

Wave-front sets of distinguished representations

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following Harris-Weich, Prasad, Sakellaridis

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- F : local field, $\mathbf{G}$: reductive group over F , $\mathbf{H} \subset \mathbf{G}$ algebraic subgroup, $G := \mathbf{G}(F)$, $H := \mathbf{H}(F)$, $\mathfrak{g} := \text{Lie}(\mathbf{G})$, $\mathfrak{g}_0 := \text{Lie}(G)$, $\mathcal{N}(\mathfrak{g}^*) := \text{nilpotent cone}$, $\mathfrak{h} := \text{Lie}(\mathbf{H})$, $\mathfrak{h}_0 := \text{Lie}(H)$.
- $\pi \in \text{Irr}(G)$ is called *H-distinguished* if $(\pi^*)^H \neq 0$.
- $\text{Irr}_H(G) := \text{all smooth admissible } H\text{-distinguished irreps of } G$.
- $WF^o(\pi) \subset \mathcal{N}(\mathfrak{g}_0^*) := \text{union of top } G\text{-orbits in the wave-front set of } \pi$.

Conjecture

For any $\pi \in \text{Irr}_H(G)$, $WF^o(\pi)|_{\mathfrak{h}_0} \ni 0$.

Theorem (G.-Sayag 2020)

- ❶ If F is Archimedean and $\mathbf{H} \subset \mathbf{G}$ is spherical then

$$\mathbf{G}(WF^o(\pi))|_{\mathfrak{h}} \ni 0.$$

- ❷ If F is p -adic then $WF^o(\pi) \subset \overline{G\mathfrak{h}_0^\perp} \subset \mathfrak{g}_0^*$

$\mathbf{H}$ is called spherical if it has an open orbit on the flag variety $\mathbf{G}/\mathbf{B}$.

Uniform applications

- $\text{Irr}_H(G)$:= all smooth admissible H -distinguished irreps.
- $WF^\circ(\pi) \subset \mathcal{N}(\mathfrak{g}_0^*)$:= union of top G -orbits in the wave-front set of π .

Theorem (G.-Sayag 2020)

- (i) If F is Archimedean and $\mathbf{H} \subset \mathbf{G}$ is spherical then

$$\mathbf{G}(WF^\circ(\pi))|_{\mathfrak{h}} \ni 0.$$

- (ii) If F is p -adic then $WF^\circ(\pi) \subset \overline{G\mathfrak{h}_0^\perp}$

Assume $\text{char } F = 0$.

Corollary (following Prasad - Sakellaridis)

$\forall$ unimodular spherical $H \subset G$,

$$\max_{\pi \in \text{Irr}_H(G)} \dim WF^\circ(\pi) = \dim \mathcal{N}(\mathfrak{g}_0^*) \cap G\mathfrak{h}_0^\perp \quad \text{and}$$

- (i) $\exists$ inf. dim. $\pi \in \text{Irr}_H(G) \iff \exists$ unipotent $u \in G - H$.
- (ii) for quasisplit G : $\exists$ generic $\pi \in \text{Irr}_H(G) \iff H \cap U = \{1\}$ for some maximal unipotent subgroup $U \subset G$.

Archimedean setting & applications to symmetric pairs

From now on $F := \mathbb{R}$.

Theorem (Rossmann 1995, using Borho-Brylinski + Joseph)

$\forall \pi \in \text{Irr}(G)$, $\text{WF}^o(\pi)$ lies in a unique $\mathbf{G}$ -orbit $\mathcal{O}(\pi) \subset \mathcal{N}(\mathfrak{g}^*)$.

Let $\mathbf{H} \subset \mathbf{G}$ be symmetric sbgrp, and $G_H^{\mathbb{R}}$ the corresponding real form of $\mathbf{G}$.
Kostant-Sekiguchi: $\mathbf{H}$ -orbits on $\mathfrak{h}^\perp \cong (\mathfrak{g}/\mathfrak{h})^* \leftrightarrow$ real $G_H^{\mathbb{R}}$ -orbits on $\mathfrak{g}^*$.
Prasad philosophy: $G_H^{\mathbb{R}}$ describes H -distinguished rep^{ns} of G .

Corollary (of our thm)

$\forall \pi \in \text{Irr}_H(G)$, $\mathcal{O}(\pi)$ includes a real $G_H^{\mathbb{R}}$ -orbit.

Real orbits are classified by Ohta and Djokovic.

Conjecture (Prasad)

For split G, H : $\exists$ tempered $\pi \in \text{Irr}_H(G) \Rightarrow \exists$ generic $\pi \in \text{Irr}_H(G)$.

Our theorem + result of Harris $\Rightarrow$ conjecture holds for most pairs.

Applications to strongly spherical pairs

Corollary

Assume that $\mathbf{H}$ is reductive and $\Delta\mathbf{H} \subset \mathbf{G} \times \mathbf{H}$ spherical.

Let $\pi \in \text{Irr}(\mathbf{G})$, $\tau \in \text{Irr}(\mathbf{H})$ with $\text{Hom}_{\mathbf{H}}(\pi|_{\mathbf{H}}, \tau) \neq 0$. Then

$$\mathcal{O}(\tau) \subset \mathcal{O}(\pi)|_{\mathfrak{h}}.$$

For classical G , $\mathcal{O}(\pi)$ is described by its Jordan partition $\lambda(\pi)$.

Corollary

Let $\mathbf{G}$ be GL_{n+1} , O_{n+1} or U_{n+1} and $\mathbf{H}$ be GL_n , O_n or U_n respectively. Let $\pi \in \text{Irr}(\mathbf{G})$, $\tau \in \text{Irr}(\mathbf{H})$ with $\text{Hom}_{\mathbf{H}}(\pi|_{\mathbf{H}}, \tau) \neq 0$. Then for any index i :

$$|\lambda(\pi)_i^t - \lambda(\tau)_i^t| \leq 1.$$

Applications to Jacquet quotients

Corollary

Let $P = LU \subset G$ be parabolic subgroup, $\pi \in \text{Irr}(G)$, and $\tau \in \text{Irr}(L)$ be a quotient of the Jacquet module $r_U(\pi)$. Then $\mathcal{O}(\tau) \subset \mathcal{O}(\pi)|_{\mathfrak{p}^*}$.

$$\mathfrak{l} = \mathfrak{p}/\mathfrak{u}, \quad \mathcal{O}(\tau) \subset \mathfrak{l}^* \subset \mathfrak{p}^* \leftarrow \mathfrak{g}^* \supset \mathcal{O}(\pi)$$

Proof.

Let $H := \{(p, pU) \in G \times L\} \subset G \times L$. Then $\text{Hom}_H(\pi \hat{\otimes} \tilde{\tau}, \mathbb{C}) \neq 0$. □

For a maximal parabolic of GL_n :

$$\text{if } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{O}(\tau) \text{ then } \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{O}(\pi) \text{ for some } C.$$

Proposition (Zhang, work in progress)

$\exists C$ s. t. $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ has given Jordan partition $\mu \iff c_{\lambda(A), \lambda(B)}^{\mu} \neq 0$
(Littlewood-Richardson coefficient)

Twisted induction of spherical pairs

Let $P = LU \subset G$ parabolic subgroup, $\mathbf{S} \subset \mathbf{L}$ be a spherical subgroup. Let $H := SU$. Fix algebraic character $\psi : U \rightarrow \mathbb{R}$, and let $\chi : H \rightarrow \mathbb{C}^\times$ be a character s.t. $\chi(u) := \exp(i\psi(u)) \forall u \in U$. Consider $d\psi \in \mathfrak{u}^* \subset \mathfrak{h}^*$.

$$\mathrm{Irr}_{H,\chi}(G) := \{\pi \in \mathrm{Irr}(G) : \mathrm{Hom}_H(\pi|_H, \chi) \neq 0\}$$

Theorem (G.-Sayag 2020)

For all $\pi \in \mathrm{Irr}(G)_{H,\chi}$, $d\psi \in \mathcal{O}(\pi)|_{\mathfrak{h}}$

Shalika models: $\mathbf{G} = \mathrm{GL}_{2n}$, $\mathbf{L} = \mathrm{GL}_n \times \mathrm{GL}_n$, $\mathbf{S} = \Delta \mathrm{GL}_n$, $\mathfrak{u} = \mathrm{Mat}_{n \times n}$,

$$H = \left\{ \begin{pmatrix} g & A \\ 0 & g \end{pmatrix} \right\}, \quad \psi(A) = \mathrm{tr}(A).$$

Corollary

For all $\pi \in \mathrm{Irr}_{H,\chi}(G)$, $\lambda(\pi)$ consists of even parts.

$P = LU \subset G$ parabolic, $\mathbf{S} \subset \mathbf{L}$ spherical, $H := SU, \psi : U \rightarrow \mathbb{R}$,
 $\chi : H \rightarrow \mathbb{C}^\times$ character s.t. $\chi(u) := \exp(i\psi(u)) \forall u \in U$.

Theorem (G.-Sayag 2020)

For all $\pi \in \text{Irr}(G)_{H,\chi}$, $d\psi \in \mathcal{O}(\pi)|_{\mathfrak{h}}$

- Shalika models: $\mathbf{G} = \text{GL}_{2n}$, $\mathbf{L} = \text{GL}_n \times \text{GL}_n$, $\mathbf{S} = \Delta \text{GL}_n$, $\mathfrak{u} = \text{Mat}_{n \times n}$,

$$H = \left\{ \begin{pmatrix} g & A \\ 0 & g \end{pmatrix} \right\}, \psi(A) = \text{tr}(A).$$

Cor. For all $\pi \in \text{Irr}_{H,\chi}(G)$, $\lambda(\pi)$ consists of even parts.

- Klyachko models: $\mathbf{G} = \text{GL}_{n+k}$, $\mathbf{L} = \text{GL}_n \times (\text{GL}_1)^k$, $\mathbf{S} = \text{Sp}_n$,

$$H = \left\{ \begin{pmatrix} g & A \\ 0 & B \end{pmatrix} : g \in \text{Sp}_n, B \text{ upper-uni-triangular} \right\}, \psi(B) = \sum B_{i,i+1}$$

Corollary

$\forall \pi \in \text{Irr}_{H,\chi}(G)$, $\#\{i : \lambda(\pi)_i^t \text{ is odd}\} = k$.

G.-Offen-Sahi-Sayag: for unitary π ,

$$\pi \in \text{Irr}_{H,\chi}(G) \iff \#\{i : \lambda(\pi)_i^t \text{ is odd}\} = k.$$

Twisted induction of strongly spherical pairs

$P = LU \subset G$ parabolic, $\mathbf{S} \subset \mathbf{L}$ with $\Delta \mathbf{S} \subset \mathbf{S} \times \mathbf{L}$ spherical,
 $H := SU, \psi : U \rightarrow \mathbb{R}, \chi : H \rightarrow \mathbb{C}^\times$ char s.t. $\chi(u) := \exp(i\psi(u)) \forall u \in U$.

Corollary (G.-Sayag 2020)

Let $\pi \in \text{Irr}(G), \tau \in \text{Irr}(S)$ with $\text{Hom}_H(\pi|_H, \tau \otimes \chi) \neq 0$. Then

$$\mathcal{O}(\tau) + d\psi \subset \mathcal{O}(\pi)|_{\mathfrak{h}}$$

Rankin-Selberg models: $\mathbf{G} = \text{GL}_{n+k}, \mathbf{L} = \text{GL}_{n+1} \times (\text{GL}_1)^{k-1}, \mathbf{S} = \text{GL}_n$,

$$H = \left\{ \begin{pmatrix} g & 0 & * & \dots & * \\ 0 & 1 & \underline{*} & \dots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \underline{*} \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} \right\}, \quad \psi = \text{sum of underlined entries}$$

Corollary

If $\text{Hom}_H(\pi|_H, \tau \otimes \chi) \neq 0$ then $|\lambda(\pi)_i^t - \lambda(\tau)_i^t| \leq 1$ for any index i .

$P = LU \subset G$ parabolic, $\mathbf{S} \subset \mathbf{L}$ with $\Delta\mathbf{S} \subset \mathbf{S} \times \mathbf{L}$ spherical,
 $H := SU, \psi : U \rightarrow \mathbb{R}, \chi : H \rightarrow \mathbb{C}^\times$ char s.t. $\chi(u) := \exp(i\psi(u)) \forall u \in U$.

Corollary (G.-Sayag 2020)

Let $\pi \in \text{Irr}(G), \tau \in \text{Irr}(S)$ with $\text{Hom}_H(\pi|_H, \tau \otimes \chi) \neq 0$. Then

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- Rankin-Selberg: $\mathbf{G} = \text{GL}_{n+k}, \mathbf{L} = \text{GL}_{n+1} \times (\text{GL}_1)^{k-1}, \mathbf{S} = \text{GL}_n,$

$$H = \left\{ \begin{pmatrix} g & 0 & * & \dots & * \\ 0 & 1 & \underline{*} & \dots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \underline{*} \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} \right\}, \quad \psi = \text{sum of underlined entries}$$

- Bessel models: analogues for $O(p, q), U(p, q), O_n(\mathbb{C})$.

Corollary (holds for both kinds of models: Rankin-Selberg and Bessel)

If $\text{Hom}_H(\pi|_H, \tau \otimes \chi) \neq 0$ then $|\lambda(\pi)_i^t - \lambda(\tau)_i^t| \leq 1$ for any index i .

Related to the non-tempered Gan-Gross-Prasad conjecture.

Proof ingredients

Theorem (Wen-Wei Li 2019)

For any nilpotent orbit $\mathcal{O}$, $2 \dim \mathcal{O} \cap \mathfrak{h}^\perp \leq \dim \mathcal{O}$.

- $\mathcal{U}_n(\mathfrak{g})$ - PBW filtration on universal enveloping algebra.
- $\text{gr} \mathcal{U}(\mathfrak{g}) \cong S(\mathfrak{g}) \cong \text{Pol}(\mathfrak{g}^*)$.
- For a $\mathfrak{g}$ -module M , $\text{Ann}(M) \subset \mathcal{U}(\mathfrak{g})$ - annihilator, and $\text{An}\mathcal{V}(M) \subset \mathfrak{g}^*$ - annihilator variety := zero set of symbols of $\text{Ann}(M)$

Theorem (Rossmann 1995)

For all $\pi \in \text{Irr}(G)$, $\text{An}\mathcal{V}(\pi) = \overline{\mathcal{O}(\pi)}$.

For finitely-generated M , define $\text{As}\mathcal{V}(M)$ to be the set of common zeros of symbols of annihilators of the generators.

Theorem (Gabber-Joseph)

$2 \dim \text{As}\mathcal{V}(M) \geq \dim \text{An}\mathcal{V}(M)$.

- For any nilpotent orbit $\mathcal{O}$, $2 \dim \mathcal{O} \cap \mathfrak{h}^\perp \leq \dim \mathcal{O}$.
- $\text{An}\mathcal{V}(\pi) = \overline{\mathcal{O}(\pi)}$
- For f.g. M , $2 \dim \text{As}\mathcal{V}(M) \geq \dim \text{An}\mathcal{V}(M)$.
- If M is generated by an $\mathfrak{h}$ -finite vector then $\text{As}\mathcal{V}(M) \subset \text{An}\mathcal{V}(M) \cap \mathfrak{h}^\perp$.

Theorem (G. - Sayag 2020)

For any $\pi \in \text{Irr}(G)_H$, $\mathcal{O}(\pi) \cap \mathfrak{h}^\perp \neq \emptyset$.

Proof.

Let $\eta \neq 0 \in (\pi^*)^H$ and $M := \mathcal{U}(\mathfrak{g})\eta \subset \pi^*$. M is non-degenerately paired with π , thus $\text{Ann}(M) = \text{Ann}(\pi)$ and $\text{An}\mathcal{V}(M) = \text{An}\mathcal{V}(\pi)$. From

$$2 \dim \text{As}\mathcal{V}(M) \geq \dim \text{An}\mathcal{V}(M) = \dim \text{An}\mathcal{V}(\pi) = \dim \mathcal{O}(\pi), \quad (1)$$

$$\text{As}\mathcal{V}(M) \subset \text{An}\mathcal{V}(M) \cap \mathfrak{h}^\perp, \text{ and} \quad (2)$$

$$\text{intersection with } \mathfrak{h}^\perp \text{ reduces dimension times 2,} \quad (3)$$

we get that $\mathcal{O}(\pi)$ intersects $\mathfrak{h}^\perp$. □

Twisted version uses Kazhdan filtration + other ideas from W -algebras.

Definition of the wave front set

Assume $\text{char } F = 0$.

Theorem (Howe, Harish-Chandra, Barbasch-Vogan, 70s)

Near $e \in G$, the character distribution (asymptotically) equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

$$\exp^*(\chi_\pi) \approx \sum c_{\mathcal{O}} \mathcal{F}(\mu_{\mathcal{O}})$$

- Define $\text{WF}(\pi) = \{\mathcal{O} : c_{\mathcal{O}} \neq 0\}$.
- Say $\mathcal{O} < \mathcal{O}'$ if $\mathcal{O} \subset \overline{\mathcal{O}'}$.
- $\text{WF}^\circ(\pi) :=$ union of maximal elements of $\text{WF}(\pi)$.

Evidence for the p -adic case, for $\mathbf{G} = \mathrm{GL}_n$

- Klyachko models: Offen-Sayag
- Jacquet quotients of ladder representations: Lapid - Minguez
- Restriction from GL_{n+1} to GL_n for representations of Arthur type:
Maxim Gurevich, Kei Yuen Chan

p -adic theorem

$\forall$ nilpotent $\varphi \in \mathfrak{g}^*$, choose $\mathfrak{sl}_2$ -triple $f, h, e \in \mathfrak{g}$ using Killing form + Jacobson-Morozov; let $\mathfrak{v}_\varphi := \mathfrak{g}_{\geq 2}^h \subset \mathfrak{g}$, $V_\varphi := \text{Exp}(\mathfrak{v}_\varphi) \subset G$.

Definition

For any rep- n Π , let $\text{WO}(\Pi) := \{\varphi \in \mathfrak{g}^* : \Pi|_{V_\varphi, \varphi} \neq 0\}$.

Theorem (Mœglin-Waldspurger 1987)

If F is p -adic then $\forall \pi \in \text{Irr}(G)$, $\overline{\text{WO}(\pi)} = \overline{\text{WF}^\circ(\pi)}$.

Theorem (G.-Sayag 2020)

$G\mathfrak{h}^\perp \cap \mathcal{N} \subset \text{WO}(\mathcal{S}(G/H)) \subset \overline{G\mathfrak{h}^\perp} \cap \mathcal{N}$.

Ingredients and sketch of proof

$\forall$ nilpotent $\varphi \in \mathfrak{g}^*$, choose $\mathfrak{sl}_2$ -triple $f, h, e \in \mathfrak{g}$ using Killing form + Jacobson-Morozov; let $\mathfrak{v}_\varphi := \mathfrak{g}_{\geq 2}^h \subset \mathfrak{g}$, $V_\varphi := \text{Exp}(\mathfrak{v}_\varphi) \subset G$.

Definition

For any rep-n Π , let $\text{WO}(\Pi) := \{\varphi \in \mathfrak{g}^* : \Pi_{V_\varphi, \varphi} \neq 0\}$.

- Kazhdan action of $F^\times$ on $\mathfrak{g}^*$: $t \cdot \psi := t^2 \exp(th)\psi$. Fixes φ .
- $\mathcal{S}(G/H)_{V_\varphi, \varphi} \neq 0 \iff \varphi(\mathfrak{v}_\varphi \cap \mathfrak{g}(\mathfrak{h})) = \{0\}$ for some $g \in G$.

Theorem (G.-Sayag 2020)

$$G\mathfrak{h}^\perp \cap \mathcal{N} \subset \text{WO}(\mathcal{S}(G/H)) \subset \overline{G\mathfrak{h}^\perp} \cap \mathcal{N}.$$

Proof.

- (i) $\varphi \in \mathfrak{g}(\mathfrak{h}^\perp) \Rightarrow \varphi(\mathfrak{v}_\varphi \cap \mathfrak{g}(\mathfrak{h})) = \{0\} \Rightarrow \mathcal{S}(G/H)_{V_\varphi, \varphi} \neq 0$.
- (ii) $\mathcal{S}(G/H)_{V_\varphi, \varphi} \neq 0 \Rightarrow \varphi(\mathfrak{v}_\varphi \cap \mathfrak{g}(\mathfrak{h})) = \{0\} \Rightarrow \varphi \in (\mathfrak{g}^*)_{>-2}^h + \mathfrak{g}(\mathfrak{h}^\perp)$.
- Kazhdan action preserves φ and $G\mathfrak{h}^\perp$, contracts $(\mathfrak{g}^*)_{>-2}^h$ thus $\varphi \in \overline{G\mathfrak{h}^\perp}$ $\square$