LIE ALGEBRAS: HOMEWORK 12
DUE: 29 JUNE 2010

(1) Let $\mathfrak{g}$ be a Lie algebra and let δ be a nilpotent derivation of $\mathfrak{g}$, say $\delta^{m+1} = 0$. Prove that e^δ is an automorphism of $\mathfrak{g}$, where

$$e^\delta(x) := \sum_{n=0}^{m} \frac{1}{n!}\delta^n(x).$$

(Hint: First show that $\delta^n([x,y]) = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!}[\delta^i(x), \delta^{n-i}(y)]$ for $x, y \in \mathfrak{g}$.)

(2) Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{b}$ be a Borel subalgebra (maximal solvable subalgebra) of $\mathfrak{g}$. Prove that $N_{\mathfrak{g}}(\mathfrak{b}) = \mathfrak{b}$, i.e. $\mathfrak{b}$ is equal to its normalizer.

(3) Prove that for each inclusion of Dynkin diagrams (e.g. $E_6 \subset E_7 \subset E_8$) there is a natural inclusion of the corresponding semisimple Lie algebras.

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