrm{filt}} \to D(\pi) := \mathcal{D}_\pi\text{-mod}$$

$$M \mapsto \bigoplus_\lambda M_{\leq \lambda} z^\lambda$$

WEAKLY EQUIVARIANT D-MODULES

Let K be any linear algebraic group acting on an algebraic variety X. We have:

$$X \xleftarrow{\text{act}} K \times X \xrightarrow{\pi_2} X$$

Definition. A K-equivariant quasicoherent sheaf on X is the data of a quasicoherent sheaf $\mathcal{F}$ on X, together with an isomorphism

$$\Phi : \text{act}^* \mathcal{F} \xrightarrow{\sim} \pi_2^* \mathcal{F},$$

satisfying an associativity condition.

Fact: The sheaf $\mathcal{D}_X$ of differential operators carries a natural K-equivariant structure

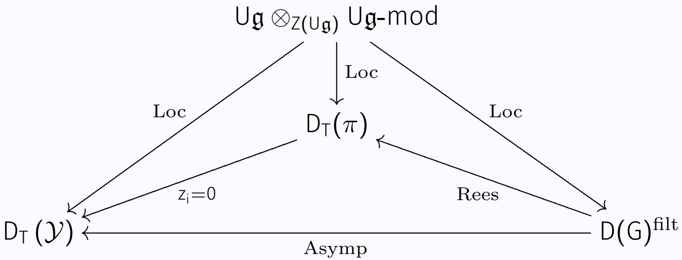
Definition. A weakly K-equivariant D-module on X is a D-module $\mathcal{M}$ on X equipped with an isomorphism $\Phi : \text{act}^* \mathcal{M} \xrightarrow{\sim} \pi_2^* \mathcal{M}$ of $\mathcal{O}_K \boxtimes \mathcal{D}_X$-modules.

MAIN RESULT

Recall that the spaces $\mathcal{Y}$ and $\mathbb{V}_G^\circ$ carry actions of T. Let $D_T(\bullet)$ denote the category of weakly T-equivariant D-modules, i.e. D-modules with a compatible T-equivariant structure.

Theorem (Ben-Zvi – G.)

There are well-defined functors that fit into a commutative diagram



IDEA BEHIND THE COMMUTATIVITY OF THE DIAGRAM

Note that $\mathcal{D}_\pi|_{\pi^{-1}(a)} = \mathcal{D}_{\pi^{-1}(a)}$ for all $a \in \mathbb{A}^r$, and $\mathrm{Rees}(\mathcal{D}_G)|_{\mathbb{V}_G^\circ} = \mathcal{D}_\pi$.

Now, for a $U\mathfrak{g} \otimes U\mathfrak{g}$-module M, we have:

$$\begin{aligned}
\mathrm{Asymp}(\mathrm{Loc}_G(M)) &= \mathrm{Rees}(\mathcal{D}_G \otimes_{U\mathfrak{g}\otimes U\mathfrak{g}} M)|_{\mathcal{Y}} \\
&= \bigoplus_{\lambda\in\Lambda} \left((\mathcal{D}_G)_{\leq\lambda} \otimes_{U\mathfrak{g}\otimes U\mathfrak{g}} M\right) z^\lambda|_{\mathcal{Y}} \\
&= \mathrm{Rees}(\mathcal{D}_G)|_{\mathcal{Y}} \otimes_{U\mathfrak{g}\otimes U\mathfrak{g}} M \\
&= \mathcal{D}_{\mathcal{Y}} \otimes_{U\mathfrak{g}\otimes U\mathfrak{g}} M = \mathrm{Loc}_{\mathcal{Y}}(M)
\end{aligned}$$

PARABOLIC RESTRICTION

Fix $I \subseteq \Delta = \{\alpha_1, \ldots, \alpha_r\}$. Correspondingly, we have:

- $P_I \supseteq B$ a parabolic subgroup of G and $N_I = R_u(P_I)$ its unipotent radical.
- $L = L_I$ the Levi quotient with Lie algebra $\mathfrak{l}_I$ and center $Z_I = Z(L_I)$.
- A point $e_I \in \mathbb{A}^r$ with ith coordinate 1 if $\alpha_i \in I$ and zero otherwise.
- $\mathcal{Y}_I = G/N_I \times_{L_I} N_I^- \backslash G$ the partial horocycle space, which includes in fiber of $\mathbb{V}_G \xrightarrow{\pi} \mathbb{A}^r$ over e_I.

Fibration:

$$\begin{array}{ccc} L_I & \longrightarrow & \mathcal{Y}_I \\ & & \big\downarrow q_I \\ & & G/P_I \times P_I^- \backslash G \end{array}$$

Fix $y \in \mathcal{Y}_I$ and let $q_I(y) = (P, P') \in G/P_I \times P_I^- \backslash G$. Choice of y gives an inclusion $L_I \simeq q_I^{-1}(q_I(y)) \hookrightarrow \mathcal{Y}_I$. Parabolic restriction functor for $\mathfrak{p} = \mathrm{Lie}(P)$:

$$\mathrm{res}_{\mathfrak{p}} : U\mathfrak{g} \to U\mathfrak{l}_I$$
$$V \mapsto (V)_{\mathfrak{n}_P} = V/\mathfrak{n}_P V$$

Theorem (Ben-Zvi – G.)

For any parabolics P, P' conjugate to P_I, P_I^- respectively, there is a functor $\overline{\mathrm{Asymp}_I} : D(G)^{\mathrm{filt}} \to D_{Z_I}(L_I)$ that fits into the following commutative diagram:

$$\begin{array}{ccc} U\mathfrak{g} \otimes U\mathfrak{g}\text{-mod} & \xrightarrow{\mathrm{res}_{\mathfrak{p}} \otimes \mathrm{res}_{\mathfrak{p}'}} & U\mathfrak{l}_I \otimes U\mathfrak{l}_I\text{-mod} \\ \downarrow{\scriptstyle \mathrm{Loc}_G} & & \downarrow{\scriptstyle \mathrm{Loc}_L} \\ D(G)^{\mathrm{filt}} & \xrightarrow{\overline{\mathrm{Asymp}_I}} & D_{Z_I}(L_I) \end{array}$$

Ingredients:

(1) **Fact**: If G acts on X transitively, the localization of a $U\mathfrak{g}$-module V to X is the coinvariants of $V \otimes \mathcal{O}_X$ with respect to the kernel of $\mathfrak{g} \otimes \mathcal{O}_X \to \Gamma(X, \mathcal{T}_X)$.

(2) Commutative diagram:

$$\begin{array}{ccccc} & & U\mathfrak{g} \otimes_{Z(U\mathfrak{g})} U\mathfrak{g}\text{-mod} & & \\ & \swarrow{\scriptstyle \mathrm{Loc}} & & \searrow{\scriptstyle \mathrm{Loc}} & \\ D_{Z_I}(\mathcal{Y}_I) & & \xleftarrow[\mathrm{Asymp}_I]{} & & D(G)^{\mathrm{filt}} \end{array}$$

Assume for simplicity that $Z(G) = 1$. Let $j : G \hookrightarrow \overline{G}$ be the inclusion of the open orbit in the wonderful compactification, and let $Z_1, \ldots, Z_r$ be the boundary divisors of $\overline{G}$. We have:

- A Λ-filtration on $\mathcal{D}_{\overline{G}}$ coming from the V-filtrations with respect to the Z_i's.
- An induced Λ-filtration on $j_*\mathcal{D}_G$, and on its global sections D_G.
- For any regular holonomic D-module $\mathcal{M}$ on G, a Λ-filtration on $\Gamma(\overline{G}, j_*\mathcal{M}) = \Gamma(G, \mathcal{M})$, coming from Kashwara–Malgrange.

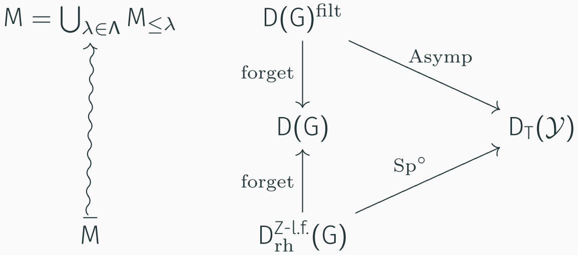
Proposition

1. The V-filtration on D_G coincides with the matrix coefficients filtration on D_G.
2. For any regular holonomic D-module $\mathcal{M}$ on G, the Kashiwara–Malgrange filtration on its global section is compatible with the matrix coefficients filtration on D_G.

RELATION TO BEZRUKAVNIKOV FINKELBERG OSTRIK

Definition. Let $D^{Z\text{-l.f.}}_{\mathrm{rh}}(G)$ denote the category of regular holonomic D-modules on G with the property that the action of $Z(U\mathfrak{g})$ is locally finite.

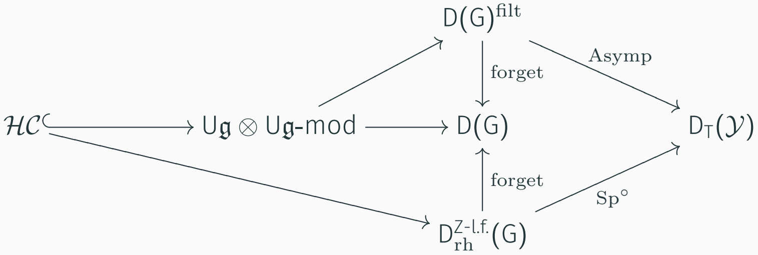
By [BFO][3], there is a functor $\mathrm{Sp}^\circ : D^{Z\text{-l.f.}}_{\mathrm{rh}}(G) \to D_T(\mathcal{Y})$. This is related to our asymptotics functor in the following way:



[3] Bezrukavnikov, Finkelberg, Ostrik, Character D-modules via Drinfeld center of Harish-Chandra bimodules, Inventiones math. 188 (2012), no. 3, 589–620

RELATION TO HARISH-CHANDRA BIMODULES

Definition. A finitely-generated $U\mathfrak{g} \otimes U\mathfrak{g}$-module V is called a Harish-Chandra bimodule if it is locally finite as a $U(\mathfrak{g}_\Delta)$-module, and locally finite as a $Z\mathfrak{g} \otimes Z\mathfrak{g}$-module. Notation: $\mathcal{HC}$.



Bottom left diagonal arrow is due to Ginzburg[4].

[4] Ginzburg, Admissible modules on a symmetric space, Astérisque 173-74 (1989), 199–255.

RELATION TO BEILINSON-BERNSTEIN LOCALIZATION

Given $\lambda \in \Lambda = X^*(T)$ there are two relevant constructions:

Central character	Line bundle
$\chi_\lambda : Z(U\mathfrak{g}) \to \mathbb{C}$	$\mathcal{L}_\lambda \to G/B$
Central reduction	**Twisted D-modules**
$U\mathfrak{g}\text{-mod}_\lambda = U\mathfrak{g}/\langle \ker(\chi_\lambda)\rangle\text{-mod}$	$D_\lambda(G/B)$

$$\begin{array}{ccc} U\mathfrak{g} \otimes_{Z(U\mathfrak{g})} U\mathfrak{g}\text{-mod} & \xrightarrow{\quad \mathrm{Loc} \quad} & D_T(\mathcal{Y}) \\ \uparrow & & \uparrow \\ U\mathfrak{g}\text{-mod}_\lambda \boxtimes U\mathfrak{g}\text{-mod}_\lambda & \xrightarrow{\quad \mathrm{Loc}_\lambda \quad} & D_\lambda(G/B) \boxtimes D_\lambda(B^-\backslash G) \end{array}$$

If λ is regular, then Loc_λ is an abelian equivalence [BB][5]. On the level of dg categories, the functor Loc gives an equivalence up to a categorical version of Weyl group symmetries on the $D_T(\mathcal{Y})$ side [BN][6].

[5] Beilinson and Bernstein, Localisation de $\mathfrak{g}$-modules, C. R. Acad. Sci. Paris (1981), 15–18.

[6] Ben-Zvi and Nadler, Beilinson–Bernstein localization over the Harish-Chandra center, arXiv:1209.0188 (2012).

We expect our results to generalize in two directions:

- **Arbitrary symmetric spaces**: Replace $(G \times G)/G_\Delta$ with the quotient G/K of G by the fixed points K of an involution. Then the horocycle space $\mathcal{Y}$ is replaced by G/MN where MN a satellite subgroup of K.
- **Quantum groups**: There are quantum versions of the wonderful compactification and Vinberg semigroup. Established notions of quantum differential operators on the group should lead to a quantum version of the localization diagram with interesting degenerations in the classical limit.

Thanks!