Algebraic Structures on Automorphic L-Functions

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Introduction

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- When the L-functions are equal: get deep connections across mathematics.
    - Modularity Theorem $\implies$ Fermat's Last Theorem.
- In this talk: only automorphic L-functions.

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- Two common ways to associate L-functions to representation of $\mathrm{GL}(2)$:
  - Godement-Jacquet zeta integrals – uses pairs of vectors,
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$\Longrightarrow$ Exotic multiplicative structure on *p*-adic and automorphic representations.

- Part of PhD thesis, under supervision of J. Bernstein.

Overview

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- Category $\mathrm{Mod}(G)$ of *smooth* G-modules:
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  - Contragradient duality $V \mapsto \widetilde{V}$: smooth vectors in dual vector space.
- An irreducible G-module is generic iff it has a Kirillov model.

L-functions à la Godement-Jacquet

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  3. Integrate:

$$Z_{\mathrm{GJ}}(\Psi, \beta, s) = \int_{\mathrm{GL}_2(F)} \Psi(g)\beta(g)|\det(g)|^{s+\frac{1}{2}} \, \mathrm{d}^\times g.$$

- Get meromorphic zeta integral. Use GCD.

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$$Z_{\mathrm{JL}}(W, v, s) = \int_{F^\times} W_v(y)|y|^{s-\frac{1}{2}} \, \mathrm{d}^\times y.$$

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Claim

For generic irreducible V, the two spaces are canonically isomorphic:

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- Moreover – isomorphism respects zeta integrals.
- Remarkable – non-linear in V!

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Hidden action exists: Y is a $G \times G \times G$-module.

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- Have Weil representation $S(U \otimes U \otimes e_2) = S(\mathrm{M}_2(F))$.
- Compact induction of $S(\mathrm{M}_2(F))$ from $\mathrm{GL}_2(F)^{3,\det=1}$ to $\mathrm{GL}_2(F)^3$ gives Y.

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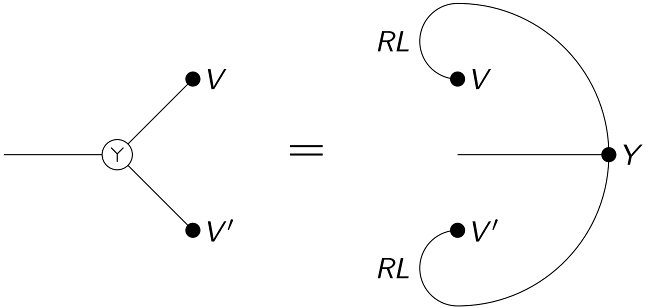
- Think of it as a bi-functor:

$$V \textcircled{\scriptsize Y} V' = V \otimes_G Y \otimes_G V'$$

- Saw stuff like this before: tensor products.



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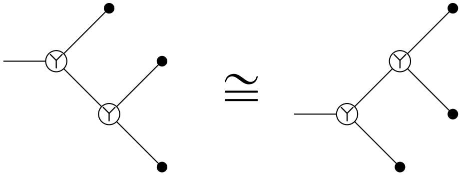

The unit

- Unit is given by Whittaker space:

$$\mathbb{1}_{\Upsilon} = \left\{ f : \mathrm{GL}_2(F) \to \mathbb{C} \;\middle|\; \begin{array}{c} f \text{ is locally const and} \\ \text{compact supp mod } U_2(F) \end{array}, \quad f\left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} g \right) = e(u) \cdot f(g) \right\}.$$

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$$\mathbb{1}_{\curlyvee} = \left\{ f : \mathrm{GL}_2(F) \to \mathbb{C} \;\middle|\; \begin{array}{c} f \text{ is locally const and} \\ \text{compact supp mod } U_2(F) \end{array}, \quad f\left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} g \right) = e(u) \cdot f(g) \right\}.$$

- Important takeaway:
  - Irreducible V has $\dim \mathrm{Hom}(\mathbb{1}_{\curlyvee}, V) = 0, 1$, exactly if V is generic.
  - Choice of map $\mathbb{1}_{\curlyvee} \to V$ is same data as Kirillov model.

Representations as algebras

- GJ vs JL is now

$$V \otimes \mathrm{Hom}(\mathbb{1}_{\curlyvee}, V) = V \circledcurlyvee \mathbb{1}_{\curlyvee} \otimes \mathrm{Hom}(\mathbb{1}_{\curlyvee}, V) \to V \circledcurlyvee V$$

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is an isomorphism.
- Turns out generic V are commutative algebras (in fact, idempotents).
- Follows because $\mathbb{1}_{\curlyvee} \twoheadrightarrow V$ is surjective.

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- New product Υ behaves like second case, despite $\mathrm{GL}_2(F)$ not being commutative.

Apology

- Global theory deserves a whole lecture on its own.
- We will give a sample instead...

Global theory

- Let F be a global function field, $\mathrm{char} F \neq 2$. Let $\mathbb{A} = \mathbb{A}_F$, $G = \mathrm{GL}_2(\mathbb{A})$.
- Category $\mathrm{Mod}(G)$ of smooth G-modules.

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  - This time, the category is huge!
  - Only care about *automorphic* representations, but it is not clear how they might fit into a category.
- Have global symmetric monoidal $\curlyvee$ induced from the local theory, given by

$$V \curlyvee V' = V \otimes_G S(\mathrm{M}_2(\mathbb{A}) \times \mathbb{A}^\times) \otimes_G V'.$$

- Unit is global Whittaker space $\mathbb{1}_\curlyvee$.

Algebra of automorphic functions

- Let
$$\mathcal{I} \subseteq S(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}))$$
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- The theta functional can be used to construct a product
$$\mathcal{J} \circledcirc \mathcal{J} \to \mathcal{J}.$$
- The product makes $\mathcal{J}$ into a commutative $\circledcirc$-algebra.

Abstractly automorphic representations

- We therefore have a category of $\mathcal{J}$-modules in the symmetric monoidal category $\mathsf{Mod}(G)$:

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- Forgetful functor

$$\mathrm{Mod}^{\mathrm{aut}}(G) \to \mathrm{Mod}(G)$$

is fully faithful.

Properties of abstract automorphicity

- Being abstractly automorphic is a *property*.
- For irreducible representations:
  - abstractly automorphic $\iff$ automorphic.
- Examples: $S(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}))$, $\mathcal{J}$.
- Closed under: limits, colimits, subquotients, contragradients, etc.
- ... And many more properties!

Questions?

Abstract

Consider the function field F of a smooth curve over $\mathbb{F}_q$, with $q \neq 2$. L-functions of automorphic representations of $\mathrm{GL}(2)$ over F are important objects for studying the arithmetic properties of the field F. Unfortunately, they can be defined in two different ways: one by Godement-Jacquet, and one by Jacquet-Langlands. Classically, one shows that the resulting L-functions coincide using a complicated computation. I will present a conceptual proof that the two families coincide, by categorifying the question. This correspondence will necessitate comparing two very different sets of data, which will have significant implications for the representation theory of $\mathrm{GL}(2)$. In particular, we will obtain an exotic symmetric monoidal structure on the category of representations of $\mathrm{GL}(2)$.
It turns out that an appropriate space of automorphic functions is a commutative algebra with respect to this symmetric monoidal structure. I will outline this construction, and show how it can be used to construct a category of automorphic representations.